Contemporary Mathematics
https://ojs.wiserpub.com/index.php/CM/

Research Article

Coupled Hybrid Proportional Fractional Systems of Caputo-Type in Generalized Banach Spaces

Yasir A. Madani[1], Oualid Zentar[2], Sidi Mohammed Benhabi[2], Tayeb Mahrouz[2], Abdelkader Moumen[1*]

[1] Department of Mathematics, College of Science, University of Hail, Hail, 55473, Saudi Arabia
[2] Department of Computer Science and Laboratory of Research in Artificial Intelligence and Systems (LRAIS), University of Tiaret, Tiaret, P.O. Box 78, Algeria
E-mail: mo.abdelkader@uoh.edu.sa

Received: 4 August 2025; **Revised:** 30 October 2025; **Accepted:** 5 November 2025

Abstract: This paper presents new quantitative results for a class of coupled systems involving the hybrid proportional-Caputo fractional derivative. First, a variant of Perov's fixed point theorem, combined with the Bielecki norm, is employed to establish existence and uniqueness. Second, an additional existence result is derived using Krasnosel'skii's fixed point theorem within the framework of generalized Banach spaces. Finally, Urs's approach is employed to analyze the Hyers-Ulam stability of solutions to the proposed problem. To validate the theoretical results, several illustrative examples are presented.

***Keywords*:** hybrid proportional-Caputo derivative, coupled system, generalized Banach space

MSC: 26A33, 34A08, 47H08

1. Introduction

Although Fractional Calculus (FC) was originally proposed over three centuries ago, its emergence represents a pivotal development bridging classical and modern calculus. Once considered unconventional, this concept has since undergone substantial development and found wide-ranging applications across numerous disciplines [1, 2]. In recent years, research in FC has grown considerably, with numerous studies investigating diverse results for various classes of fractional differential equations and inclusions under a wide range of assumptions [3–10].

To capture the nonlocal behavior of differential systems, various formulations of fractional derivatives have been proposed, including the widely used Riemann-Liouville, Caputo, Erdélyi-Kober, and Hadamard derivatives. In [11], the authors replace the classical derivative–whether in the Riemann-Liouville or Caputo sense–with a more general proportional derivative. This substitution enhances the flexibility of modeling in various contexts, such as the temperature dynamics of buildings [12] and the unsteady, incompressible flow of viscous fluids [13], magnetohydrodynamic viscous fluids [14], and Brinkman–type fluids containing hybrid nanoparticles [15]. Within this framework, Abbes [16, 17] studied the existence, uniqueness, and Ulam stability of fractional Cauchy problems governed by hybrid proportional-Caputo derivatives. The analysis was conducted by applying Banach's and Schaefer's fixed-point theorems to the model:

Copyright ©2026 Abdelkader Moumen, et al.
DOI: https://doi.org/10.37256/cm.7220268130
This is an open-access article distributed under a CC BY license
(Creative Commons Attribution 4.0 International License)
https://creativecommons.org/licenses/by/4.0/

$$\begin{cases} {}^{pc}\mathscr{D}_t^{\alpha} x(t) = g(t), \quad t \in [0, T],\ \alpha \in (0, 1), \\ \\ x(0) = x_0 \in \mathbb{R}, \end{cases}$$

where ${}^{pc}\mathscr{D}_t^{\alpha}$ is defined in Definition 1. Furthermore, the methodology in [16, 17] combines the Laplace transform technique with a generalized Gronwall inequality. Using the Gramian matrix approach, the authors in [18] established a controllability result for fractional dynamical systems with multiple control delays, employing generalized proportional fractional derivatives in the Caputo sense. Moreover, further studies on the discrete setting of this type of operator can be found in [19].

On the other hand, natural phenomena in fields such as chemistry, physics, computer networks, and biology are often modeled by various coupled fractional differential systems [20–22]. Our aim is to extend the study initiated in [16, 17] to a more general setting, particularly when the nonlinearities exhibit strong coupling.

Motivated by the preceding discussion, this paper considers the following coupled system:

$$\begin{cases} {}^{pc}\mathscr{D}_0^{\kappa_1} \mathfrak{r}_1(\vartheta) = \mathbb{A}_1 \mathfrak{r}_1(\vartheta) + \Omega_1(\vartheta, \mathfrak{r}_1(\vartheta), \mathfrak{r}_2(\vartheta)), \quad \vartheta \in \mathfrak{J} := [0, b], \\ \\ {}^{pc}\mathscr{D}_0^{\kappa_2} \mathfrak{r}_2(\vartheta) = \mathbb{A}_2 \mathfrak{r}_2(\vartheta) + \Omega_2(\vartheta, \mathfrak{r}_1(\vartheta), \mathfrak{r}_2(\vartheta)), \quad \vartheta \in \mathfrak{J} := [0, b], \\ \\ \mathfrak{r}_1(0) = R_1, \quad \mathfrak{r}_1(0) = R_2, \end{cases} \tag{1}$$

where ${}^{pc}\mathscr{D}_0^{\kappa_i}$ denotes the hybrid proportional–Caputo fractional derivative of order $0 < \kappa_i < 1$, $\Omega_i \colon \mathfrak{J} \times \mathbb{R}^n \to \mathbb{R}^n$ are a functions satisfying certain hypotheses to be specified later, $R_i \in \mathbb{R}^n$ and $\mathbb{A}_i \in \mathbb{R}^{n \times n}$, $n \in \mathbb{N}$, $i = 1, 2$.

The novelty of our results can be summarized as follows:

• We propose a hybrid proportional fractional differential system of Caputo type.

• A Perov-type fixed point principle is employed to establish a new existence and uniqueness result, while Krasnosel'skii's fixed point principle, within the framework of a generalized Banach space, is used to derive additional existence results for the problem under consideration.

• To illustrate the theoretical results, two examples are presented.

• The results established in [16, 17] can be regarded as a special case of our findings.

This manuscript is structured as follows. Section 2 presents some basic background material used in the sequel. The main results are stated and proved in Section 3. Our findings are validated through two illustrative examples in Section 4. Finally, the last section is devoted to concluding remarks.

2. Preliminaries

We endow the space $C(\mathfrak{J}, \mathbb{R}^n)$ of continuous functions $z \colon \mathfrak{J} \to \mathbb{R}^n$ by the norm:

$$\|z\| = \sup_{\vartheta \in \mathfrak{J}} \|z(\vartheta)\|, \quad \forall z \in C(\mathfrak{J}, \mathbb{R}^n).$$

$L^{\infty}(\mathfrak{J}, \mathbb{R}^n)$ is the Banach space of all equivalence classes of essentially bounded measurable functions on $\mathfrak{J}$, equipped with the norm:

$$\|z\|_{L^\infty} = {}_{t\in\mathfrak{J}}\|z(t)\| = \inf\{M > 0;\ \|z(t)\| \le M \text{ for almost every } t \in \mathfrak{J}\}.$$

Definition 1 [11] The Hybrid Proportional-Caputo Fractional Derivative (HPCFD) of order $\kappa \in (0,\ 1)$ of a differentiable function $z(\vartheta)$ can be defined in one of two possible ways. The following general way:

$$\begin{aligned} {}^{pc}\mathscr{D}_0^{\kappa} z(\vartheta) &= \frac{1}{\Gamma(1-\alpha)} \int_0^{\vartheta} \left(\eta_1(\kappa,\ \tau) z(\vartheta) + \eta_0(\eta,\ \tau) z'(\vartheta)\right) (\vartheta - \tau)^{-\kappa} d\tau \\ &= {}^{RL}\mathscr{I}_0^{1-\kappa} \left(\eta_1(\kappa,\ \tau) z(\vartheta) + \eta_0(\kappa,\ \tau) z'(\vartheta)\right), \end{aligned} \tag{2}$$

where ${}^{RL}\mathscr{I}_0^{1-\kappa}$ is the Riemann-Liouville fractional integral difined by for $1-\kappa > 0$,

$${}^{RL}\mathscr{I}_0^{1-\kappa} z(\vartheta) = \frac{1}{\Gamma(1-\kappa)} \int_0^{\vartheta} (\vartheta - \tau)^{\kappa} z(\tau) d\tau,$$

and $\eta_0,\ \eta_1 \colon [0,\ 1] \times \mathbb{R} \to [0,\ \infty)$ are continuous functions of the variable ϑ and the parameter $\kappa \in [0,\ 1]$. They satisfy the following conditions for all $\vartheta \in \mathbb{R}$:

$$\lim_{\kappa\to 0^+} \eta_0(\kappa,\ \vartheta) = 0, \quad \lim_{\kappa\to 1^-} \eta_0(\kappa,\ \vartheta) = 1, \quad \eta_0(\kappa,\ \vartheta) \neq 0, \quad \kappa \in (0,\ 1],$$

$$\lim_{\kappa\to 0^+} \eta_1(\kappa,\ \vartheta) = 1, \quad \lim_{\kappa\to 1^-} \eta_1(\kappa,\ \vartheta) = 0, \quad \eta_1(\kappa,\ \vartheta) \neq 0, \quad \kappa \in [0,\ 1).$$

Definition 2 [11] The inverse operator of the fractional Proportional-Caputo (PC) derivative (2) is given by

$${}^{pc}\mathscr{I}_0^{\kappa} z(\vartheta) = \int_0^{\vartheta} \exp\left(-\int_y^{\vartheta} \frac{\eta_1(\kappa,\ s)}{\eta_0(\kappa,\ s)} ds\right) \frac{{}^{RL}\mathscr{D}_y^{1-\kappa} z(y)}{\eta_0(\kappa,\ y)} dy,$$

where ${}^{RL}\mathscr{D}_y^{1-\kappa}$ denotes the Riemann–Liouville fractional derivative of order $1-\kappa$ given by

$${}^{RL}\mathscr{D}_y^{1-\kappa} z(y) = \frac{1}{\Gamma(\kappa)} \frac{d}{dy} \int_0^{y} (y - \tau)^{\kappa-1} z(\tau) d\tau.$$

Now, for $\varsigma > 0$, we endow the space $C(\mathfrak{J},\ \mathbb{R}^n)$ by Bielecki norm:

$$\|z\|_B = \sup_{\vartheta\in\mathfrak{J}} e^{-\varsigma\vartheta} \|z(\vartheta)\|. \tag{3}$$

Lemma 1 [24] The norms $\|\cdot\|_B$ defined by (3) and $\|\cdot\|_\infty$ are equivalent, i.e; there exist $\ell \in (0,\ \infty)$ such that

$$\|\cdot\|_B \leq \|\cdot\| \leq \ell\|\cdot\|_B.$$

Let $p, q \in \mathbb{R}^m$ with $p = (p_1, p_2, \ldots, p_m)$, $q = (q_1, q_2, \ldots, q_m)$. By $p \leq q$ we mean $p_i \leq q_i$, $i = 1, \ldots, m$. Also,

$$|p| = (|p_1|, |p_2|, \ldots, |p_m|),$$

$$\max(p, q) = (\max(p_1, q_1), \max(p_2, q_2), \ldots, \max(p_m, q_m)),$$

and

$$\mathbb{R}^m_+ = \{p \in \mathbb{R}^m : p_i \in \mathbb{R}_+, i = 1, \ldots, m\}.$$

If $c \in \mathbb{R}$, then $p \leq c$ means $p_i \leq c$, $i = 1, \ldots, m$.

Definition 3 [25] Let $\mathbb{V}$ be a nonempty set. By a vector-valued metric on $\mathbb{V}$ we mean a map $d: \mathbb{V} \times \mathbb{V} \to \mathbb{R}^m$ with the following properties:

1. $\mathrm{d}(p, q) \geq 0$ for all $p, q \in \mathbb{V}$, and if $\mathrm{d}(p, q) = 0$, then $p = q$;
2. $\mathrm{d}(p, q) = \mathrm{d}(q, p)$ for all $p, q \in \mathbb{V}$;
3. $\mathrm{d}(p, w) \leq \mathrm{d}(p, q) + \mathrm{d}(q, w)$ for all $p, q, w \in \mathbb{V}$.

We call the pair $(\mathbb{V}, \mathrm{d})$ a generalized metric space with

$$\mathrm{d}(p, q) := \begin{pmatrix} \mathrm{d}_1(p, q) \\ \mathrm{d}_2(p, q) \\ \vdots \\ \mathrm{d}_m(p, q) \end{pmatrix}.$$

Notice that d is a generalized metric space on $\mathbb{V}$ if and only if d_i, $i = 1, \ldots, m$, are metrics on $\mathbb{V}$.

Definition 4 [23] A square matrix $\mathfrak{M}$ of real numbers is said to be convergent to zero if and only if its spectral radius $\rho(\mathfrak{M})$ is strictly less than 1. In other words, this means that all the eigenvalues of $\mathbb{A}$ are in the open unit disc, i.e., $|\mu| < 1$ for every $\mu \in \mathbb{C}$ with $\det(\mathfrak{M} - \mu I) = 0$, where I denotes the unit matrix of $\mathfrak{M}_{m\times m}(\mathbb{R})$.

Theorem 1 [26] For any nonnegative square matrix $\mathfrak{M}$, the following properties are equivalent

1. $\mathfrak{M}$ is convergent to zero;
2. $\rho(\mathfrak{M}) < 1$;
3. the matrix $I - \mathfrak{M}$ is nonsingular and

$$(I - \mathfrak{M})^{-1} = I + \mathfrak{M} + \cdots + \mathfrak{M}^n + \cdots$$

4. $I - \mathfrak{M}$ is nonsingular and $(I - \mathfrak{M})^{-1}$ is a nonnegative matrix.

Now, we recall some fixed point theorems that will be used later.

Definition 5 [27, 28] Let $(\mathbb{X}, d)$ be a generalized metric space. An operator $\mathcal{N}: \mathbb{X} \to \mathbb{X}$ is said to be contractive if there exists a matrix $\mathfrak{M}$ convergent to zero such that:

$$d(\mathcal{N}(u_1), \mathcal{N}(u_2)) \leq \mathfrak{M}\, d(u_1, u_2), \quad \text{for all} \quad u_1, u_2 \in \mathbb{X}.$$

Theorem 2 [27] Let $(\mathbb{X}, d)$ be a complete generalized metric space and $\mathcal{N}: \mathbb{X} \to \mathbb{X}$ be a contractive operator with Lipschitz matrix $\mathfrak{M}$. Then $\mathcal{N}$ has a unique fixed point u_0, and for each $u \in \mathbb{X}$, we have:

$$d(\mathcal{N}^k(u), u_0) \leq \mathfrak{M}^k (I - \mathfrak{M})^{-1} d(u, \mathcal{N}(u)), \quad \text{for all} \quad k \in \mathbb{N}.$$

Theorem 3 [25] Let X be a closed, convex and non-empty subset of a generalized Banach spaces $\mathbb{X}$. Suppose that $\mathscr{U}$ and $\mathscr{V}$ map X into $\mathbb{X}$ and:

1. $\mathscr{U}u + \mathscr{V}v \in X$ for all $u, v \in X$;
2. $\mathscr{U}$ is compact and continuous;
3. $\mathscr{V}$ is an $\mathfrak{M}$-contraction mapping.

Then the operator equation $\mathscr{U}w + \mathscr{V}w = w$ has at least one solution on X.

3. Main results

Let the generalized Banach space $\mathbb{K} := C(\mathfrak{J}, \mathbb{R}^n) \times C(\mathfrak{J}, \mathbb{R}^n)$ equipped with the Bielecki vector-valued norm:

$$\|(z_1, z_2)\|_B = \begin{pmatrix} \|z_1\|_B \\ \\ \|z_2\|_B \end{pmatrix}.$$

Theorem 4 Let's impose the assumptions:

(H1) The functions $\Omega_1, \Omega_2: \mathfrak{J} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ are continuous.

(H2) There exists functions $\varpi_{i,1}, \varpi_{i,2} \in L^\infty(\mathfrak{J}, [0, \infty))$, $i = 1, 2$ such that

$$\|\Omega_i(\vartheta, x_1, x_2) - \Omega_i(\vartheta, u_1, u_2)\| \leq \sum_{j=1}^{2} \varpi_{i,j}(\vartheta) \|x_j - u_j\|, \tag{4}$$

for any $x_1, x_2, u_1, u_2 \in \mathbb{R}^n$ and $\vartheta \in \mathfrak{J}$.

Then, the coupled system (1) admits a unique solution on $\mathfrak{J}$.

For computational convenience, we denote $\eta_i^* = \sup_{\vartheta \in \mathfrak{J}} \frac{1}{\eta_0(\kappa_i, \vartheta)}$, $i = 1, 2$.

Proof. According to [16, Lemma 4.1], let us introduce the operator $\mathscr{Z}: \mathbb{K} \to \mathbb{K}$ given by

$$(\mathscr{Z}(\mathfrak{r}_1, \mathfrak{r}_2))(\vartheta) = ((\mathscr{Z}_1(\mathfrak{r}_1, \mathfrak{r}_2))(\vartheta), (\mathscr{Z}_2(\mathfrak{r}_1, \mathfrak{r}_2))(\vartheta)), \tag{5}$$

where, the operators $\mathscr{Z}_i: \mathbb{K} \to C(\mathfrak{J}, \mathbb{R}^n)$, are defined by

$$
\begin{aligned}
(\mathscr{Z}_i(\mathfrak{r}_1, \mathfrak{r}_2))(\vartheta) = &\exp\left(-\int_0^{\vartheta} \frac{\eta_1(\kappa_i, s)}{\eta_0(\kappa_i, s)} ds\right) R_i + \frac{1}{\Gamma(\kappa_i - 1)} \int_0^{\vartheta} \int_0^{y} \exp\left(-\int_y^{\vartheta} \frac{\eta_1(\kappa_i, s)}{\eta_0(\kappa_i, s)} ds\right) \\
&\times \frac{(y-\tau)^{\kappa_i - 2}}{\eta_0(\kappa_i, y)} \left[\mathbb{A}_i \mathfrak{r}_i(\vartheta) + \Omega_i(\tau, \mathfrak{r}_1(\tau), \mathfrak{r}_2(\tau))\right] d\tau dy, \quad \vartheta \in \mathfrak{J},\ i = 1, 2.
\end{aligned} \tag{6}
$$

Clearly, the set of all solutions to (1) coincides with the set of fixed points of the operator $\mathscr{Z}$. Therefore, it is enough to show that $\mathscr{Z}$ is a $\mathfrak{M}_\varsigma$-contraction via the Bielecki's norm.

For all $(\mathfrak{r}_1, \mathfrak{r}_2), (\mathfrak{s}_1, \mathfrak{s}_2) \in \mathbb{K}$ and all $\vartheta \in \mathfrak{J}$, by (H2) combined the fact

$$
\exp\left(-\int_y^{\vartheta} \frac{\eta_1(\kappa_i, s)}{\eta_0(\kappa_i, s)} ds\right) \leq 1, \quad \text{for all } \vartheta \in \mathfrak{J},\ i = 1, 2, \tag{7}
$$

we can get

$$
\begin{aligned}
&\|(\mathscr{Z}_i(\mathfrak{r}_1, \mathfrak{r}_2))(\vartheta) - (\mathscr{Z}_i(\mathfrak{s}_1, \mathfrak{s}_2))(\vartheta)\| \\
\leq &\frac{\eta_i^*}{\Gamma(\kappa_i - 1)} \int_0^{\vartheta} \int_0^{y} \frac{(y-\tau)^{\kappa_i - 2}}{|\eta_0(\kappa_i, y)|} \|\Omega_i(\tau, \mathfrak{r}_1(\tau), \mathfrak{r}_2(\tau)) - \Omega_i(\tau, \mathfrak{s}_1(\tau), \mathfrak{s}_2(\tau))\| d\tau dy \\
&+ \frac{\eta_i^*}{\Gamma(\kappa_i - 1)} \int_0^{\vartheta} \int_0^{y} \frac{(y-\tau)^{\kappa_i - 2}}{|\eta_0(\kappa_i, y)|} \mathbb{A}_i \|\mathfrak{r}_i(\tau) - \mathfrak{s}_i(\tau)\| d\tau dy \\
\leq &\frac{\eta_i^*}{\Gamma(\kappa_i - 1)} \left(\|\mathbb{A}_i\| \|\mathfrak{r}_i - \mathfrak{s}_i\|_B + \sum_{j=1}^{2} \|\varpi_{i,j}\|_{L^\infty} \|\mathfrak{r}_j - \mathfrak{s}_j\|_B\right) \int_0^{\vartheta} \int_0^{y} (y-\tau)^{\kappa_i - 2} e^{\varsigma\tau} d\tau dy, \quad i = 1, 2.
\end{aligned}
$$

Using Dirichlet's formula [2], we have

$$
\begin{aligned}
&\|(\mathscr{Z}_i(\mathfrak{r}_1, \mathfrak{r}_2))(\vartheta) - (\mathscr{Z}_i(\mathfrak{s}_1, \mathfrak{s}_2))(\vartheta)\| \\
\leq &\frac{\eta_i^*}{\Gamma(\kappa_i - 1)} \left(\|\mathbb{A}_i\| \|\mathfrak{r}_i - \mathfrak{s}_i\|_B + \sum_{j=1}^{2} \|\varpi_{i,j}\|_{L^\infty} \|\mathfrak{r}_j - \mathfrak{s}_j\|_B\right) \int_0^{\vartheta} e^{\varsigma y} \int_y^{\vartheta} (\tau - y)^{\kappa_i - 2} d\tau dy \\
\leq &\frac{\eta_i^*}{\Gamma(\kappa_i)} \left(\|\mathbb{A}_i\| \|\mathfrak{r}_i - \mathfrak{s}_i\|_B + \sum_{j=1}^{2} \|\varpi_{i,j}\|_{L^\infty} \|\mathfrak{r}_j - \mathfrak{s}_j\|_B\right) \int_0^{\vartheta} e^{\varsigma y} (\vartheta - y)^{\kappa_i - 1} dy,\ i = 1, 2.
\end{aligned} \tag{8}
$$

Potting $t = \vartheta - y$, one gets

$$\int_0^{\vartheta} e^{\varsigma y}(\vartheta - y)^{\kappa_i - 1} dy = e^{\varsigma \vartheta} \int_0^{\vartheta} t^{\kappa_i - 1} e^{-\varsigma t} dt, \quad i = 1, 2. \tag{9}$$

By making the substitution $x = \varsigma t$ in (9), we obtain the following inequality:

$$\int_0^{\vartheta} e^{\varsigma y}(\vartheta - y)^{\kappa_i - 1} dy = \frac{e^{\varsigma \vartheta}}{\varsigma^{\kappa_i}} \int_0^{\vartheta} x^{\kappa_i - 1} e^{-x} dx \leq \frac{e^{\varsigma \vartheta}}{\varsigma^{\kappa_i}} \int_0^{\infty} x^{\kappa_i - 1} e^{-x} dx \leq \frac{e^{\varsigma \vartheta} \Gamma(\kappa_i)}{\varsigma^{\kappa_i}}, \quad i = 1, 2. \tag{10}$$

So,

$$\| \mathscr{Z}_i(\mathfrak{r}_1, \mathfrak{r}_2) - \mathscr{Z}_i(\mathfrak{s}_1, \mathfrak{s}_2) \|_B \leq \frac{\eta_i^*}{\varsigma^{\kappa_i}} \left(\|\mathbb{A}_i\| \|\mathfrak{r}_i - \mathfrak{s}_i\|_B + \sum_{j=1}^{2} \|\varpi_{i,j}\|_{L^\infty} \|\mathfrak{r}_j - \mathfrak{s}_j\|_B \right). \tag{11}$$

This implies that

$$\| \mathscr{Z}_i(\mathfrak{r}_1, \mathfrak{r}_2) - \mathscr{Z}_i(\mathfrak{s}_1, \mathfrak{s}_2) \|_B \leq \quad \mathfrak{M}_\varsigma \| (\mathfrak{r}_1, \mathfrak{r}_2) - (\mathfrak{s}_1, \mathfrak{s}_2) \|_B,$$

where

$$\mathfrak{M}_\varsigma = \begin{pmatrix} \frac{\eta_1^*}{\varsigma^{\kappa_1}} \|\mathbb{A}_1\| + \frac{\eta_1^*}{\varsigma^{\kappa_1}} \|\varpi_{1,1}\|_{L^\infty} & \frac{\eta_1^*}{\varsigma^{\kappa_1}} \|\varpi_{1,2}\|_{L^\infty} \\ \\ \frac{\eta_2^*}{\varsigma^{\kappa_2}} \|\varpi_{2,1}\|_{L^\infty} & \frac{\eta_2^*}{\varsigma^{\kappa_2}} \|\mathbb{A}_2\| + \frac{\eta_2^*}{\varsigma^{\kappa_2}} \|\varpi_{2,2}\|_{L^\infty} \end{pmatrix}. \tag{12}$$

By taking ς sufficiently large, it follows that the matrix $\mathfrak{M}_\varsigma$ converges to zero. Consequently, an application of Theorem 2 shows that $\mathscr{Z}$ has a unique fixed point. Therefore, the coupled system (1) has a unique solution in $\mathbb{K}$.

Next, we present the second result of this paper, which is based on Krasnosel'skii–type fixed point Theorem 3.

Theorem 5 Suppose that (H1) hold. If

(H3) There exist functions $\Psi_i \in L^\infty(\mathfrak{J}, \mathbb{R}_+)$ and real constants $\mho_i > 0$, $(i = 1, 2)$ such that

$$\|\Omega_i(\vartheta, x, u)\| \leq \Psi_i(\vartheta)(\mho_i + \|x\| + \|u\|), \quad i = 1, 2,$$

for all $x, u \in \mathbb{R}^n$, and $\vartheta \in \mathfrak{J}$.

Then, the coupled system (1) has at least one solution.

Proof. We define a closed, convex and bounded subset $\mathbb{J}_T$ of $\mathbb{K}$ by

$$\mathbb{J}_T = \{(\mathfrak{r}_1, \mathfrak{r}_2) \in \mathbb{K} : \ \|(\mathfrak{r}_1, \mathfrak{r}_2)\|_B < T\},$$

with $T = (T_1, T_2) \in \mathbb{R}_+^2$ such that

$$\begin{pmatrix} \delta_1 & \delta_2 \\ \\ \delta_3 & \delta_4 \end{pmatrix} \begin{pmatrix} \omega_1 \\ \\ \omega_2 \end{pmatrix} \leq \begin{pmatrix} T_1 \\ \\ T_2 \end{pmatrix}, \tag{13}$$

where ω_1, ω_2, δ_1, δ_2, δ_3 and δ_4 are positive real numbers that will be specified later.

Moreover, construct the operators $\mathscr{U} = (\mathscr{U}_1, \mathscr{U}_2)$ and $\mathscr{V} = (\mathscr{V}_1, \mathscr{V}_2)$ on $\mathbb{J}_T$ as:

$$\mathscr{U}_i(\mathfrak{r}_1, \mathfrak{r}_2)(\vartheta) = \frac{1}{\Gamma(\kappa_i - 1)} \int_0^{\vartheta} \int_0^{y} \exp\left(- \int_y^{\vartheta} \frac{\eta_1(\kappa_i, s)}{\eta_0(\kappa_i, s)} ds \right)$$

$$\times \frac{(y-\tau)^{\kappa_i - 2}}{\eta_0(\kappa_i, y)} \Omega_i(\tau, \mathfrak{r}_1(\tau), \mathfrak{r}_2(\tau)) d\tau dy, \quad i = 1, 2,$$

and

$$\mathscr{V}_i(\mathfrak{r}_1, \mathfrak{r}_2)(\vartheta) = \exp\left(- \int_0^{\vartheta} \frac{\eta_1(\kappa_i, s)}{\eta_0(\kappa_i, s)} ds \right) R_i + \frac{1}{\Gamma(\kappa_i - 1)} \int_0^{\vartheta} \int_0^{y} \exp\left(- \int_y^{\vartheta} \frac{\eta_1(\kappa_i, s)}{\eta_0(\kappa_i, s)} ds \right)$$

$$\times \frac{(y-\tau)^{\kappa_i - 2}}{\eta_0(\kappa_i, y)} \mathbb{A}_i \mathfrak{r}_i(\vartheta) d\tau dy, \quad i = 1, 2.$$

Obviously, both $\mathscr{U}$ and $\mathscr{V}$ are well defined due to (H1) and (H3). Furthermore, by [16, Lemma 4.1] the operator (5) may be written as:

$$(\mathfrak{r}_1, \mathfrak{r}_2) = (\mathscr{U}_1(\mathfrak{r}_1, \mathfrak{r}_2), \mathscr{U}_2(\mathfrak{r}_1, \mathfrak{r}_2)) + (\mathscr{V}_1(\mathfrak{r}_1, \mathfrak{r}_2), \mathscr{V}_2(\mathfrak{r}_1, \mathfrak{r}_2)) := \mathscr{Z}(\mathfrak{r}_1, \mathfrak{r}_2). \tag{14}$$

Thus, the fixed point of operator $\mathscr{Z}$ coincides with the solution of (1).

Step 1: We show that $\mathscr{U}(\mathfrak{r}_1, \mathfrak{r}_2) + \mathscr{V}(\mathfrak{s}_1, \mathfrak{s}_2) \in \mathbb{J}_T$, for any $(\mathfrak{r}_1, \mathfrak{r}_2)$, $(\mathfrak{s}_1, \mathfrak{s}_2) \in \mathbb{J}_T$.

For all $(\mathfrak{r}_1, \mathfrak{r}_2)$, $(\mathfrak{s}_1, \mathfrak{s}_2) \in \mathbb{K}$ and for each $\vartheta \in \mathfrak{J}$, by using (7) and (H3), we can get

$$\|(\mathscr{U}_i(\mathfrak{r}_1, \mathfrak{r}_2))(\vartheta)\|$$

$$\leq \frac{1}{\Gamma(\kappa_i - 1)} \int_0^{\vartheta} \int_0^{y} \frac{(y-\tau)^{\kappa_i - 2}}{\eta_0(\kappa_i, y)} \Psi_i(\vartheta)(\mho_i + \|\mathfrak{r}_1(\tau)\| + \|\mathfrak{r}_2(\tau)\|) d\tau dy$$

$$\leq \frac{\eta_i^*}{\Gamma(\kappa_i - 1)} \|\Psi_i\|_{L^\infty} \left(\mho_i + \|\mathfrak{r}_1\|_B + \|\mathfrak{r}_2\|_B \right) \int_0^{\vartheta} \int_0^{y} (y-\tau)^{\kappa_i - 2} e^{\varsigma\tau} d\tau dy, \quad i = 1, 2.$$

Analogously to proof of Theorem 4, we have

$$\|\mathscr{U}_i(\mathfrak{r}_1, \mathfrak{r}_2)\|_B \leq \frac{\eta_i^*}{\varsigma^{\kappa_i}} \|\Psi_i\|_{L^\infty} \Big(\mho_i + \|\mathfrak{r}_1\|_B + \|\mathfrak{r}_2\|_B \Big), \quad i = 1, 2.$$

Thus, the above inequalities can be expressed in the following vector form:

$$\|\mathscr{U}(\mathfrak{r}_1, \mathfrak{r}_2)\|_B := \begin{pmatrix} \|\mathscr{U}_1(\mathfrak{r}_1, \mathfrak{r}_2)\|_B \\ \\ \|\mathscr{U}_2(\mathfrak{r}_1, \mathfrak{r}_2)\|_B \end{pmatrix} \leq \mathfrak{N}_\varsigma \begin{pmatrix} \|\mathfrak{r}_1\|_B \\ \\ \|\mathfrak{r}_2\|_B \end{pmatrix} + \begin{pmatrix} \frac{\eta_1^* \mho_1}{\varsigma^{\kappa_1}} \|\Psi_1\|_{L^\infty} \\ \\ \frac{\eta_2^* \mho_2}{\varsigma^{\kappa_2}} \|\Psi_2\|_{L^\infty} \end{pmatrix}, \tag{15}$$

where

$$\mathfrak{N}_\varsigma = \begin{pmatrix} \frac{\eta_1^*}{\varsigma^{\kappa_1}} \|\Psi_1\|_{L^\infty} & \frac{\eta_1^*}{\varsigma^{\kappa_1}} \|\Psi_1\|_{L^\infty} \\ \\ \frac{\eta_2^*}{\varsigma^{\kappa_2}} \|\Psi_2\|_{L^\infty} & \frac{\eta_2^*}{\varsigma^{\kappa_2}} \|\Psi_2\|_{L^\infty} \end{pmatrix}.$$

Using the same technique, we can also obtain

$$\|\mathscr{V}_i(\mathfrak{s}_1, \mathfrak{s}_2)\|_B \leq \frac{\eta_i^*}{\varsigma^{\kappa_i}} \|\mathbb{A}_i\| \|\mathfrak{s}_i\|_B + R_i, \quad i = 1, 2.$$

Hence

$$\|\mathscr{V}(\mathfrak{s}_1, \mathfrak{s}_2)\|_B := \begin{pmatrix} \|\mathscr{V}_1(\mathfrak{s}_1, \mathfrak{s}_2)\|_B \\ \\ \|\mathscr{V}_2(\mathfrak{s}_1, \mathfrak{s}_2)\|_B \end{pmatrix} \leq \mathfrak{D}_\varsigma \begin{pmatrix} \|\mathfrak{s}_1\|_B \\ \\ \|\mathfrak{s}_2\|_B \end{pmatrix} + \begin{pmatrix} R_1 \\ \\ R_2 \end{pmatrix}, \tag{16}$$

where

$$\mathfrak{D}_\varsigma = \begin{pmatrix} \frac{\eta_1^*}{\varsigma^{\kappa_1}} \|\mathbb{A}_1\| & 0 \\ \\ 0 & \frac{\eta_2^*}{\varsigma^{\kappa_2}} \|\mathbb{A}_2\| \end{pmatrix}. \tag{17}$$

Combining (15) and (16), we get

$$\|\mathscr{U}(\mathfrak{r}_1, \mathfrak{r}_2)\|_B + \|\mathscr{V}(\mathfrak{s}_1, \mathfrak{s}_2)\|_B \leq \mathfrak{N}_\varsigma \begin{pmatrix} \|\mathfrak{r}_1\|_B \\ \\ \|\mathfrak{r}_2\|_B \end{pmatrix} + \mathfrak{D}_\varsigma \begin{pmatrix} \|\mathfrak{s}_1\|_B \\ \\ \|\mathfrak{s}_2\|_B \end{pmatrix} + \begin{pmatrix} \frac{\eta_1^* \mho_1}{\varsigma^{\kappa_1}} \|\Psi_1\|_{L^\infty} + R_1 \\ \\ \frac{\eta_2^* \mho_2}{\varsigma^{\kappa_2}} \|\Psi_2\|_{L^\infty} + R_2 \end{pmatrix}. \tag{18}$$

Now, we look for $T=(T_1,\,T_2)\in\mathbb{R}^2_+$, such that $\mathscr{U}(\mathfrak{r}_1,\,\mathfrak{r}_2)+\mathscr{V}(\mathfrak{s}_1,\,\mathfrak{s}_2)\in\mathbb{J}_T$ for any $(\mathfrak{r}_1,\,\mathfrak{r}_2)$, $(\mathfrak{s}_1,\,\mathfrak{s}_2)\in\mathbb{J}_T$. To this end, according to (18), it is sufficient to show

$$\mathfrak{B}_\varsigma\begin{pmatrix}T_1\\T_2\end{pmatrix}+\begin{pmatrix}\omega_1\\\omega_2\end{pmatrix}\leq\begin{pmatrix}T_1\\T_2\end{pmatrix},$$

where

$$\mathfrak{B}_\varsigma=\mathfrak{N}_\varsigma+\mathfrak{D}_\varsigma \text{ and } \begin{pmatrix}\omega_1\\\omega_2\end{pmatrix}=\begin{pmatrix}\frac{\eta_1^*\mho_1}{\varsigma^{\kappa_1}}\|\Psi_1\|_{L^\infty}+R_1\\\frac{\eta_2^*\mho_2}{\varsigma^{\kappa_2}}\|\Psi_2\|_{L^\infty}+R_2\end{pmatrix}.$$

Equivalently

$$\begin{pmatrix}\omega_1\\\omega_2\end{pmatrix}\leq(I-\mathfrak{B}_\varsigma)\begin{pmatrix}T_1\\T_2\end{pmatrix}. \tag{19}$$

For a sufficiently large ς, matrix $\mathfrak{B}_\varsigma$ is convergent to zero. It yields, from Theorem 1 that the matrix $(I-\mathfrak{B}_\varsigma)$ is nonsingular and $(I-\mathfrak{B}_\varsigma)^{-1}$ has nonnegative elements. Therefore, (19) is equivalent to

$$(I-\mathfrak{B}_\varsigma)^{-1}\begin{pmatrix}\omega_1\\\omega_2\end{pmatrix}\leq\begin{pmatrix}T_1\\T_2\end{pmatrix}.$$

Moreover, if we denote

$$(I-\mathfrak{B}_\varsigma)^{-1}=\begin{pmatrix}\delta_1&\delta_2\\\delta_3&\delta_4\end{pmatrix},$$

then we obtain

$$\begin{cases}\delta_1\omega_1+\delta_2\omega_2\leq T_1,\\\delta_3\omega_1+\delta_4\omega_2\leq T_2.\end{cases}$$

Which means that $\mathscr{U}(\mathfrak{r}_1,\,\mathfrak{r}_2)+\mathscr{V}(\mathfrak{s}_1,\,\mathfrak{s}_2)\in\mathbb{J}_T$.

Step 2: $\mathscr{V}$ is $\mathfrak{D}_\varsigma$-contraction mapping on $\mathbb{J}_T$.

In fact, for each $\vartheta \in \mathfrak{J}$ and for any $(\mathfrak{r}_1, \mathfrak{r}_2)$, $(\mathfrak{s}_1, \mathfrak{s}_2) \in \mathbb{J}_T$. By the same way of the proof of Theorem 4, we can easily show that

$$\|\mathscr{V}(\mathfrak{r}_1, \mathfrak{r}_2) - \mathscr{V}(\mathfrak{s}_1, \mathfrak{s}_2)\|_B \leq \mathfrak{D}_\varsigma \|(\mathfrak{r}_1, \mathfrak{r}_2) - (\mathfrak{s}_1, \mathfrak{s}_2)\|_B,$$

where $\mathfrak{D}_\varsigma$ is (17). Taking ς large enough it follows that the matrix $\mathfrak{D}_\varsigma$ is convergent to zero and thus, $\mathscr{V}$ is and $\mathfrak{D}_\varsigma$-contraction mapping on $\mathbb{J}_T$ with respect to the Bielecki norm.

Step 3: $\mathscr{U}$ is compact and continuous.

Firstly, let $\{(\mathfrak{r}_{1,n}, \mathfrak{r}_{2,n})\}$ in $\mathbb{J}_T$ be a sequence such that $(\mathfrak{r}_{1,n}, \mathfrak{r}_{2,n}) \to (\mathfrak{r}_1, \mathfrak{r}_2)$ in $\mathbb{J}_T$ as $n \to \infty$. For each $\vartheta \in \mathfrak{J}$, we have

$$\begin{aligned}
&\|(\mathscr{U}_i(\mathfrak{r}_{1,n}, \mathfrak{r}_{2,n}))(\vartheta) - (\mathscr{U}_i(\mathfrak{r}_1, \mathfrak{r}_2))(\vartheta)\| \\
&\leq \eta_i^* \int_0^\vartheta \int_0^y \exp\left(-\int_y^\vartheta \frac{\eta_1(\kappa_i, s)}{\eta_0(\kappa_i, s)} ds\right) \frac{(y-\tau)^{\kappa_i - 2}}{\Gamma(\kappa_i - 1)} \|\Omega_i(\tau, \mathfrak{r}_{1,n}(\tau), \mathfrak{r}_{2,n}(\tau)) - \Omega_i(\tau, \mathfrak{r}_1(\tau), \mathfrak{r}_2(\tau))\| d\tau dy \\
&\leq \eta_i^* \|\Omega_i(\cdot, \mathfrak{r}_{1,n}, \mathfrak{r}_{2,n}) - \Omega_i(\cdot, \mathfrak{r}_1, \mathfrak{r}_2)\|_B \int_0^\vartheta \int_0^y \frac{(y-\tau)^{\kappa_i - 2}}{\Gamma(\kappa_i - 1)} e^{\varsigma\tau} d\tau dy, \quad i = 1, 2.
\end{aligned}$$

Analogously to (10), we get

$$\|\mathscr{U}_i(\mathfrak{r}_{1,n}, \mathfrak{r}_{2,n}) - \mathscr{U}_i(\mathfrak{r}_1, \mathfrak{r}_2)\|_B \leq \frac{\eta_i^*}{\varsigma^{\kappa_i}} \|\Omega_i(\cdot, \mathfrak{r}_{1,n}, \mathfrak{r}_{2,n}) - \Omega_i(\cdot, \mathfrak{r}_1, \mathfrak{r}_2)\|_B, \quad i = 1, 2.$$

Since Ω_i, $i = 1, 2$ are continuous (H1), we have

$$\|\mathscr{U}_i(\mathfrak{r}_{1,n}, \mathfrak{r}_{2,n}) - \mathscr{U}_i(\mathfrak{r}_1, \mathfrak{r}_2)\|_B \to 0 \quad \text{as} \quad n \to \infty.$$

So, $\mathscr{U}$ is continuous.

Next we prove that $\mathscr{U}$ is uniformly bounded on $\mathbb{J}_T$. From (15), and for each $(\mathfrak{r}_1, \mathfrak{r}_2) \in \mathbb{J}_T$ we can get

$$\|\mathscr{U}(\mathfrak{r}_1, \mathfrak{r}_2)\|_B < \infty.$$

This proves that $\mathscr{U}$ is uniformly bounded.

Step 4: It remains to show that $\mathscr{U}(\mathbb{J}_T)$ is equicontinuous.

Let $(\mathfrak{r}_1, \mathfrak{r}_2) \in \mathbb{J}_T$ and any $\vartheta_1, \vartheta_2 \in \mathfrak{J}$, with $\vartheta_1 \leq \vartheta_2$, then

$$\|(\mathscr{U}_i(\mathfrak{r}_1, \mathfrak{r}_2))(\vartheta_2) - (\mathscr{U}_i(\mathfrak{r}_1, \mathfrak{r}_2))(\vartheta_1)\| \leq \quad \mathscr{Y}_{i,1} + \mathscr{Y}_{i,2}, \quad i = 1, 2,$$

where

$$\mathscr{Y}_{i,1}=\frac{1}{\Gamma(\kappa_i-1)}\int_{\vartheta_1}^{\vartheta_2}\int_0^y \exp\left(-\int_y^{\vartheta_2}\frac{\eta_1(\kappa_i,s)}{\eta_0(\kappa_i,s)}ds\right)\frac{(y-\tau)^{\kappa_i-2}}{\eta_0(\kappa_i,y)}\|\Omega_i(\tau,\mathfrak{r}_1(\tau),\mathfrak{r}_2(\tau))\|d\tau dy,\ i=1,2,$$

and

$$\mathscr{Y}_{i,2}=\frac{1}{\Gamma(\kappa_i-1)}\int_0^{\vartheta_1}\int_0^y\left|\exp\left(-\int_y^{\vartheta_2}\frac{\eta_1(\kappa_i,s)}{\eta_0(\kappa_i,s)}ds\right)-\exp\left(-\int_y^{\vartheta_1}\frac{\eta_1(\kappa_i,s)}{\eta_0(\kappa_i,s)}ds\right)\right|$$

$$\times\frac{(y-\tau)^{\kappa_i-2}}{\eta_0(\kappa_i,y)}\|\Omega_i(\tau,\mathfrak{r}_1(\tau),\mathfrak{r}_2(\tau))\|d\tau dy,\ i=1,2.$$

From one hand, by (H3) and (7), we can find

$$\mathscr{Y}_{i,1}\leq\frac{\eta_i^*}{\Gamma(\kappa_i-1)}\int_{\vartheta_1}^{\vartheta_2}\int_0^y(y-\tau)^{\kappa_i-2}\Psi_i(\vartheta)(\mho_i+\|\mathfrak{r}_1(\tau)\|+\|\mathfrak{r}_2(\tau)\|)d\tau dy$$

$$\leq\frac{\eta_i^*\|\Psi_i\|_{L^\infty}}{\Gamma(\kappa_i-1)}\Big(\mho_i+\|\mathfrak{r}_1\|+\|\mathfrak{r}_2\|\Big)\int_{\vartheta_1}^{\vartheta_2}\int_0^y(y-\tau)^{\kappa_i-2}d\tau dy$$

$$\leq\frac{\eta_i^*\|\Psi_i\|_{L^\infty}}{\Gamma(\kappa_i)}\Big(\mho_i+\|\mathfrak{r}_1\|+\|\mathfrak{r}_2\|\Big)\int_{\vartheta_1}^{\vartheta_2}y^{\kappa_i-1}dy$$

$$\leq\frac{\eta_i^*\|\Psi_i\|_{L^\infty}}{\Gamma(\kappa_i+1)}\Big(\mho_i+\|\mathfrak{r}_1\|+\|\mathfrak{r}_2\|\Big)\left[\vartheta_2^{\kappa_i}-\vartheta_1^{\kappa_i}\right]$$

$$\leq\frac{\eta_i^*\|\Psi_i\|_{L^\infty}}{\Gamma(\kappa_i+1)}\Big(\mho_i+\ell\|\mathfrak{r}_1\|_B+\ell\|\mathfrak{r}_2\|_B\Big)\left[\vartheta_2^{\kappa_i}-\vartheta_1^{\kappa_i}\right]$$

$$\leq\frac{\eta_i^*\|\Psi_i\|_{L^\infty}}{\Gamma(\kappa_i+1)}\Big(\mho_i+\ell T_1+\ell T_2\Big)\left[\vartheta_2^{\kappa_i}-\vartheta_1^{\kappa_i}\right],\ i=1,2.$$

Hence

$$\mathscr{Y}_{i,1}\to 0\quad\text{as}\quad\vartheta_1\to\vartheta_2,\quad i=1,2.$$

On the other hand, let $h(\vartheta)=\exp\left(-\int_0^{\vartheta}\frac{\eta_1(\kappa_i,s)}{\eta_0(\kappa_i,s)}ds\right)$. Using Lagrange Mean value theorem, there exist $t\in[\vartheta_1,\vartheta_2]\subset\mathfrak{J}$, such that

$$\frac{h(\vartheta_2)-h(\vartheta_1)}{\vartheta_2-\vartheta_1}\leq h'(t).$$

Then

$$|h(\vartheta_2)-h(\vartheta_1)| \leq \left|\frac{\eta_1(\kappa_i,t)}{\eta_0(\kappa_i,t)}\exp\left(-\int_0^{\vartheta}\frac{\eta_1(\kappa_i,s)}{\eta_0(\kappa_i,s)}ds\right)(\vartheta_2-\vartheta_1)\right| \leq \left|\frac{\eta_1(\kappa_i,t)}{\eta_0(\kappa_i,t)}\right|(\vartheta_2-\vartheta_1). \tag{20}$$

So

$$\mathscr{Y}_{i,2} \leq \frac{\eta_i^*\|\Psi_i\|_{L^\infty}}{\Gamma(\kappa_i+1)}\left(\upsilon_i+\ell\|\mathfrak{r}_1\|_B+\ell\|\mathfrak{r}_2\|_B\right)\left|\frac{\eta_1(\kappa_i,t)}{\eta_0(\kappa_i,t)}\right|(\vartheta_2-\vartheta_1)\vartheta_2^{\kappa_i}$$

$$\leq \frac{\eta_i^*\|\Psi_i\|_{L^\infty}}{\Gamma(\kappa_i+1)}\left(\upsilon_i+\ell T_1+\ell T_2\right)\left|\frac{\eta_1(\kappa_i,t)}{\eta_0(\kappa_i,t)}\right|(\vartheta_2-\vartheta_1)\vartheta_2^{\kappa_i},\ i=1,2.$$

Hence

$$\mathscr{Y}_{i,2}\to 0 \quad \text{as} \quad \vartheta_1\to\vartheta_2, \quad i=1,2.$$

Therefore,

$$\|(\mathscr{U}_i(\mathfrak{r}_1,\mathfrak{r}_2))(\vartheta_2)-(\mathscr{U}_i(\mathfrak{r}_1,\mathfrak{r}_2))(\vartheta_1)\|\to 0 \quad \text{as} \quad \vartheta_1\to\vartheta_2, \quad i=1,2.$$

Hence, by Lemma 1, we get

$$\|(\mathscr{U}(\mathfrak{r}_1,\mathfrak{r}_2))(\vartheta_2)-(\mathscr{U}(\mathfrak{r}_1,\mathfrak{r}_2))(\vartheta_1)\|\to 0 \quad \text{as} \quad \vartheta_1\to\vartheta_2.$$

We conclude that $\mathscr{U}(\mathbb{J}_T)$ is equicontinuous. By Arzelá-Ascoli's theorem, we deduce that $\mathscr{U}$ is a compact operator. Thus, all the hypotheses of Theorem 3 are satisfied. Hence, the operator $\mathscr{Z}$ has at least one fixed point $(\mathfrak{r}_1,\mathfrak{r}_2)\in\mathbb{J}_T$, which is just the solution of system (1).

Hyers–Ulam stability: We now close this section by studying the Hyers–Ulam stability of system (1). Define the following nonlinear operators $\Phi_1,\Phi_2\colon \mathbb{K}\to C(\mathfrak{J},\mathbb{R}^n)$:

$$\begin{cases} {}^{pc}\mathscr{D}_0^{\kappa_1}\mathfrak{r}_1(\vartheta)-\mathbb{A}_1\mathfrak{r}_1(\vartheta)-\Omega_1(\vartheta,\mathfrak{r}_1(\vartheta),\mathfrak{r}_2(\vartheta))=\Phi_1(\mathfrak{r}_1(\vartheta),\mathfrak{r}_2(\vartheta)), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \vartheta\in\mathfrak{J}:=[0,b], \\ {}^{pc}\mathscr{D}_0^{\kappa_2}\mathfrak{r}_2(\vartheta)-\mathbb{A}_2\mathfrak{r}_2(\vartheta)-\Omega_2(\vartheta,\mathfrak{r}_1(\vartheta),\mathfrak{r}_2(\vartheta))=\Phi_2(\mathfrak{r}_1(\vartheta),\mathfrak{r}_2(\vartheta)), \end{cases}$$

For some $\varepsilon_1,\varepsilon_2>0$, we consider the following inequality:

$$\begin{cases} \|\Phi_1(\mathfrak{r}_1(\vartheta),\mathfrak{r}_2(\vartheta))\|\leq\varepsilon_1, \\ \qquad\qquad\qquad\qquad \vartheta\in\mathfrak{J}. \\ \|\Phi_2(\mathfrak{r}_1(\vartheta),\mathfrak{r}_2(\vartheta))\|\leq\varepsilon_2, \end{cases} \tag{21}$$

Definition 6 The coupled system (1) is Hyers-Ulam stable if there exists a positive constant γ_i, $i = \overline{1, 4}$ such that for every $\varepsilon_1, \varepsilon_2 > 0$ and for each solution $(\mathfrak{z}_1, \mathfrak{z}_2) \in \mathbb{K}$ of inequality (21), there exists a solution $(\mathfrak{r}_1, \mathfrak{r}_2) \in \mathbb{K}$ of (1) with

$$\begin{cases} \|\mathfrak{z}_1(\vartheta) - \mathfrak{r}_1(\vartheta)\| \leq \gamma_1\varepsilon_1 + \gamma_2\varepsilon_2, \\ \\ \|\mathfrak{z}_2(\vartheta) - \mathfrak{r}_2(\vartheta)\| \leq \gamma_3\varepsilon_1 + \gamma_4\varepsilon_2, \end{cases} \quad \vartheta \in \mathfrak{J}.$$

Theorem 6 (Stability results) Assume that

1. The conditions of Theorem 4 are satisfied.
2. The inequalities (21) have at least one solution.

Then, the system (1) is Hyers-Ulam stable with respect to the Bielecki's norm.

Proof. Let $(\mathfrak{r}_1, \mathfrak{r}_2) \in \mathbb{K}$ be the solution of problem (1) satisfying (6) and let $(\mathfrak{z}_1, \mathfrak{z}_2)$ be any solution satisfying (21), then

$$\mathfrak{z}_i(\vartheta) = \mathscr{Z}_i(\mathfrak{z}_1, \mathfrak{z}_2)(\vartheta) + {}^{pc}\mathscr{I}_0^{\kappa_i}\Phi_i(\mathfrak{z}_1(\vartheta), \mathfrak{z}_2(\vartheta)), \quad i = 1, 2. \tag{22}$$

From (22), it follows that

$$\|\mathfrak{z}_i(\vartheta) - \mathscr{Z}_i(\mathfrak{z}_1, \mathfrak{z}_2)(\vartheta)\| \leq {}^{pc}\mathscr{I}_0^{\kappa_i}\|\Phi_i(\mathfrak{z}_1(\vartheta), \mathfrak{z}_2(\vartheta))\| \leq \left({}^{pc}\mathscr{I}_0^{\kappa_i}1\right)(\vartheta)\varepsilon_i, \quad i = 1, 2. \tag{23}$$

Thus, by (H2), and inequalities (8), (10), (23), we get

$$\begin{aligned} \|\mathfrak{z}_i(\vartheta) - \mathfrak{r}_i(\vartheta)\| =& \|\mathfrak{z}_i(\vartheta) - \mathscr{Z}_i(\mathfrak{z}_1, \mathfrak{z}_2)(\vartheta) + \mathscr{Z}_i(\mathfrak{z}_1, \mathfrak{z}_2)(\vartheta) - \mathfrak{r}_i(\vartheta)\| \\ \\ \leq& \|\mathfrak{z}_i(\vartheta) - \mathscr{Z}_i(\mathfrak{z}_1, \mathfrak{z}_2)(\vartheta)\| + \|\mathscr{Z}_i(\mathfrak{z}_1, \mathfrak{z}_2)(\vartheta) - \mathscr{Z}_i(\mathfrak{r}_1, \mathfrak{r}_2)(\vartheta)\| \\ \\ \leq& \left({}^{pc}\mathscr{I}_0^{\kappa_i}1\right)(\vartheta)\varepsilon_i + \frac{\eta_i^* e^{\varsigma\vartheta}}{\varsigma^{\kappa_i}}\left(\|\mathbb{A}_i\|\|\mathfrak{z}_i - \mathfrak{r}_i\|_B + \sum_{j=1}^{2}\|\varpi_{i,j}\|_{L^\infty}\|\mathfrak{z}_j - \mathfrak{r}_j\|_B\right), \ i = 1, 2. \end{aligned}$$

Hence, we get

$$\|\mathfrak{z}_i - \mathfrak{r}_i\|_B \leq \left({}^{pc}\mathscr{I}_0^{\kappa_i}1\right)(b)\varepsilon_i + \frac{\eta_i^*}{\varsigma^{\kappa_i}}\left(\|\mathbb{A}_i\|\|\mathfrak{z}_i - \mathfrak{r}_i\|_B + \sum_{j=1}^{2}\|\varpi_{i,j}\|_{L^\infty}\|\mathfrak{z}_j - \mathfrak{r}_j\|_B\right), \ i = 1, 2. \tag{24}$$

Inequalities (24) can be rewritten in matrix form as

$$(I-\mathfrak{M}_{\varsigma})\begin{pmatrix}\|\mathfrak{z}_1-\mathfrak{r}_1\|_B\\ \\ \|\mathfrak{z}_2-\mathfrak{r}_2\|_B\end{pmatrix}\leq\begin{pmatrix}\left({}^{pc}\mathscr{I}_0^{\kappa_1}1\right)(b)\varepsilon_1\\ \\ \left({}^{pc}\mathscr{I}_0^{\kappa_2}1\right)(b)\varepsilon_2\end{pmatrix} \tag{25}$$

where $\mathfrak{M}_{\varsigma}$ is the matrix given by (12). Since the matrix $\mathfrak{M}_{\varsigma}$ is convergent to zero for sufficiently large ς, it yields, from Theorem 1 that the matrix $(I-\mathfrak{M}_{\varsigma})$ is nonsingular and $(I-\mathfrak{M}_{\varsigma})^{-1}$ has nonnegative elements. Therefore, (25) is equivalent to

$$\begin{pmatrix}\|\mathfrak{z}_1-\mathfrak{r}_1\|_B\\ \\ \|\mathfrak{z}_2-\mathfrak{r}_2\|_B\end{pmatrix}\leq(I-\mathfrak{M}_{\varsigma})^{-1}\begin{pmatrix}\left({}^{pc}\mathscr{I}_0^{\kappa_1}1\right)(b)\varepsilon_1\\ \\ \left({}^{pc}\mathscr{I}_0^{\kappa_1}1\right)(b)\varepsilon_2\end{pmatrix},$$

which yields that

$$\begin{cases}\|\mathfrak{z}_1-\mathfrak{r}_1\|_B\leq\rho_1\left({}^{pc}\mathscr{I}_0^{\kappa_1}1\right)(b)\varepsilon_1+\rho_2\left({}^{pc}\mathscr{I}_0^{\kappa_2}1\right)(b)\varepsilon_2,\\ \\ \|\mathfrak{z}_2-\mathfrak{r}_2\|_B\leq\rho_3\left({}^{pc}\mathscr{I}_0^{\kappa_1}1\right)(b)\varepsilon_1+\rho_4\left({}^{pc}\mathscr{I}_0^{\kappa_2}1\right)(b)\varepsilon_2,\end{cases}\quad\vartheta\in\mathfrak{J}.$$

where ρ_i, $i=\overline{1,4}$ are the elements of the matrix $(I-\mathfrak{M}_{\varsigma})^{-1}$.

Hence, the coupled system (1) is Hyers-Ulam stable with respect to Bielecki's norm.

4. Examples

In this section, we present examples to illustrate the results established in the previous two sections.

Example 1: Consider the following fractional systems:

$$\begin{cases}{}^{pc}\mathscr{D}_0^{\frac{1}{2}}\mathfrak{r}_1(\vartheta)=\mathbb{A}_1\mathfrak{r}_1(\vartheta)+\Omega_1(\vartheta,\mathfrak{r}_1(\vartheta),\mathfrak{r}_2(\vartheta)),\quad\vartheta\in[0,1],\\ \\ {}^{pc}\mathscr{D}_0^{\frac{1}{3}}\mathfrak{r}_2(\vartheta)=\mathbb{A}_2\mathfrak{r}_2(\vartheta)+\Omega_2(\vartheta,\mathfrak{r}_1(\vartheta),\mathfrak{r}_2(\vartheta)),\quad\vartheta\in[0,1],\\ \\ \mathfrak{r}_1(0)=\mathfrak{r}_1(0)=(0,0),\end{cases} \tag{26}$$

where

$$n=2,\quad\kappa_1=\frac{1}{2},\quad\kappa_2=\frac{1}{3},\quad\mathbb{A}_1=\begin{pmatrix}1&2\\ \\ 3&4\end{pmatrix},\quad\mathbb{A}_2=\begin{pmatrix}3/2&5/4\\ \\ 1/2&1/4\end{pmatrix},$$

and Ω_1, Ω_2: $[0,1]\times\mathbb{R}^2\times\mathbb{R}^2\to\mathbb{R}^2$ such that, for $\mathfrak{r}_1=(r_1,r_2)$ and $\mathfrak{r}_2=(z_1,z_2)$, we have

$$\Omega_1(\vartheta, \mathfrak{r}_1(\vartheta), \mathfrak{r}_2(\vartheta)) = \begin{pmatrix} (\vartheta+1)r_1(\vartheta)+(\vartheta+1)r_2(\vartheta) \\ \\ e^{2\vartheta}\log_e(1+|z_1(\vartheta)|+|z_2(\vartheta)|) \end{pmatrix},$$

$$\Omega_2(\vartheta, \mathfrak{r}_1(\vartheta), \mathfrak{r}_2(\vartheta)) = \begin{pmatrix} \frac{e^{\vartheta}}{e^{z_1(\vartheta)+z_2(\vartheta)}} \\ \\ \vartheta\arctan(r_1(\vartheta)+r_2(\vartheta)) \end{pmatrix}.$$

Clearly, the functions Ω_1, Ω_2 are continuous. Moreover, for any u_1, u_2, $\bar{u}_1$, $\bar{u}_2 \in \mathbb{R}^2$ and $\vartheta \in [0, 1]$ we have

$$\|\Omega_1(\vartheta, u_1, u_2) - \Omega_1(\vartheta, \bar{u}_1, \bar{u}_2)\|_1, \leq \varpi_{1,1}(\vartheta)\|u_1 - \bar{u}_1\|_1 + \varpi_{1,2}(\vartheta)\|u_2 - \bar{u}_2\|_1$$

and

$$\|\Omega_2(\vartheta, u_1, u_2) - \Omega_1(\vartheta, \bar{u}_1, \bar{u}_2)\|_1, \leq \varpi_{2,1}(\vartheta)\|u_1 - \bar{u}_1\|_1 + \varpi_{2,2}(\vartheta)\|u_2 - \bar{u}_2\|_1$$

where $\|\cdot\|_1$ is a norm in $\mathbb{R}^2$ defined as follows

$$\|(u_1, u_2)\|_1 = |u_1| + |u_2|.$$

Hence the hypothesis (H2) is satisfied with

$$\varpi_{1,1}(\vartheta) = \vartheta + 1, \quad \varpi_{1,2}(\vartheta) = e^{2\vartheta}, \quad \varpi_{2,1}(\vartheta) = e^{\vartheta}, \quad \varpi_{2,2}(\vartheta) = \vartheta.$$

Obviously,

$$\|\varpi_{1,1}\|_{L^\infty} = 2, \quad \|\varpi_{1,2}\|_{L^\infty} = e^2, \quad \|\varpi_{2,1}\|_{L^\infty} = e, \quad \|\varpi_{2,2}\|_{L^\infty} = 1.$$

and

$$\|\mathbb{A}_1\|_{\max} = 4, \quad \|\mathbb{A}_2\|_{\max} = \frac{3}{2},$$

where $\|\cdot\|_{\max}$ is a norm of matrix $A = (a_{k,j})$ defined as follows

$$\|A\|_{\max} = \max_{k,j} |a_{k,j}|.$$

Furthermore, the matrix $\mathfrak{m}_\varsigma$ given by (12) has the following form

$$\mathfrak{M}_\varsigma = \begin{pmatrix} \frac{6\eta_1^*}{\varsigma^{\kappa_1}} & \frac{\eta_1^* e^2}{\varsigma^{\kappa_1}} \\ \\ \frac{\eta_2^* e}{\varsigma^{\kappa_2}} & \frac{5\eta_2^*}{2\varsigma^{\kappa_2}} \end{pmatrix}.$$

Taking ς large enough it follows that the matrix $\mathfrak{M}_\varsigma$ is convergent to zero and thus, an application of Theorem 4 shows that the coupled system (26) has a unique solution and is Hyers–Ulam stable.

Example 2. Consider the systems

$$\begin{cases} {}^{pc}\mathscr{D}_0^{\frac{1}{3}}\mathfrak{r}_1(\vartheta) = \frac{1}{4}\mathfrak{r}_1(\vartheta) + \Omega_1(\vartheta, \mathfrak{r}_1(\vartheta), \mathfrak{r}_2(\vartheta)), \quad \vartheta \in [0, e], \\ \\ {}^{pc}\mathscr{D}_0^{\frac{1}{3}}\mathfrak{r}_2(\vartheta) = \frac{1}{5}\mathfrak{r}_2(\vartheta) + \Omega_2(\vartheta, \mathfrak{r}_1(\vartheta), \mathfrak{r}_2(\vartheta)), \quad \vartheta \in [0, e], \\ \\ \mathfrak{r}_1(0) = \mathfrak{r}_1(0) = 0, \end{cases} \tag{27}$$

where

$$n = 1, \quad \kappa_1 = \frac{1}{3}, \quad \kappa_2 = \frac{1}{5}, \quad \mathbb{A}_1 = \frac{1}{4}, \quad \mathbb{A}_2 = \frac{1}{5},$$

and $\Omega_1, \Omega_2 \colon [0, e] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that,

$$\Omega_1(\vartheta, \mathfrak{r}_1(\vartheta), \mathfrak{r}_2(\vartheta)) = \frac{\log_e(1 + |\mathfrak{r}_1(\vartheta)| + |\mathfrak{r}_2(\vartheta)|)}{e^\vartheta + 13} + \frac{5}{e^\vartheta + 13},$$

$$\Omega_1(\vartheta, \mathfrak{r}_1(\vartheta), \mathfrak{r}_2(\vartheta)) = \frac{\arctan(\mathfrak{r}_1(\vartheta))}{\vartheta + 13} + \frac{|\mathfrak{r}_2(\vartheta)|}{(\vartheta + 13)(1 + |\mathfrak{r}_2(\vartheta)|)} + \frac{5}{\vartheta + 13}.$$

Clearly, the hypothesis (H1) of Theorem 5 is satisfied. Furthermore,

$$\|\Omega_1(\vartheta, \mathfrak{r}_1, \mathfrak{r}_2)\| \leq \frac{1}{e^\vartheta + 13}\left(5 + \|\mathfrak{r}_1\| + \|\mathfrak{r}_2\|\right),$$

and

$$\|\Omega_2(\vartheta, \mathfrak{r}_1, \mathfrak{r}_2)\| \leq \frac{1}{\vartheta + 13}\left(5 + \|\mathfrak{r}_1\| + \|\mathfrak{r}_2\|\right).$$

Consequently, the inequalities above ensure that condition (H3) of Theorem 5 is fulfilled, where

$$\Psi_1(\vartheta) = \frac{1}{e^{\vartheta}+13}, \quad \Psi_2(\vartheta) = \frac{1}{\vartheta+13}, \quad \vartheta \in [0, e], \quad \mho_1 = \mho_2 = 5.$$

It follows from Theorem 5 that the system (27), has at least one solution.

5. Concluding

Using Perov-type and Krasnosel'skii fixed point principles within the framework of generalized Banach spaces, we investigated a class of nonlinear hybrid proportional fractional differential systems. We established the existence, uniqueness, and Hyers-Ulam stability of solutions under minimal assumptions, thereby generalizing previous results in the literature. In a forthcoming paper, we aim to introduce a new hybrid proportional fractional derivative involving a non-singular kernel.

Author contributions

All authors contributed to the content and writing of the main manuscript. All authors reviewed the manuscript.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this research.

Confilict of interest

The authors declare no competing financial interest.

References

[1] Kilbas AA, Srivastava HM, Trujillo JJ. *Theory and Applications of Fractional Differential Equations*. North-Holland Mathematics Studies 204. Amsterdam, Netherlands: Elsevier Science; 2006.

[2] Samko SG, Kilbas AA, Marichev OI. *Fractional Integrals and Derivatives: Theory and Applications*. New York: Gordon and Breach; 1993.

[3] Norouzi F, N'Guérékata G. A study of ψ–Hilfer fractional differential system with application in financial crisis. *Chaos Solitons Fractals: X*. 2021; 6: 100056.

[4] Aghajani A, Pourhadi E, Trujillo J. Application of measure of noncompactness to a Cauchy problem for fractional differential equations in Banach spaces. *Fractional Calculus and Applied Analysis*. 2013; 16: 962-977.

[5] Das A, Hazarika B, Agarwal RP, Nashine HK. Solvability of infinite systems of fractional differential equations in the spaces of tempered sequences. *Filomat*. 2019; 33(17): 5519-5530.

[6] Haque I, Ali J, Mursaleen M. Existence of solutions for an infinite system of Hilfer fractional boundary value problems in tempered sequence spaces. *Alexandria Engineering Journal*. 2023; 65: 575-583.

[7] Tamilvanan K, Özger F, Mohiuddine SA, Ahmad N, Kabeto MJ. Fixed point technique: stability analysis of quadratic functional equation in various quasi-Banach spaces. *Journal of Mathematics*. 2025; 2025: 6689441.

[8] Micula S. Iterative numerical methods for a Fredholm-Hammerstein integral equation with modified argument. *Symmetry*. 2023; 15(1): 66. Available from: https://doi.org/10.3390/sym15010066.

[9] Alharbi MH, Ahmad J. A fresh look at Volterra integral equations: a fixed point approach in bipolar metric spaces. *AIMS Mathematics*. 2025; 10(4): 8926-8945. Available from: https://doi.org/10.3934/math.2025409.
[10] Özger F, Temizer Ersoy M, Ödemiş Özger Z. Existence of solutions: investigating Fredholm integral equations via a fixed-point theorem. *Axioms*. 2024; 13: 261. Available from: https://doi.org/10.3390/axioms13040261.
[11] Baleanu D, Fernandez A, Akgül A. On a fractional operator combining proportional and classical differintegrals. *Mathematics*. 2020; 8: 360.
[12] Acay B, Inc M. Fractional modeling of temperature dynamics of a building with singular kernels. *Chaos Solitons Fractals*. 2021; 142: 110482.
[13] Asjad MI, Ikram MD, Akgül A. Analysis of MHD viscous fluid flow through porous medium with novel power law fractional differential operator. *Physica Scripta*. 2020; 95(11): 115209.
[14] Ahmad M, Imran MA, Baleanu D, Alshomrani AS. Thermal analysis of magnetohydrodynamic viscous fluid with innovative fractional derivative. *Thermal Science*. 2020; 24(1): 351-359.
[15] Ikram MD, Asjad MI, Akgül A, Baleanu D. Effects of hybrid nanofluid on novel fractional model of heat transfer flow between two parallel plates. *Alexandria Engineering Journal*. 2021; 60(4): 3593-3604.
[16] Abbas M. Existence results and the Ulam stability for fractional differential equations with hybrid proportional–Caputo derivatives. *Journal of Nonlinear Functional Analysis*. 2020; 2020: 1-14.
[17] Abbas M. Ulam stability and existence results for fractional differential equations with hybrid proportional–Caputo derivatives. *Journal of Interdisciplinary Mathematics*. 2021; 25(2): 213-231.
[18] Vishnukumar KS, Govindaraj V. Controllability of the generalised proportional fractional dynamical systems having delays in control. *Journal of Control and Decision*. 2024; 1-16. Available from: https://doi.org/10.1080/23307706.2024.2441853.
[19] Lizama C, Murillo-Arcila M. On proportional hybrid operators in the discrete setting. *Mathematical Methods in the Applied Sciences*. 2025; 48(4): 4344-4364.
[20] Zentar O, Ziane M, Al Horani M. Theoretical study of a ϕ-Hilfer fractional differential system in Banach spaces. *Canadian Mathematical Bulletin*. 2024; 67(3): 742-759.
[21] Zentar O, Al Horani M, Ziane M. A study of a coupled system involving tempered Caputo derivatives with respect to functions. *Proceedings of the Mathematical Circle of Palermo*. 2025; 74(1): 17.
[22] Abbas S, Benchohra M, N'Guérékata GM. *Advanced Fractional Differential and Integral Equations*. New York: Nova Science Publishers; 2015.
[23] Varga RS. *Matrix Iterative Analysis*. Berlin: Springer-Verlag; 2000.
[24] Sousa J, Oliveira E. Existence, uniqueness, estimation and continuous dependence of the solutions of a nonlinear integral and an integrodifferential equations of fractional order. *arXiv:1806.01441*. 2018. Available from: https://doi.org/10.48550/arXiv.1806.01441.
[25] Petre I, Petrusel A. Krasnoselskii's theorem in generalized Banach spaces and application. *Electronic Journal of Qualitative Theory of Differential Equations*. 2012; 2012(85): 1-20.
[26] Rus IA. The theory of a metrical fixed point theorem: theoretical and applicative relevances. *Fixed Point Theory*. 2008; 9(2): 541-559.
[27] Precup R, Viorel A. Existence results for systems of nonlinear evolution equations. *International Journal of Pure and Applied Mathematics*. 2008; 47: 199-206.
[28] Rus I. *Generalized Contractions and Applications*. Cluj University Press; 2001.