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Research Article

# Probabilistic Evaluation of Industrial Systems Under Daily Operational Cycles with Constraints on Maximum Repair Time

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**Abstract:** This article focuses on the performance of a single-active-unit repairable industrial system with three different states of unit: active unit ($N$), partial failure unit ($P$), and complete failure unit ($F$). The system operates continuously, 24 hours a day, without interruption, except for two cases: a complete failure of the active unit or maintenance at partial failure ($P$). The system always has one service available, which is kept for maintenance and repair. In the case of complete failure, the faulty unit is replaced with a new one if the repair duration exceeds the maximum allowed time predefined. However, the nighttime replacement operations have a lot of challenges that may include limited availability of spare part suppliers or replacement must be performed at a higher cost since options are limited. In this paper, it is assumed that all the times in the system are negative exponentially distributed. The supplementary variable technique and Markov process theory have been employed to evaluate the reliability of the system. Further sensitivity and relative sensitivity analyses are performed on some system parameters in order to study the effect of these parameters on the proposed system. These results are presented with numerical examples that provide useful insights contributing to the enhancement of the systems efficiency and operational reliability.

***Keywords***: single-unit system, reliability analysis, sensitivity analysis, maximum repair times, day and night hours, supplementary variable technique

**MSC:** 90B25, 60K10, 62N05

## 1. Introduction

Reliability stands as a cornerstone of success in modern industrial systems, representing a system's capacity to perform required functions without failure or interruption for a specified duration. This reliability directly impacts production efficiency, cost reduction, and operational sustainability. In critical sectors such as power generation or pharmaceuticals, unexpected failures can trigger severe economic losses, production downtime, and even life-threatening hazards.

Consequently, reliability has become a primary focus for engineers and facility managers. Advanced strategies-including predictive and corrective maintenance-are implemented alongside mathematical modeling and data analysis


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techniques to enhance system performance. Investment in reliability improvements not only boosts operational efficiency but also strengthens companies' competitive reputation in the marketplace.

Thus, prior literature on industrial system modeling abounds with studies analyzing reliability and cost-benefit under diverse conditions and assumptions. This necessitates reviewing key contributions and clarifying how our methodology distinguishes itself from existing approaches. Numerous studies have been conducted in recent years investigating the behavior of industrial systems under varying operating conditions during their warranty period.

In 2024, an industrial model was developed [1] to evaluate system performance based on three essential indicators: availability, reliability, and Mean Time to System Failure (MTSF). Following this, a comprehensive sensitivity analysis was conducted to investigate the impact of key parameters-such as preparation time, repair time, and warranty policy-on the overall efficiency of the system. Numerical simulations were employed to illustrate the causal relationships among these parameters, thereby providing deeper insights into their influence on system behavior. In 2021 study [2] conducted a reliability and sensitivity analysis of a repairable system with warranty policies and administrative delays, showing through simulations how these factors affect system efficiency and key performance metrics. Tseni et al. [3] developed the first optimization model integrating inspection, maintenance, and warranty policies for micromachines under wear, aiming to enhance reliability and reduce total quality-related costs. Moreover, several studies have focused on analyzing the behavior of various industrial models under varying environmental or climatic conditions, highlighting the impact of such factors on system performance and reliability.

A substantial body of research has explored how environmental fluctuations affect the performance and economic aspects of industrial systems. Among the early works, Goel et al. [4] examined in 1985 a single-unit system with an operator whose condition alternated due to changing weather. They later extended their analysis to a cost evaluation of a two-unit cold standby system under different climatic scenarios [5]. Building on this, Gupta and Goel [6] analyzed the profit function of a system with two non-identical units operating in diverse environmental conditions.

In subsequent years, especially in 2012 and 2015, several studies [7–9] applied stochastic modeling to investigate repairable systems with heterogeneous components subject to climate variability. Barak et al. [10] introduced a stochastic reliability model for a single-unit system, explicitly considering weather effects during inspections. Further, Barak and Barak [11] assessed how abnormal weather impacts reliability metrics in inspectable and repairable systems. These investigations have collectively contributed to a deeper understanding of the operational and economic challenges posed by environmental variability. In 2023, Kamal et al. [12] investigated a two-unit warm standby system with dissimilar components operating under normal and abnormal weather conditions, using regenerative point techniques to assess reliability, repair workload, and cost-effectiveness. Complementing this, Zhang and Liu [13] underscored the significance of predictive maintenance and real-time monitoring-powered by machine learning-in minimizing unplanned downtime and enhancing system availability. In a related context, Shan and Wang [14] examined a K-out-of-G mixed standby system that incorporates both warm and cold standby components, along with an unreliable repair facility. Their analysis utilized Markov process modeling and Laplace transform techniques to assess key performance indicators, including system reliability, steady-state availability, and the impact of preventive maintenance and repair delays on cost-effectiveness.

Although numerous studies have addressed the reliability and performance of repairable systems under varying operational and environmental conditions, most have focused on binary-state systems (fully operational or complete failure) or standby systems with limited operational dynamics. Moreover, many practical challenges-such as maintenance delays, spare part replacement constraints, and time-limited repair protocols-have often been overlooked in favor of more idealized modeling assumptions.

The present study offers a distinctive contribution by modeling a single-active-unit industrial repairable system with a detailed classification of three operational states: normal operation, partial failure, and complete failure, providing a more accurate and realistic representation of system behavior. The model also incorporates operational constraints, including limited nighttime replacement capabilities and a predefined maximum repair time, reflecting actual industrial conditions. Utilizing the supplementary variable technique and Markov process theory, the analysis delivers precise evaluations of key performance indicators, supported by both absolute and relative sensitivity analyses to understand the impact of varying operational parameters on system performance. In doing so, this study bridges the gap between theoretical models and practical realities, enriching the literature with a robust framework for reliability analysis in real-world industrial environments.

Before proceeding to the detailed exposition of the article, Table 1 provides a rigorous and systematic analysis of the research gap through a comprehensive comparison between the present study and the prior works reviewed in the introduction, which outlined the evolutionary development of industrial systems over the past years. This table clearly highlights the novel aspects that distinguish the current study and underscores the scientific value it contributes to the field of industrial system reliability. Building on this foundation, Section 2 presents the fundamental assumptions of the system, defines its operational states, and clarifies the analytical notation adopted throughout the study. Section 2 reviews the fundamental assumptions of the system, clarifies the definitions of the various states, and introduces the notation used throughout the analysis. Building on this foundation, Section 3 develops a mathematical framework based on Markov processes to derive key reliability metrics such as the reliability function, mean time to failure, and steady-state availability. Section 4 offers an in-depth examination of the impact of changes in system parameters on performance through both absolute and relative sensitivity analyses. To demonstrate the practical application of the model, Section 5 presents numerical examples that highlight important behavioral patterns. The study concludes in Section 6 by summarizing the main findings and discussing their broader implications.

**Table 1.** Research gap identification and contribution of the present study

| Author (s)/Year | Focus of study | Identified gap/Limitation | Contribution of the present work |
|---|---|---|---|
| Hussien and El-Sherbeny [1] | Stochastic analysis of an industrial system with preparation-time repair under warranty. | Did not consider multi-state unit operation. | Models a single active-unit system with three states ($N$, $P$, $F$). |
| El-Sherbeny and Hussien [2] | Reliability and sensitivity of a repairable system with warranty and administrative delays. | Binary-state assumption; no repair time or replacement constraints. | Introduces maximum repair time policy and day/night replacement conditions. |
| Tseni et al. [3] | Optimization of warranty and inspection for micromachines. | Limited to micromachines; not scalable to industrial systems. | Extends framework to industrial single-unit systems with practical repair/replacement policies. |
| Goel et al. [4] | Weather effect on man-machine systems. | Did not integrate repair time or partial failure states. | Considers both partial and complete failures under real operational settings. |
| Goel et al. [5] | Cost analysis of two-unit cold standby under weather variation | Focused on standby systems, not single active-unit. | Provides a single-unit continuous operation model. |
| Gupta and Goel [6] | Profit analysis of cold standby systems under weather. | Concentrated on economic cost; ignored time-limited repair. | Combines reliability with cost implications of delayed nighttime replacement. |
| Malik and Deswal [7–9] | Reliability of heterogeneous units under weather. | Environmental focus; no operational constraints (repair deadlines, spare part limits). | Incorporates spare-part availability and cost-sensitive replacement (day vs night). |
| Barak et al. [10] | Reliability with inspection under weather variability. | Ignored repair time limits and spare-part supply constraints. | Adds predefined maximum repair time and supplier-based constraints. |
| Barak and Barak [11] | Impact of abnormal weather on inspectable systems. | Focus only on environmental uncertainty. | Extends to operational challenges: nighttime repairs and economic penalties. |
| Kamal et al. [12] | Profitability of two-unit warm standby system. | Multi-unit context; no single active-unit modeling. | Models realistic single active-unit with continuous operation. |
| Wang et al. [13] | Predictive maintenance for industrial robots using AI. | Data-driven approach; no analytical reliability measures. | Provides analytical evaluation using supplementary variable technique. |
| Shan and Wang [14] | K-out-of-G mixed standby with unreliable repair. | Standby-focused; does not reflect continuous industrial unit. | Bridges theory and practice by modeling continuous, constrained, single-unit operation. |
| Present study | Reliability and sensitivity of single active-unit system with three states ($N$, $P$, $F$). | | Bridges the gap by incorporating: (i) multi-state operation, (ii) maximum repair time, (iii) day/night replacement constraints, (iv) both sensitivity & relative sensitivity analysis for real-world insights. |

## 2. Description of the system

The proposed system consists of a single active unit operating continuously 24 hours a day without interruption. It only stops under two conditions: when a complete failure of the unit occurs or when maintenance is needed for a unit that has undergone a decline in operational efficiency (partial failure). A single server is assigned to perform maintenance and repair tasks. If the repairman is unable to complete the repair of the completely failed unit within the predefined maximum repair time, the failed unit must be replaced with a new active unit. These procedures are carried out during both daytime or nighttime hours without distinction. In the light of this explanation, the proposed system can be described under the following assumptions:

1. The system operates with a single active unit at any given time.

2. The "first system state" refers to the unit's operation during morning working hours, ensuring continuous activity throughout this period.

3. The unit, whether functioning during the day or night, can experience two possible failure states: partial failure or complete failure.

4. In the event of a partial failure that reduces operational efficiency, immediate preventive maintenance must be conducted without delay, regardless of whether it occurs during daytime or nighttime operation.

5. If the active unit undergoes a complete failure, causing a total system shutdown, immediate repair procedures must be initiated to restore functionality.

6. If the repair technician exceeds the allocated maximum repair time, the faulty unit must be replaced immediately to ensure system efficiency and continuous operation.

7. Unit replacement occurs during one of two time periods: morning or evening work hours. The morning period is preferred due to better supplier availability and greater price flexibility. In contrast, the evening period presents two choices: either postponing replacement until morning to benefit from more favorable market conditions or proceeding with immediate replacement to minimize downtime, albeit at a higher cost due to limited supplier availability and increased procurement expenses.

8. All time intervals in the system follow a negative exponential distribution.

9. After each repair or maintenance, the unit becomes like a new one.

*State Specification:*

To examine how the system behaves at any specific virtual time $t$, the states of the system can be outlined as follows:

$S_0/S_1$: The unit is active during/morning working hours/night working hours.

$S_2/S_3$: The partial failure unit is stopped from operation and is undergoing preventive maintenance during/morning working hours/night working hours.

$S_4/S_5$: The failed unit under repair during/morning working hours/night working hours.

$S_6/S_7$: The failed unit replaced during/morning working hours/night working hours.

$S_8$: The failed unit witting replaced during night working hours.

*Notations:*

$\alpha_1/\alpha_2$: A constant rate of change from the morning work period to the evening work period/from the evening work period to the morning work period.

$\theta$: The repair rate of the failed unit during the morning work period.

$\theta_1$: The maintenance rate of the partially failed unit during the morning work period.

$\theta_2$: The maintenance rate of the partially failed unit during the evening work period.

$\theta_3$: The replacement rate of the failed unit during the evening work period.

$\theta_4$: The repair rate of the failed unit during the evening work period.

$\theta_5$: The replacement rate of the completely failed unit during the morning work period.

$\eta/\eta_1$: The partially failed rate during the morning work period/the evening work period.

$\beta/\beta_1$: The maximum repair rate after complete failure during the evening work period/the morning work period.

$\lambda_1/\lambda_2$: The failure rate from the normal state to failure during the morning work period/the evening work period.

$\kappa$: The waiting replacement rate of the failed unit during the evening work period.

$\pi_0(t)/\pi_1(t)$: The probability that at time $t$ the system is in good state during the morning work period/the evening work period.

$\pi_2(t)/\pi_3(t)$: The probability that the system at time $t$ is in a state of partial failure and is undergoing preventive maintenance during the morning work period/the evening work period.

$\pi_4(t)/\pi_5(t)$: The probability that at time $t$ the system is in a failed state and getting the repairman available in the morning work period/the evening work period.

$\pi_6(t)/\pi_8(t)$: The probability that at time $t$ the system is in a replacement of the failed unit in the morning work period/the evening work period.

$\pi_7(t)$: The probability that at time $t$ the system is in a waiting replacement of the failed unit in the evening work period.

Using these symbols, the possible states of the system being analyzed are as follows:

$O_M/O_N$: The unit is in normal operation mode and operative during/morning working hours/night working hours.

$PF_{pm,\, M}/PF_{pm,\, N}$: The unit is in partial failure mode and undergoing preventive maintenance during/morning working hours/night working hours.

$F_{ur,\, M}/F_{ur,\, N}$: The unit is in failure mode and undergoing repairs during/morning working hours/night working hours.

$F_{wrp,\, N}$: The unit is in failure mode and awaiting replacement during night working hours.

$F_{urp,\, M}/PF_{urp,\, N}$: The unit is in failure mode and undergoing replaced during/morning working hours/night working hours.

Possible states of the system are as follows:

Up states: $S_0 = (O_M)$, $S_1 = (O_N)$.

Down state: $S_2 = (PF_{pm,\, M})$, $S_3 = (PF_{pm,\, N})$, $S_4 = (F_{ur,\, M})$, $S_5 = (F_{ur,\, N})$, $S_6 = (F_{urp,\, M})$, $S_8 = (F_{wrp,\, N})$, $S_7 = (F_{urp,\, N})$.

Possible states and transitions are shown in Figure 1.

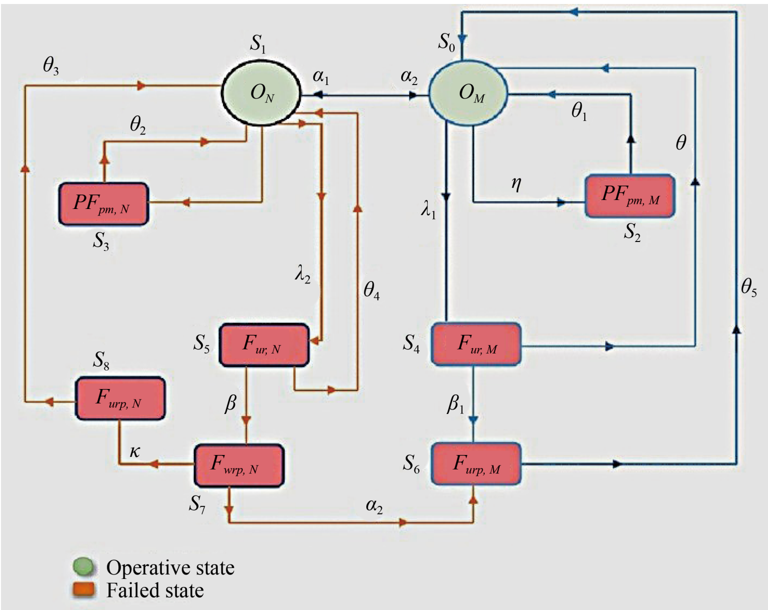


**Figure 1.** State transition diagram

## 3. Analysis of the system

### 3.1 *Reliability*

The system reliability $R(s)$ represents the probability that the system will function properly over the time interval $(0,\ t)$. To evaluate system performance measures, the Markov process is utilized to model the operational states, namely $S_0$, and $S_1$. The reliability function is derived based on the proposed equations.

$$\left\{\frac{d}{dt}+\alpha_1+\lambda_1+\eta\right\}\pi_0(t)=\alpha_2\pi_1(t), \tag{1}$$

$$\left\{\frac{d}{dt}+\alpha_2+\lambda_2+\eta_1\right\}\pi_1(t)=\alpha_1\pi_0(t). \tag{2}$$

The initial conditions are as follows:

$$\pi_i(0)=\begin{cases}1 & i\neq 0\\ 0 & i=0.\end{cases} \tag{3}$$

We apply the Laplace transform to Equations (1)-(3), incorporating the initial condition, and obtain the following result:

$$\{s+\alpha_1+\lambda_1+\eta\}\pi_0(s)=1+\alpha_2\pi_1(s), \tag{4}$$

$$\{s+\alpha_2+\lambda_2+\eta_1\}\pi_1(s)=\alpha_1\pi_0(s), \tag{5}$$

From Equations (4) and (5), we obtain

$$\pi_0(s)=\frac{\{s+\alpha_2+\lambda_2+\eta_1\}}{\{s+\alpha_1+\lambda_1+\eta\}\{s+\alpha_2+\lambda_2+\eta_1\}-\alpha_1\alpha_2}, \tag{6}$$

$$\pi_1(s)=\frac{\alpha_1}{\{s+\alpha_1+\lambda_1+\eta\}\{s+\alpha_2+\lambda_2+\eta_1\}-\alpha_1\alpha_2}. \tag{7}$$

From Equations (6) and (7), we derive the Laplace transform formula for the system's reliability as follows:

$$R(s)=\pi_0(s)+\pi_1(s), \tag{8}$$

wherefore,

$$R(s)=\frac{\{s+\alpha_2+\lambda_2+\eta_1\}+\alpha_1}{\{s+\alpha_1+\lambda_1+\eta\}\{s+\alpha_2+\lambda_2+\eta_1\}-\alpha_1\alpha_2}. \tag{9}$$

If we use the inverse Laplace transform for Equation (9), we obtain the reliability function as follows:

$$R(t) = \frac{\left((\eta - \alpha_1 - \alpha_2 - \eta_1 + \lambda_1 - \lambda_2)\left(e^{\left(H_1 - \frac{1}{2}H\right)t} - e^{\left(H_1 + \frac{1}{2}H\right)t}\right) + H\left(e^{\left(H_1 - \frac{1}{2}H\right)t} + e^{\left(H_1 + \frac{1}{2}H\right)t}\right)\right)}{2H}, \tag{10}$$

where,

$$H = \sqrt{(\eta + \eta_1 + \alpha_1 + \alpha_2 + \lambda_1 + \lambda_2)^2 - 4\left(\alpha_2(\eta + \lambda_1) + (\eta_1 + \lambda_2)(\eta + \alpha_1 + \lambda_1)\right)}$$

$$H_1 = \left(\frac{-\eta}{2} - \frac{\eta_1}{2} - \frac{\alpha_1}{2} - \frac{\alpha_2}{2} - \frac{\lambda_1}{2} - \frac{\lambda_2}{2}\right).$$

## 3.2 *MTSF*

The MTSF represents the expected duration a system operates before it fails. This metric is one of the fundamental measures in reliability analysis. The MTSF can be determined using Equation (10) as shown below.

$$\text{MTSF} = \int_0^\infty R(t)dt. \tag{11}$$

Based on equation (11), the MTSF can be expressed in the following form.

$$\text{MTSF} = \frac{(\alpha_1 + \alpha_2 + \eta_1 + \lambda_2)}{\alpha_2(\eta + \lambda_1) + (\eta + \alpha_1 + \lambda_1)(\eta_1 + \lambda_2)}. \tag{12}$$

## 3.3 *System availability*

Availability refers to the probability that a system or component is operational and functioning correctly when needed, considering both operational time (uptime) and downtime due to failures or maintenance. we can represent the system's states using differential equations as follows:

$$\left\{\frac{d}{dt} + \alpha_1 + \lambda_1 + \eta\right\}\pi_0(t) = \alpha_2\pi_1(t) + \theta_1\pi_2(t) + \theta\pi_4(t) + \theta_5\pi_6(t) \tag{13}$$

$$\left\{\frac{d}{dt} + \alpha_2 + \lambda_2 + \eta_1\right\}\pi_1(t) = \alpha_1\pi_0(t) + \theta_2\pi_3(t) + \theta_4\pi_5(t) + \theta_3\pi_8(t) \tag{14}$$

$$\left\{\frac{d}{dt} + \theta_1\right\}\pi_2(t) = \eta\pi_0(t) \tag{15}$$

$$\left\{\frac{d}{dt} + \theta_2\right\}\pi_3(t) = \eta_1\pi_1(t) \tag{16}$$

$$\left\{\frac{d}{dt}+\theta+\beta_1\right\}\pi_4(t)=\lambda_1\pi_0(t) \tag{17}$$

$$\left\{\frac{d}{dt}+\theta_4+\beta\right\}\pi_5(t)=\lambda_2\pi_1(t) \tag{18}$$

$$\left\{\frac{d}{dt}+\theta_5\right\}\pi_6(t)=\beta_1\pi_4(t)+\alpha_2\pi_7(t) \tag{19}$$

$$\left\{\frac{d}{dt}+\kappa+\alpha_2\right\}\pi_7(t)=\beta\pi_5(t) \tag{20}$$

$$\left\{\frac{d}{dt}+\theta_3\right\}\pi_8(t)=\kappa\pi_7(t). \tag{21}$$

The initial conditions are

$$\pi_i(0)=\begin{cases}1 & i\neq 0\\ 0 & i=0.\end{cases} \tag{22}$$

By applying the Laplace transformation to all the preceding equations, we derive the following expressions:

$$\{s+\alpha_1+\lambda_1+\eta\}\,\pi_0(s)=1+\alpha_2\pi_1(s)+\theta_1\pi_2(s)+\theta\pi_4(s)+\theta_5\pi_6(s), \tag{23}$$

$$\{s+\alpha_2+\lambda_2+\eta_1\}\,\pi_1(s)=\alpha_1\pi_0(s)+\theta_2\pi_3(s)+\theta_4\pi_5(s)+\theta_3\pi_8(s), \tag{24}$$

$$\{s+\theta_1\}\,\pi_2(s)=\eta\pi_0(s) \tag{25}$$

$$\{s+\theta_2\}\,\pi_3(s)=\eta_1\pi_1(s) \tag{26}$$

$$\{s+\theta+\beta_1\}\,\pi_4(s)=\lambda_1\pi_0(s) \tag{27}$$

$$\{s+\theta_4+\beta\}\,\pi_5(s)=\lambda_2\pi_1(s) \tag{28}$$

$$\{s+\theta_5\}\,\pi_6(s)=\beta_1\pi_4(s)+\alpha_2\pi_7(s) \tag{29}$$

$$\{s+\kappa+\alpha_2\}\,\pi_7(s)=\beta\pi_5(s) \tag{30}$$

$$\{s+\theta_3\}\,\pi_8(s) = \kappa\pi_7(s). \tag{31}$$

We also note the following:

$$\sum_{i=0}^{8} \pi_i(s) = \frac{1}{s}. \tag{32}$$

From Equations (23)-(32) we obtain

$$\pi_0(s) = \frac{B(s)}{1-(B_1(s)+B_2(s)+B_3(s)+B_4(s)+B_5(s))}, \tag{33}$$

$$\pi_1(s) = \frac{C(s)B(s)}{1-(B_1(s)+B_2(s)+B_3(s)+B_4(s)+B_5(s))}, \tag{34}$$

$$\pi_2(s) = \left(\frac{\eta}{s+\theta_1}\right)\pi_0(s), \tag{35}$$

$$\pi_3(s) = \left(\frac{\eta_1}{s+\theta_2}\right)\pi_1(s), \tag{36}$$

$$\pi_4(s) = \left(\frac{\lambda_1}{s+\theta+\beta_1}\right)\pi_0(s), \tag{37}$$

$$\pi_5(s) = \left(\frac{\lambda_2}{s+\theta_4+\beta}\right)\pi_1(s), \tag{38}$$

$$\pi_6(s) = \left(\frac{\beta_1\lambda_1\pi_0(s)}{(s+\theta+\beta_1)(s+\theta_5)} + \frac{\alpha_2\lambda_2\beta\pi_1(s)}{(s+\theta_5)(s+\theta_4+\beta)(s+\kappa+\alpha_2)}\right), \tag{39}$$

$$\pi_7(s) = \frac{\lambda_2\beta\pi_1(s)}{(s+\theta_4+\beta)(s+\kappa+\alpha_2)}, \tag{40}$$

$$\pi_8(s) = \frac{\lambda_2\beta\kappa\pi_1(s)}{(s+\theta_4+\beta)(s+\kappa+\alpha_2)(s+\theta_3)}, \tag{41}$$

where,

$$B(s) = \frac{1}{(s+\alpha_1+\lambda_1+\eta)},\ B_1(s) = \frac{\alpha_2 C(s)}{(s+\alpha_1+\lambda_1+\eta)},\ B_2(s) = \frac{\theta_1\eta}{(s+\alpha_1+\lambda_1+\eta)(s+\theta_1)},$$

$$B_3(s) = \frac{\theta\lambda_1}{(s+\alpha_1+\lambda_1+\eta)(s+\theta+\beta_1)},\ B_4(s) = \frac{\theta_5\beta_1\lambda_1}{(s+\theta_5)(s+\theta+\beta_1)(s+\alpha_1+\lambda_1+\eta)},$$

$$B_5(s) = \frac{(\theta_5\lambda_2\alpha_2\beta)C(s)}{(s+\theta_5)(s+\theta_4+\beta)(s+\kappa+\alpha_2)(s+\alpha_1+\lambda_1+\eta)},\ C(s) = \frac{A(s)}{1-(A_1(s)+A_2(s)+A_3(s))},$$

$$A(s) = \frac{\alpha_1}{(s+\alpha_2+\lambda_2+\eta_1)},\ A_1(s) = \frac{\theta_2\eta_1}{(s+\alpha_2+\lambda_2+\eta_1)(s+\theta_2)},$$

$$A_2(s) = \frac{\theta_4\lambda_2}{(s+\alpha_2+\lambda_2+\eta_1)(s+\theta_4+\beta)},\ A_3(s) = \frac{\theta_3\lambda_2\kappa\beta}{(s+\alpha_2+\lambda_2+\eta_1)(s+\theta_4+\beta)(s+\kappa+\alpha_2)(s+\theta_3)}.$$

Based on Equations (21)-(29), we derive the Laplace transform expression for the availability $Av(s)$ as follows:

$$Av(s) = \pi_0(s) + \pi_1(s). \tag{42}$$

Consequently, the steady-state availability $Av(\infty)$ of the system is given by:

$$Av(\infty) = \lim_{s\to 0}\{sAv(s)\} \tag{43}$$

$$Av(\infty) = \frac{H}{H_1}, \tag{44}$$

where,

$$H = \theta_1\theta_2\theta_3\theta_5(\theta+\beta_1)((\kappa+\alpha_2)(\alpha_1+\alpha_2)(\beta+\theta_4)+\beta\alpha_2\lambda_2),$$

$$H_1 = \alpha_2\theta_2\theta_3(\theta_5(\theta+\beta_1)(\eta+\theta_1)+\theta_1\lambda_1(\beta_1+\theta_5))((\kappa+\alpha_2)(\beta+\theta_4)+\beta\lambda_2)+\theta_1\alpha_1(\beta_1+\theta)$$

$$(\theta_5\theta_3(\kappa+\alpha_2)(\eta_1+\theta_2)(\beta+\theta_4)+\theta_2\lambda_2(\beta\alpha_2\theta_3+\theta_5(\beta\kappa+\theta_3(\beta+\kappa+\alpha_2)))).$$

## 4. Sensitivity and relative sensitivity analysis

In this section, we explore the concept of sensitivity analysis and relative sensitivity, highlighting their crucial role in evaluating system impact. Accordingly, it is essential first to define sensitivity analysis for the proposed system, followed by clarifying its importance in understanding system behavior and enhancing its performance.

Sensitivity analysis is a technique used to evaluate the impact of changes in system parameters on its reliability. In other words, this analysis identifies how sensitive system performance is to variations in its inputs, such as failure rates, repair times, or other factors affecting reliability system.

Impotence of sensitivity analysis in reliability:

1. Identifying key influencing factors: Helps recognize which parameters have the most significant impact on reliability, allowing for targeted improvements.

2. Enhancing design and maintenance: Understanding these effects enables better engineering designs or maintenance strategies to extend system lifespan.

3. Risk management: Reduces unexpected failures by predicting the impact of specific changes.

### 4.1 *Sensitivity and relative sensitivity analysis for reliability function*

By differentiating Equation (10) with respect to parameter $\tau$, we first obtain the sensitivity analysis, as shown in the following equation.

$$\varphi_\tau = \frac{\partial R(t)}{\partial \tau}, \tag{45}$$

where, $\tau = \eta,\ \eta_1,\ \alpha_1,\ \alpha_2,\ \lambda_1,\ \lambda_2$.

Next, by performing sensitivity analysis calculations in Equation (45), we can also determine the relative sensitivity, as illustrated below.

$$\psi_\tau = \frac{\varphi_\tau . \tau}{R(t)}. \tag{46}$$

### 4.2 *Sensitivity and relative sensitivity analysis for MTSF*

We can also determine the sensitivity analysis of MTSF based on the parameters that affect it.

By differentiating Equation (12) with respect to parameter $\tau$, we obtain

$$\sigma_\tau = \frac{\partial MTSF}{\partial \tau} = \int_0^\infty \frac{\partial R(t)}{\partial \tau} dt. \tag{47}$$

The relative sensitivity of the Mean Time to System Failure can be represented as follows:

$$\omega_\tau = \frac{\sigma_\tau . \tau}{\text{MTSF}}. \tag{48}$$

## 5. Computational findings

This section presents a precise analytical framework for the numerical evaluation of three key system reliability metrics: the reliability function $R(t)$, the mean time to first failure MTSF, and steady-state availability $Av(\infty)$. The evaluation is based on a systematic analysis of system parameter variations within defined operational limits, while keeping another parameters constant. Through well-structured computational experiments, the impact of each parameter is assessed in terms of reliability degradation patterns, fault tolerance, and long-term operational readiness. Subsequently, sensitivity and relative sensitivity analyses are performed to evaluate the impact of parameter variations on system reliability $R(t)$ and the Mean Time to System Failure MTSF, contributing to enhanced system performance under dynamic operational constraints and ensuring an optimal balance between cost-efficiency and functional longevity.

## 5.1 *Computational analysis of R(t), MTSF, and Av(∞)*

This section provides a detailed numerical analysis examining the impact of variations in system parameters on key performance indicators, including the reliability function $R(t)$, MTSF, and steady-state availability $Av(\infty)$. The analysis aims to deliver a comprehensive assessment of how parameter fluctuations affect the overall behavior and effectiveness of the system across diverse operational scenarios

The analysis begins with a comprehensive evaluation of the system reliability function $R(t)$, followed by a systematic study of the impact of variations in system parameters on key performance indicators. In the reference scenario, the parameters were held constant at $\lambda_2 = 0.3,\ \eta = 0.6,\ \eta_1 = 0.4,\ \alpha_1 = 0.4$, and $\alpha_2 = 0.3$, while the parameter $\lambda_1$ was varied across the values 0.1, 0.4, 0.6, and 0.9. The temporal variable $t$ was examined over the range from 1 to 10 units with a step size of one unit, enabling a detailed investigation of the dynamic behavior of the system under diverse operational conditions. This systematic approach ensures clarity of results and facilitates reproducibility in future numerical studies.

Table 2 presents the numerical values of the reliability function $R(t)$ over the time horizon $0 \le t \le 10$, with the model parameters fixed at $\lambda_2 = 0.3,\ \eta = 0.6,\ \eta_1 = 0.4,\ \alpha_1 = 0.4$ and $\alpha_2 = 0.3$ while the failure-rate parameter $\lambda_1$ is varied at three levels ($\lambda_1 = 0.1,\ 0.4,\ 0.6$). All cases start with $R(0) = 1.0000$, which is consistent with the mathematical definition of the reliability function. At time $t = 1$, the reliability values are $R(t) = 0.496585$ for $\lambda_1 = 0.1$, $R(t) = 0.387471$ for $\lambda_1 = 0.4$, and $R(t) = 0.329869$ for $\lambda_1 = 0.6$. The transition from $\lambda_1 = 0.1$ to $\lambda_1 = 0.6$ thus corresponds to a relative reduction of approximately 33.6% in reliability. At $t = 5$, the values decrease further to $R(t) = 0.0301974$ for $\lambda_1 = 0.1$, $R(t) = 0.0138206$ for $\lambda_1 = 0.4$, and $R(t) = 0.0119421$ for $\lambda_1 = 0.6$, indicating that the system at $\lambda_1 = 0.1$ maintains more than twice the reliability of the system at $\lambda_1 = 0.6$. By $t = 10$, the reliability values approach zero across all cases, yet they remain distinct: 0.000911882 for $\lambda_1 = 0.1$, 0.000251625 for $\lambda_1 = 0.4$, and 0.000134612 for $\lambda_1 = 0.6$. These findings confirm that the failure-rate parameter $\lambda_1$ is the dominant driver of reliability degradation, and that reducing its value represents an effective means of extending system lifetime and sustaining long-term performance.

**Table 2.** Variation of reliability values $R(t)$ for different values of "$t$" at $\eta = 0.6;\ \eta_1 = 0.4;\ \alpha_1 = 0.4;\ \alpha_2 = 0.3;\ \lambda_2 = 0.3$

| $t$ | $R(t)\ (\lambda_1 = 0.1)$ | $R(t)\ (\lambda_1 = 0.4)$ | $R(t)\ (\lambda_1 = 0.6)$ |
|---|---|---|---|
| 0 | 1.0000 | 1.0000 | 1.0000 |
| 1 | 0.496585 | 0.387471 | 0.329869 |
| 2 | 0.246597 | 0.161613 | 0.124738 |
| 3 | 0.122456 | 0.0700959 | 0.0507542 |
| 4 | 0.0608101 | 0.030987 | 0.0213555 |
| 5 | 0.0301974 | 0.0138206 | 0.00911415 |
| 6 | 0.0149956 | 0.00618924 | 0.00391245 |
| 7 | 0.00744658 | 0.00277682 | 0.00168346 |
| 8 | 0.00369786 | 0.00124686 | 0.000725044 |
| 9 | 0.0018363 | 0.000560079 | 0.000312386 |
| 10 | 0.000911882 | 0.000251625 | 0.000134612 |

Table 3 presents the variation of the reliability function $R(t)$ for different values of the parameter $\lambda_2 = (0.2,\ 0.4,\ 0.6)$, across a time horizon $t = 0$ to $t = 10$. The analysis is conducted under the fixed conditions $\eta = 0.6,\ \eta_1 = 0.4,\ \alpha_1 = 0.4,\ \alpha_2 = 0.3$, and $\lambda_1 = 0.4$. At $t = 0$, the system reliability starts at its maximum value, $R(0) = 1.000$, for all considered values of $\lambda_2$ , as expected from the definition of reliability functions. As time increases, $R(t)$ decreases monotonically, reflecting the natural degradation in system performance with elapsed time. For instance, at $t = 1$, the reliability values are $R(1) = 0.394925$ for $\lambda_2 = 0.2$, $R(1) = 0.380501$ for $\lambda_2 = 0.4$, and $R(1) = 0.367879$ for $\lambda_2 = 0.6$. This trend clearly shows that higher values of $\lambda_2$ accelerate the decay rate of reliability, leading to lower survival probabilities. The same behavior persists across the time interval. For example, at $t = 5$, the reliability reduces to $R(5) = 0.0183242$ ($\lambda_2 = 0.2$), $R(5) = 0.010643$ ($\lambda_2 = 0.4$), and $R(5) = 0.00673795$ ($\lambda_2 = 0.6$). By the end of the observation period ($t = 10$), the values become extremely small: $R(10) = 0.000490485$, $R(10) = 0.00013504$, and $R(10) = 0.0000453999$, respectively. Overall, the results emphasize that reliability $R(t)$ decreases more rapidly as $\lambda_2$ increases, highlighting the significant influence of this parameter on system degradation dynamics. This sensitivity suggests that $\lambda_2$ plays a critical role in shaping long-term system performance, and its proper estimation is vital for predictive reliability modeling.

**Table 3.** Variation of reliability values $R(t)$ for different values of "$t$" at $\eta = 0.6;\ \eta_1 = 0.4;\ \alpha_1 = 0.4;\ \alpha_2 = 0.3;\ \lambda_1 = 0.4$

| $t$ | $R(t)$ ($\lambda_2 = 0.2$) | $R(t)$ ($\lambda_2 = 0.4$) | $R(t)$ ($\lambda_2 = 0.6$) |
|---|---|---|---|
| 0 | 1.0000 | 1.0000 | 1.0000 |
| 1 | 0.394925 | 0.380501 | 0.367879 |
| 2 | 0.173007 | 0.151666 | 0.135335 |
| 3 | 0.0801166 | 0.0619385 | 0.0497871 |
| 4 | 0.0380907 | 0.0256008 | 0.0183156 |
| 5 | 0.0183242 | 0.010643 | 0.00673795 |
| 6 | 0.00886041 | 0.00443678 | 0.00247875 |
| 7 | 0.00429374 | 0.00185199 | 0.000911882 |
| 8 | 0.00208269 | 0.000773529 | 0.000335463 |
| 9 | 0.00101062 | 0.000323176 | 0.00012341 |
| 10 | 0.000490485 | 0.00013504 | 0.0000453999 |

**Table 4.** Variation of reliability values $R(t)$ for different values of "$t$" at $\eta_1 = 0.4;\ \alpha_1 = 0.4;\ \alpha_2 = 0.3;\ \lambda_1 = 0.4;\ \lambda_2 = 0.3$

| $t$ | $R(t)$ ($\eta = 0.1$) | $R(t)$ ($\eta = 0.5$) | $R(t)$ ($\eta = 0.8$) |
|---|---|---|---|
| 0 | 1.0000 | 1.0000 | 1.0000 |
| 1 | 0.58828 | 0.420519 | 0.329869 |
| 2 | 0.333611 | 0.185245 | 0.124738 |
| 3 | 0.186021 | 0.0836083 | 0.0507542 |
| 4 | 0.10289 | 0.0381919 | 0.0213555 |
| 5 | 0.0566847 | 0.0175473 | 0.00911415 |
| 6 | 0.0311686 | 0.00808434 | 0.00391245 |
| 7 | 0.0171218 | 0.00372945 | 0.00168346 |
| 8 | 0.00940108 | 0.00172152 | 0.000725044 |
| 9 | 0.00516062 | 0.000794886 | 0.000312386 |
| 10 | 0.00283254 | 0.000367076 | 0.000134612 |

The analysis of Tables 4 and 5 provides a comprehensive understanding of the combined effect of parameters $\eta$ and $\eta_1$ on the system's reliability function $R(t)$. Both tables confirm the general property that reliability starts at $R(0) = 1.000$ for all parameter settings and decreases monotonically over time. Distinct patterns emerge when the results are compared. In Table 4, with $\eta_1 = 0.4$, reliability decays more rapidly as $\eta$ increases. For instance, at $t = 5$, the reliability values are $R(5) = 0.0566847$ ($\eta = 0.1$), $R(5) = 0.0175473$ ($\eta = 0.5$), and $R(5) = 0.00911415$ ($\eta = 0.8$). In Table 5, where $\eta = 0.6$, the corresponding values are lower in magnitude but the early-stage decay is slower: $R(5) = 0.0341047$, 0.010643, and 0.00554037, respectively. This indicates that increasing $\eta$ slightly improves short-term survival, while the long-term downward trend remains. At later times ($t = 10$), both tables show very low reliability values, such as $R(10) = 0.00283254$ ($\eta_1 = 0.4;\ \eta = 0.1$) and $R(10) = 0.00208712$ ($\eta_1 = 0.1;\ \eta = 0.6$), highlighting that long-term system survival is limited regardless of the chosen parameter values. Overall, two key insights are drawn from the analysis:

Effect of $\eta$: Higher $\eta$ consistently accelerates the decay of reliability, reducing survival probability across all time intervals.

Effect of $\eta_1$: Increasing $\eta_1$ slows down early-stage reliability decline, enhancing short-term robustness without altering the long-term trend.

**Table 5.** Variation of reliability values $R(t)$ for different values of "$t$" at $\eta = 0.6$; $\alpha_1 = 0.4$; $\alpha_2 = 0.3$; $\lambda_1 = 0.4$; $\lambda_2 = 0.3$

| $t$ | $R(t)$ ($\eta_1 = 0.1$) | $R(t)$ ($\eta_1 = 0.5$) | $R(t)$ ($\eta_1 = 0.8$) |
|---|---|---|---|
| 0 | 1.0000 | 1.0000 | 1.0000 |
| 1 | 0.411445 | 0.380501 | 0.362165 |
| 2 | 0.20113 | 0.151666 | 0.128632 |
| 3 | 0.107798 | 0.0619385 | 0.0452619 |
| 4 | 0.0601493 | 0.0256008 | 0.0158536 |
| 5 | 0.0341047 | 0.010643 | 0.00554037 |
| 6 | 0.0194565 | 0.00443678 | 0.00193401 |
| 7 | 0.0111255 | 0.00185199 | 0.000674738 |
| 8 | 0.00636723 | 0.000773529 | 0.000235336 |
| 9 | 0.00364521 | 0.000323176 | 0.0000820695 |
| 10 | 0.00208712 | 0.00013504 | 0.0000286183 |

Figure 2 illustrates the relationship between the MTSF, the failure rate ($\lambda_1$), and the transition rate from the morning to the evening work period ($\alpha_1$). The analysis reveals a strong inverse correlation, demonstrating a predictable, sharp decline in MTSF as the failure rate ($\lambda_1$) increases. However, the more significant finding is the critical role of the transition rate ($\alpha_1$). For any fixed value of $\lambda_1$, a higher $\alpha_1$ value results in a marked improvement in system reliability, evidenced by a higher MTSF curve. This behavior can be interpreted through the premise that a faster transition out of the morning work period reduces the system's exposure time to a high-stress operational state, thereby granting it greater resilience and extending its overall operational lifespan. Consequently, system reliability is not merely a function of its intrinsic component failure rates but is fundamentally influenced by its temporal patterns and operational dynamics. This insight provides a crucial strategic framework for system designers, suggesting that optimizing operational management policies and scheduling transitions between operational modes can be an effective method for enhancing system durability and maximizing long-term performance.

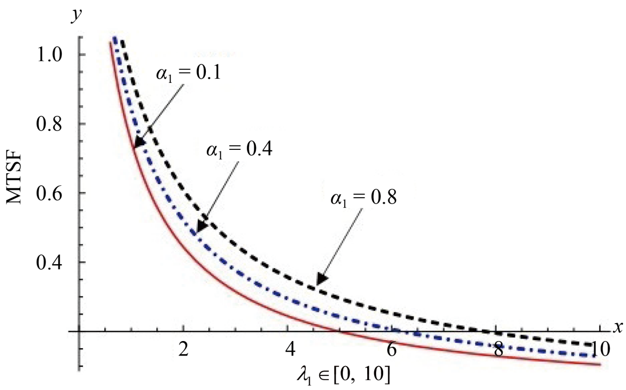


**Figure 2.** MTSF versus failure rate $\lambda_1$ and rate of change from the morning work period to the evening work period $\alpha_1$

Figure 3 illustrates the effect of the failure rate ($\lambda_2$) and the transition rate from the evening to the morning work period ($\alpha_2$) on the MTSF. The results show a clear inverse relationship between $\lambda_2$ and MTSF, confirming that higher failure rates shorten system lifetime. At the same time, increasing $\alpha_2$ improves reliability by raising MTSF values for any fixed $\lambda_2$. This improvement stems from reducing exposure to high-stress conditions through faster transitions, which enhances system resilience and prolongs operational lifespan.

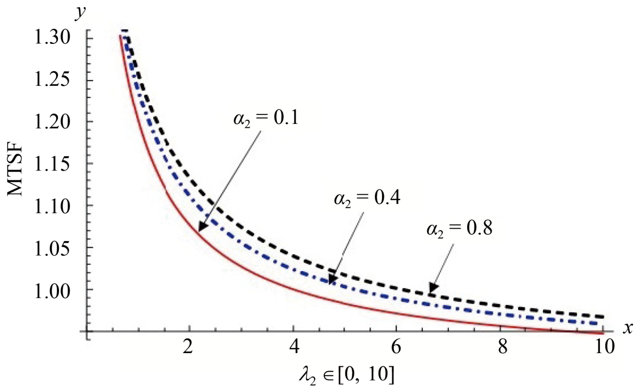


**Figure 3.** MTSF versus failure rate $\lambda_2$ and rate of change from the evening work period to the morning work period $\alpha_2$

Figure 4 demonstrates the critical relationship between steady-state availability and both failure rate ($\lambda_1$) and maximum repair rate ($\beta_1$). While increasing $\lambda_1$ reduces availability due to more frequent failures, higher $\beta_1$ values significantly compensate for this effect through enhanced repair efficiency during morning operations. This highlights that optimizing repair capabilities during critical work periods is essential for maintaining system availability despite rising failure rates.

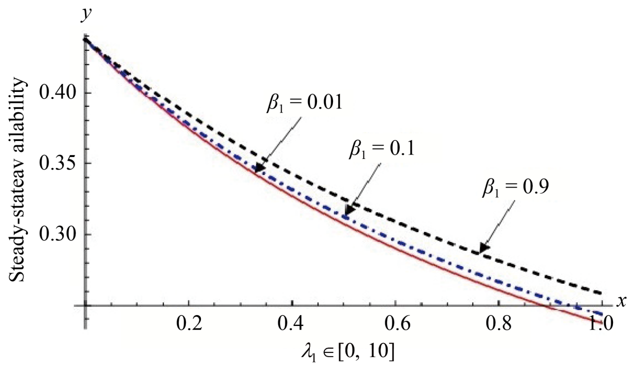


**Figure 4.** The steady-state availability versus failure rate $\lambda_1$ and rate of the maximum repair rate after complete failure during the morning work period $\beta_1$

Figure 5 reveals the essential interplay between failure rate ($\lambda_2$) and maximum repair rate ($\beta$) in determining system availability. The analysis confirms that while elevated failure rates inevitably diminish availability, this degradation can be effectively counterbalanced by strengthening repair capabilities during morning operations. Specifically, higher $\beta$ values demonstrate a compensatory effect, substantially mitigating the negative impacts of increased failure rates. These findings underscore the strategic importance of optimizing morning repair capacity as a crucial mechanism for sustaining system availability amidst growing reliability challenges.

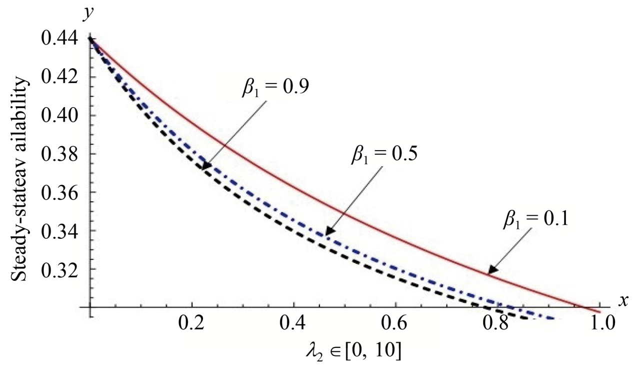


**Figure 5.** The steady-state availability versus failure rate $\lambda_1$ and rate of the maximum repair rate after complete failure during the evening work period $\beta$

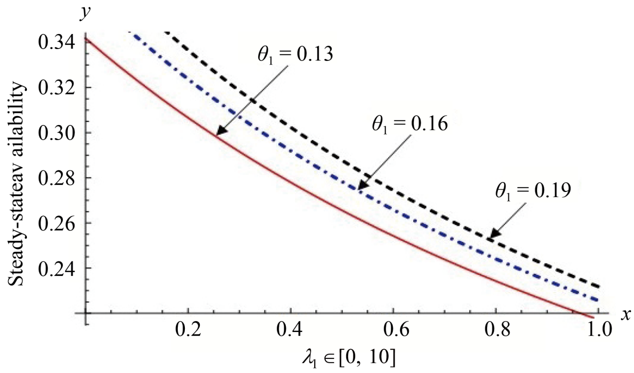


**Figure 6.** The steady-state availability versus failure rate $\lambda_1$ and rate of the maintenance rate of the partially failed unit during the morning work period $\theta_1$

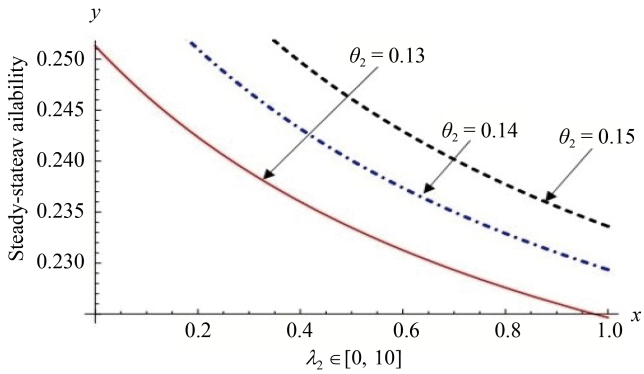


**Figure 7.** The steady-state availability versus failure rate $\lambda_2$ and rate of the maintenance rate of partially failed unit during the evening work period $\theta_2$

Figures 6 and 7 collectively demonstrate the combined impact of maintenance rates ($\theta_1$ for morning, $\theta_2$ for evening) and their corresponding failure rates ($\lambda_1$, $\lambda_2$) on system steady-state availability. The comparative analysis reveals that increasing maintenance rates during both periods significantly enhances system availability by effectively counteracting the negative effects of rising failure rates. The results show with greater precision that the effectiveness of morning maintenance ($\theta_1$) in combating its corresponding failure rate ($\lambda_1$) substantially surpasses that of evening maintenance ($\theta_2$) in addressing its specific failure rate ($\lambda_2$), particularly under high-stress operational conditions. These findings confirm the strategic importance of implementing balanced maintenance strategies, with higher priority given to optimizing morning maintenance capabilities ($\theta_1$) to handle failure rates ($\lambda_1$), while maintaining robust evening maintenance capacities ($\theta_2$) to control failure rates ($\lambda_2$). This integrated approach ensures maximum operational availability and 24/7 system reliability.

Figure 8 presents a comprehensive sensitivity analysis of the system reliability function $R(t)$ over the time interval $t \in [0,\ 10]$ , examining how variations in different system parameters affect reliability. The analysis reveals distinct sensitivity patterns among the parameters: The partial derivatives $\frac{\partial R(t)}{\partial \lambda_1}$, $\frac{\partial R(t)}{\partial \alpha_1}$, $\frac{\partial R(t)}{\partial \eta_1}$, $\frac{\partial R(t)}{\partial \lambda_2}$, and $\frac{\partial R(t)}{\partial \eta}$ demonstrate negative values, indicating that increasing these parameters leads to a reduction in system reliability. This is particularly pronounced for the failure rate $\lambda_1 = \eta$, which shows the most significant negative sensitivity. Conversely, the partial derivative $\frac{\partial R(t)}{\partial \alpha_2}$ exhibits a positive value, suggesting that increasing this transition rate $\alpha_2$ contributes to enhancing system reliability. These findings provide crucial quantitative insights for reliability optimization, clearly identifying which parameters require careful control and which can be leveraged for system improvement.

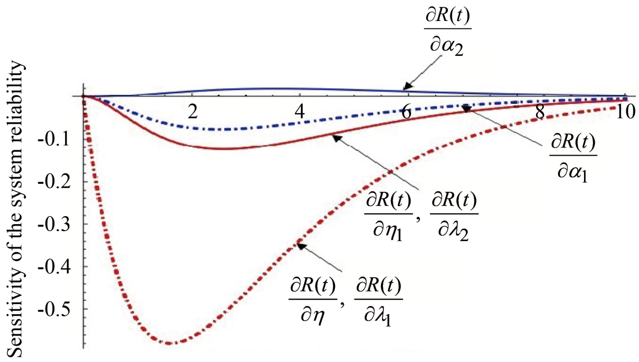


**Figure 8.** Sensitivity analysis with different parameters

## 5.2 *Numerical findings of sensitivity and relative sensitivity analyses*

Tables 6 and 7 provide an integrated and advanced analysis of the sensitivity of reliability for industrial system. Table 6 focuses on measuring sensitivity of the system reliability $\varphi$, which estimates the instantaneous rate of change in reliability with respect to a slight variation in critical operational parameters, such as failure rates ($\lambda_i$; $\eta_i$) and transition rates ($\alpha_i$). The results show that sensitivity values at the start of operation $t = 0$ are nearly negligible, which is expected at the beginning of any industrial system's life cycle. As operational time progresses, the negative value for parameters ($\varphi_{\alpha_1}$, $\varphi_{\eta_1}$, $\varphi_{\eta}$, $\varphi_{\lambda_1}$, $\varphi_{\lambda_2}$) indicate that their increase lead to a degradation of system reliability, whereas the positive sign for $\varphi_{\alpha_2}$ suggests a potentially positive role, possibly representing the efficiency of repair process or the activation of a backup system. It is noted that the peak negative absolute impact is achieved at $t = 2$ for the two parameters $\eta$ and $\lambda_1$ with a value of (-0.575535), identifying the most influential operational parameters in the medium-term performance of system. While Table 6 define the dimension of the immediate impact, Table 7 transitions to a more in-depth analysis via "Relative Sensitivity" $\psi$, a dimensionless quantity that measures the percentage change in reliability resulting from a 1% relative change in a parameter. The paramount engineering value of this table lies in its revelation of a long-term dynamic; contrary to the absolute sensitivity which declines, we observe that relative sensitivity increases steadily with advancing operational age for all parameters. This clarifies that as the industrial system ages and its fundamental reliability begins to erode, its performance becomes relatively more sensitive to any variances in design and operational parameters. Crucially, the failure rate $\lambda_1$ emerges as the most critical parameter overall, with its value $\psi_{\lambda_1}$ reaching (-1.99619) at $t = 10$. This means, in practice, that a 1% increase in $\lambda_1$ would cause an approximately 2% decrease in system reliability at this stage. In conclusion, these analyses provide strategic insights for ensuring quality, robustness, and durability. While Table 6 identifies immediate operational pressure points, Table 7 reveals long-term cumulative threats, confirming that $\lambda_1$

is the decisive parameter that must be the central focus of preventive maintenance strategies, quality control, and design reliability optimization to ensure the optimal operational lifespan of industrial systems.

**Table 6.** Sensitivity of the system reliability at $\eta = 0.2;\ \eta_1 = 0.3;\ \alpha_1 = 0.3;\ \alpha_2 = 0.35;\ \lambda_1 = 0.3;\ \lambda_2 = 0.4$

| $t$ | $\varphi_{\alpha_1}$ | $\varphi_{\alpha_2}$ | $\varphi_{\eta_1}$ | $\varphi_{\eta}$ | $\varphi_{\lambda_1}$ | $\varphi_{\lambda_2}$ |
|---|---|---|---|---|---|---|
| 0 | $2.220510^{-16}$ | $2.220510^{-16}$ | $5.551110^{-17}$ | $-5.551110^{-17}$ | $-5.551110^{-17}$ | $5.551110^{-17}$ |
| 1 | -0.0217564 | 0.00210746 | -0.0694841 | -0.52987 | -0.52987 | -0.0694841 |
| 2 | -0.0391911 | 0.00726344 | -0.1321 | -0.575535 | -0.575535 | -0.1321 |
| 3 | -0.0409799 | 0.0107883 | -0.144516 | -0.477388 | -0.477388 | -0.144516 |
| 4 | -0.0348138 | 0.0114853 | -0.127412 | -0.356573 | -0.356573 | -0.127412 |
| 5 | -0.0266271 | 0.0102697 | -0.100421 | -0.252005 | -0.252005 | -0.100421 |
| 6 | -0.0191561 | 0.00827013 | -0.0740086 | -0.17211 | -0.17211 | -0.0740086 |
| 7 | -0.0132515 | 0.00622087 | -0.0521961 | -0.114823 | -0.114823 | -0.0521961 |
| 8 | -0.00892338 | 0.00446455 | -0.0356993 | -0.0752993 | -0.0752993 | -0.0356993 |
| 9 | -0.00589269 | 0.00309758 | -0.0238732 | -0.0487326 | -0.0487326 | -0.0238732 |
| 10 | -0.00383413 | 0.00209575 | -0.0156939 | -0.0312091 | -0.0312091 | -0.0156939 |

**Table 7.** Relative Sensitivity of the system reliability at $\eta = 0.2;\ \eta_1 = 0.3;\ \alpha_1 = 0.3;\ \alpha_2 = 0.35;\ \lambda_1 = 0.3;\ \lambda_2 = 0.4$

| $t$ | $\psi_{\alpha_1}$ | $\psi_{\alpha_2}$ | $\psi_{\eta_1}$ | $\psi_{\eta}$ | $\psi_{\lambda_1}$ | $\psi_{\lambda_2}$ |
|---|---|---|---|---|---|---|
| 0 | $6.6614 \times 10^{-17}$ | $7.7716 \times 10^{-17}$ | $1.6653 \times 10^{-17}$ | $-1.1102 \times 10^{-17}$ | $-1.6653 \times 10^{-17}$ | $1.6653 \times 10^{-17}$ |
| 1 | -0.0108899 | 0.00123068 | -0.0347795 | -0.176814 | -0.265221 | -0.0347795 |
| 2 | -0.0332299 | 0.00718507 | -0.112007 | -0.325329 | -0.487993 | -0.112007 |
| 3 | -0.0593049 | 0.0182146 | -0.20914 | -0.460574 | -0.69086 | -0.20914 |
| 4 | -0.0863179 | 0.033223 | -0.315907 | -0.589395 | -0.884093 | -0.315907 |
| 5 | -0.113331 | 0.0509953 | -0.427413 | -0.715058 | -1.07259 | -0.427413 |
| 6 | -0.140099 | 0.0705647 | -0.541266 | -0.839156 | -1.25873 | -0.541266 |
| 7 | -0.166617 | 0.091254 | -0.656285 | -0.962476 | -1.44371 | -0.656285 |
| 8 | -0.192941 | 0.112621 | -0.771886 | -1.08541 | -1.62811 | -0.771886 |
| 9 | -0.219132 | 0.134389 | -0.887776 | -1.20815 | -1.81222 | -0.887776 |
| 10 | -0.245238 | 0.156389 | -1.00381 | -1.33079 | -1.99619 | -1.00381 |

The sensitivity and relative sensitivity analysis of the MTSF with respect to various system parameters is presented in Tables 8-10. These results provide significant insights into how changes in the model parameters influence system reliability, thereby highlighting critical factors that should be carefully controlled to ensure optimal system performance.

Table 8 illustrates the effect of varying $\alpha_1$ and $\alpha_2$ while keeping the other parameters constant. The sensitivity values associated with $\alpha_1$ are negative, ranging from -0.804149 at $\alpha_1 = 0.1$ to -0.205721 at $\alpha_1 = 0.9$, while the relative sensitivity $\omega_{\alpha_1}$ varies between -0.0421374 and -0.11696. This indicates that higher values of $\alpha_1$ reduce the MTSF, with a relatively stable effect at larger parameter values. In contrast, the sensitivity values of $\alpha_2$ are positive, decreasing from 0.189036 at $\alpha_2 = 0.1$ to 0.0757512 at $\alpha_2 = 0.9$, while the relative sensitivity increases from 0.0108696 to 0.037156. Thus, $\alpha_2$ exerts a positive influence on MTSF, which becomes more significant at higher values. Overall, $\alpha_1$ negatively affects system reliability, whereas $\alpha_2$ contributes positively to enhancing it.

**Table 8.** Sensitivity and relative sensitivity analysis of MTSF for different values of $\alpha_1$, $\alpha_2$

| $\eta = 0.2;\ \eta_1 = 0.3;\ \alpha_2 = 0.35;\ \lambda_1 = 0.3;\ \lambda_2 = 0.5$ | | | $\eta = 0.2;\ \eta_1 = 0.3;\ \alpha_1 = 0.3;\ \lambda_1 = 0.3;\ \lambda_2 = 0.5$ | | |
|---|---|---|---|---|---|
| $\alpha_1$ | $\delta_{\alpha_1}$ | $\omega_{\alpha_1}$ | $\alpha_2$ | $\delta_{\alpha_2}$ | $\omega_{\alpha_2}$ |
| 0.1 | -0.804149 | -0.0421374 | 0.1 | 0.189036 | 0.0108696 |
| 0.2 | -0.638623 | -0.0695389 | 0.2 | 0.164354 | 0.018711 |
| 0.3 | -0.519402 | -0.087582 | 0.3 | 0.144208 | 0.0244123 |
| 0.4 | -0.430698 | -0.0994774 | 0.4 | 0.127551 | 0.0285714 |
| 0.5 | -0.362919 | -0.107226 | 0.5 | 0.113622 | 0.0316011 |
| 0.6 | -0.309966 | -0.112119 | 0.6 | 0.101856 | 0.0337922 |
| 0.7 | -0.26781 | -0.115014 | 0.7 | 0.0918274 | 0.0353535 |
| 0.8 | -0.233704 | -0.116493 | 0.8 | 0.0832101 | 0.0364372 |
| 0.9 | -0.205721 | -0.11696 | 0.9 | 0.0757512 | 0.037156 |

Table 9 reports the sensitivity analysis results for $\eta_1$ and $\eta$. The sensitivity values for $\eta_1$ are negative, decreasing from -0.804149 at $\eta_1 = 0.1$ to -0.205721 at $\eta_1 = 0.9$, with relative sensitivity values between -0.0421374 and -0.11696. This confirms that increasing $\eta_1$ leads to a reduction in MTSF, though with a relatively stable effect at higher values. For $\eta$, however, the sensitivity values are much larger in magnitude, ranging from -3.40306 at $\eta = 0.1$ to -0.635383 at $\eta = 0.9$, while the relative sensitivity increases in magnitude from -0.164286 to -0.638889. These results demonstrate that $\eta$ has a much stronger negative impact on MTSF compared to $\eta_1$, indicating that system reliability is highly sensitive to variations in $\eta$.

**Table 9.** Sensitivity and relative sensitivity analysis of MTSF for different values of $\eta_1$, $\eta$

| $\eta = 0.2;\ \alpha_1 = 0.3;\ \alpha_2 = 0.35;\ \lambda_1 = 0.3;\ \lambda_2 = 0.5$ | | | $\eta_1 = 0.3;\ \alpha_1 = 0.3;\ \alpha_2 = 0.35;\ \lambda_1 = 0.3;\ \lambda_2 = 0.5$ | | |
|---|---|---|---|---|---|
| $\eta_1$ | $\delta_{\eta_1}$ | $\omega_{\eta_1}$ | $\eta$ | $\delta_{\eta}$ | $\omega_{\eta}$ |
| 0.1 | -0.804149 | -0.0421374 | 0.1 | -3.40306 | -0.164286 |
| 0.2 | -0.638623 | -0.0695389 | 0.2 | -2.51044 | -0.282209 |
| 0.3 | -0.519402 | -0.087582 | 0.3 | -1.92797 | -0.370968 |
| 0.4 | -0.430698 | -0.0994774 | 0.4 | -1.52698 | -0.440191 |
| 0.5 | -0.362919 | -0.107226 | 0.5 | -1.23922 | -0.49569 |
| 0.6 | -0.309966 | -0.112119 | 0.6 | -1.02576 | -0.541176 |
| 0.7 | -0.26781 | -0.115014 | 0.7 | -0.863051 | -0.579137 |
| 0.8 | -0.233704 | -0.116493 | 0.8 | -0.736195 | -0.611296 |
| 0.9 | -0.205721 | -0.11696 | 0.9 | -0.635383 | -0.638889 |

Table 10 presents the sensitivity analysis results for $\lambda_1$ and $\lambda_2$. The sensitivity values for $\lambda_1$ are negative and relatively large, decreasing from -4.87253 at $\lambda_1 = 0.1$ to -0.736195 at $\lambda_1 = 0.9$, while the relative sensitivity changes from -0.196581 to -0.687708. This indicates that increasing $\lambda_1$ leads to a significant reduction in MTSF, with the relative impact becoming stronger at higher values. Similarly, the sensitivity values of $\lambda_2$ are also negative but smaller in magnitude compared to $\lambda_1$, ranging from -1.40802 at $\lambda_2 = 0.1$ to -0.26781 at $\lambda_2 = 0.9$, while the relative sensitivity increases from -0.0663781 to -0.147875. This suggests that both $\lambda_1$ and $\lambda_2$ negatively affect MTSF, but the impact of $\lambda_1$ is significantly stronger, making it the most critical parameter to control in order to enhance system reliability.

**Table 10.** Sensitivity and relative sensitivity analysis of MTSF for different values of $\lambda_1$, $\lambda_2$

| $\eta = 0.2;\ \alpha_1 = 0.3;\ \alpha_2 = 0.35;\ \eta_1 = 0.3;\ \lambda_2 = 0.5$ | | | $\eta_1 = 0.3;\ \alpha_1 = 0.3;\ \alpha_2 = 0.35;\ \lambda_1 = 0.3;\ \eta = 0.2$ | | |
|---|---|---|---|---|---|
| $\lambda_1$ | $\delta_{\lambda_1}$ | $\omega_{\lambda_1}$ | $\lambda_2$ | $\delta_{\lambda_2}$ | $\omega_{\lambda_2}$ |
| 0.1 | -4.87253 | -0.196581 | 0.1 | -1.40802 | -0.0663781 |
| 0.2 | -3.40306 | -0.328571 | 0.2 | -1.04348 | -0.104348 |
| 0.3 | -2.51044 | -0.423313 | 0.3 | -0.804149 | -0.126412 |
| 0.4 | -1.92797 | -0.494624 | 0.4 | -0.638623 | -0.139078 |
| 0.5 | -1.52698 | -0.550239 | 0.5 | -0.519402 | -0.14597 |
| 0.6 | -1.23922 | -0.594828 | 0.6 | -0.430698 | -0.149216 |
| 0.7 | -1.02576 | -0.631373 | 0.7 | -0.362919 | -0.150117 |
| 0.8 | -0.863051 | -0.661871 | 0.8 | -0.309966 | -0.149492 |
| 0.9 | -0.736195 | -0.687708 | 0.9 | -0.26781 | -0.147875 |

In summary, the sensitivity analysis highlights the contrasting roles of the parameters under consideration. While $\alpha_2$ positively contributes to system performance, parameters such as $\alpha_1$, $\eta_1$, $\eta$, $\lambda_1$, and $\lambda_2$ negatively influence the MTSF, with $\eta$ and $\lambda_1$ emerging as the most dominant factors. These findings emphasize the importance of carefully regulating these parameters to improve system dependability.

## 6. Summary

This study provides a comprehensive reliability analysis of a single-active-unit industrial repairable system, explicitly considering three operational states: normal operation, partial failure, and complete failure. The proposed model incorporates practical operational constraints, including limited nighttime replacement capabilities and a predefined maximum repair time, reflecting real industrial conditions. Using the supplementary variable technique combined with Markov process theory, key performance indicators-such as the reliability function, MTSF, and steady-state availability-are systematically evaluated. Sensitivity and relative sensitivity analyses further highlighted the influence of critical parameters on system performance, offering practical insights for maintenance planning and operational decision-making.

## 7. Conclusions

This study has explored the reliability and sensitivity analyses of a single-unit engineering system operating in both morning and evening shifts, under the constraint of a maximum allowable repair time and preventive maintenance for partial failures of the active unit. By employing a state transition diagram, Laplace transforms of the state probabilities were successfully derived. Based on these, key reliability measures of the system were obtained. Furthermore, both sensitivity and relative sensitivity analyses were conducted for two critical system performance indices. The results provide valuable insights into the impact of system parameters on reliability, which can guide design optimization and maintenance planning in practical engineering applications. Finally, a set of numerical results was obtained, leading to the following key conclusions:

1) Tables 2-5 demonstrate that the reliability function $R(t)$ of the system decreases gradually with the increase in failure rates ($\lambda_1,\ \lambda_2,\ \eta,\ \eta_1$) and over time ($t$), assuming other parameters remain constant. The results also indicate the absence of abrupt changes in the behavior of the reliability function, suggesting that the system maintains a high level of reliability over an extended period.

2) Figures 2 and 3, illustrate the behavior of the MTSF for the proposed industrial system. The results indicate that MTSF increases as the parameters $\alpha_1$ and $\alpha_2$ increase within the interval $\lambda_1 \in [0,\ 10]$ and $\lambda_2 \in [0,\ 10]$, respectively, suggesting an enhancement in the system's reliability under these conditions.

3) The curves presented in Figures 4 and 6 indicate that the steady-state availability $Av(\infty)$ increases significantly with the rise in both the preventive maintenance rate $\theta_1$ and maximum repair time rate $\beta_1$ of the failed unit during the morning shift. This improvement is observed under the condition of varying the total failure rate of the active unit within the time interval $\lambda_1 \in [0, 1]$.

4) Figures 5 and 7, illustrate that the steady-state availability $Av(\infty)$ improves as the preventive maintenance rate $\theta_2$ for the partially failing unit increases and as the maximum repair time rate $\beta$ during the evening shift decreases. This trend is observed under the condition that the total failure rate of the unit varies within the interval $\lambda_2 \in [0, 1]$.

As shown in Figure 8, the parameters $\lambda_1 = \eta$, $\lambda_2 = \eta_1$ and $\alpha_1$ demonstrate significant sensitivity with respect to the reliability function $R(t)$. Furthermore, the sensitivity and relative sensitivity of the MTSF with respect to the system parameters are ranked as follows:

1) $\lambda_1 > \eta > \lambda_2 > \eta_1 = \alpha_1 > \alpha_2$.

2) $\lambda_1 > \eta > \lambda_2 > \eta_1 = \alpha_1 > \alpha_2$.

Based on the previously established relationships, it is evident that the parameters $\lambda_1$, $\eta$, $\lambda_2$, $\eta_1$ and $\alpha_1$ are the most influential factors affecting both performance metrics. Consequently, careful monitoring of their values is essential.

## Conflict of interest

The author declares no competing financial interest.

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## Appendix

At any time $t$, if the system is in state $S_i$, then the probability of the system to be that state is defined as the probability that the system is in state $S_i$ at time $t$ and remains there in interval $(t,\ t+\Delta t)$, or/ and if it is in some other state at time $t$, provided that transition exists between the states and $\Delta t \to 0$.

Accordingly, (13)-(21) are interpreted as follows.

The probability of the system to be in state $S_0$ in the interval $(t,\ t+\Delta t)$ is given by

$$\pi_0(t+\Delta t) = (1-(\alpha_1+\lambda_1+\eta)\Delta t)\pi_0(t) + \alpha_2\Delta t\pi_1(t) + \theta_1\Delta t\pi_2(t) + \theta\Delta t\pi_4(t) + \theta_5\Delta t\pi_6(t)$$

$$\Rightarrow \lim_{\Delta t\to 0}\frac{\pi_0(t+\Delta t)-\pi_0(t)}{\Delta t} + (\alpha_1+\lambda_1+\eta)\pi_0(t) = \alpha_2\pi_1(t) + \theta_1\pi_2(t) + \theta\pi_4(t) + \theta_5\pi_6(t)$$

$$\Rightarrow \left\{\frac{d}{dt} + \alpha_1 + \lambda_1 + \eta\right\}\pi_0(t) = \alpha_2\pi_1(t) + \theta_1\pi_2(t) + \theta\pi_4(t) + \theta_5\pi_6(t)$$

**For state $S_1$**

$$\pi_1(t+\Delta t) = (1-(\alpha_2+\lambda_2+\eta_1)\Delta t)\pi_1(t) + \alpha_1\Delta t\pi_0(t) + \theta_2\Delta t\pi_3(t) + \theta_4\Delta t\pi_5(t) + \theta_3\Delta t\pi_8(t)$$

$$\Rightarrow \lim_{\Delta t\to 0}\frac{\pi_1(t+\Delta t)-\pi_1(t)}{\Delta t} + (\alpha_2+\lambda_2+\eta_1)\pi_1(t) = \alpha_1\pi_0(t) + \theta_2\pi_3(t) + \theta_4\pi_5(t) + \theta_3\pi_8(t)$$

$$\Rightarrow \left\{\frac{d}{dt} + \alpha_2 + \lambda_2 + \eta_1\right\}\pi_1(t) = \alpha_1\pi_0(t) + \theta_2\pi_3(t) + \theta_4\pi_5(t) + \theta_3\pi_8(t)$$

**For state $S_2$**

$$\pi_2(t+\Delta t) = (1-\theta_1\Delta t)\pi_2(t) + \eta\Delta t\pi_0(t)$$

$$\Rightarrow \lim_{\Delta t\to 0}\frac{\pi_2(t+\Delta t)-\pi_2(t)}{\Delta t} + \theta_1\pi_2(t) = \eta\pi_0(t)$$

$$\Rightarrow \left\{\frac{d}{dt} + \theta_1\right\}\pi_2(t) = \eta\pi_0(t)$$

**For state $S_3$**

$$\pi_3(t+\Delta t) = (1-\theta_2\Delta t)\pi_3(t) + \eta_1\Delta t\pi_1(t)$$

$$\Rightarrow \lim_{\Delta t\to 0}\frac{\pi_3(t+\Delta t)-\pi_3(t)}{\Delta t} + \theta_2\pi_3(t) = \eta_1\pi_1(t)$$

$$\Rightarrow \left\{ \frac{d}{dt} + \theta_2 \right\} \pi_3(t) = \eta_1 \pi_1(t)$$

**For state $S_4$**

$$\pi_4(t + \Delta t) = (1 - (\theta + \beta_1)\Delta t)\pi_4(t) + \lambda_1 \Delta t \pi_0(t)$$

$$\Rightarrow \lim_{\Delta t \to 0} \frac{\pi_4(t + \Delta t) - \pi_4(t)}{\Delta t} + (\theta + \beta_1)\pi_4(t) = \lambda_1 \pi_0(t)$$

$$\Rightarrow \left\{ \frac{d}{dt} + \theta + \beta_1 \right\} \pi_4(t) = \lambda_1 \pi_0(t)$$

**For state $S_5$**

$$\pi_5(t + \Delta t) = (1 - (\theta_4 + \beta)\Delta t)\pi_5(t) + \lambda_2 \Delta t \pi_1(t)$$

$$\Rightarrow \lim_{\Delta t \to 0} \frac{\pi_5(t + \Delta t) - \pi_5(t)}{\Delta t} + (\theta_4 + \beta)\pi_5(t) = \lambda_2 \pi_1(t)$$

$$\Rightarrow \left\{ \frac{d}{dt} + \theta + \beta_1 \right\} \pi_4(t) = \lambda_1 \pi_0(t)$$

**For state $S_6$**

$$\pi_6(t + \Delta t) = (1 - \theta_5 \Delta t)\pi_6(t) + \beta_1 \Delta t \pi_4(t) + \alpha_2 \Delta t \pi_7(t)$$

$$\Rightarrow \lim_{\Delta t \to 0} \frac{\pi_6(t + \Delta t) - \pi_6(t)}{\Delta t} + \theta_5 \pi_6(t) = \beta_1 \pi_4(t) + \alpha_2 \pi_7(t)$$

$$\Rightarrow \left\{ \frac{d}{dt} + \theta_5 \right\} \pi_6(t) = \beta_1 \pi_4(t) + \alpha_2 \pi_7(t)$$

**For state $S_7$**

$$\pi_7(t + \Delta t) = (1 - (\kappa + \alpha_2)\Delta t)\pi_7(t) + \beta \Delta t \pi_5(t)$$

$$\Rightarrow \lim_{\Delta t \to 0} \frac{\pi_7(t + \Delta t) - \pi_7(t)}{\Delta t} + (\kappa + \alpha_2)\pi_7(t) = \beta \pi_5(t)$$

$$\Rightarrow \left\{ \frac{d}{dt} + \kappa + \alpha_2 \right\} \pi_7(t) = \beta \pi_5(t)$$

**For state $S_8$**

$$\pi_8(t + \Delta t) = (1 - \theta_3 \Delta t)\, \pi_8(t) + \kappa \Delta t\, \pi_7(t)$$

$$\Rightarrow \lim_{\Delta t \to 0} \frac{\pi_8(t + \Delta t) - \pi_8(t)}{\Delta t} + \theta_3 \pi_8(t) = \kappa \pi_7(t)$$

$$\Rightarrow \left\{ \frac{d}{dt} + \theta_3 \right\} \pi_8(t) = \kappa \pi_7(t)$$