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Research Article

Existence of Solutions and Ulam Stability Analysis of Implicit (*p*, *q*)-Fractional Difference Equations

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Abstract: This paper studies the existence theorems and Ulam stability results of solutions for implicit (p, q)-fractional difference equations. By applying Banach and Schauder fixed-point principles, we derive results related to the existence and uniqueness of solutions. Additionally, we analyze generalized Ulam-Hyers stability under (p, q)-Gronwall inequality. Key results are supported with illustrative examples, demonstrating the applicability of the proposed framework. Compared to previous studies restricted to the standard q-calculus, the present work introduces the (p, q)-Caputo fractional difference setting, which offers a more flexible and generalized approach. This novelty extends existing results and provides new perspectives for the analysis of stability and solvability of fractional systems.

***Keywords*:** implicit equation, (p, q)-fractional difference calculus, fixed point theorem, (p, q)-Gronwall inequality, generalized Ulam-Hyers-Rassias stability

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1. Introduction

The advancement of fractional calculus has profoundly influenced the study of differential equations, particularly in modeling complex systems across diverse scientific fields. In this context, (p, q)-calculus has emerged as an extension of q-calculus, introducing two parameters p and q that provide greater analytical flexibility. Several theoretical contributions have laid the foundation of (p, q)-calculus, including integral inequalities [1], operator theory applications [2, 3], combinatorial identities related to binomial coefficients and Stirling polynomials [4], Hermite-Hadamard type inequalities [5], Taylor formulas [6], and extensions of gamma and beta functions [7, 8].

Building on these theoretical developments, fractional and quantum calculus have been widely applied to solvability and stability problems. Houas and Samei [9] examined solvability and Ulam-Hyers-Rassias stability for generalized sequential quantum fractional pantograph equations, while Etemad et al. [10] applied quantum Laplace transforms to q-difference equations. AlMutairi et al. [11] investigated Hyers-Ulam, Rassias, and Mittag-Leffler stability in the β-calculus framework, and Alzabut et al. [12] considered coupled hybrid differential systems with sequential Caputo fractional q-derivatives.

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Regarding $(p,\ q)$-calculus, several works have addressed existence and stability results. Mesmouli et al. [13–15] studied integral boundary problems, integro-difference systems, and initial boundary value problems in the $(p,\ q)$-fractional setting. Boundary conditions were also analyzed by [16–18], focusing on nonlocal, separated, and anti-periodic cases, respectively. Further developments include coupled $(p,\ q)$-differential systems with Lipschitzian matrices studied by Boutiara et al. [19], as well as local existence and Ulam stability for nonlinear and neutral fractional differential equations explored by Khochemane et al. [20, 21].

The investigation of Ulam stability in fractional differential equations has attracted considerable interest due to its importance in validating the reliability of approximate solutions. Benchohra and Lazreg [22, 23] explored the stability properties of nonlinear implicit fractional differential equations with various fractional derivatives, contributing to the understanding of solution behaviors under perturbations. Abbas et al. [24] extended these ideas to q-difference equations, providing insights into stability within discrete frameworks. Recently, Dai and Zhang [25] investigated nonlinear implicit equations involving the Caputo-Katugampola derivative, further enriching the theoretical foundation of fractional stability analysis.

Despite these valuable contributions, the analysis of $(p,\ q)$-Caputo fractional difference equations remains relatively limited, highlighting the need for further research in this direction. So, in this paper, we consider the following implicit $(p,\ q)$-fractional difference equation

$$\begin{cases} {}^{c}\mathrm{D}_{p,\ q}^{\varkappa} w(\phi) = F\left(\phi,\ w(\phi),{}^{c}\mathrm{D}_{p,\ q}^{\varkappa} w(\phi)\right),\ \ 0 \le \phi \le 1, \\ w(0) = w_0, \end{cases} \tag{1}$$

where $0 < q < p \le 1$, ${}^{c}\mathrm{D}_{p,\ q}^{\varkappa}$ denotes the Caputo-type fractional $(p,\ q)$-derivative of order $\varkappa \in (0,\ 1]$ and $F : [0,\ 1] \times \mathbb{R}\mathbb{R} \to \mathbb{R}$ is given continuous functions.

The main contributions of this paper can be summarized as follows: (i) we establish new existence and uniqueness results for implicit $(p,\ q)$-fractional difference equations using Banach and Schauder fixed-point theorems; (ii) we extend Ulam-Hyers-Rassias stability to the $(p,\ q)$-Caputo setting by employing a suitable $(p,\ q)$-Gronwall inequality; (iii) we provide illustrative examples, including a numerical illustration, to demonstrate the applicability of our results. These contributions extend previous works in q-calculus and fractional calculus by incorporating the two-parameter $(p,\ q)$-approach, which allows for more refined analysis.

This paper is outlined as follows: Section 2 introduces key mathematical concepts and essential definitions relevant to $(p,\ q)$-calculus and fractional derivatives. Section 3 presents the existence results for implicit $(p,\ q)$-fractional difference equations utilizing fixed-point theorems. Section 4 delves into the stability analysis of these equations, focusing on Ulam stability. In section 5, several examples are provided to illustrate the theoretical findings. Finally, section 6 concludes the paper by summarizing the results and offering directions for future research.

2. Essential materials

This section introduces key concepts and materials essential for our analysis. We start by reviewing some foundational definitions and results of q-calculus and $(p,\ q)$-calculus, which can be found in [6–8, 26].

Let $C[0,\ 1]$ denote the space of continuous functions $w : [0,\ 1] \to \mathbb{R}$. This space is a complete metric space under the given metric

$$d(w,\ v) = \sup_{\phi \in [0,\ 1]} |w(\phi) - v(\phi)|.$$

The corresponding norm, often referred to as the supremum or uniform norm, is defined by:

$$\|w\|_\infty = \sup_{\phi \in [0,\ 1]} |w(\phi)|.$$

Thus, $C[0,\ 1]$ equipped with this uniform norm is a Banach space.

Moreover, let $L^1[0,\ 1]$ denote the space of measurable functions $w : I \to \mathbb{R}$ that are Lebesgue integrable. The norm in this space is given by:

$$\|w\|_1 = \int_0^1 |w(\phi)|\ d\phi.$$

Let $0 < q < p \leq 1$ be constants. We now present the following relations in $(p,\ q)$-calculus

$$[\Theta]_{p,\ q} := \begin{cases} \dfrac{p^\Theta - q^\Theta}{p-q} = p^{\Theta-1}[\Theta]_{\frac{q}{p}}, & \Theta \in \mathbb{N}^+, \\ 1, & \Theta = 0, \end{cases}$$

where

$$[\Theta]_{\frac{q}{p}} := \frac{1-\left(\dfrac{q}{p}\right)^\Theta}{1-\dfrac{q}{p}},$$

$$[\Theta]_{p,\ q}! := \begin{cases} [\Theta]_{p,\ q}[\Theta-1]_{p,\ q} \cdots [1]_{p,\ q} = \prod_{i=1}^{\Theta} \dfrac{p^i - q^i}{p-q}, & \Theta \in \mathbb{N}^+, \\ 1, & \Theta = 0. \end{cases}$$

The q-analogue of the power function $(\eta - \tau)_q^{(\mathfrak{n})}$ is given by

$$(\eta - \tau)_q^{(\mathfrak{n})} := \begin{cases} \prod_{i=0}^{\mathfrak{n}-1} \left(\eta - \tau q^i\right), & \mathfrak{n} \in \mathbb{N}^+,\ \eta,\ \tau \in \mathbb{R}, \\ 1, & \mathfrak{n} = 0. \end{cases}$$

The $(p,\ q)$-analogue of the power function $(\eta - \tau)_{p,\ q}^{(\mathfrak{n})}$ is defined by:

$$(\eta - \tau)_{p,\ q}^{(\mathfrak{n})} := \begin{cases} \prod_{i=0}^{\mathfrak{n}-1} \left(\eta p^i - \tau q^i\right), & \mathfrak{n} \in \mathbb{N}^+,\ \eta,\ \tau \in \mathbb{R}, \\ 1, & \mathfrak{n} = 0, \end{cases}$$

and for $\varkappa \in \mathbb{R}$, the general form of the above is given by:

$$(\eta-\tau)_{p,\,q}^{(\varkappa)} := p^{\binom{\varkappa}{2}}\,(\eta-\tau)_{\frac{q}{p}}^{(\varkappa)} = \eta^{\varkappa} p^{\binom{\varkappa}{2}} \prod_{i=0}^{\infty} \frac{\eta-\tau\left(\dfrac{q}{p}\right)^{i}}{\eta-\tau\left(\dfrac{q}{p}\right)^{\varkappa+i}}, \quad 0<\tau<\eta,$$

where $\binom{\varkappa}{2} := \dfrac{\varkappa(\varkappa-1)}{2}$.

Definition 1 [6] For $0<q<p\leq 1$, the $(p,\,q)$-derivative of a function w is defined as

$$\mathrm{D}_{p,\,q} w(\phi) := \frac{w(p\phi)-w(q\phi)}{(p-q)\,\phi}, \quad \phi\neq 0,$$

and $(\mathrm{D}_{p,\,q}w)(0) = \lim_{\phi\to 0}(\mathrm{D}_{p,\,q}w)(\phi)$, provided that w is differentiable at 0. Furthermore, the higher-order $(p,\,q)$-derivative, denotede as $\mathrm{D}^{n}_{p,\,q}w(\phi)$ is defined as follows:

$$\mathrm{D}^{n}_{p,\,q} w(\phi) = \begin{cases} \mathrm{D}_{p,\,q}\mathrm{D}^{n-1}_{p,\,q} w(\phi), & n\in\mathbb{N}^{+}, \\ w(\phi), & n=0. \end{cases}$$

Definition 2 [6] For $0<q<p\leq 1$, let w represent any function of a real variable ϕ. The $(p,\,q)$-integral of the function w is expressed as

$$\mathrm{I}_{p,\,q} w(\phi) := \int_{0}^{\phi} w(s)\,\mathrm{d}_{p,\,q}s = (p-q)\,\phi \sum_{i=0}^{\infty} \frac{q^{i}}{p^{i+1}} w\left(\frac{q^{i}}{p^{i+1}}\phi\right),$$

where the series on the right-hand side converges. In this case, w is said to be $(p,\,q)$-integrable over $[0,\,\phi]$.

Definition 3 [8] The $(p,\,q)$-Gamma function for $\varkappa\in\mathbb{R}$ is defined as

$$\Gamma_{p,\,q}(\varkappa) = \frac{(p-q)_{p,\,q}^{(\varkappa-1)}}{(p-q)^{\varkappa-1}},$$

with $\Gamma_{p,\,q}(\varkappa+1) = [\varkappa]_{p,\,q}\Gamma_{p,\,q}(\varkappa)$.

The $(p,\,q)$-Beta function for $\varkappa,\,\mu\in\mathbb{R}$ is expressed as

$$B_{p,\,q}(\mu,\,\varkappa) = \int_{0}^{1} s^{\mu-1}\,(1-qs)_{p,\,q}^{(\varkappa-1)}\,\mathrm{d}_{p,\,q}s, \tag{2}$$

and (2) can also be written as

$$B_{p,\,q}(\mu,\,\varkappa) = p^{(\varkappa-1)(2\mu+\varkappa-2)/2}\,\frac{\Gamma_{p,\,q}(\mu)\,\Gamma_{p,\,q}(\varkappa)}{\Gamma_{p,\,q}(\mu+\varkappa)}.$$

Definition 4 [8] For $\varkappa > 0$ and $0 < q < p \leq 1$, let $w : [0,\ 1] \to \mathbb{R}$ be an arbitrary function. The fractional $(p,\ q)$-integral of order $\varkappa$ is defined by

$$\mathrm{I}^{\varkappa}_{p,\ q} w(\phi) = \frac{1}{p^{\binom{\varkappa}{2}} \Gamma_{p,\ q}(\varkappa)} \int_0^{\phi} (\phi - qs)^{(\varkappa-1)}_{p,\ q} w\left(\frac{s}{p^{\varkappa-1}}\right) \mathrm{d}_{p,\ q} s,$$

and $\mathrm{I}^{0}_{p,\ q} w(\phi) = w(\phi)$.

Definition 5 [8] For $\varkappa \in (0,\ 1]$ and $0 < q < p \leq 1$. The Caputo-type fractional $(p,\ q)$-derivative of order $\varkappa$ of an arbitrary function w over $[0,\ 1]$is defined as

$$\begin{aligned} {}^{\mathrm{c}}\mathrm{D}^{\varkappa}_{p,\ q} w(\phi) &= \mathrm{I}^{1-\varkappa}_{p,\ q} \mathrm{D}_{p,\ q} w(\phi) \\ &= \frac{1}{p^{\binom{1-\varkappa}{2}} \Gamma_{p,\ q}(1-\varkappa)} \int_0^{\phi} (\phi - qs)^{(-\varkappa)}_{p,\ q} \mathrm{D}_{p,\ q} w\left(\frac{s}{p^{-\varkappa}}\right) \mathrm{d}_{p,\ q} s, \end{aligned}$$

and ${}^{\mathrm{c}}\mathrm{D}^{0}_{p,\ q} w(\phi) = w(\phi)$.

Lemma 1 [8] Let $\varkappa \in (\Theta - 1,\ \Theta]$, $\Theta \in \mathbb{N}$ and $0 < q < p \leq 1$. Let $w : [0,\ 1] \to \mathbb{R}$, then we have

$$\mathrm{I}^{\varkappa}_{p,\ q} \left({}^{\mathrm{c}}\mathrm{D}^{\varkappa}_{p,\ q} w(\phi)\right) = w(\phi) - \sum_{k=0}^{\Theta-1} \frac{\phi^k}{p^{\binom{\varkappa}{2}} \Gamma_{p,\ q}(k+1)} \mathrm{D}^{k}_{p,\ q} w(0), \tag{3}$$

Indeed, for equation ${}^{\mathrm{c}}\mathrm{D}^{\varkappa}_{p,\ q} w(\phi) = 0$, the general solution is expressed as

$$w(\phi) = c_0 + c_1 \phi + c_2 \phi^2 + \ldots + c_{\Theta-1} \phi^{\ \Theta-1},$$

where $c_0,\ c_1,\ c_2,\ \ldots,\ c_{\Theta-1} \in \mathbb{R}$. In addition,

$${}^{\mathrm{c}}\mathrm{D}^{\varkappa}_{p,\ q} \mathrm{I}^{\varkappa}_{p,\ q} w(\phi) = w(\phi). \tag{4}$$

Lemma 2 For $0 < q < p \leq 1$, let $\digamma : [0,\ 1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that $\digamma(\cdot,\ w,\ z) \in C[0,\ 1]$ for all $w,\ z \in \mathbb{R}$. The equivalence between the problem (1) and the problem of finding the solutions to the integral equation

$$w(\phi) = \digamma\left(\phi,\ w_0 + \mathrm{I}^{\varkappa}_{p,\ q} z(p^{\varkappa}\phi),\ z(p^{\varkappa}\phi)\right), \tag{5}$$

is established. If $z(\cdot) \in C[0,\ 1]$ solves this equation, then the function $w(\phi)$, defined as

$$w(\phi) = w_0 + \left(\mathrm{I}^{\varkappa}_{p,\ q} z\right)(p^{\varkappa}\phi),$$

is a solution to the original problem.

Proof. Let w be a solution to problem (1), and define

$$z(p^{\varkappa}\phi) = {}^{\mathrm{c}}\mathrm{D}^{\varkappa}_{p,\, q} w(\phi) \text{ for } \phi \in [0,\ 1].$$

We aim to show that $w(\phi) = w_0 + \left(\mathrm{I}^{\varkappa}_{p,\, q}\right) z(p^{\varkappa}\phi)$ and that z satisfies the integral equation. By Lemma 1, $w(\phi)$ can be expressed as

$$w(\phi) = w_0 + \left(\mathrm{I}^{\varkappa}_{p,\, q}\right) z(p^{\varkappa}\phi),$$

and verifying this form satisfies Eq. (5) is straightforward. Conversely, if $w(\phi)$ satisfies the integral equation $w(\phi) = w_0 + \left(\mathrm{I}^{\varkappa}_{p,\, q}\right) z(p^{\varkappa}\phi)$ and z satisfies Eq. (5), it follows that w is indeed a solution to problem (1). □

Lemma 3 [26] Let $0 < q < p \leq 1$, and $0 < \varkappa < 1$.

Then

$$\int_0^{\phi} (\phi - qs)^{(\varkappa-1)}_{p,\, q}\, \mathrm{d}_{p,\, q}s = \phi^{\varkappa} \int_0^1 (1 - qs)^{(\varkappa-1)}_{p,\, q}\, \mathrm{d}_{p,\, q}s = \phi^{\varkappa} B_{p,\, q}(1,\ \varkappa),$$

and

$$\int_0^{\phi} (\phi - qs)^{(\varkappa-1)}_{p,\, q}\, s^{-\varkappa} \mathrm{d}_{p,\, q}s = \int_0^1 (1 - qs)^{(\varkappa-1)}_{p,\, q}\, s^{-\varkappa} \mathrm{d}_{p,\, q}s = B_{p,\, q}(\varkappa,\ 1-\varkappa).$$

Theorem 1 (Banach Fixed-Point Theorem). Consider a complete metric space $(X,\ d)$. Suppose a mapping $\mathscr{P} : X \to X$ satisfies the following condition:

$$d(\mathscr{P}(w),\ \mathscr{P}(v)) \leq kd(w,\ v),$$

where $0 < k < 1$ (a contraction condition). Under these assumptions, $\mathscr{P}$ admits exactly one fixed point in X.

Theorem 2 (Schauder Fixed-Point Theorem [27]). Let X be a Banach space, and let B a bounded, closed, and convex subset of X. If a mapping $\mathscr{P} : B \to B$ is both compact and continuous, then $\mathscr{P}$ is guaranteed to have at least one fixed point in B.

3. Existence results

This section examines the criteria required to ensure both the existence and uniqueness of solutions to the problem specified by equation (1).

Definition 6 A continuous function $w \in C[0,\ 1]$ is considered a solution to the problem described by equation (1) that satisfies equation

$${}^{\mathrm{c}}\mathrm{D}^{\varkappa}_{p,\, q} w(\phi) = F\left(\phi,\, w(p^{\varkappa}\phi),\, {}^{\mathrm{c}}\mathrm{D}^{\varkappa}_{p,\, q} w(\phi)\right),$$

on $[0,\ 1]$ along with the initial condition $w(0) = w_0$.

Theorem 3 Suppose that assumption;

(C1) The function $\digamma$ satisfies the Lipschitz condition:

$$|\digamma(\phi,\ w_1,\ v_1) - \digamma(\phi,\ w_2,\ v_2)| \leq \sigma_1 |w_1 - w_2| + \sigma_2 |v_1 - v_2|,$$

for all $\phi \in [0,\ 1]$ and $w_1,\ w_2,\ v_1,\ v_2 \in \mathbb{R}$, where σ_1 and σ_2 are Lipschitz constants satisfy

$$\frac{(1-\sigma_2)^{-1}\sigma_1}{\Gamma_{p,\ q}(\varkappa+1)} < 1.$$

Then there exists a unique solution of the problem (1) on the interval $[0,\ 1]$.

Proof. By applying Lemma 2, we transform the problem described by equation (1) into an equivalent fixed-point formulation. So, we define the operator $\mathscr{P} : C[0,\ 1] \to C[0,\ 1]$ as

$$(\mathscr{P}w)(\phi) = w(\phi) = w_0 + \mathrm{I}^{\varkappa}_{p,\ q} z(p^{\varkappa}\phi), \quad \phi \in [0,\ 1], \tag{6}$$

where $z \in C[0,\ 1]$ satisfies

$$z(p^{\varkappa}\phi) = \digamma(\phi,\ w(\phi),\ z(p^{\varkappa}\phi)), \quad \text{or} \quad z(p^{\varkappa}\phi) = \digamma(\phi,\ w_0 + \mathrm{I}^{\varkappa}_{p,\ q} z(p^{\varkappa}\phi),\ z(p^{\varkappa}\phi)).$$

Let $w,\ v \in C[0,\ 1]$. Then, for $\phi \in [0,\ 1]$, we obtain

$$\begin{aligned}|(\mathscr{P}w)(\phi) - (\mathscr{P}v)(\phi)| &\leq \frac{1}{p^{\binom{\varkappa}{2}}\Gamma_{p,\ q}(\varkappa)} \int_0^{\phi} (\phi - qs)^{(\varkappa-1)}_{p,\ q} \left| z\left(\frac{p^{\varkappa}s}{p^{\varkappa-1}}\right) - y\left(\frac{p^{\varkappa}s}{p^{\varkappa-1}}\right) \right| \mathrm{d}_{p,\ q}s \\ &= \frac{1}{p^{\binom{\varkappa}{2}}\Gamma_{p,\ q}(\varkappa)} \int_0^{\phi} (\phi - qs)^{(\varkappa-1)}_{p,\ q} |z(ps) - y(ps)| \mathrm{d}_{p,\ q}s, \end{aligned} \tag{7}$$

where $z,\ y \in C[0,\ 1]$ satisfy

$$z(p\phi) = \digamma(\phi,\ w(\phi),\ z(p\phi)),$$

and

$$y(p\phi) = \digamma(\phi,\ v(\phi),\ y(p\phi)).$$

From **(**C1**)**, it follows that

$$|z(p\phi)-y(p\phi)| \leq \sigma_1 |w(\phi)-v(\phi)| + \sigma_2 |z(p\phi)-y(p\phi)|.$$

Thus, we can write

$$|z(p\phi)-y(p\phi)| \leq (1-\sigma_2)^{-1}\sigma_1 |w(\phi)-v(\phi)|,$$

From (7) and Lemma 3, we derive

$$\begin{aligned}
|(\mathscr{P}w)(\phi)-(\mathscr{P}v)(\phi)| &\leq \frac{1}{p^{\binom{\varkappa}{2}}\Gamma_{p,\, q}(\varkappa)} \int_0^{\phi} (\phi-qs)_{p,\, q}^{(\varkappa-1)} |z(ps)-y(ps)| \, \mathrm{d}_{p,\, q}s \\
&\leq \frac{\phi^{\varkappa} B_{p,\, q}(1,\, \varkappa)}{p^{\binom{\varkappa}{2}}\Gamma_{p,\, q}(\varkappa)} (1-\sigma_2)^{-1}\sigma_1 |w(\phi)-v(\phi)| \\
&\leq \frac{(1-\sigma_2)^{-1}\sigma_1}{\Gamma_{p,\, q}(\varkappa+1)} |w(\phi)-v(\phi)|,
\end{aligned}$$

since

$$\frac{(1-\sigma_2)^{-1}\sigma_1}{\Gamma_{p,\, q}(\varkappa+1)} < 1.$$

Hence, we obtain

$$d(\mathscr{P}(w),\, \mathscr{P}(v)) \leq \frac{(1-\sigma_2)^{-1}\sigma_1}{\Gamma_{p,\, q}(\varkappa+1)} d(w,\, v).$$

Using Theorem 1, we conclude that the operator $\mathscr{P}$ has a unique fixed point, corresponding to the unique solution of problem (1). □

Remark 1 The above contraction condition in Theorem 3 guarantees that the operator is a strict contraction in the Banach space under the supremum norm, which ensures the uniqueness of the fixed point and, consequently, the uniqueness of the solution.

Theorem 4 Assume that the condition

(C2) There exist functions $\alpha,\ \beta,\ \gamma \in C([0,\ 1],\ (0,\ \infty))$ with $\gamma(\phi) < 1$ such that

$$|F(\phi,\ w,\ v)| \leq \alpha(\phi) + \beta(\phi)|w| + \gamma(\phi)|v|,$$

for every $\phi \in [0,\ 1]$ and $w,\ v \in \mathbb{R}$, where

$$\alpha^* = \sup_{\phi\in[0,\ 1]} \alpha(\phi), \quad \beta^* = \sup_{\phi\in[0,\ 1]} \beta(\phi), \quad \gamma^* = \sup_{\phi\in[0,\ 1]} \beta(\phi).$$

is satisfied. If the inequality

$$\gamma^* + \frac{\beta^*}{\Gamma_{p,\ q}(\varkappa+1)} < 1,$$

holds, then the problem (1) has at least one solution on the interval $[0,\ 1]$.

Proof. The proof starts by defining the operator $\mathscr{P}$ as given in equation (6). Let

$$R \geq \frac{\overline{\dfrac{\alpha^*}{\Gamma_{p,\ q}(\varkappa+1)}}}{1-\gamma^* - \dfrac{\beta^*}{\Gamma_{p,\ q}(\varkappa+1)}},$$

and define the closed and convex set $B_R = \{w \in C[0,\ 1] : \|w\|_\infty \leq R\}$.

For $w \in B_R$ and for any $\phi \in [0,\ 1]$, we have

$$|(\mathscr{P}w)(\phi)| \leq \frac{1}{p^{\binom{\varkappa}{2}}\Gamma_{p,\ q}(\varkappa)} \int_0^\phi (\phi - qs)_{p,\ q}^{(\varkappa-1)} \left| z\left(\frac{p^\varkappa s}{p^{\varkappa-1}}\right)\right| \mathrm{d}_{p,\ q}s$$

$$= \frac{1}{p^{\binom{\varkappa}{2}}\Gamma_{p,\ q}(\varkappa)} \int_0^\phi (\phi - qs)_{p,\ q}^{(\varkappa-1)} |z(ps)| \mathrm{d}_{p,\ q}s,$$

where $z \in C[0,\ 1]$ satisfies

$$z(p\phi) = F(\phi,\ w(\phi),\ z(p\phi)),$$

Using (C2), for every $\phi \in [0,\ 1]$, we have

$$|z(p\phi)| \leq \alpha(\phi) + \beta(\phi)|w(\phi)| + \gamma(\phi)\,|z(p\phi)|,$$

which implies

$$|z(p\phi)| \leq \alpha^* + \beta^*\,\|w\|_\infty + \gamma^*\,|z(p\phi)|,$$

Thus,

$$|z(p\phi)| \leq \frac{\alpha^* + \beta^* R}{1 - \gamma^*}.$$

Therefore,

$$\|\mathscr{P}(w)\|_\infty \leq \frac{(\alpha^* + \beta^* R)}{(1-\gamma^*)\,\Gamma_{p,\,q}(\varkappa+1)},$$

which shows that

$$\|\mathscr{P}(w)\|_\infty \leq R,$$

because

$$R \geq \frac{\dfrac{\alpha^*}{\Gamma_{p,\,q}(\varkappa+1)}}{1-\gamma^* - \dfrac{\beta^*}{\Gamma_{p,\,q}(\varkappa+1)}}.$$

This demonstrates that $\mathscr{P}$ maps the ball B_R into itself. We now aim to show that the operator $\mathscr{P}$ is both continuous and compact from B_R to B_R.

Let $\{w_n\}_{n\in\mathbb{N}}$ be a sequence such that $w_n \to u$ in B_R. Then, for each $\phi \in [0,\ 1]$, we have

$$\begin{aligned}
&|(\mathscr{P}w_n)(\phi) - (\mathscr{P}w)(\phi)| \\
&\leq \frac{1}{p^{\binom{\varkappa}{2}}\Gamma_{p,\,q}(\varkappa)} \int_0^\phi (\phi - qs)_{p,\,q}^{(\varkappa-1)} \left| z_n\left(\frac{p^\varkappa s}{p^{\varkappa-1}}\right) - z\left(\frac{p^\varkappa s}{p^{\varkappa-1}}\right)\right| \mathrm{d}_{p,\,q}s \\
&= \frac{1}{p^{\binom{\varkappa}{2}}\Gamma_{p,\,q}(\varkappa)} \int_0^\phi (\phi - qs)_{p,\,q}^{(\varkappa-1)} |z_n(ps) - z(ps)|\, \mathrm{d}_{p,\,q}s \\
&= \frac{1}{p^{\binom{\varkappa}{2}}\Gamma_{p,\,q}(\varkappa)} \int_0^\phi (\phi - qs)_{p,\,q}^{(\varkappa-1)} |\digamma(s,\ w_n(s),\ z_n(ps)) - \digamma(s,\ w(s),\ z(ps))|\, \mathrm{d}_{p,\,q}s,
\end{aligned}$$

where $z_n,\ z \in C[0,\ 1]$ satisfy

$$z_n(p\phi) = \digamma(\phi,\ w_n(\phi),\ z_n(p\phi)),$$

and

$$z(p\phi) = \digamma(\phi,\ w(\phi),\ z(p\phi)).$$

Since $w_n \to w$ as $n \to \infty$ and $\digamma$ is continuous, we deduce that

$$z_n(p\phi) \to z(p\phi) \quad \text{as} \quad n \to \infty, \quad \text{for each} \quad \phi \in [0,\ 1].$$

Therefore,

$$\|\mathscr{P}(w_n) - \mathscr{P}(w)\|_\infty \leq \frac{1}{\Gamma_{p,\ q}(\varkappa+1)}\|z_n - z\|_\infty \to 0 \quad \text{as} \quad n \to \infty.$$

Hence $\mathscr{P}$ is continuous.

Now, since $\mathscr{P}(B_R) \subseteq B_R$, and B_R is bounded. Then $\mathscr{P}(B_R)$ is bounded.

On the other hand. Let $\phi_1,\ \phi_2 \in [0,\ 1]$ such that $\phi_1 < \phi_2$ and $w \in B_R$. Then, we have

$$|(\mathscr{P}w)(\phi_1) - (\mathscr{P}w)(\phi_2)| \leq \frac{1}{p^{\binom{\varkappa}{2}}\Gamma_{p,\ q}(\varkappa)} \int_0^{\phi_1} \left|(\phi_2 - qs)_{p,\ q}^{(\varkappa-1)} - (\phi_1 - qs)_{p,\ q}^{(\varkappa-1)}\right| |z(ps)|\, \mathrm{d}_{p,\ q}s$$

$$+ \int_{\phi_1}^{\phi_2} (\phi_2 - qs)_{p,\ q}^{(\varkappa-1)} |z(ps)|\, \mathrm{d}_{p,\ q}s,$$

where $z \in C[0,\ 1]$ satisfies $z(p\phi) = \digamma(\phi,\ w(\phi),\ z(p\phi))$. Thus, we can write

$$|(\mathscr{P}w)(\phi_1) - (\mathscr{P}w)(\phi_2)| \leq \frac{\alpha^* + \beta^* R}{1-\gamma^*} \frac{1}{p^{\binom{\varkappa}{2}}\Gamma_{p,\ q}(\varkappa)} \int_0^{\phi_1} \left|(\phi_2 - qs)_{p,\ q}^{(\varkappa-1)} - (\phi_1 - qs)_{p,\ q}^{(\varkappa-1)}\right| \mathrm{d}_{p,\ q}s$$

$$+ \frac{\alpha^* + \beta^* R}{1-\gamma^*} \int_{\phi_1}^{\phi_2} (\phi_2 - qs)_{p,\ q}^{(\varkappa-1)}\, \mathrm{d}_{p,\ q}s.$$

As $\phi_1 \to \phi_2$, the right-hand side of the inequality approaches zero. Hence, $\mathscr{P}$ maps bounded sets into equicontinuous sets in B_R.

Using the three steps above, along with the Arzel-Ascoli theorem, we conclude that $\mathscr{P} : B_R \to B_R$ is continuous and compact.

Applying Theorem 2, we deduce that $\mathscr{P}$ has at least one fixed point, which corresponds to a solution of the problem (1). □

Remark 2 For clarity, we expand the proof. The continuity of the operator $\mathscr{P}$ follows from the continuity of z, while compactness is obtained by combining boundedness and equicontinuity of the image set $\mathscr{P}(B_R)$, as guaranteed by the Arzelà-Ascoli theorem. These steps ensure that all requirements of Schauder's theorem are satisfied.

3.1 *Ulam stability*

In this side, we focus on the generalized Ulam-Hyers-Rassias stability for the problem (1) using $(p,\ q)$-Gronwall inequality as a handy tool. Let $\varepsilon > 0$ and consider a continuous function $\Psi : I \to \mathbb{R}_+$. The following inequalities are examined:

$$\left|{}^{\mathrm{c}}\mathrm{D}^{\varkappa}_{p,\ q}w(\phi) - F\left(\phi,\ w(\phi),\ {}^{\mathrm{c}}\mathrm{D}^{\varkappa}_{p,\ q}w(\phi)\right)\right| \leq \varepsilon, \tag{8}$$

$$\left|{}^{\mathrm{c}}\mathrm{D}^{\varkappa}_{p,\ q}w(\phi) - F\left(\phi,\ w(\phi),\ {}^{\mathrm{c}}\mathrm{D}^{\varkappa}_{p,\ q}w(\phi)\right)\right| \leq \Psi(\phi), \tag{9}$$

$$\left|{}^{\mathrm{c}}\mathrm{D}^{\varkappa}_{p,\ q}w(\phi) - F\left(\phi,\ w(\phi),\ {}^{\mathrm{c}}\mathrm{D}^{\varkappa}_{p,\ q}w(\phi)\right)\right| \leq \varepsilon\Psi(\phi), \tag{10}$$

for $0 \leq \phi \leq 1$.

These inequalities establish different conditions for examining the stability properties of the given problem.

Definition 7 [28, 29] The problem defined by (1) is said to be Ulam-Hyers stable if there exists a constant $c_f > 0$ such that, for every $\varepsilon > 0$ and for each solution $w \in C[0,\ 1]$ satisfying inequality (8), there exists a solution $v \in C[0,\ 1]$ of (1) satisfying

$$|w(\phi) - v(\phi)| \leq \varepsilon c_f, \text{ for } \phi \in [0,\ 1].$$

Definition 8 [28, 29] The problem represented by (1)) is said to be generalized Ulam-Hyers stable if there exists a function $c_f : \mathbb{R}_+ \to \mathbb{R}_+$, with $c_f(0) = 0$, such that for each $\varepsilon > 0$ and every solution $w \in C[0,\ 1]$ of inequality (8), there exists a solution $v \in C[0,\ 1]$ of (1) with

$$|w(\phi) - v(\phi)| \leq c_f(\varepsilon), \text{ for } \phi \in [0,\ 1].$$

Definition 9 [28, 29] The problem (1) is considered Ulam-Hyers-Rassias stable with respect to Ψ if there exists a constant $c_{f,\ \Psi} > 0$ such that for every $\varepsilon > 0$ and each solution $w \in C[0,\ 1]$ of inequality (10), there exists a solution $v \in C[0,\ 1]$ of (1) satisfying

$$|w(\phi) - v(\phi)| \leq \varepsilon c_{f,\ \Psi}\Psi(\phi), \text{ for } \phi \in [0,\ 1].$$

Definition 10 [28, 29] The problem (1) is said to be generalized Ulam-Hyers-Rassias stable with respect to Ψ if there exists a constant $c_{f,\ \Psi} > 0$ such that for every solution $w \in C[0,\ 1]$ of inequality (9), there exists a solution $v \in C[0,\ 1]$ of (1) satisfying

$$|w(\phi) - v(\phi)| \leq c_{f,\ \Psi}\Psi(\phi), \quad \phi \in [0,\ 1].$$

Remark 3 The following logical relationships can be observed:

(i) Definition 7 leads to Definition 8.

(ii) Definition 9 leads to Definition 10.

(iii) Definition 9, with $\Psi(\cdot) = 1$, leads to Definition 7.

Similar observations can be made for the inequalities corresponding to (8) and (10).

We need the following lemma.

Lemma 4 (The $(p,\ q)$-Gronwall inequality [26]). Suppose that $\Psi(\phi) \geq 0$, and $\sigma \geq 0$. Assume that function $w : [0,\ \infty) \to \mathbb{R}^+$ is continuous and satisfies

$$w(\phi) \leq \Psi(\phi) + \frac{\sigma}{p^{\binom{\varkappa}{2}}\Gamma_{p,\ q}(\varkappa)} \int_0^\phi (\phi - qs)_{p,\ q}^{(\varkappa-1)} w(s)\, \mathrm{d}_{p,\ q}s.$$

Then, the following inequality

$$w(\phi) \leq \Psi(\phi) E_{p,\ q}(\sigma,\ \phi),$$

holds, where $E_{p,\ q}(\sigma,\ \phi) = \sum_{k=0}^{\infty} \frac{\sigma^k \phi^{k\varkappa}}{\Gamma_{p,\ q}(k\varkappa+1)}$.

Theorem 5 Suppose that assumptions (C1) and (C3) hold, where (C3) states that there exists $\theta_\Psi > 0$ such that for every $\phi \in [0,\ 1]$, we have

$$\mathrm{I}_{p,\ q}^{\varkappa} \Psi(\phi) \leq \theta_\Psi \Psi(\phi).$$

Under these conditions, the problem (1) has at least one solution and exhibits generalized Ulam-Hyers-Rassias stability.

Proof. Let w be a solution to inequality (9), and assume that v solves the problem (1). Then, we have

$$v(\phi) = w_0 + \mathrm{I}_{p,\ q}^{\varkappa} y(p^{\varkappa}\phi),$$

where $y \in C[0,\ 1]$ such that $y(p\phi) = \digamma(\phi,\ v(\phi),\ y(p\phi))$.

From inequality (9), for each $\phi \in [0,\ 1]$, we get

$$\left| w(\phi) - w_0 - \mathrm{I}_{p,\ q}^{\varkappa} z(p^{\varkappa}\phi) \right| \leq \mathrm{I}_{p,\ q}^{\varkappa} \Psi(\phi), \quad 0 \leq \phi \leq 1,$$

where $z \in C[0,\ 1]$ such that $z(p\phi) = \digamma(\phi,\ w(\phi),\ z(p\phi))$.

Using assumptions (C1), (C3), and Lemma 4, for every $\phi \in [0,\ 1]$, we obtain

$$\begin{aligned} |w(\phi) - v(\phi)| &\leq \left| w(\phi) - w_0 - \mathrm{I}_{p,\ q}^{\varkappa} z(p^{\varkappa}\phi) + \mathrm{I}_{p,\ q}^{\varkappa}(z - y)(p^{\varkappa}\phi) \right| \\ &\leq \left| w(\phi) - w_0 - \mathrm{I}_{p,\ q}^{\varkappa} z(p^{\varkappa}\phi) \right| + \left| \mathrm{I}_{p,\ q}^{\varkappa}(z - y)(p^{\varkappa}\phi) \right| \\ &\leq \mathrm{I}_{p,\ q}^{\varkappa} \Psi(\phi) + \frac{1}{p^{\binom{\varkappa}{2}}\Gamma_{p,\ q}(\varkappa)} \int_0^\phi (\phi - qs)_{p,\ q}^{(\varkappa-1)} (|z(ps) - y(ps)|)\, \mathrm{d}_{p,\ q}s \end{aligned}$$

$$\leq \theta_\Psi \Psi(\phi) + \frac{(1-\sigma_2)^{-1}\sigma_1}{p^{\binom{\varkappa}{2}}\Gamma_{p,\, q}(\varkappa)} \int_0^\phi (\phi - qs)_{p,\, q}^{(\varkappa-1)} |w(\phi) - v(\phi)| \mathrm{d}_{p,\, q}s$$

$$\leq \theta_\Psi \Psi(\phi) + \theta_\Psi E_{p,\, q}\left((1-\sigma_2)^{-1}\sigma_1,\, \phi\right)\Psi(\phi)$$

$$\left[1 + E_{p,\, q}\left((1-\sigma_2)^{-1},\, \phi\right)\right]\theta_\Psi \Psi(\phi)$$

$$= c_{f,\, \Psi}\Psi(\phi).$$

Hence, the problem (1) is proven to be generalized Ulam-Hyers-Rassias stable. □

4. Examples

Example 1 Let us examine the following implicit $\left(\frac{1}{4},\, \frac{1}{5}\right)$-fractional difference equation

$$\begin{cases} {}^{c}\mathrm{D}_{\frac{1}{4},\, \frac{1}{5}}^{\frac{1}{2}} w(\phi) = \digamma\left(\phi, w(\phi), {}^{c}\mathrm{D}_{\frac{1}{4},\, \frac{1}{5}}^{\frac{1}{2}} w(\phi)\right), & 0 \leq \phi \leq 1, \\ w(0) = 1, \end{cases} \tag{11}$$

where the function $\digamma(\phi,\, w,\, v)$ is defined by

$$\digamma(\phi,\, w,\, v) = \sin(\phi) + \phi^2|w| + \frac{e^\phi}{5+\phi^2}|v|.$$

For this function, we can choose

$$\alpha(\phi) = |\sin(\phi)|, \quad \beta(\phi) = \phi^2, \quad \gamma(\phi) = \frac{e^\phi}{5+\phi^2}.$$

So, for every $\phi \in [0,\, 1]$ and $w,\, v \in \mathbb{R}$, we have

$$|\digamma(\phi,\, w,\, v)| = |\sin(\phi)| + \phi^2\,|w| + \frac{e^\phi}{5+\phi^2}|v|.$$

Since $|\sin(\phi)| \leq 1$, $\phi^2 \geq 0$, and $\frac{e^\phi}{5+\phi^2} > 0$ for $\phi \in [0,\, 1]$, we have

$$|F(\phi,\ w,\ v)| \leq |\sin(\phi)| + \phi^2|w| + \frac{e^{\phi}}{5+\phi^2}|v|$$

$$= \alpha(\phi) + \beta(\phi)|w| + \gamma(\phi)|v|.$$

Therefore the condition (C2) is hold. Additionally, observe that

$$\beta^* = \sup_{\phi\in[0,\ 1]} \beta(\phi) = \sup_{\phi\in[0,\ 1]} \phi^2 = 1$$

and

$$\gamma^* + \frac{\beta^*}{\Gamma_{p,\ q}(\varkappa+1)} = \frac{e}{6} + \frac{1}{\Gamma_{\frac{1}{4},\ \frac{1}{5}}\left(\frac{3}{2}\right)} = \frac{e}{6} + \frac{1}{2.2361} < 1.$$

Hence, by using Theorem 4 then the problem (11) admits at least one solution on $[0,\ 1]$.

Example 2 Let us examine the implicit $\left(\frac{1}{4},\ \frac{1}{5}\right)$-fractional difference equation

$$\begin{cases} {}^{c}\mathrm{D}^{\frac{1}{2}}_{\frac{1}{4},\ \frac{1}{5}} w(\phi) = F\left(\phi, w(\phi), {}^{c}\mathrm{D}^{\frac{1}{2}}_{\frac{1}{4},\ \frac{1}{5}} w(\phi)\right), \quad 0 \leq \phi \leq 1, \\ w(0) = 1, \end{cases} \tag{12}$$

where the function $F(\phi,\ w,\ v)$ is defined by:

$$F(\phi,\ w,\ v) = \frac{\phi}{10}(\cos w - w\sin\phi) + \frac{v}{10+v},$$

Then

$$|F(\phi,\ w_1,\ v_1) - F(\phi,\ w_2,\ v_2)| = \frac{1}{10}|\cos w_1 - \cos w_2| + \frac{1}{10}|\sin\phi|\,|w_1 - w_2|$$

$$+ \frac{|v_1 - v_2|}{(10+v_1)(10+v_2)}$$

$$\leq \frac{1}{10}|w_1 - w_2| + \frac{1}{10}|w_1 - w_2| + \frac{1}{10}|v_1 - v_2|$$

$$= \frac{1}{5}|w_1 - w_2| + \frac{1}{10}|v_1 - v_2|.$$

which satisfies the Lipschitz condition (C1) with $\sigma_1 = \frac{1}{5}$ and $\sigma_2 = \frac{1}{10}$. In addition

$$\frac{(1-\sigma_2)^{-1}\sigma_1}{\Gamma_{p,\,q}(\varkappa+1)} = \frac{\left(1-\frac{1}{10}\right)^{-1}\frac{1}{5}}{\Gamma_{\frac{1}{4},\,\frac{1}{5}}\left(\frac{3}{2}\right)} < 1.$$

Hence, by Theorem 3, there exists a unique solution to the problem (12) on $[0,\ 1]$.

Alternatively, we may select any function $\Psi(\phi)$ that satisfies condition (C3). Consequently, based on Theorem 5, the problem (12) has a unique solution and exhibits the generalized Ulam-Hyers-Rassias stability.

5. Conclusion

In this paper, we established existence and uniqueness results for implicit $(p,\ q)$-fractional difference equations using Banach and Schauder fixed-point theorems. We also proved generalized Ulam-Hyers-Rassias stability by employing a $(p,\ q)$-Gronwall inequality. Compared to previous works restricted to q-calculus, our approach with the $(p,\ q)$-Caputo operator provides a more general and flexible setting.

The findings contribute to the theory of fractional difference equations and open new perspectives for further research. In particular, promising directions include the development of numerical methods for approximating solutions, the investigation of boundary value problems, and the analysis of multi-term and state-dependent fractional difference systems. Moreover, the introduction of the $(p,\ q)$-Caputo calculus enriches the dynamics of fractional systems by enabling finer adjustments through two parameters, thereby generalizing traditional calculus approaches.

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Conflict of interest

The authors declare no competing financial interest.

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