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Research Article

Exploring Soliton Behavior in Nonlinear Schrödinger Equation with Higher-Order Terms Via Extended Direct Algebraic Method

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Abstract: We study the propagation of optical solitons in nonlinear media using a nonlinear Schrödinger equation with higher-order terms. We achieve accurate solutions as brilliant and dark solitons using the Extended Direct Algebraic Method (EDAM). The approach enables us to efficiently manage the dispersive and higher-order nonlinear factors. The impact of the coefficients on the soliton solutions is also investigated. Visual insights into the behaviour of the solitons are provided by the graphic analysis of the generated solutions using 2D and 3D plots. Furthermore, contour plots are created to show how the solutions vary depending on the different parameters, exposing complex structures and patterns. A greater comprehension of the physical phenomena that the equation describes is made possible by these graphical representations.

***Keywords*:** nonlinear Schrödinger equation, higher-order terms, optical solitons, Extended Direct Algebraic Method (EDAM), exact solutions

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1. Introduction

Nonlinear Partial Differential Equations (NLPDEs) are used in research and engineering, which are essential. They were employed to describe a variety of occurrences by simulating complex interactions and correlations between variables. These equations are used in many scientific fields, such as quantum physics, mathematical biology, fluid dynamics, and materials science [1–7]. By resolving NLPDEs, scientists can generate predictions, get important insights, and provide original solutions to issues that arise in the real world. Consequently, a number of techniques have been created to find precise solutions for NLPDEs. The modified extended direct algebraic methodology [8–12], the extended F-expansion technique [13], the modified extended mapping technique [14], the Riccati equation mapping technique [15], and the Kudryashov approach [16] are some of these techniques. Wave packets that are able to sustain themselves while

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travelling and keep their shape and speed are referred to as solitary waves, or solitons [17–22]. These solitary waves can occur in multiple modes in multi-mode fibres, each with unique properties. Solitary waves are created as a result of complex nonlinear dynamics resulting from the interaction between the modes. These solitary waves are a promising field for study in nonlinear optics and fibre optics because they can display complex behaviours like fission, fusion, and collision. For instance, Deng et al. [23] examined the features of multimode fibre propagation of soliton pulses and demonstrated how to efficiently control optical soliton properties. Additionally, they demonstrated how parameter selection can be used to influence soliton properties. According to research by Sun et al. [24], multi-mode soliton interactions in graded-index optical fibres cause the trailing soliton, which then runs into the front-running soliton. Others offered an understanding of the intricate dynamics of nonlinear graded-index solitons in multi-mode fibres and advances in knowledge regarding their behavior. Mayteevarunyoo et al. [25] created the dispersion management (DM) in a multi-mode system for three-dimensional (3D) solitons fibre optics. It is found that the parameter for DM strength influences quasi-stability in the three-dimensional spatiotemporal solitons noticed at reduced concentrations, and total stability at increased values. The research also looked at 3D solitons colliding, demonstrating quasielastic relationships. The Resonance nonlinear Schrödinger equation that reads as [26]:

$$\begin{cases} i\dfrac{\partial U}{\partial t}+a\dfrac{\partial^2 U}{\partial x^2}+(b_1|U|+b_2|U|^2)U \\ =i\sigma\dfrac{\partial U}{\partial x}+\beta|U|^2\dfrac{\partial U}{\partial x}+\gamma|U|^2U+\theta_1|U|^2\dfrac{\partial^2 U}{\partial x^2} \\ +\theta_2|U|^2\dfrac{\partial^2 U}{\partial x^2}+\theta_3U^2U^*\dfrac{\partial^2}{\partial x^2}. \end{cases} \tag{1}$$

$U(x,t)$ is the complex-valued dependent variable in this model, and its complex conjugate is $U^*(x,t)$; a represents the group velocity dispersion, and b_1 and b_2 collectively make up the quadratic–cubic nonlinearity. The self-steepening and nonlinear dispersion are represented by β and γ, respectively, on the right-hand side, whereas σ represents the intermodal dispersion. Lastly, the concepts that have θ_i for $i=1,2,3$ are mentioned in relation to metamaterials. This model simulates how solitons are activated using multi-mode fibres. Numerous scholarly investigations examined the proposed framework. For instance, Zhou et al. [27] employed the Hirota approach to produce dark solitons, dark double-hump solitons, and anti-dark solitons. They examined these solitons' transmission characteristics and discussed the conditions under which states transition between them. This work will employ the enhanced modified extended tanh function technique to develop various novel solutions for the suggested model. Solitons that are solitary, black, and brilliant are among these solutions. It is also possible to obtain hyperbolic and singular periodic solutions, Weierstrass elliptic functions, Jacobi elliptic functions, and other solutions.

This document is organized as follows: In Section 2, we define a conformable derivative. In Section 3, we describe the generalized extended direct algebraic method. In Section 4, we illustrate the proposed procedure for finding the exact solutions to the underlying equation. Section 5 gives graphical representations for individual outcomes. Finally, we provide a summary of the study carried out in Section 6.

2. Description of the conformable derivative

Definition 1 The conformable derivative of a function $f(t)$ of order $\alpha\in(0,1]$ at $t=t_0$ is defined as [28–30]:

$$C^t_\alpha f(t_0)=\lim_{\varepsilon\to 0}\frac{f(t_0+\varepsilon t^{1-\alpha})-f(t_0)}{\varepsilon}$$

Properties 1 The conformable derivative has several important properties, including:
1. Linearity: $C^t_\alpha(c_1 f + c_2 g) = c_1 C^t_\alpha(f) + c_2 C^t_\alpha(g)$.
2. Homogeneity: $C^t_\alpha(t^c) = ct^{c-\alpha}$.
3. Product rule: $C^t_\alpha(fg) = fC^t_\alpha(g) + gC^t_\alpha(f)$.
4. Quotient rule: $C^t_\alpha\left(\frac{f}{g}\right) = \frac{gC^t_\alpha(f) - fC^t_\alpha(g)}{g^2}$.
5. Relation to classical derivative: If $f(t)$ is a differentiable function (in the standard sense), then $C^t_\alpha(f)(t) = t^{1-\alpha}\frac{df}{dt}$.

These properties make the conformable derivative a useful tool for modeling and analyzing complex systems that exhibit fractional-order behavior.

Theorem 1 Let $f : (0, 1] \to \mathbb{R}$ be a function such that it is classically and conformably differentiable. Moreover, suppose that $g(t)$ is a differentiable function defined on the range of $f(t)$. Then we have:

$$C^t_\alpha(f \circ g)(t) = t^{1-\alpha} g'(f(t)) \cdot f'(t) \tag{2}$$

where primes stand for standard derivatives with respect to t.

3. Operational procedure of EDAM

The EDAM approach is described in this section. Examine the following FPDE [31–35]:

$$P(w, \partial_t^\alpha w, \partial_{y_1}^\beta w, \partial_{y_2}^\gamma w, w^2, \ldots) = 0, \;\; 0 < \alpha, \beta, \gamma \leq 1, \tag{3}$$

In this case, w is a function of t, $y_1, y_2, y_3, \ldots, y_r$.

Equation (3) can be solved using the steps listed below:

The variables $w(y_1, y_2, y_3, \ldots, y_r)$ are first transformed into $W(\xi)$, where ξ is defined in a number of ways. This transformation converts (3) into the following nonlinear ODE:

$$Q(W, W'W, W', \ldots) = 0, \tag{4}$$

where the derivatives of W with respect to ξ are given in (4). The constant(s) of integration can be obtained by integrating (4) once or more times.

2. Next, we'll assume that the answer to (4) is as follows:

$$W(\Omega) = \sum_{l=-M}^{M} d_l(\zeta(\Omega))^l, \tag{5}$$

where $\zeta(\Omega)$ is the general solution of the following ODE, and $d_l (l = -M, \cdots, 0, 1, 2, \cdots, M)$ are constants to be found:

$$\zeta'(\xi) = \ln(\mho)(d + e\zeta(\xi) + f(\zeta(\xi))^2), \tag{6}$$

where $\mho \neq 0, 1$ and the constants d, e, and f are present.

3. The positive integer M presented in (5) is obtained by establishing the homogeneous balance between the biggest nonlinear term and the highest order derivative in (4). More specifically, the two equations [34] can be used to estimate the balance number:

$$D\left(\frac{d^kW}{d\Omega^k}\right)=M+k \text{ and } D\left(W^j\left(\frac{d^kW}{d\Omega^k}\right)^l\right)=Mj+l(k+M),$$

where j,k, and l are positive integers and D denotes the degree of $W(\Omega)$ since $D[W(\Omega)]=M$.

4. Next, we integrate (4) to obtain the equation that follows, or we insert (5) into (4) and put all of the terms in $\zeta(\Omega)$ in the same order. After setting all of the coefficients of the ensuing polynomial to zero, an algebraic system of equations for d_l $(l=-M,\ldots,0,1,2,\ldots,M)$ and extra parameters is produced.

5. MAPLE is used to solve this set of algebraic problems.

6. After figuring out the unknown values, they are added to (5) along with the $\zeta(\Omega)$ (solution of equation (6)) to obtain the analytical solutions to (3).

4. Results

To solve the main partial differential equation, we first need to consider the following new complex transformations:

$$\Phi(x,t)=\Phi(\varsigma)\times exp^{i\left(\left(\frac{-q}{\zeta}\right)x^{\zeta}+\left(\frac{w}{\zeta}t^{\zeta}\right)\right)}, \tag{7}$$

where

$$\varsigma=\left(\frac{l}{\zeta}\right)x^{\zeta}-\left(\frac{l-\sigma-2aq}{\zeta}\right)t^{\zeta}$$

where l, σ, q, and w are unknown values which need to be determined. Inserting Eq. (8) into Eq. (7), after doing some cumbersome manipulations, reduces the original PDE to two equations from the real and imaginary parts.

The imaginary part yields

$$v=-\sigma-2aq, \quad 3B+2\gamma-2q(3\theta_1+\theta_2-\theta_3)=0. \tag{8}$$

$$v=-\sigma-2aq \tag{9}$$

and

$$3\beta+2\gamma-2q(3\theta_1+\theta_2-B_3)=0 \tag{10}$$

also

$$
\begin{cases} al^2\Phi'' - \left[w + aq^2 + \sigma q\right]\Phi + b_1\Phi^2 + \left[b_2 - Bq + q^2\theta_1 + q^2\theta_2 + q^2\theta_3\right]\Phi^3 \\ \\ - l^2(3\theta_1 + \theta_2 + \theta_3)\Phi^2\Phi'' - 6l^2\theta_1\Phi\Phi''^2 = 0 \end{cases} \tag{11}
$$

Then imposing the conditions $\theta_1 = 0$ and $\theta_2 = -\theta_3$ in (11) and (12), we get respectively [9]:

$$
q = \frac{-3B + 2\gamma}{4\theta_3} \tag{12}
$$

and

$$
al^2\Phi'' - \left[w + aq^2 + \sigma q\right]\Phi + b_1\Phi^2 + (b_2 - qB)\Phi^3 = 0 \tag{13}
$$

Now, applying the balancing principle to (14) for Φ^3 and Φ'' suggests

$$
3n_0 = n_0 + 2,
$$

therefore

$$
n_0 = 1.
$$

Case. 1

$$
\begin{aligned}
& B = 1/9\,\frac{9\,ab_2q^2 + 9\,aqb_2 + 2\,{b_1}^2 + 9\,wb_2}{q\,(w + aq^2 + aq)},\ \ a = a,\ \ l = \sqrt{\frac{w + aq^2 + aq}{a\Pi}}\,(\ln(\mu))^{-1}, \\
& d_{-1} = 3\,\frac{\sqrt{\Pi^{-1}}\,\left(w + aq^2 + \sigma q\right)\alpha}{b_1},\ \ \sigma = \sigma \\
& d_0 = 3/2\,\frac{aq^2 + aq + \beta\,\sqrt{\Pi^{-1}}w + \beta\,\sqrt{\Pi^{-1}}aq^2 + \beta\,\sqrt{\Pi^{-1}}aq + w}{b_1},\ \ d_1 = 0.
\end{aligned} \tag{14}
$$

Case. 2

$$B = 1/9\,\frac{9\,ab_2q^2+9\,aqb_2+2\,{b_1}^2+9\,wb_2}{q\,(w+aq^2+aq)},\;\; a=a,\;\; \sigma=\sigma,\;\; l=\sqrt{-\frac{w+aq^2+aq}{4\,a\delta\,\alpha-a\beta^2}}\,(\ln(\mu))^{-1},$$

$$d_{-1}=0,\;\; d_0 = 3/2\,\frac{aq^2+aq+\beta\,\sqrt{\Pi^{-1}}w+\beta\,\sqrt{\Pi^{-1}}aq^2+\beta\,\sqrt{\Pi^{-1}}aq+w}{b_1}, \tag{15}$$

$$d_1 = 3\,\frac{\sqrt{\Pi^{-1}}\left(w+aq^2+\sigma q\right)\delta}{b_1}.$$

Assuming case 1.

Family. 1.1: When $\Pi<0;\delta\neq 0$, we have

$$\Phi_{1,1}(x,t) = \exp^{i\left(\left(\frac{-q}{\zeta}\right)x^{\zeta}+\left(\frac{w}{\zeta}t^{\zeta}\right)\right)}\left[3\,\sqrt{-\Pi^{-1}}\left(w+aq^2+\sigma q\right)\alpha {b_1}^{-1}\left(-\frac{1}{2}\frac{\beta}{\delta}+\frac{1}{2}\frac{\sqrt{-\Pi}\tan\left(\frac{1}{2}\sqrt{-\Pi}\varsigma\right)}{\delta}\right)^{-1}\right.$$
$$\left.+\frac{3}{2}\frac{aq^2+aq+\beta\,\sqrt{-\Pi^{-1}}w+\beta\,\sqrt{-\Pi^{-1}}aq^2+\beta\,\sqrt{-\Pi^{-1}}aq+w}{b_1}\right], \tag{16}$$

$$\Phi_{1,2}(x,t) = exp^{i\left(\left(\frac{-q}{\zeta}\right)x^{\zeta}+\left(\frac{w}{\zeta}t^{\zeta}\right)\right)}\left[3\,\sqrt{-\Pi^{-1}}\left(w+aq^2+\sigma q\right)\alpha {b_1}^{-1}\left(-\frac{1}{2}\frac{\beta}{\delta}-\frac{1}{2}\frac{\sqrt{-\Pi}\cot\left(\frac{1}{2}\sqrt{-\Pi}\varsigma\right)}{\delta}\right)^{-1}\right.$$
$$\left.+\frac{3}{2}\frac{aq^2+aq+\beta\,\sqrt{\Pi^{-1}}w+\beta\,\sqrt{-\Pi^{-1}}aq^2+\beta\,\sqrt{-\Pi^{-1}}aq+w}{b_1}\right], \tag{17}$$

$$\Phi_{1,3}(x,t) = exp^{i\left(\left(\frac{-q}{\zeta}\right)x^{\zeta}+\left(\frac{w}{\zeta}t^{\zeta}\right)\right)}\left[3\,\sqrt{-\Pi^{-1}}\left(w+aq^2+\sigma q\right)\alpha {b_1}^{-1}\right.$$
$$\left(-\frac{1}{2}\frac{\beta}{\delta}+\frac{1}{2}\frac{\sqrt{-\Pi}\left(\tan\left(\sqrt{-\Pi}\varsigma\right)\pm\left(\sec\left(\sqrt{-\Pi}\varsigma\right)\right)\right)}{\delta}\right)^{-1} \tag{18}$$
$$\left.+\frac{3}{2}\frac{aq^2+aq+\beta\,\sqrt{-\Pi^{-1}}w+\beta\,\sqrt{-\Pi^{-1}}aq^2+\beta\,\sqrt{-\Pi^{-1}}aq+w}{b_1}\right],$$

$$\Phi_{1,4}(x,t) = exp^{i\left(\left(\frac{-q}{\zeta}\right)x^{\zeta}+\left(\frac{w}{\zeta}t^{\zeta}\right)\right)}\Bigg[3\sqrt{\Pi^{-1}}\left(w+aq^2+\sigma q\right)\alpha {b_1}^{-1}$$

$$\left(-\frac{1}{2}\frac{\beta}{\delta}+\frac{1}{2}\frac{\sqrt{-\Pi}\left(\cot\left(\sqrt{-\Pi}\varsigma\right)\pm\left(\csc\left(\sqrt{-\Pi}\varsigma\right)\right)\right)}{\upsilon}\right)^{-1} \tag{19}$$

$$+\frac{3}{2}\frac{aq^2+aq+\beta\sqrt{-\Pi^{-1}}w+\beta\sqrt{-\Pi^{-1}}aq^2+\beta\sqrt{-\Pi^{-1}}aq+w}{b_1}\Bigg],$$

and

$$\Phi_{1,5}(x,t) = exp^{i\left(\left(\frac{-q}{\zeta}\right)x^{\zeta}+\left(\frac{w}{\zeta}t^{\zeta}\right)\right)}\Bigg[3\sqrt{-\Pi^{-1}}\left(w+aq^2+\sigma q\right)\alpha {b_1}^{-1}$$

$$\left(-\frac{1}{2}\frac{\beta}{\delta}+1/4\,\frac{\sqrt{-\Pi}\left(\tan\left(1/4\sqrt{-\Pi}\varsigma\right)-\cot\left(1/4\sqrt{-\Pi}\varsigma\right)\right)}{\upsilon}\right)^{-1} \tag{20}$$

$$+\frac{3}{2}\frac{aq^2+aq+\beta\sqrt{-\Pi^{-1}}w+\beta\sqrt{-\Pi^{-1}}aq^2+\beta\sqrt{-\Pi^{-1}}aq+w}{b_1}\Bigg].$$

Family. 1.2: When $\Pi > 0; \delta \neq 0$, we have

$$\begin{aligned}\Phi_{1,6}(x,t) = exp^{i\left(\left(\frac{-q}{\zeta}\right)x^{\zeta}+\left(\frac{w}{\zeta}t^{\zeta}\right)\right)}\Bigg[3\sqrt{\Pi^{-1}}\left(w+aq^2+\sigma q\right)\alpha {b_1}^{-1}\left(-\frac{1}{2}\frac{\beta}{\delta}-\frac{1}{2}\frac{\sqrt{\Pi}\tanh\left(\frac{1}{2}\sqrt{\Pi}\varsigma\right)}{\upsilon}\right)^{-1} \\ +\frac{3}{2}\frac{aq^2+aq+\beta\sqrt{\Pi^{-1}}w+\beta\sqrt{\Pi^{-1}}aq^2+\beta\sqrt{\Pi^{-1}}aq+w}{b_1}\Bigg],\end{aligned} \tag{21}$$

$$\begin{aligned}\Phi_{1,7}(x,t) = exp^{i\left(\left(\frac{-q}{\zeta}\right)x^{\zeta}+\left(\frac{w}{\zeta}t^{\zeta}\right)\right)}\Bigg[3\sqrt{\Pi^{-1}}\left(w+aq^2+\sigma q\right)\alpha {b_1}^{-1}\left(-\frac{1}{2}\frac{\beta}{\delta}-1/2\,\frac{\sqrt{\Pi}\coth\left(\frac{1}{2}\sqrt{\Pi}\varsigma\right)}{\delta}\right)^{-1} \\ +\frac{3}{2}\frac{aq^2+aq+\beta\sqrt{\Pi^{-1}}w+\beta\sqrt{\Pi^{-1}}aq^2+\beta\sqrt{\Pi^{-1}}aq+w}{b_1}\Bigg],\end{aligned} \tag{22}$$

$$\Phi_{1,8}(x,t) = exp^{i\left(\left(\frac{-q}{\zeta}\right)x^{\zeta}+\left(\frac{w}{\zeta}t^{\zeta}\right)\right)}\left[3\sqrt{\Pi^{-1}}\left(w+aq^2+\sigma q\right)\alpha b_1{}^{-1}\right.$$

$$\left(-\frac{1}{2}\frac{\beta}{\delta}-\frac{1}{2}\frac{\sqrt{\Pi}\left(\tanh\left(\sqrt{\Pi}\varsigma\right)\pm\left(\operatorname{sech}\left(\sqrt{\Pi}\varsigma\right)\right)\right)}{\upsilon}\right)^{-1} \tag{23}$$

$$\left.+\frac{3}{2}\frac{aq^2+aq+\beta\sqrt{\Pi^{-1}}w+\beta\sqrt{\Pi^{-1}}aq^2+\beta\sqrt{\Pi^{-1}}aq+w}{b_1}\right],$$

$$\Phi_{1,9}(x,t) = exp^{i\left(\left(\frac{-q}{\zeta}\right)x^{\zeta}+\left(\frac{w}{\zeta}t^{\zeta}\right)\right)}\left[3\sqrt{\Pi^{-1}}\left(w+aq^2+\sigma q\right)\alpha b_1{}^{-1}\right.$$

$$\left(-\frac{1}{2}\frac{\beta}{\delta}-\frac{1}{2}\frac{\sqrt{\Pi}\left(\coth\left(\sqrt{-\Pi}\varsigma\right)\pm\left(sech\left(\sqrt{-\Pi}\varsigma\right)\right)\right)}{\delta}\right)^{-1} \tag{24}$$

$$\left.+\frac{3}{2}\frac{aq^2+aq+\beta\sqrt{\Pi^{-1}}w+\beta\sqrt{\Pi^{-1}}aq^2+\beta\sqrt{\Pi^{-1}}aq+w}{b_1}\right],$$

$$\Phi_{1,10}(x,t) = exp^{i\left(\left(\frac{-q}{\zeta}\right)x^{\zeta}+\left(\frac{w}{\zeta}t^{\zeta}\right)\right)}\left[3\sqrt{\Pi^{-1}}\left(w+aq^2+\sigma q\right)\alpha b_1{}^{-1}\right.$$

$$\left(-\frac{1}{2}\frac{\beta}{\delta}-1/4\,\frac{\sqrt{\Pi}\left(\tanh\left(1/4\sqrt{\Pi}\varsigma\right)-\coth\left(1/4\sqrt{\Pi}\varsigma\right)\right)}{\upsilon}\right)^{-1} \tag{25}$$

$$\left.+\frac{3}{2}\frac{aq^2+aq+\beta\sqrt{\Pi^{-1}}w+\beta\sqrt{\Pi^{-1}}aq^2+\beta\sqrt{\Pi^{-1}}aq+w}{b_1}\right].$$

Family. 1.3: If $\delta\alpha > 0$; $\Pi > 0$ and $\beta = 0$, we have

$$\Phi_{1,11}(x,t) = exp^{i\left(\left(\frac{-q}{\zeta}\right)x^{\zeta}+\left(\frac{w}{\zeta}t^{\zeta}\right)\right)}\left[3\sqrt{\Pi^{-1}}\left(w+aq^2+\sigma q\right)\alpha\frac{1}{\sqrt{\frac{\alpha}{\delta}}}\left(\tan\left(\sqrt{\delta\alpha}\varsigma\right)\right)^{-1}b_1{}^{-1}\right.$$

$$\left.+\frac{3}{2}\frac{w+aq^2+aq}{b_1}\right], \tag{26}$$

$$\Phi_{1,12}(x,t) = exp^{i\left(\left(\frac{-q}{\zeta}\right)x^{\zeta}+\left(\frac{w}{\zeta}t^{\zeta}\right)\right)}\left[-3\sqrt{\Pi^{-1}}\left(w+aq^2+\sigma q\right)\alpha\frac{1}{\sqrt{\frac{\alpha}{\delta}}}\left(\cot\left(\sqrt{\delta\,\alpha}\varsigma\right)\right)^{-1}{b_1}^{-1} + \frac{3}{2}\frac{w+aq^2+aq}{b_1}\right], \tag{27}$$

$$\Phi_{1,13}(x,t) = exp^{i\left(\left(\frac{-q}{\zeta}\right)x^{\zeta}+\left(\frac{w}{\zeta}t^{\zeta}\right)\right)}\left[3\sqrt{\Pi^{-1}}\left(w+aq^2+\sigma q\right)\alpha\frac{1}{\sqrt{\frac{\alpha}{\delta}}}\left(\tan\left(2\sqrt{\delta\,\alpha}\varsigma\right) \pm\left(\sec\left(2\sqrt{\delta\,\alpha}\varsigma\right)\right)\right)^{-1}{b_1}^{-1} + \frac{3}{2}\frac{w+aq^2+aq}{b_1}\right], \tag{28}$$

$$\Phi_{1,14}(x,t) = exp^{i\left(\left(\frac{-q}{\zeta}\right)x^{\zeta}+\left(\frac{w}{\zeta}t^{\zeta}\right)\right)}\left[-3\sqrt{\Pi^{-1}}\left(w+aq^2+\sigma q\right)\alpha\frac{1}{\sqrt{\frac{\alpha}{\delta}}}\left(\cot\left(2\sqrt{\delta\,\alpha}\varsigma\right) \pm\left(\csc\left(2\sqrt{\delta\,\alpha}\varsigma\right)\right)\right)^{-1}{b_1}^{-1} + \frac{3}{2}\frac{w+aq^2+aq}{b_1}\right], \tag{29}$$

and

$$\Phi_{1,15}(x,t) = exp^{i\left(\left(\frac{-q}{\zeta}\right)x^{\zeta}+\left(\frac{w}{\zeta}t^{\zeta}\right)\right)}\left[6\sqrt{\Pi^{-1}}\left(w+aq^2+\sigma q\right)\alpha\frac{1}{\sqrt{\frac{\alpha}{\delta}}}\left(\tan\left(1/2\sqrt{\delta\,\alpha}\varsigma\right) -\cot\left(1/2\sqrt{\delta\,\alpha}\varsigma\right)\right)^{-1}{b_1}^{-1} + \frac{3}{2}\frac{w+aq^2+aq}{b_1}\right]. \tag{30}$$

Family. 1.4: When $\beta = 0$; $\Pi > 0$ and $\delta\alpha < 0$, we have

$$\Phi_{1,16}(x,t) = exp^{i\left(\left(\frac{-q}{\zeta}\right)x^{\zeta}+\left(\frac{w}{\zeta}t^{\zeta}\right)\right)}\left[-3\sqrt{\Pi^{-1}}\left(w+aq^2+\sigma q\right)\alpha\frac{1}{\sqrt{-\frac{\alpha}{\delta}}}\left(\tanh\left(\sqrt{-\delta\,\alpha}\varsigma\right)\right)^{-1}{b_1}^{-1} + \frac{3}{2}\frac{w+aq^2+aq}{b_1}\right], \tag{31}$$

$$\Phi_{1,17}(x,t)=exp^{i\left((\frac{-q}{\zeta})x^{\zeta}+(\frac{w}{\zeta}t^{\zeta})\right)}\Bigg[-3\sqrt{\Pi^{-1}}\left(w+aq^{2}+\sigma q\right)\alpha\frac{1}{\sqrt{-\frac{\alpha}{\delta}}}\left(\coth\left(\sqrt{-\delta\,\alpha}\varsigma\right)\right)^{-1}{b_1}^{-1} \\ +\frac{3}{2}\frac{w+aq^{2}+aq}{b_1}\Bigg], \tag{32}$$

$$\Phi_{1,18}(x,t)=exp^{i\left((\frac{-q}{\zeta})x^{\zeta}+(\frac{w}{\zeta}t^{\zeta})\right)}\Bigg[-3\sqrt{\Pi^{-1}}\left(w+aq^{2}+\sigma q\right)\alpha\frac{1}{\sqrt{-\frac{\alpha}{\delta}}}\Bigg(\tanh\left(2\sqrt{-\delta\,\alpha}\varsigma\right) \\ \pm\left(isech\left(2\sqrt{-\delta\,\alpha}\varsigma\right)\right)\Bigg)^{-1}{b_1}^{-1}+\frac{3}{2}\frac{w+aq^{2}+aq}{b_1}\Bigg], \tag{33}$$

$$\Phi_{1,19}(x,t)=exp^{i\left((\frac{-q}{\zeta})x^{\zeta}+(\frac{w}{\zeta}t^{\zeta})\right)}\Bigg[-3\sqrt{\Pi^{-1}}\left(w+aq^{2}+\sigma q\right)\alpha\frac{1}{\sqrt{-\frac{\alpha}{\delta}}}\Bigg(\coth\left(2\sqrt{-\delta\,\alpha}\varsigma\right) \\ \pm\left(\operatorname{csch}\left(2\sqrt{-\delta\,\alpha}\varsigma\right)\right)\Bigg)^{-1}{b_1}^{-1}+\frac{3}{2}\frac{w+aq^{2}+aq}{b_1}\Bigg], \tag{34}$$

and

$$\Phi_{1,20}(x,t)=exp^{i\left((\frac{-q}{\zeta})x^{\zeta}+(\frac{w}{\zeta}t^{\zeta})\right)}\Bigg[-6\sqrt{\Pi^{-1}}\left(w+aq^{2}+\sigma q\right)\alpha\frac{1}{\sqrt{-\frac{\alpha}{\delta}}}\Bigg(\tanh\left(\frac{1}{2}\sqrt{-\delta\,\alpha}\varsigma\right) \\ +\coth\left(\frac{1}{2}\sqrt{-\delta\,\alpha}\varsigma\right)\Bigg)^{-1}{b_1}^{-1}+\frac{3}{2}\frac{w+aq^{2}+aq}{b_1}\Bigg], \tag{35}$$

Family. 1.5: When $\delta=\alpha$; $\Pi>0$ and $\beta=0$, we have

$$\Phi_{1,21}(x,t)=exp^{i\left((\frac{-q}{\zeta})x^{\zeta}+(\frac{w}{\zeta}t^{\zeta})\right)}\left[3\frac{\sqrt{\Pi^{-1}}\left(w+aq^{2}+\sigma q\right)\alpha}{b_1\tan\left(\alpha\,\varsigma\right)}+\frac{3}{2}\frac{w+aq^{2}+aq}{b_1}\right], \tag{36}$$

$$\Phi_{1,22}(x,t)=exp^{i\left((\frac{-q}{\zeta})x^{\zeta}+(\frac{w}{\zeta}t^{\zeta})\right)}\left[-3\frac{\sqrt{\Pi^{-1}}\left(w+aq^{2}+\sigma q\right)\alpha}{\cot\left(\alpha\,\varsigma\right)b_1}+\frac{3}{2}\frac{w+aq^{2}+aq}{b_1}\right], \tag{37}$$

$$\Phi_{1,23}(x,t)=exp^{i\left((\frac{-q}{\zeta})x^{\zeta}+(\frac{w}{\zeta}t^{\zeta})\right)}\left[3\frac{\sqrt{\Pi^{-1}}\left(w+aq^{2}+\sigma q\right)\alpha}{b_1\left(\tan\left(2\,\alpha\,\varsigma\right)\pm\left(\sec\left(2\,\alpha\,\varsigma\right)\right)\right)}+\frac{3}{2}\frac{w+aq^{2}+aq}{b_1}\right], \tag{38}$$

$$\Phi_{1,24}(x,t)=exp^{i\left((\frac{-q}{\zeta})x^{\zeta}+(\frac{w}{\zeta}t^{\zeta})\right)}\left[3\,\frac{\sqrt{\Pi^{-1}}\left(w+aq^{2}+\sigma q\right)\alpha}{b_1\left(-\cot\left(2\,\alpha\,\varsigma\right)\pm\left(\csc\left(2\,\alpha\,\varsigma\right)\right)\right)}+\frac{3}{2}\,\frac{w+aq^{2}+aq}{b_1}\right], \tag{39}$$

and

$$\Phi_{1,25}(x,t)=exp^{i\left((\frac{-q}{\zeta})x^{\zeta}+(\frac{w}{\zeta}t^{\zeta})\right)}\left[3\,\frac{\sqrt{\Pi^{-1}}\left(w+aq^{2}+\sigma q\right)\alpha}{b_1\left(\frac{1}{2}\tan\left(\frac{1}{2}\alpha\,\varsigma\right)-\frac{1}{2}\cot\left(\frac{1}{2}\alpha\,\varsigma\right)\right)}+\frac{3}{2}\,\frac{w+aq^{2}+aq}{b_1}\right]. \tag{40}$$

Family. 1.6: When $\delta=-\alpha$; $\Pi>0$ and $\beta=0$, we have

$$\Phi_{1,26}(x,t)=exp^{i\left((\frac{-q}{\zeta})x^{\zeta}+(\frac{w}{\zeta}t^{\zeta})\right)}\left[3\,\frac{\sqrt{\Pi^{-1}}\left(w+aq^{2}+\sigma q\right)\alpha}{\tanh\left(\alpha^{2}\right)b_1}+\frac{3}{2}\,\frac{w+aq^{2}+aq}{b_1}\right], \tag{41}$$

$$\Phi_{1,27}(x,t)=exp^{i\left((\frac{-q}{\zeta})x^{\zeta}+(\frac{w}{\zeta}t^{\zeta})\right)}\left[-3\,\frac{\sqrt{\Pi^{-1}}\left(w+aq^{2}+\sigma q\right)\alpha}{\coth\left(\alpha\,\varsigma\right)b_1}+\frac{3}{2}\,\frac{w+aq^{2}+aq}{b_1}\right], \tag{42}$$

$$\Phi_{1,28}(x,t)=exp^{i\left((\frac{-q}{\zeta})x^{\zeta}+(\frac{w}{\zeta}t^{\zeta})\right)}\left[3\,\frac{\sqrt{\Pi^{-1}}\left(w+aq^{2}+\sigma q\right)\alpha}{b_1\left(-\tanh\left(2\,\alpha\,\varsigma\right)\pm\left(isech\left(2\,\alpha\,\varsigma\right)\right)\right)}+\frac{3}{2}\,\frac{w+aq^{2}+aq}{b_1}\right], \tag{43}$$

$$\Phi_{1,29}(x,t)=exp^{i\left((\frac{-q}{\zeta})x^{\zeta}+(\frac{w}{\zeta}t^{\zeta})\right)}\left[3\,\frac{\sqrt{\Pi^{-1}}\left(w+aq^{2}+\sigma q\right)\alpha}{b_1\left(-\coth\left(2\,\alpha\,\varsigma\right)\pm\left(cech\left(2\,\alpha\,\varsigma\right)\right)\right)}+\frac{3}{2}\,\frac{w+aq^{2}+aq}{b_1}\right], \tag{44}$$

and

$$\Phi_{1,30}(x,t)=exp^{i\left((\frac{-q}{\zeta})x^{\zeta}+(\frac{w}{\zeta}t^{\zeta})\right)}\left[3\,\frac{\sqrt{\Pi^{-1}}\left(w+aq^{2}+\sigma q\right)\alpha}{b_1\left(-\frac{1}{2}\tanh\left(\frac{1}{2}\alpha\,\varsigma\right)-\frac{1}{2}\coth\left(\frac{1}{2}\alpha\,\varsigma\right)\right)}+\frac{3}{2}\,\frac{w+aq^{2}+aq}{b_1}\right]. \tag{45}$$

Family. 1.7: When $\beta=\rho$; $\Pi>0;\alpha=\pi\rho\,(\pi\neq0);\delta=0$, we have

$$\begin{aligned}\Phi_{1,31}(x,t)=exp^{i\left((\frac{-q}{\zeta})x^{\zeta}+(\frac{w}{\zeta}t^{\zeta})\right)}&\left[3\,\frac{\sqrt{\Pi^{-1}}\left(w+aq^{2}+\sigma q\right)\pi\,\rho}{b_1\left(\mu^{\rho\,\varsigma}-\pi\right)}\right.\\&\left.+\frac{3}{2}\,\frac{aq^{2}+aq+\rho\,\sqrt{\Pi^{-1}}w+\rho\,\sqrt{\Pi^{-1}}aq^{2}+\rho\,\sqrt{\Pi^{-1}}aq+w}{b_1}\right].\end{aligned} \tag{46}$$

Family. 1.8: When $\beta=0$; $\Pi>0$ and $\delta=0$, we have

$$\Phi_{1,32}(x,t)=exp^{i\left((\frac{-q}{\zeta})x^{\zeta}+(\frac{w}{\zeta}t^{\zeta})\right)}\left[3\,\frac{\sqrt{\Pi^{-1}}\left(w+aq^{2}+\sigma q\right)}{\varsigma\,\ln\left(\mu\right)b_1}+\frac{3}{2}\,\frac{w+aq^{2}+aq}{b_1}\right]. \tag{47}$$

Family. 1.9: When $\beta = 0; \alpha = 0$, we have

$$\Phi_{1,33}(x,t) = exp^{i\left(\left(\frac{-q}{\zeta}\right)x^{\zeta}+\left(\frac{w}{\zeta}t^{\zeta}\right)\right)}\left[\frac{3}{2}\frac{w+aq^2+aq}{b_1}\right]. \tag{48}$$

Family. 1.10: When $\alpha = 0; \beta \neq 0$; $\Pi > 0$ and $\delta \neq 0$, we have

$$\Phi_{1,34}(x,t) = exp^{i\left(\left(\frac{-q}{\zeta}\right)x^{\zeta}+\left(\frac{w}{\zeta}t^{\zeta}\right)\right)}\left[\frac{3}{2}\frac{aq^2+\sigma q+\beta\sqrt{\Pi^{-1}}w+\beta\sqrt{\Pi^{-1}}aq^2+\beta\sqrt{\Pi^{-1}}aq+w}{b_1}\right]. \tag{49}$$

Family. 1.11: $\beta = \rho; \delta = \Pi\rho$; $\Pi > 0$ and $\alpha = 0$, we have

$$\Phi_{1,35}(x,t) = exp^{i\left(\left(\frac{-q}{\zeta}\right)x^{\zeta}+\left(\frac{w}{\zeta}t^{\zeta}\right)\right)}\left[\frac{3}{2}\frac{aq^2+\sigma q+\rho\sqrt{\Pi^{-1}}w+\rho\sqrt{\Pi^{-1}}aq^2+\rho\sqrt{\Pi^{-1}}aq+w}{b_1}\right]. \tag{50}$$

Assuming case 2, we have
Family. 2.1: When $\Pi < 0; \delta \neq 0$, we have

$$\begin{aligned}\Phi_{2,1}(x,t) = exp^{i\left(\left(\frac{-q}{\zeta}\right)x^{\zeta}+\left(\frac{w}{\zeta}t^{\zeta}\right)\right)}&\left[3\sqrt{-\Pi^{-1}}\left(w+aq^2+\sigma q\right)\delta\left(-\frac{1}{2}\frac{\beta}{\delta}+\frac{1}{2}\frac{\sqrt{-\Pi}\tan\left(\frac{1}{2}\sqrt{-\Pi}\varsigma\right)}{\delta}\right){b_1}^{-1}\right.\\&\left.+\frac{3}{2}\frac{aq^2+aq+\beta\sqrt{-\Pi^{-1}}w+\beta\sqrt{-\Pi^{-1}}aq^2+\beta\sqrt{-\Pi^{-1}}aq+w}{b_1}\right],\end{aligned} \tag{51}$$

$$\begin{aligned}\Phi_{2,2}(x,t) = exp^{i\left(\left(\frac{-q}{\zeta}\right)x^{\zeta}+\left(\frac{w}{\zeta}t^{\zeta}\right)\right)}&\left[3\sqrt{-\Pi^{-1}}\left(w+aq^2+\sigma q\right)\delta\left(-\frac{1}{2}\frac{\beta}{\delta}-\frac{1}{2}\frac{\sqrt{-\Pi}\cot\left(\frac{1}{2}\sqrt{-\Pi}\varsigma\right)}{\delta}\right){b_1}^{-1}\right.\\&\left.+\frac{3}{2}\frac{aq^2+aq+\beta\sqrt{-\Pi^{-1}}w+\beta\sqrt{-\Pi^{-1}}aq^2+\beta\sqrt{-\Pi^{-1}}aq+w}{b_1}\right],\end{aligned} \tag{52}$$

$$\begin{aligned}\Phi_{2,3}(x,t) = exp^{i\left(\left(\frac{-q}{\zeta}\right)x^{\zeta}+\left(\frac{w}{\zeta}t^{\zeta}\right)\right)}&\left[3\sqrt{-\Pi^{-1}}\left(w+aq^2+\sigma q\right)\delta\right.\\&\left(-\frac{1}{2}\frac{\beta}{\delta}+\frac{1}{2}\frac{\sqrt{-\Pi}\left(\tan\left(\sqrt{-\Pi}\varsigma\right)\pm\left(\sec\left(\sqrt{-\Pi}\varsigma\right)\right)\right)}{\delta}\right){b_1}^{-1}\\&\left.+\frac{3}{2}\frac{aq^2+aq+\beta\sqrt{-\Pi^{-1}}w+\beta\sqrt{-\Pi^{-1}}aq^2+\beta\sqrt{-\Pi^{-1}}aq+w}{b_1}\right],\end{aligned} \tag{53}$$

$$\Phi_{2,4}(x,t)=exp^{i\left(\left(\frac{-q}{\zeta}\right)x^{\zeta}+\left(\frac{w}{\zeta}t^{\zeta}\right)\right)}\left[3\sqrt{-\Pi^{-1}}\left(w+aq^2+\sigma q\right)\delta\right.$$

$$\left(-\frac{1}{2}\frac{\beta}{\delta}+\frac{1}{2}\frac{\sqrt{-\Pi}\left(\cot\left(\sqrt{-\Pi}\varsigma\right)\pm\left(\csc\left(\sqrt{-\Pi}\varsigma\right)\right)\right)}{\upsilon}\right){b_1}^{-1} \tag{54}$$

$$\left.+\frac{3}{2}\frac{aq^2+aq+\beta\sqrt{-\Pi^{-1}}w+\beta\sqrt{-\Pi^{-1}}aq^2+\beta\sqrt{-\Pi^{-1}}aq+w}{b_1}\right],$$

$$\Phi_{2,5}(x,t)=exp^{i\left(\left(\frac{-q}{\zeta}\right)x^{\zeta}+\left(\frac{w}{\zeta}t^{\zeta}\right)\right)}\left[3\sqrt{-\Pi^{-1}}\left(w+aq^2+\sigma q\right)\delta\right.$$

$$\left(-\frac{1}{2}\frac{\beta}{\delta}+1/4\,\frac{\sqrt{-\Pi}\left(\tan\left(1/4\sqrt{-\Pi}\varsigma\right)-\cot\left(1/4\sqrt{-\Pi}\varsigma\right)\right)}{\upsilon}\right){b_1}^{-1} \tag{55}$$

$$\left.+\frac{3}{2}\frac{aq^2+aq+\beta\sqrt{-\Pi^{-1}}w+\beta\sqrt{-\Pi^{-1}}aq^2+\beta\sqrt{-\Pi^{-1}}aq+w}{b_1}\right],$$

Family. 2.2: When $\Pi>0;\delta\neq 0$, we have

$$\Phi_{2,6}(x,t)=exp^{i\left(\left(\frac{-q}{\zeta}\right)x^{\zeta}+\left(\frac{w}{\zeta}t^{\zeta}\right)\right)}\left[3\sqrt{\Pi^{-1}}\left(w+aq^2+\sigma q\right)\delta\right.$$

$$\left(-\frac{1}{2}\frac{\beta}{\delta}-\frac{1}{2}\frac{\sqrt{\Pi}\tanh\left(\frac{1}{2}\sqrt{\Pi}\varsigma\right)}{\upsilon}\right){b_1}^{-1} \tag{56}$$

$$\left.+\frac{3}{2}\frac{aq^2+aq+\beta\sqrt{\Pi^{-1}}w+\beta\sqrt{\Pi^{-1}}aq^2+\beta\sqrt{\Pi^{-1}}aq+w}{b_1}\right],$$

$$\Phi_{2,7}(x,t)=exp^{i\left(\left(\frac{-q}{\zeta}\right)x^{\zeta}+\left(\frac{w}{\zeta}t^{\zeta}\right)\right)}\left[3\sqrt{\Pi^{-1}}\left(w+aq^2+\sigma q\right)\delta\right.$$

$$\left(-\frac{1}{2}\frac{\beta}{\delta}-\frac{1}{2}\frac{\sqrt{\Pi}\coth\left(\frac{1}{2}\sqrt{\Pi}\varsigma\right)}{\delta}\right){b_1}^{-1} \tag{57}$$

$$\left.+\frac{3}{2}\frac{aq^2+aq+\beta\sqrt{\Pi^{-1}}w+\beta\sqrt{\Pi^{-1}}aq^2+\beta\sqrt{\Pi^{-1}}aq+w}{b_1}\right],$$

$$\Phi_{2,8}(x,t) = exp^{i\left(\left(\frac{-q}{\zeta}\right)x^{\zeta} + \left(\frac{w}{\zeta}t^{\zeta}\right)\right)}\Bigg[3\sqrt{\Pi^{-1}}\left(w + aq^2 + \sigma q\right)\delta$$

$$\left(-\frac{1}{2}\frac{\beta}{\delta} - \frac{1}{2}\frac{\sqrt{\Pi}\left(\tanh\left(\sqrt{\Pi}\varsigma\right) \pm \left(\operatorname{sech}\left(\sqrt{\Pi}\varsigma\right)\right)\right)}{\upsilon}\right){b_1}^{-1} \tag{58}$$

$$+3/2\,\frac{aq^2 + aq + \beta\sqrt{\Pi^{-1}}w + \beta\sqrt{\Pi^{-1}}aq^2 + \beta\sqrt{\Pi^{-1}}aq + w}{b_1}\Bigg],$$

$$\Phi_{2,9}(x,t) = exp^{i\left(\left(\frac{-q}{\zeta}\right)x^{\zeta} + \left(\frac{w}{\zeta}t^{\zeta}\right)\right)}\Bigg[3\sqrt{\Pi^{-1}}\left(w + aq^2 + \sigma q\right)\delta$$

$$\left(-\frac{1}{2}\frac{\beta}{\delta} - \frac{1}{2}\frac{\sqrt{\Pi}\left(\coth\left(\sqrt{\Pi}\varsigma\right) \pm \left(\operatorname{sech}\left(\sqrt{\Pi}\varsigma\right)\right)\right)}{\delta}\right){b_1}^{-1} \tag{59}$$

$$+3/2\,\frac{aq^2 + aq + \beta\sqrt{\Pi^{-1}}w + \beta\sqrt{\Pi^{-1}}aq^2 + \beta\sqrt{\Pi^{-1}}aq + w}{b_1}\Bigg],$$

and

$$\Phi_{2,10}(x,t) = exp^{i\left(\left(\frac{-q}{\zeta}\right)x^{\zeta} + \left(\frac{w}{\zeta}t^{\zeta}\right)\right)}\Bigg[3\sqrt{\Pi^{-1}}\left(w + aq^2 + \sigma q\right)\delta$$

$$\left(-\frac{1}{2}\frac{\beta}{\delta} - 1/4\,\frac{\sqrt{\Pi}\left(\tanh\left(1/4\sqrt{\Pi}\varsigma\right) - \coth\left(1/4\sqrt{\Pi}\varsigma\right)\right)}{\upsilon}\right){b_1}^{-1} \tag{60}$$

$$+3/2\,\frac{aq^2 + aq + \beta\sqrt{\Pi^{-1}}w + \beta\sqrt{\Pi^{-1}}aq^2 + \beta\sqrt{\Pi^{-1}}aq + w}{b_1}\Bigg].$$

Family. 2.3: When $\delta\alpha > 0$; $\Pi > 0$ and $\beta = 0$, we have

$$\Phi_{2,11}(x,t) = exp^{i\left(\left(\frac{-q}{\zeta}\right)x^{\zeta} + \left(\frac{w}{\zeta}t^{\zeta}\right)\right)}\Bigg[3\sqrt{\Pi^{-1}}\left(w + aq^2 + \sigma q\right)\delta\sqrt{\frac{\alpha}{\delta}}\tan\left(\sqrt{\alpha\delta}\varsigma\right){b_1}^{-1}$$

$$+3/2\,\frac{aq^2 + aq + \beta\sqrt{\Pi^{-1}}w + \beta\sqrt{\Pi^{-1}}aq^2 + \beta\sqrt{\Pi^{-1}}aq + w}{b_1}\Bigg], \tag{61}$$

$$\Phi_{2,12}(x,t)=exp^{i\left((\frac{-q}{\zeta})x^{\zeta}+(\frac{w}{\zeta}t^{\zeta})\right)}\left[-3\sqrt{\Pi^{-1}}\left(w+aq^2+\sigma q\right)\delta\sqrt{\frac{\alpha}{\delta}}\cot\left(\sqrt{\alpha\delta}\varsigma\right){b_1}^{-1}\right.$$
$$\left.+3/2\,\frac{aq^2+aq+\beta\sqrt{\Pi^{-1}}w+\beta\sqrt{\Pi^{-1}}aq^2+\beta\sqrt{\Pi^{-1}}aq+w}{b_1}\right], \tag{62}$$

$$\Phi_{2,13}(x,t)=exp^{i\left((\frac{-q}{\zeta})x^{\zeta}+(\frac{w}{\zeta}t^{\zeta})\right)}\left[3\sqrt{\Pi^{-1}}\left(w+aq^2+\sigma q\right)\delta\right.$$
$$\sqrt{\frac{\alpha}{\delta}}\left(\tan\left(2\sqrt{\alpha\delta}\varsigma\right)\pm\left(\sec\left(2\sqrt{\alpha\delta}\varsigma\right)\right)\right){b_1}^{-1} \tag{63}$$
$$\left.+3/2\,\frac{aq^2+aq+\beta\sqrt{\Pi^{-1}}w+\beta\sqrt{\Pi^{-1}}aq^2+\beta\sqrt{\Pi^{-1}}aq+w}{b_1}\right],$$

$$\Phi_{2,14}(x,t)=exp^{i\left((\frac{-q}{\zeta})x^{\zeta}+(\frac{w}{\zeta}t^{\zeta})\right)}\left[-3\sqrt{\Pi^{-1}}\left(w+aq^2+\sigma q\right)\delta\right.$$
$$\sqrt{\frac{\alpha}{\delta}}\left(\cot\left(2\sqrt{\alpha\delta}\varsigma\right)\pm\left(\csc\left(2\sqrt{\alpha\delta}\varsigma\right)\right)\right){b_1}^{-1} \tag{64}$$
$$\left.+3/2\,\frac{aq^2+aq+\beta\sqrt{\Pi^{-1}}w+\beta\sqrt{\Pi^{-1}}aq^2+\beta\sqrt{\Pi^{-1}}aq+w}{b_1}\right],$$

and

$$\Phi_{2,15}(x,t)=exp^{i\left((\frac{-q}{\zeta})x^{\zeta}+(\frac{w}{\zeta}t^{\zeta})\right)}\left[3/2\sqrt{\Pi^{-1}}\left(w+aq^2+\sigma q\right)\delta\right.$$
$$\sqrt{\frac{\alpha}{\delta}}\left(\tan\left(\frac{1}{2}\sqrt{\alpha\delta}\varsigma\right)-\cot\left(\frac{1}{2}\sqrt{\alpha\delta}\varsigma\right)\right){b_1}^{-1} \tag{65}$$
$$\left.+3/2\,\frac{aq^2+aq+\beta\sqrt{\Pi^{-1}}w+\beta\sqrt{\Pi^{-1}}aq^2+\beta\sqrt{\Pi^{-1}}aq+w}{b_1}\right].$$

Family. 2.4: When $\beta=0$; $\Pi>0$ and $\delta\alpha<0$, we have

$$\Phi_{2,16}(x,t)=exp^{i\left((\frac{-q}{\zeta})x^{\zeta}+(\frac{w}{\zeta}t^{\zeta})\right)}\left[-3\sqrt{\Pi^{-1}}\left(w+aq^2+\sigma q\right)\delta\sqrt{-\frac{\alpha}{\delta}}\tanh\left(\sqrt{-\alpha\,\delta}\varsigma\right){b_1}^{-1}\right.$$
$$\left.+3/2\,\frac{w+aq^2+aq}{b_1}\right], \tag{66}$$

$$\Phi_{2,17}(x,t)=exp^{i\left((\frac{-q}{\zeta})x^{\zeta}+(\frac{w}{\zeta}t^{\zeta})\right)}\left[-3\sqrt{\Pi^{-1}}\left(w+aq^2+\sigma q\right)\delta\sqrt{-\frac{\alpha}{\delta}}\coth\left(\sqrt{-\alpha\,\delta}\varsigma\right){b_1}^{-1}\right.$$
$$\left.+3/2\,\frac{w+aq^2+aq}{b_1}\right], \tag{67}$$

$$\Phi_{2,18}(x,t)=exp^{i\left((\frac{-q}{\zeta})x^{\zeta}+(\frac{w}{\zeta}t^{\zeta})\right)}\left[-3\sqrt{\Pi^{-1}}\left(w+aq^2+\sigma q\right)\delta\sqrt{-\frac{\alpha}{\delta}}\left(\tanh\left(2\sqrt{-\alpha\,\delta}\varsigma\right)\right.\right.$$
$$\left.\left.\pm\left(isech\left(2\sqrt{-\alpha\,\delta}\varsigma\right)\right)\right){b_1}^{-1}+3/2\,\frac{w+aq^2+aq}{b_1}\right], \tag{68}$$

$$\Phi_{2,19}(x,t)=exp^{i\left((\frac{-q}{\zeta})x^{\zeta}+(\frac{w}{\zeta}t^{\zeta})\right)}\left[-3\sqrt{\Pi^{-1}}\left(w+aq^2+\sigma q\right)\delta\sqrt{-\frac{\alpha}{\delta}}\left(\coth\left(2\sqrt{-\alpha\,\delta}\varsigma\right)\right.\right.$$
$$\left.\left.\pm\left(csch\left(2\sqrt{-\alpha\,\delta}\varsigma\right)\right)\right){b_1}^{-1}+3/2\,\frac{w+aq^2+aq}{b_1}\right], \tag{69}$$

and

$$\Phi_{2,20}(x,t)=exp^{i\left((\frac{-q}{\zeta})x^{\zeta}+(\frac{w}{\zeta}t^{\zeta})\right)}\left[-3/2\sqrt{\Pi^{-1}}\left(w+aq^2+\sigma q\right)\delta\right.$$
$$\sqrt{-\frac{\alpha}{\delta}}\left(\tanh\left(1/2\sqrt{-\alpha\,\delta}\varsigma\right)+\coth\left(1/2\sqrt{-\alpha\,\delta}\varsigma\right)\right){b_1}^{-1} \tag{70}$$
$$\left.+3/2\,\frac{w+aq^2+aq}{b_1}\right],$$

Family. 2.5: When $\delta=\alpha$; $\Pi>0$ and $\beta=0$, we have

$$\Phi_{2,21}(x,t) = exp^{i\left(\left(\frac{-q}{\zeta}\right)x^{\zeta}+\left(\frac{w}{\zeta}t^{\zeta}\right)\right)}\left[3\,\frac{\sqrt{\Pi^{-1}}\left(w+aq^{2}+\sigma q\right)\alpha\,\tan\left(\alpha\,\varsigma\right)}{b_{1}}+3/2\,\frac{w+aq^{2}+aq}{b_{1}}\right], \tag{71}$$

$$\Phi_{2,22}(x,t) = exp^{i\left(\left(\frac{-q}{\zeta}\right)x^{\zeta}+\left(\frac{w}{\zeta}t^{\zeta}\right)\right)}\left[-3\,\frac{\sqrt{\Pi^{-1}}\left(w+aq^{2}+\sigma q\right)\alpha\,\cot\left(\alpha\,\varsigma\right)}{b_{1}}+3/2\,\frac{w+aq^{2}+aq}{b_{1}}\right], \tag{72}$$

$$\begin{aligned}\Phi_{2,23}(x,t) = exp^{i\left(\left(\frac{-q}{\zeta}\right)x^{\zeta}+\left(\frac{w}{\zeta}t^{\zeta}\right)\right)}&\left[3\,\frac{\sqrt{\Pi^{-1}}\left(w+aq^{2}+\sigma q\right)\alpha\,\left(\tan\left(2\,\alpha\,\varsigma\right)\pm\left(\sec\left(2\,\alpha\,\varsigma\right)\right)\right)}{b_{1}}\right.\\ &\left.+3/2\,\frac{w+aq^{2}+aq}{b_{1}}\right],\end{aligned} \tag{73}$$

$$\begin{aligned}\Phi_{2,24}(x,t) = exp^{i\left(\left(\frac{-q}{\zeta}\right)x^{\zeta}+\left(\frac{w}{\zeta}t^{\zeta}\right)\right)}&\left[3\,\frac{\sqrt{\Pi^{-1}}\left(w+aq^{2}+\sigma q\right)\alpha\,\left(-\cot\left(2\,\alpha\,\varsigma\right)\pm\left(\csc\left(2\,\alpha\,\varsigma\right)\right)\right)}{b_{1}}\right.\\ &\left.+3/2\,\frac{w+aq^{2}+aq}{b_{1}}\right],\end{aligned} \tag{74}$$

and

$$\begin{aligned}\Phi_{2,25}(x,t) = exp^{i\left(\left(\frac{-q}{\zeta}\right)x^{\zeta}+\left(\frac{w}{\zeta}t^{\zeta}\right)\right)}&\left[3\,\frac{\sqrt{\Pi^{-1}}\left(w+aq^{2}+\sigma q\right)\alpha\,\left(\frac{1}{2}\tan\left(1/2\,\alpha\,\varsigma\right)-\frac{1}{2}\cot\left(\frac{1}{2}\alpha\,\varsigma\right)\right)}{b_{1}}\right.\\ &\left.+3/2\,\frac{w+aq^{2}+aq}{b_{1}}\right].\end{aligned} \tag{75}$$

Family. 2.6: When $\delta = -\alpha$; $\Pi > 0$ and $\beta = 0$, we have

$$\Phi_{2,26}(x,t) = exp^{i\left(\left(\frac{-q}{\zeta}\right)x^{\zeta}+\left(\frac{w}{\zeta}t^{\zeta}\right)\right)}\left[-3\,\frac{\sqrt{\Pi^{-1}}\left(w+aq^{2}+\sigma q\right)\alpha\,\tanh\left(\alpha^{2}\right)}{b_{1}}+3/2\,\frac{w+aq^{2}+aq}{b_{1}}\right], \tag{76}$$

$$\Phi_{2,27}(x,t) = 3\,\frac{\sqrt{\Pi^{-1}}\left(w+aq^{2}+\sigma q\right)\alpha\,\coth\left(\alpha\,\varsigma\right)}{b_{1}}+3/2\,\frac{w+aq^{2}+aq}{b_{1}}\Bigg], \tag{77}$$

$$\Phi_{2,28}(x,t) = exp^{i\left(\left(\frac{-q}{\zeta}\right)x^{\zeta}+\left(\frac{w}{\zeta}t^{\zeta}\right)\right)}\left[-3\,\frac{\sqrt{\Pi^{-1}}\left(w+aq^{2}+\sigma q\right)\alpha\left(-\tanh\left(2\,\alpha\,\varsigma\right)\pm\left(isech\left(2\,\alpha\,\varsigma\right)\right)\right)}{b_{1}}\right.$$
$$\left.+3/2\,\frac{w+aq^{2}+aq}{b_{1}}\right], \tag{78}$$

$$\Phi_{2,29}(x,t) = exp^{i\left(\left(\frac{-q}{\zeta}\right)x^{\zeta}+\left(\frac{w}{\zeta}t^{\zeta}\right)\right)}\left[-3\,\frac{\sqrt{\Pi^{-1}}\left(w+aq^{2}+\sigma q\right)\alpha\left(-\coth\left(2\,\alpha\,\varsigma\right)\pm\left(cech\left(2\,\alpha\,\varsigma\right)\right)\right)}{b_{1}}\right.$$
$$\left.+3/2\,\frac{w+aq^{2}+aq}{b_{1}}\right], \tag{79}$$

and

$$\Phi_{2,30}(x,t) = exp^{i\left(\left(\frac{-q}{\zeta}\right)x^{\zeta}+\left(\frac{w}{\zeta}t^{\zeta}\right)\right)}\left[-3\,\frac{\sqrt{\Pi^{-1}}\left(w+aq^{2}+\sigma q\right)\alpha\left(-\frac{1}{2}\tanh\left(\frac{1}{2}\alpha\,\varsigma\right)-\frac{1}{2}\coth\left(\frac{1}{2}\alpha\,\varsigma\right)\right)}{b_{1}}\right.$$
$$\left.+3/2\,\frac{w+aq^{2}+aq}{b_{1}}\right], \tag{80}$$

Family. 2.7: When $\beta=\rho;\alpha=\pi\rho(\pi\neq 0)$; $\Pi>0$ and $\delta=0$, we have

$$\Phi_{2,31}(x,t) = exp^{i\left(\left(\frac{-q}{\zeta}\right)x^{\zeta}+\left(\frac{w}{\zeta}t^{\zeta}\right)\right)}\left[3/2\,\frac{aq^{2}+\sigma q+\rho\sqrt{\Pi^{-1}}w+\rho\sqrt{\Pi^{-1}}aq^{2}+\rho\sqrt{\Pi^{-1}}aq+w}{b_{1}}\right]. \tag{81}$$

Family. 2.8: When $\beta=0;\delta=0$, we have

$$\Phi_{2,32}(x,t) = exp^{i\left(\left(\frac{-q}{\zeta}\right)x^{\zeta}+\left(\frac{w}{\zeta}t^{\zeta}\right)\right)}\left[3/2\,\frac{w+aq^{2}+aq}{b_{1}}\right]. \tag{82}$$

Family. 2.9: When $\beta=0$; $\Pi>0$ and $\alpha=0$, we have

$$\Phi_{2,33}(x,t) = exp^{i\left(\left(\frac{-q}{\zeta}\right)x^{\zeta}+\left(\frac{w}{\zeta}t^{\zeta}\right)\right)}\left[-3\,\frac{\sqrt{\Pi^{-1}}\left(w+aq^{2}+\sigma q\right)}{\varsigma\ln\left(\mu\right)b_{1}}+3/2\,\frac{w+aq^{2}+aq}{b_{1}}\right]. \tag{83}$$

Family. 2.10: When $\alpha=0;\beta\neq 0$; $\Pi>0$ and $\delta\neq 0$,

$$\Phi_{2,34}(x,t) = exp^{i\left(\left(\frac{-q}{\zeta}\right)x^{\zeta}+\left(\frac{w}{\zeta}t^{\zeta}\right)\right)}\left[-3\,\frac{\sqrt{\Pi^{-1}}\left(w+aq^2+\sigma q\right)\delta\beta}{\delta\left(\cosh\left(\beta\varsigma\right)-\sinh\left(\beta\varsigma\right)+1\right)b_1}\right.$$
$$\left.+3/2\,\frac{aq^2+aq+\beta\sqrt{\Pi^{-1}}w+\beta\sqrt{\Pi^{-1}}aq^2+\beta\sqrt{\Pi^{-1}}aq+w}{b_1}\right]. \tag{84}$$

Family. 2.11: When $\beta = \rho; \delta = \pi\rho$; $\Pi > 0$ and $\alpha = 0$,

$$\Phi_{2,35}(x,t) = exp^{i\left(\left(\frac{-q}{\zeta}\right)x^{\zeta}+\left(\frac{w}{\zeta}t^{\zeta}\right)\right)}\left[3\,\frac{\sqrt{\Pi^{-1}}\left(w+aq^2+\sigma q\right)\pi\rho\,\mu^{\rho\varsigma}}{\left(1-\pi\,\mu^{\rho\varsigma}\right)b_1}\right.$$
$$\left.+3/2\,\frac{aq^2+aq+\rho\sqrt{\Pi^{-1}}w+\rho\sqrt{\Pi^{-1}}aq^2+\rho\sqrt{\Pi^{-1}}aq+w}{b_1}\right]. \tag{85}$$

5. Graphical representation of some obtained solutions

The Schrödinger Equation is a powerful tool in quantum mechanics that helps us understand how tiny particles like atoms and electrons behave. In this study, we used a special method to visualize the patterns that emerge from this equation. Our results are presented in a series of detailed graphs that showcase the intricate structures and patterns of the wave solutions. These graphs provide a detailed look at how the wave solutions behave, highlighting their repeating patterns, symmetry, and complexity. The 2D graphs show the wave patterns in a flat space, while the 3D graphs offer a more complete view of the wave patterns in three-dimensional space. These visualizations allow us to see complex structures and patterns that are not visible in simpler representations. In order to comprehend and forecast the behaviour of complex systems, it is essential to comprehend the wave patterns in the Resonance Schrödinger Equation. This understanding has important ramifications for a number of disciplines, such as condensed matter physics, optical communications, and quantum mechanics. Our graphs illustrate this equation's rich dynamics, which offer important insights into how wave solutions behave in different physical systems. In conclusion, by offering a thorough illustration of the wave solutions and their complex structures, this work advances our knowledge of wave patterns in the Resonance Schrödinger Equation. Our findings underline the significance of further investigation into the behaviour of complex systems and have important ramifications for a range of physical applications. Our findings can be utilised to better understand and forecast the behaviour of complex systems, and this study opens up new directions for research in quantum mechanics and related fields. We can learn more about the behaviour of matter and energy at the quantum level by investigating and comprehending the wave patterns in the Resonance Schrödinger Equation.

Figures 1–8 present the graphical behavior of the obtained analytical solutions through 3D surface, contour, and 2D profile plots under different parameter settings. The figures demonstrate that the derived solutions exhibit a variety of localized wave structures, including bright, dark, and singular soliton profiles. The 3D plots illustrate the propagation and localization of the waves, the contour plots reveal their spatial distribution and symmetry, while the 2D profiles clearly show the variations in amplitude and waveform. It is observed from Figures 1–8 that changing the parameters α, β, δ, q, and w significantly influences the amplitude, width, and localization of the soliton solutions without altering their fundamental wave characteristics. In particular, increasing the nonlinear parameter β produces more pronounced changes in the wave profiles, confirming the strong influence of higher-order nonlinear effects on the dynamics of the nonlinear Schrödinger equation.

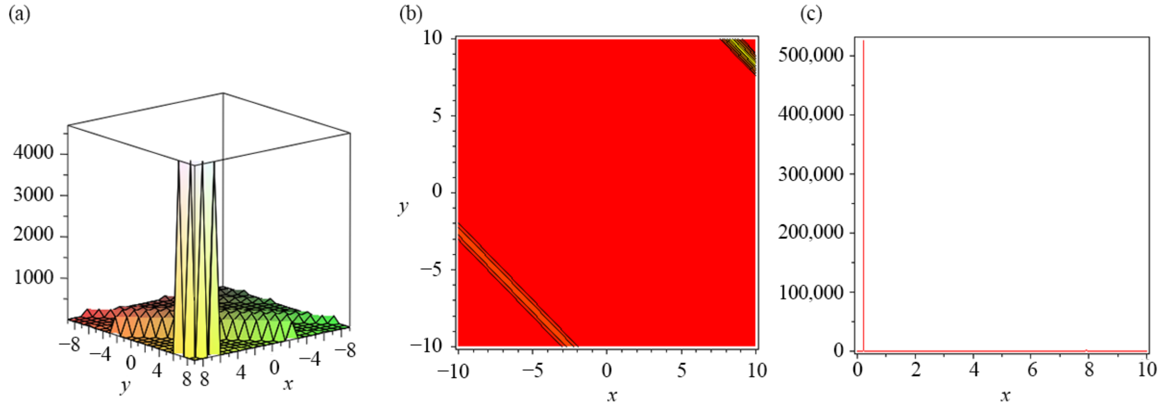

Figure 1. The 3D, contour & 2D plots for $\beta = 0.4$, $\alpha = 1$, $\delta = 11$, $\mu = \exp(1)$, $w = 5$, $a = 0.10$, $q = 0.2$, $\beta = 0$, $\alpha = -1$, $\delta = 1$, $\mu = \exp(1)$, $w = 9$, $a = 0.10$, $q = 2$, $b_1 = 0.93$, $r = 0.2$, $l = 0.10$, $r = 0.11$, $l = 0.19$, given by (16)

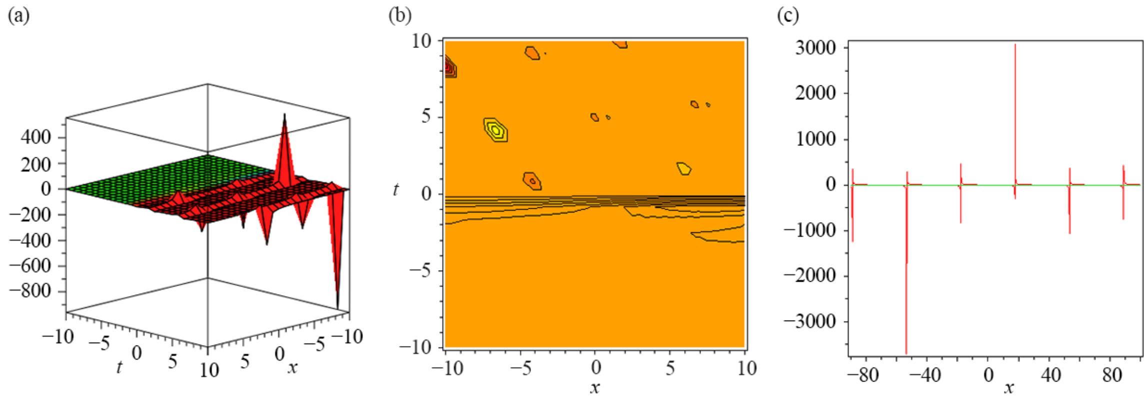

Figure 2. The 3D, contour & 2D plots for $\beta = 4$, $\alpha = 11$, $\delta = 11$, $\mu = \exp(1)$, $w = 5$, $a = 0.10$, $q = 2$, $\beta = 0$, $\alpha = -1$, $\delta = 1$, $\mu = \exp(1)$, $w = 9$, $a = 0.10$, $q = 2$, $b_1 = 0.93$, $r = 0.2$, $l = 0.10$, $r = 2$, $l = 1$. While a two-for $t = 5$ given by (17)

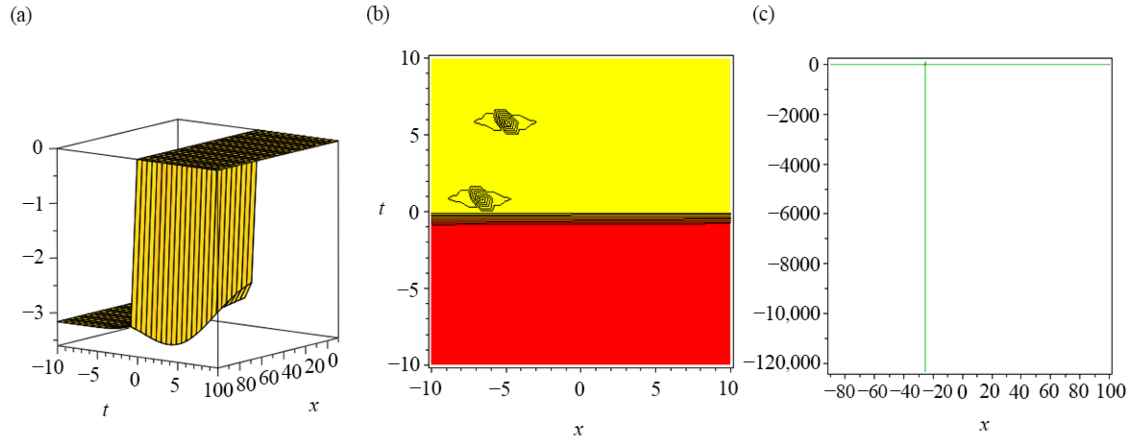

Figure 3. The 3D, contour & 2D plots for $\beta = 4$, $\alpha = 1$, $\delta = 1$, $\mu = \exp(1)$, $w = 5$, $a = 0.10$, $q = 2$, $\beta = 0$, $\alpha = -1$, $\delta = 1$, $\mu = \exp(1)$, $w = 9$, $a = 0.10$, $q = 2$, $b_1 = 0.93$, $r = 0.2$, $l = 0.10$, $r = 2$, $l = 1$. r $t = 5$ given by (21)

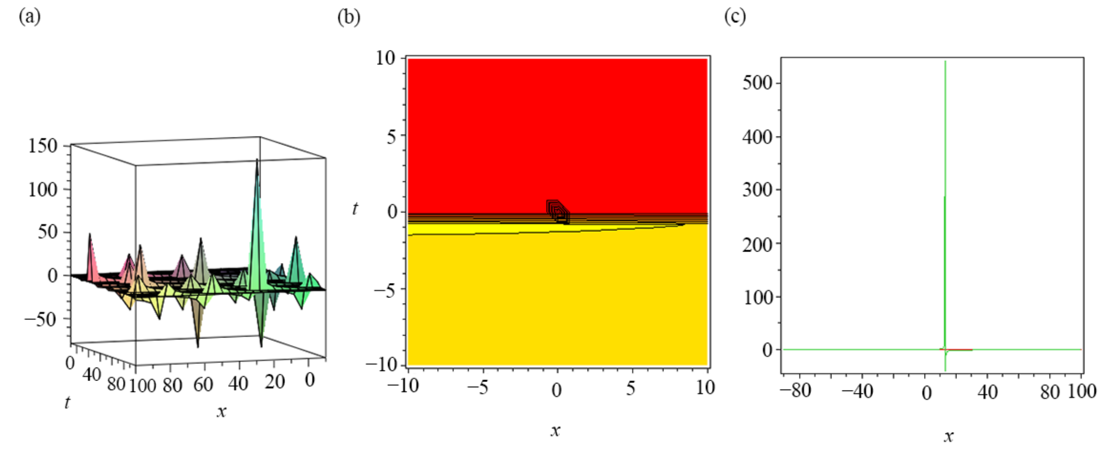

Figure 4. The 3D, contour & 2D plots for $\beta = 4$, $\alpha = 1$, $\delta = 1$, $\mu = \exp(1)$, $w = 5$, $a = 0.10$, $q = 2$, $\beta = 0$, $\alpha = -1$, $\delta = 1$, $\mu = \exp(1)$, $w = 9$, $a = 0.10$, $q = 2$, $b_1 = 0.93$, $r = 0.2$, $l = 0.10$, $l = 0.3$, $r = 0.2$, $l = 2$. $t = 5$ given by (22)

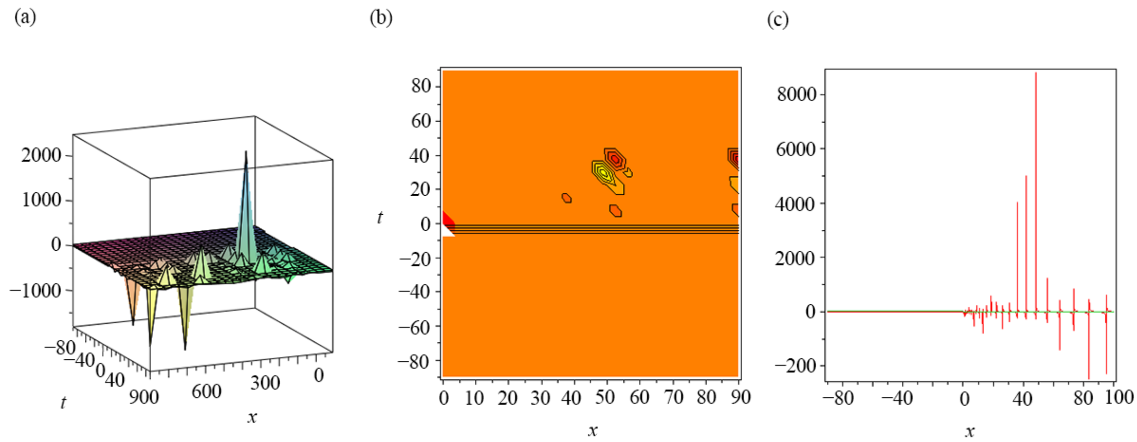

Figure 5. The 3D, contour & 2D plots for $\beta = 0$, $\alpha = 1$, $\delta = 1$, $\mu = \exp(1)$, $w = 9$, $a = 0.10$, $q = 2$, $b_1 = 0.93$, $r = 0.2$, $l = 10$. $t = 5$ given by (51)

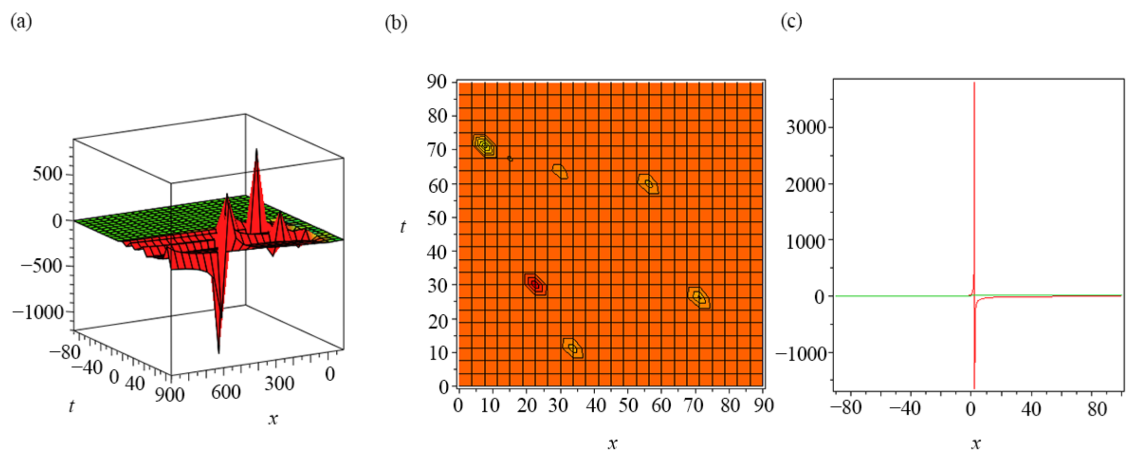

Figure 6. The 3D, contour & 2D plots for $\beta = 0$, $\alpha = 1$, $\delta = 1$, $\mu = \exp(1)$, $w = 9$, $a = 0.10$, $q = 2$, $\beta = 0$, $\alpha = -1$, $\delta = 1$, $\mu = \exp(1)$, $w = 9$, $a = 0.10$, $q = 2$, $b_1 = 0.93$, $r = 0.2$, $l = 0.10$, $l = 0.93$, $r = 0.2$, $l = 0.2$. r $t = 5$ given by (52)

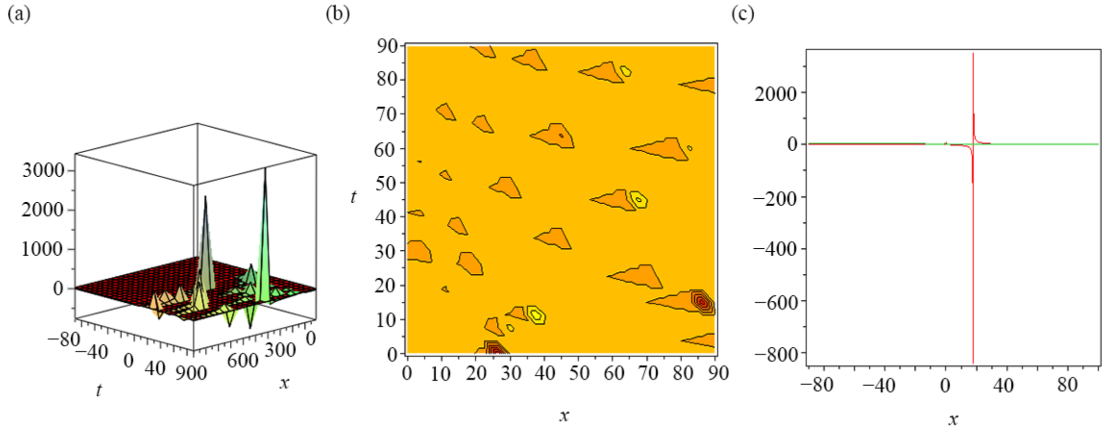

Figure 7. The 3D, contour & 2D plots for $\beta = 10$, $\alpha = 1$, $\delta = 1$, $\mu = \exp(1)$, $w = 9$, $a = 0.10$, $q = 2$, $\beta = 0$, $\alpha = -1$, $\delta = 1$, $\mu = \exp(1)$, $w = 9$, $a = 0.10$, $q = 2$, $b_1 = 0.93$, $r = 0.2$, $l = 0.10$, $l = 0.93$, $r = 0.2$, $l = 0.10$, given by (56)

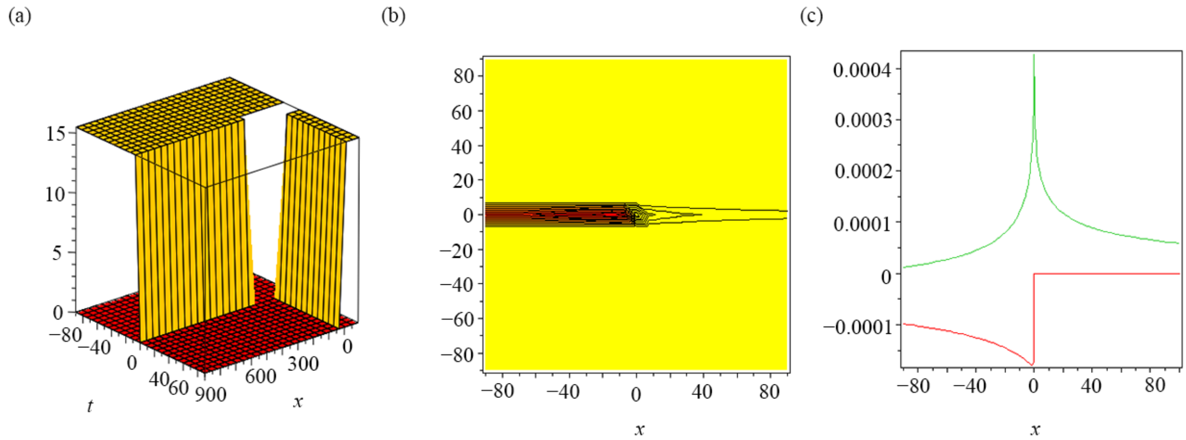

Figure 8. The 3D, contour & 2D plots for $\beta = 0$, $\alpha = -1$, $\delta = 1$, $\mu = \exp(1)$, $w = 9$, $a = 0.10$, $q = 2$, $b_1 = 0.93$, $r = 0.2$, $l = 0.10$, given by (57)

6. Conclusion

The successful application of the Extended Direct Algebraic Method (EDAM) to the Resonance Schrödinger Equation has significantly advanced our understanding of its behavior and solutions. The obtained solutions demonstrate the method's effectiveness in solving complex nonlinear equations, laying the groundwork for future research into the equation's characteristics.

Implications and applications:

The study's findings have important implications for various fields, including optics, condensed matter physics, and quantum mechanics. The solutions can be used to model and analyze complex physical phenomena, such as nonlinear optical processes and resonance effects in quantum systems. Specifically, the results can be applied to investigate quantum system behavior under resonant interactions, crucial for understanding phenomena like superconductivity and quantum entanglement.

Future research directions:

Potential avenues for future research include:

1. Exploring novel applications of the Resonance Schrödinger Equation.

2. Investigating nonlinear optical effects, such as four-wave mixing, third-harmonic generation, and second-harmonic generation.

3. Developing advanced optical materials and devices.

Author contributions

Conceptualization, A.A.L., P.O.M., and K.K.A.; Funding acquisition, P.O.M. and A.A.L.; Investigation, M.B., I.U., and M.A.Y.; Methodology, M.B., I.U., and M.A.Y.; Project administration, P.O.M. and A.A.L.; Software, M.B. and M.A.Y.; Supervision, P.O.M. and A.A.L.; Writing—original draft, M.A.Y. and M.B.; Writing—review & editing, A.A.L., P.O.M., and K.K.A. All authors have read and agreed to the published version of the manuscript.

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Data availability statement

Data is contained within the article or supplementary material.

Conflict of interest

The authors declare no competing financial interest.

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