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Research Article

Some Binomial and Multinomial Sums

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Abstract: We give expressions for $\sum_{k=-\infty}^{\infty} \binom{N}{mk-r} s^k P(k)$, where m, k, r are integers, s is real, N is any complex number, and $P(k)$ is any polynomial. Two multivariate extensions are given.

***Keywords*:** binomial, multinomial, sum

MSC: 15A06, 33E20

1. Introduction

Binomial and multinomial sums play pivotal roles in various fields of research, owing to their ability to model discrete outcomes and support combinatorial computations. These sums are instrumental in statistical methods, particularly where outcomes can be classified into two or more categories. For instance, the binomial distribution underpins binary trials, where powers of a fixed probability yield insights into success rates in fixed sample sizes. Combinatorial techniques enable researchers to derive formulas and perform calculations involving probabilities in areas such as genetics, finance, and quality control, as evidenced in the work of Jia et al. [1], where binomial sums are pivotal for understanding count data in stationary processes.

Furthermore, multinomial sums excel in modeling scenarios with multiple categories, surpassing the binomial context by allowing researchers to analyze complex situations with multiple outcomes. Annamalai's work [2, 3] highlights the utilization of multinomial coefficients in artificial intelligence and machine learning applications, demonstrating how they can inform cybersecurity mechanisms through effective processing of diverse datasets that represent different categories of threats. The richness of multinomial models extends to genomics, where they facilitate the analysis of genetic variations, enhancing our understanding of complex traits [4].

In computational mathematics, the utility of binomial and multinomial sums becomes even clearer. Annamalai's studies [5, 6] on combinatorial identities highlight how these mathematical tools help improve computational efficiency. By utilizing binomial expansions, researchers can predict outcomes and analyze vast datasets more effectively. Such expansions allow for efficient approximations and simplifications in algorithm designs, thus enhancing the speed and accuracy of computational tasks within scientific disciplines such as physics and engineering [7].

Moreover, these sums find practical usage in health sciences, particularly in modeling various outcomes for categorical data analyses, such as those found in disability studies or clinical trials. The multinomial distribution is employed to

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understand the prevalence of different health statuses within populations, offering a probabilistic approach to health outcomes [8, 9]. This application underscores the relevance of multinomial statistics in real-world contexts, where decision-making must be based on data with multiple classifications.

The intersection of theory and application extends to the implications of binomial and multinomial sums in advanced machine learning frameworks. Techniques such as Probably Approximately Correct (PAC)-Bayesian theory leverage these mathematical constructs to improve learning algorithms, optimizing prediction accuracy under uncertainty [10, 11]. As researchers continue to explore these mathematical fields, the foundational concepts of binomial and multinomial sums will remain integral to innovations across diverse scientific endeavors, from artificial intelligence to statistical modeling in biomedicine.

Some advances in related polynomial families and operational frameworks can be found in [12, 13].

In this note, we derive various expressions for sums of the form

$$\sum_{k=-\infty}^{\infty} \binom{N}{mk-r} s^k P(k), \tag{1}$$

where m, k, r are integers, s is real, N is any complex number, and $P(k)$ is any polynomial. The case, where m, k, r, s are scalars and $P(\cdot)$ is a univariate polynomial, is considered in section 2. The case, where $\mathbf{m}$, $\mathbf{k}$, $\mathbf{r}$, $\mathbf{s}$ are vectors and $P(\cdot)$ is a multivariate polynomial, is considered in sections 3 and 4. The given expressions are simple and avoid hypergeometric functions commonly used to express sums of the form (1), see Koepf [14]. A summary and future research are noted in section 5. The proofs of all results are given in an appendix.

2. Binomial type sums

Fix $m = 1, 2, \ldots$. Let r be any integer. Let N, t be any complex numbers and fix j in $\mathcal{N} = \{0, 1, 2, \ldots\}$. For n in $\mathcal{N}$ let $\binom{N}{n}$ be the usual binomial coefficient

$$\binom{N}{n} = \frac{(N)_n}{n!},$$

where

$$(N)_n = N(N-1)\cdots(N-n+1) = \frac{\Gamma(N+1)}{\Gamma(N-n+1)}.$$

For n a negative integer, set $\binom{N}{n} = 0$.

Theorem 1 gives an expression for

$$U_{N,\,r}(P_j) = \sum_{k=-\infty}^{\infty} \binom{N}{mk-r} t^{mk-r} P_j(k)$$

for $P_j(k)$ any polynomial of degree j. Our notation suppresses dependency on m and t. We express $U_{N,\,r}(P_j)$ as a sum of m terms, each a function of the mth roots of 1, $w_m = \exp\left(\frac{2\pi \mathrm{i}}{m}\right)$, where $\mathrm{i} = \sqrt{-1}$.

Theorem 1 For $P_0(k) = 1$, we have

$$U_{N,\,r,\,0} = U_{N,\,r}(1) = m^{-j-1} \sum_{\ell=0}^{m-1} w_m^{\ell r} \left(1 + w_m^{\ell} t\right)^N. \tag{2}$$

More generally, for any polynomials $P_0,\ P_1,\ \ldots$,

$$U_{N,\,r}(P_j) = m^{-j-1} \sum_{\ell=0}^{m-1} w_m^{\ell r} \left(1 + w_m^{\ell} t\right)^N V_{N,\,r,\,j}\left(\frac{w_m^{\ell} t}{1 + w_m^{\ell} t}\right), \tag{3}$$

where

$$V_{N,\,r,\,j}(t) = \sum_{i=0}^{j} T_{r,\,j,\,i} (N)_i t^i, \tag{4}$$

and the coefficients $T_{r,\,j,\,i}$ do not depend on N, t.

Once $T_{r,\,j,\,i}$ have been given for one set of polynomials, they can be computed for any other since any polynomial of degree j can be written as a linear combination of $P_0,\ \ldots,\ P_j$. We assume that $P_0 \neq 0$. Candidates for $P_j(k)$ are

$$A_j(k) = k^j,\ B_j(k) = (k)_j,\ C_j(k) = (mk - r)^j,\ D_j(k) = (mk - r)_j. \tag{5}$$

We denote V, T for these by adding a superscript $A,\ \ldots,\ D$. The last choice gives the simplest expressions for V, T:

$$T^D_{j,\,i} = \delta_{i,\,j},\ V^D_{N,\,j}(t) = (N)_j t^j, \tag{6}$$

where $\delta_{i,\,j}$ is 1 or 0 for $i = j$ or $i \neq j$. The choice C_j also gives fairly simple expressions for V, T:

$$T^C_{j,\,i} = S(j,\ i),\ V^C_{N,\,j}(t) = \sum_{i=0}^{j} S(j,\ i)(N)_i t^i. \tag{7}$$

In both cases the subscripts r have been suppressed since the expressions do not depend on r. Here, $S(j,\ i)$ is the Stirling number of the second kind tabled, for example, on page 310 of Comtet [15] and defined by

$$N^j = \sum_{i=0}^{j} (N)_i S(j,\ i) \tag{8}$$

for $j \geq i \geq 0$. The leading values are

$$S(j,\ 0) = \delta_{j,\,0},\ S(j,\ 1) = 1,\ \ S(j,\ 2) = 2^{j-1} - 1,\ S(j,\ j-1) = \binom{j}{2},\ S(j,\ j) = 1.$$

This gives

$$V^C_{N,\,0}(t) = 1,$$

$$V^C_{N,\,1}(t) = Nt,$$

$$V^C_{N,\,2}(t) = Nt + (N)_2 t^2,$$

$$V^C_{N,\,3}(t) = Nt + 3(N)_2 t^2 + (N)_3 t^3,$$

$$V^C_{N,\,4}(t) = Nt + 7(N)_2 t^2 + 6(N)_3 t^3 + (N)_4 t^4,$$

$$V^C_{N,\,5}(t) = Nt + 15(N)_2 t^2 + 25(N)_3 t^3 + 10(N)_4 t^4 + (N)_5 t^5,$$

$$V^C_{N,\,6}(t) = Nt + 31(N)_2 t^2 + 90(N)_3 t^3 + 65(N)_4 t^4 + 15(N)_5 t^5 + (N)_6 t^6,$$

and so on.

Theorem 2 shows that $U_{N,\,r,\,j}$ can be reduced for any polynomial for the case $P_j = D_j$ of (5).

Theorem 2 Consider the general polynomial

$$P_j(k) = \sum_{a=0}^{j} c_a k^a.$$

Then

$$U_{N,\,r,\,j} = c_0 U_{N,\,r,\,0} + \sum_{i=1}^{j} (N)_i t^i U_{N-i,\,r+i,\,0} \sum_{a=i}^{j} c_a m^{-a} T_{r,\,a,\,i}, \tag{9}$$

where

$$T_{r,\,a,\,i} = S(a,\,i) \sum_{j=i}^{a} \binom{a}{j} r^{a-j} \tag{10}$$

is the product of a Stirling number and a binomial partial sum. The leading $T_{r,\,a,\,i}$ are:

$$T_{r,\,a,\,0} = \delta_{a,\,0},$$

$$T_{r,\,a,\,1} = (r+1)^a - r^a,$$

$$T_{r,\,a,\,2} = S(a,\,2)\left\{(r+1)^a - r^a - ar^{a-1}\right\},$$

$$T_{r,\,a,\,a-1} = \binom{a}{2}(ar+1),$$

$$T_{r,\,a,\,a} = 1,$$

where $S(a,\,2) = 2^{a-1} - 1$.

Because of linearity we can express one set of U, V, T in terms of any other. We illustrate this though the expansion for $A_j(k) = k^j$ in terms of $\{C_i(k)\}$.

Example 1 Expanding $\left(\frac{n}{m} + r\right)^j$ we obtain A_j in terms of $\{C_i\}$ as

$$k^j = \sum_{i=0}^{j} d^{AC}_{r,\,j,\,i} n^i,$$

$$d^{AC}_{r,\,j,\,i} = \binom{j}{i} m^{-i} r^{j-i}$$

for $n = mk - r$. So,

$$U_{N,\,r}\left(k^j\right) = U^A_{N,\,r,\,j} = \sum_{i=0}^{j} d^{AC}_{r,\,j,\,i} U^C_{N,\,j},$$

$$V^A_{N,\,r,\,j} = \sum_{i=0}^{j} d^{AC}_{r,\,j,\,i} V^C_{N,\,j},$$

$$T^A_{r,\,j,\,a} = \sum_{i=a}^{j} d^{AC}_{r,\,j,\,i} T^C_{j,\,i}. \tag{11}$$

So, $U^A_{N,\,r,\,j}$ can be obtained from the first of these three equations together with (3), (4), (6) applied to D_j, or from (3) applied to A_j with V^A given by the second of these three equations, or with (3), (4) applied to A_j with T^A given by the third of these three equations. Of these three options the first seems the simplest. For example,

$$\sum_{k=-\infty}^{\infty} \binom{N}{mk-r} t^{mk-r} k^4 = \sum_{i=0}^{4} \binom{4}{i} m^{-i} r^{4-i} U^D_{N,\,i}$$

for $U^D_{N,\,i}$ given by the right hand side of (3) with $V_{N,\,r,\,j}(t) = V^D_{N,\,j}(t)$ of (7).

Similarly, from

$$C_j(k) = \sum_{i=0}^{j} d^{CD}_{j,\,i}\, D_i(k),$$

where $d^{CD}_{j,\,i} = S(j,\,i)$, the three equations (11) hold with C replaced by D, giving

$$U^{C}_{N,\,r,\,j} = \sum_{i=0}^{j} S(j,\,i) U^{D}_{N,\,j},$$

$$V^{C}_{N,\,r,\,j} = \sum_{i=0}^{j} S(j,\,i) V^{D}_{N,\,j},$$

$$T^{C}_{r,\,j,\,a} = \sum_{i=a}^{j} S(j,\,i) T^{D}_{j,\,i}.$$

For $P_j(k)$ not depending on r, one can derive $U_{N,\,r,\,j}$ from $U_{N,\,0,\,j}$ using

$$\left(\frac{d}{dt}\right)^{r} U_{N+r,\,0,\,j} = (N+r)_r U_{N,\,r,\,j}.$$

However, we shall not use this.

Example 2 Suppose that $P_j(k) = k^j$, where $j > 0$. Then

$$U_{N,\,r,\,j} = m^{-j} \sum_{i=0}^{j} T_{r,\,j,\,i} (N)_i t^i U_{N-i,\,r+i,\,0}.$$

In particular,

$$\begin{aligned}
mU_{N,\,r,\,1} &= NtU_{N-1,\,r+1,\,0},\\
m^2U_{N,\,r,\,2} &= (2r+1)NtU_{N-1,\,r+1,\,0} + (N)_2 t^2 U_{N-2,\,r+2,\,0},\\
m^3U_{N,\,r,\,3} &= \left(3r^2+3r+1\right)NtU_{N-1,\,r+1,\,0} + (3r+1)(N)_2 t^2 U_{N-2,\,r+2,\,0}\\
&\quad + (N)_3 t^3 U_{N-3,\,r+3,\,0},\\
m^4U_{N,\,r,\,4} &= \left(4r^3+6r^2+4r+1\right)NtU_{N-1,\,r+1,\,0}\\
&\quad + 7\left(6r^2+4r+1\right)(N)_2 t^2 U_{N-2,\,r+2,\,0} + 6(4r+1)(N)_3 t^3 U_{N-3,\,r+3,\,0}
\end{aligned}$$

$$+ (N)_4 t^4 U_{N-4,\, r+4,\, 0},$$

and so on. This gives the most useful result of this section: for $j \geq 0$, $U_{N,\, r,\, j}$ is given by (3), where $V_{N,\, r,\, j}(t)$ is given by (4). That is,

$$V_{N,\, r,\, 0}(t) = 1,$$

$$V_{N,\, r,\, 1}(t) = Nt,$$

$$V_{N,\, r,\, 2}(t) = (2r+1)Nt + (N)_2 t^2,$$

$$V_{N,\, r,\, 3}(t) = \left(3r^2+3r+1\right)Nt + 3(3r+1)(N)_2 t^2 + (N)_3 t^3,$$

$$V_{N,\, r,\, 4}(t) = \left(4r^3+6r^2+4r+1\right)Nt + 7\left(6r^2+4r+1\right)(N)_2 t^2$$

$$+ 6(4r+1)(N)_3 t^3 + (N)_4 t^4,$$

$$V_{N,\, r,\, 5}(t) = \left\{(r+1)^5 - r^5\right\}Nt + 15\left(10r^3+10r^2+5r+1\right)(N)_2 t^2$$

$$+ 25\left(10r^2+5r+1\right)(N)_3 t^3 + 10(5r+1)(N)_4 t^4 + (N)_5 t^5,$$

$$V_{N,\, r,\, 6}(t) = \left\{(r+1)^6 - r^6\right\}Nt + 31\left(15r^4+20r^3+15r^2+6r+1\right)(N)_2 t^2$$

$$+ 90\left(20r^3+15r^2+6r+1\right)(N)_3 t^3 + 65\left(15r^2+6r+1\right)(N)_4 t^4$$

$$+ 15(6r+1)(N)_5 t^5 + (N)_6 t^6,$$

and so on.

3. Multinomial type sums

For $\mathbf{t}$ in $\mathscr{C}^p$, $\mathbf{n}$ in $\mathscr{N}^p$ and N in $\mathscr{C}$, set

$$\mathbf{t}^{\mathbf{n}} = t_1^{n_1} \cdots t_p^{n_p},$$

$$\mathbf{t}^{\otimes \mathbf{n}} = \left(t_1^{n_1},\, \ldots,\, t_p^{n_p}\right),$$

$$|\mathbf{n}| = n_1 + \cdots + n_p,$$

$$\mathbf{tn} = (t_1 n_1, \ \ldots, \ t_p n_p),$$

$$\frac{\mathbf{t}}{\mathbf{n}} = \left(\frac{t_1}{n_1}, \ \ldots, \ \frac{t_p}{n_p}\right),$$

$$\binom{N}{\mathbf{n}} = \binom{N}{n_1 \cdots n_p} = \frac{N!}{n_1! \cdots n_p! \, (N - |\mathbf{n}|)!},$$

$$\binom{\mathbf{t}}{\mathbf{n}} = \prod_{a=1}^{p} \binom{t_a}{n_a}.$$

So, $\binom{N}{\mathbf{n}}$ is the usual multinomial coefficient while $\binom{\mathbf{t}}{\mathbf{n}}$ is a product of binomial coefficients.

For $\mathbf{j}$, $\mathbf{k}$, $\mathbf{r}$ in $\mathcal{N}^p$, let $P_{\mathbf{j}}(\mathbf{k})$ a polynomial in $\mathbf{k}$ of degree $\mathbf{j}$, that is a polynomial in k_a of degree j_a for $a = 1, \ 2, \ \ldots p$. Fix $\mathbf{m} > \mathbf{0}$ in $\mathcal{N}^p$. Inequalities in $\mathcal{N}^p$ are understood to hold componentwise with both sides having dimension p.

In this section, we give expressions for

$$U_{N, \, \mathbf{r}, \, \mathbf{j}} = \sum_{\mathbf{k}=-\boldsymbol{\infty}}^{\boldsymbol{\infty}} \binom{N}{\mathbf{n}} \mathbf{t}^{\mathbf{n}} P_{\mathbf{j}}(\mathbf{k}) \Big|_{\mathbf{n}=\mathbf{mk}-\mathbf{r}}. \tag{12}$$

We assume that $P_0(\mathbf{k}) = 1$. Since $\mathbf{t}^{\mathbf{r}} U_{N, \, \mathbf{r}, \, \mathbf{j}}$ is not altered by changing r_ℓ by m_ℓ, we may assume that $\mathbf{0} \leq \mathbf{r} < \mathbf{m}$. If N is not a positive integer we also assume that

$$\sum_{a=1}^{p} |t_a| < 1.$$

Set $\mathbf{w}_{\mathbf{m}} = \left(w_{m_1}, \ \ldots, \ w_{m_p}\right)$ and let $\mathbf{t}^T$ denote the transpose of $\mathbf{t}$. Note that for $\boldsymbol{\ell}$ in $\mathcal{N}^p$,

$$\mathbf{w}_{\mathbf{m}}^{\boldsymbol{\ell}\mathbf{r}} = \prod_{a=1}^{p} w_{m_a}^{\ell_a r_a},$$

$$\mathbf{t}^{\mathbf{n}} \mathbf{w}_{\mathbf{m}}^{\boldsymbol{\ell}\mathbf{n}} = \left(\mathbf{t}\mathbf{w}_{\mathbf{m}}^{\otimes \boldsymbol{\ell}}\right)^{\mathbf{n}},$$

$$\mathbf{t}^T \mathbf{w}_{\mathbf{m}}^{\otimes \boldsymbol{\ell}} = \sum_{a=1}^{p} w_{m_a}^{\ell_a} t_a.$$

Also $\mathbf{m}^{-\mathbf{1}_p} = (m_1 \cdots m_p)^{-1}$, where $\mathbf{1}_p = (1, \ \ldots, \ 1)$.

Theorem 3 For $\mathbf{0} \leq \mathbf{r} < \mathbf{m}$ in $\mathcal{N}^p$,

$$U_{N, \, \mathbf{r}, \, \mathbf{0}} = \mathbf{m}^{-\mathbf{1}_p} \sum_{\mathbf{0} \leq \boldsymbol{\ell} < \mathbf{m}} \mathbf{w}_{\mathbf{m}}^{\boldsymbol{\ell}\mathbf{r}} \left(1 + \mathbf{t}^T \mathbf{w}_{\mathbf{m}}^{\otimes \boldsymbol{\ell}}\right)^N.$$

The right hand side of (A5) has $m_1 \cdots m_p$ terms. For N a large positive integer, the left hand side of (A5) has $\approx \dfrac{N^p}{m_1 \cdots m_p p!}$ terms.

We now show how to reduce $U_{N,\,\mathbf{r},\,\mathbf{j}}$ to $U_{N,\,\mathbf{r},\,\mathbf{0}}$.

Lemma 1 The system

$$\sum_{\mathbf{0} \le \mathbf{s} \le \mathbf{n}} \binom{\mathbf{n}}{\mathbf{s}} A_{\mathbf{s}} = B_{\mathbf{n}},\ \mathbf{n} \text{ in } \mathscr{N}^p \tag{13}$$

has solution

$$A_{\mathbf{s}} = \sum_{\mathbf{0} \le \mathbf{n} \le \mathbf{s}} \binom{\mathbf{s}}{\mathbf{n}} (-1)^{|\mathbf{s}-\mathbf{n}|} B_{\mathbf{n}},\ \mathbf{s} \text{ in } \mathscr{N}^p. \tag{14}$$

Let $B_{\mathbf{y}}$ be a polynomial in $\mathbf{y}$ in $\mathscr{C}^p$ of degree $\mathbf{j}$. As for $p = 1$, Lemma 1 implies that

$$B_{\mathbf{y}} = \sum_{\mathbf{0} \le \mathbf{s} \le \mathbf{j}} \binom{\mathbf{y}}{\mathbf{s}} A_{\mathbf{s}}.$$

Corollary 1 We have $B_{\mathbf{y}}$ given in terms of its values at $\{\mathbf{n} : \mathbf{0} \le \mathbf{n} \le \mathbf{j}\}$ in $\mathscr{N}^p$ by

$$B_{\mathbf{y}} = \sum_{\mathbf{0} \le \mathbf{n} \le \mathbf{j}} B_{\mathbf{n}} \binom{\mathbf{y}}{\mathbf{n}} \sum_{\boldsymbol{\ell}=\mathbf{0}}^{\mathbf{j}-\mathbf{n}} (-1)^{|\boldsymbol{\ell}|} \binom{\mathbf{y}-\mathbf{n}}{\boldsymbol{\ell}}.$$

Theorem 4 Given $\mathbf{j},\ \mathbf{m},\ \mathbf{r}$ in $\mathscr{N}^p$ with $\mathbf{m} > \mathbf{0}$, and $P_{\mathbf{j}}(\mathbf{k})$ a polynomial of degree $\mathbf{j}$ in $\mathbf{k}$, set $B_{\mathbf{y}} = \mathbf{m}^{\mathbf{j}} P_{\mathbf{j}}\left(\dfrac{\mathbf{y}+\mathbf{r}}{\mathbf{m}}\right)$ and define $A_{\mathbf{s}}$ by (14). Then $U_{N,\,\mathbf{r},\,\mathbf{j}}$ of (12) satisfies

$$\mathbf{m}^{\mathbf{j}} U_{N,\,\mathbf{r},\,\mathbf{j}} = \sum_{\mathbf{s}=\mathbf{0}}^{\mathbf{j}} \binom{N}{\mathbf{s}} A_{\mathbf{s}} \mathbf{t}^{\mathbf{s}} U_{N-|\mathbf{s}|,\,\mathbf{r}+\mathbf{s},\,\mathbf{0}} \tag{15}$$

$$= \sum_{\mathbf{n}=\mathbf{0}}^{\mathbf{j}} B_{\mathbf{n}} \binom{N}{\mathbf{n}} \sum_{\boldsymbol{\ell}=\mathbf{0}}^{\mathbf{j}-\mathbf{n}} (-1)^{|\boldsymbol{\ell}|} \binom{N-\mathbf{n}-\boldsymbol{\ell}}{\boldsymbol{\ell}} \mathbf{t}^{\mathbf{n}+\boldsymbol{\ell}} U_{N-|\mathbf{n}+\boldsymbol{\ell}|,\,\mathbf{r}+\mathbf{n}+\boldsymbol{\ell},\,\mathbf{0}}. \tag{16}$$

Example 3 By (A6) for $P_{\mathbf{j}}(\mathbf{k}) = \binom{\mathbf{mk}-\mathbf{r}}{\mathbf{j}}$,

$$U_{N,\,\mathbf{r},\,\mathbf{j}} = \binom{N}{\mathbf{j}} \mathbf{t}^{\mathbf{j}} U_{N-|\mathbf{j}|,\,\mathbf{r}+\mathbf{j},\,\mathbf{0}}.$$

Example 4 Suppose that $P_{\mathbf{j}}(\mathbf{k}) = \mathbf{k}^{\mathbf{j}}$. Extend $T_{r,\,j,\,i}$ of (10) and $V_{N,\,r,\,j}(t)$ of (4) to $\mathscr{N}^p$ by setting

$$T_{\mathbf{r},\,\mathbf{j},\,\mathbf{i}} = \prod_{a=1}^{p} T_{r_a,\,j_a,\,i_a},$$

and

$$V_{N\mathbf{r},\,\mathbf{j}}(\mathbf{t}) = \sum_{\mathbf{i}=\mathbf{0}}^{\mathbf{j}} T_{\mathbf{r},\,\mathbf{j},\,\mathbf{i}} (N)_{|\mathbf{i}|} \mathbf{t}^{\mathbf{i}}.$$

Then (9) and (3) extend to

$$\begin{aligned} \mathbf{m}^{\mathbf{j}} U_{N,\,\mathbf{r},\,\mathbf{j}} &= \sum_{\mathbf{i}=\mathbf{0}}^{\mathbf{j}} T_{\mathbf{r},\,\mathbf{j},\,\mathbf{i}} (N)_{|\mathbf{i}|} \mathbf{t}^{\mathbf{i}} U_{N-|\mathbf{i}|,\,\mathbf{r}+\mathbf{i},\,\mathbf{0}} \\ &= \mathbf{m}^{-\mathbf{1}_p} \sum_{\mathbf{0} \le \boldsymbol{\ell} < \mathbf{m}} \mathbf{w}_{\mathbf{m}}^{\boldsymbol{\ell}\mathbf{r}} \left(1 + \mathbf{t}^T \mathbf{w}_{\mathbf{m}}^{\otimes \boldsymbol{\ell}}\right)^N V_{N,\,\mathbf{r},\,\mathbf{j}} \left(\mathbf{w}_{\mathbf{m}}^{\otimes \boldsymbol{\ell}} \frac{\mathbf{t}}{1 + \mathbf{t}^T \mathbf{w}_{\mathbf{m}}^{\otimes \boldsymbol{\ell}}}\right). \end{aligned} \tag{17}$$

The derivation uses the fact that (A3) and (A4) hold in $\mathscr{N}^p$ with

$$(\mathbf{m})_{\mathbf{i}} = \prod_{a=1}^{p} (m_a)_{i_a}.$$

For example, if $p = 2$ and $\mathbf{j} = (2,\ 3)$ then by (4)

$$\begin{aligned} V_{N,\,\mathbf{r},\,\mathbf{j}}(\mathbf{t}) = {} & (2r_1 + 1) \left\{ \left(3r_2^2 + 3r_2 + 1\right) (N)_2 t_1 t_2 + (3r_2 + 1) (N)_3 t_1 t_2^2 + (N)_4 t_1 t_2^3 \right\} \\ & \cdot \left(3r_2^2 + 3r_2 + 1\right) (N)_3 t_1^2 t_2 + (3r_2 + 1) (N)_4 t_1^2 t_2^2 + (N)_5 t_1^2 t_2^3. \end{aligned} \tag{18}$$

To express $V_{N,\,\mathbf{r},\,\mathbf{j}}(\mathbf{t})$ on $\mathscr{C}^p$ in terms of $V_{N,\,\mathbf{r},\,\mathbf{j}}(\mathbf{t})$ on $\mathscr{C}^{p-1}$ write $\mathbf{j} = (j_1,\ \mathbf{J})$, $\mathbf{r} = (r_1,\ \mathbf{R})$, $\mathbf{t} = (t_1,\ \mathbf{T})$ for $\mathbf{J},\ \mathbf{R}$ in $\mathscr{N}^{p-1}$ and $\mathbf{T}$ in $\mathscr{C}^{p-1}$. Then

$$V_{N,\,\mathbf{r},\,\mathbf{j}}(\mathbf{t}) = \sum_{i_1=0}^{j_1} T_{r_1,\,j_1,\,i_1} (N)_{i_1} t_1^i V_{N-i_1,\,\mathbf{R},\,\mathbf{J}}(\mathbf{T}).$$

4. Conditional multinomial type sums

In this section, we consider sums like (12) but with the restrictions that $|\mathbf{n}| = N$, where now N is a positive integer. Set

$$U'_{N,\,\mathbf{r},\,\mathbf{j}} = \sum_{\mathbf{k}=-\infty}^{\infty} \left\{ \binom{N}{\mathbf{n}} \mathbf{t}^{\mathbf{n}} P_{\mathbf{j}}(\mathbf{k}) : |\mathbf{n}| = N \right\}_{\mathbf{n}=\mathbf{m}\mathbf{k}-\mathbf{r}}$$

for $\mathbf{m}$, $\mathbf{k}$, $\mathbf{r}$, $\mathbf{t}$, $P_{\mathbf{j}}(\mathbf{k})$ as in section 3. This sum is finite since for $a = 1, \ldots p$, $0 \le m_a k_a - r_a \le N$.

Again $\mathbf{t}^{\mathbf{r}} U'_{N,\,\mathbf{r},\,\mathbf{0}}$ is not altered by changing r_a by m_a, so we can assume that $\mathbf{0} \le \mathbf{r} < \mathbf{m}$.

Theorem 5 For $\mathbf{0} \le \mathbf{r} < \mathbf{m}$,

$$U'_{N,\,\mathbf{r},\,\mathbf{0}} = \mathbf{m}^{-\mathbf{1}_p} \sum_{\mathbf{0} \le \boldsymbol{\ell} < \mathbf{m}} \mathbf{w}_{\mathbf{m}}^{\boldsymbol{\ell}\mathbf{r}} \left(\mathbf{t}^T \mathbf{w}_{\mathbf{m}}^{\otimes \boldsymbol{\ell}} \right)^N. \tag{19}$$

Similarly, we have

Theorem 6 Take $B_{\mathbf{y}}$ and $A_{\mathbf{s}}$ as in section 3. Then Theorem 4 holds with $\{U_{N,\,\mathbf{r},\,\mathbf{j}}\}$ replaced by $\left\{U'_{N,\,\mathbf{r},\,\mathbf{j}}\right\}$.

Suppose that $q > 0$ in $\mathcal{N}$ is a common factor to $\{m_1,\ r_1,\ \ldots,\ m_p,\ r_p\}$ then it must be a factor of N or else $U'_{N,\,\mathbf{r},\,\mathbf{0}} = 0$.

Note that $U'_{N,\,\mathbf{r},\,\mathbf{0}}$ sums over all solutions $\mathbf{k}$ in $\mathcal{N}^p$ to the Diophantine equation

$$\sum_{a=1}^{p} (m_a k_a - r_a) = N$$

that satisfy $0 \le m_a k_a - r_a \le N$. A heuristic argument suggests that $U'_{N,\,\mathbf{r},\,\mathbf{j}}$ sums over $\approx \dfrac{N^{p-1}}{(p-1)!m_1 \cdots m_p}$ terms for N large, while (19) sums over $m_1 \cdots m_p$ terms.

Example 5 Example 3 holds with $U_{N,\,\mathbf{r},\,\mathbf{j}}$, $U_{N,\,\mathbf{r},\,\mathbf{0}}$ replaced by $U'_{N,\,\mathbf{r},\,\mathbf{j}}$, $U'_{N,\,\mathbf{r},\,\mathbf{0}}$.

Example 6 Example 4 holds with $U_{N,\,\mathbf{r},\,\mathbf{j}}$, $U_{N,\,\mathbf{r},\,\mathbf{0}}$ replaced by $U'_{N,\,\mathbf{r},\,\mathbf{j}}$, $U'_{N,\,\mathbf{r},\,\mathbf{0}}$ and '1+' in (17) replaced by zero. That is, for $P_{\mathbf{j}}(\mathbf{k}) = \mathbf{k}^{\mathbf{j}}$ in $\mathcal{N}^p$,

$$\mathbf{m}^{\mathbf{j}} U'_{N,\,\mathbf{r},\,\mathbf{j}} = \sum_{\mathbf{i}=\mathbf{0}}^{\mathbf{j}} T_{r,\,\mathbf{j},\,\mathbf{i}}(N)_{|\mathbf{i}|} \mathbf{t}^{\mathbf{i}} U'_{N-|\mathbf{i}|,\,\mathbf{r}+\mathbf{i},\,\mathbf{0}}$$

$$= \mathbf{m}^{-\mathbf{1}_p} \sum_{\mathbf{0} \le \boldsymbol{\ell} < \mathbf{m}} \mathbf{w}_{\mathbf{m}}^{\boldsymbol{\ell}\mathbf{r}} \left(\mathbf{t}^T \mathbf{w}_{\mathbf{m}}^{\otimes \boldsymbol{\ell}} \right)^N V_{N,\,\mathbf{r},\,\mathbf{j}} \left(\mathbf{w}_{\mathbf{m}}^{\otimes \boldsymbol{\ell}} \frac{\mathbf{t}}{\mathbf{t}^T \mathbf{w}_{\mathbf{m}}^{\otimes \boldsymbol{\ell}}} \right). \tag{20}$$

A simple case of this, where $\mathbf{t}$ can be interpreted as a vector of probabilities is given by

Example 7 Suppose that $\mathbf{m} = \mathbf{1}_p$ so that

$$U'_{N,\,\mathbf{r},\,\mathbf{j}} = \sum_{\mathbf{n}=\mathbf{0}}^{\infty} \left\{ \binom{N}{\mathbf{n}} \mathbf{t}^{\mathbf{n}} P_{\mathbf{j}}(\mathbf{n}+\mathbf{r}) :\ |\mathbf{n}| = N \right\}.$$

Suppose also that $P_{\mathbf{j}}(\mathbf{k}) = \mathbf{k}^{\mathbf{j}}$ and $t_1 + \cdots + t_p = 1$. By (20),

$$U'_{N,\,\mathbf{r},\,\mathbf{j}} = V_{N,\,\mathbf{r},\,\mathbf{j}}(\mathbf{t}).$$

For example, if $p = 2$ then

$$U'_{N,\,\mathbf{r},\,\mathbf{j}} = \sum_{n_1=0}^{N} \binom{N}{n_1} t_1^{n_1} t_2^{N-n_1} (n_1 + r_1)^{j_1} (N - n_1 + r_2)^{j_2}.$$

See (18) for $V_{N,\,\mathbf{r},\,\mathbf{j}}(\mathbf{t})$ when $p = 2$ and $\mathbf{j} = (2,\ 3)$.

If also $\mathbf{t} > \mathbf{0}$, then $U'_{N,\,\mathbf{r},\,\mathbf{j}}$ is the mean of the random function $(\mathbf{X} + \mathbf{r})^{\mathbf{j}}$, where $\mathbf{X}$ is a multinomial random vector in $\mathcal{N}^p$: $\mathbf{X} \sim$ Multinomial $(N,\ \mathbf{t})$.

The above results can be extended to N not a positive integer using

$$\left(\sum_{a=1}^{p} t_a \right)^N = \sum_{\mathbf{k}=-\boldsymbol{\infty}}^{\boldsymbol{\infty}} \left\{ \binom{N}{\mathbf{k}} \mathbf{t}^{\mathbf{k}} : |\mathbf{k}| = N,\ k_2 \geq 0,\ \ldots,\ k_p \geq 0 \right\}$$

for

$$\left| \sum_{a=2}^{p} \frac{t_a}{t_1} \right| < 1,$$

where $k_1 = N - k_2 - \cdots - k_p$ is no longer an integer.

5. Summary and future research

This note presented derivations for both univariate and multivariate binomial/multinomial sums. In the univariate case (section 2), the authors express sums of the form

$$\sum_{k=-\infty}^{\infty} \binom{N}{mk - r} t^{mk-r} P_j(k)$$

as a sum of m terms involving mth roots of unity, with relatively low computational complexity. The key advantage is the simplification achieved for specific polynomial choices like $D_j(k) = (mk - r)_j$, and the avoidance of hypergeometric functions. For the multivariate case (sections 3–4), the derivation extends to vector-valued parameters, expressing sums as $m_1 m_2 \cdots m_p$ terms, resulting in exponentially higher complexity with dimension. Despite this complexity, the multivariate formulation offers the significant advantage of handling high-dimensional problems and conditional constraints (where $|\mathbf{n}| = N$), while still avoiding hypergeometric functions and providing flexible polynomial handling. Both approaches maintain the core benefit of expressing these sums in closed form without special functions, but trade off computational simplicity for dimensional flexibility.

Future research stemming from this note could significantly advance computer algebra systems by implementing the derived expressions for binomial and multinomial sums as efficient algorithms, enabling faster symbolic computations and extending capabilities for handling complex polynomial functions. In combinatorics, these results open avenues for discovering new combinatorial identities, establishing connections with graph theory and lattice structures, and developing novel generating function techniques for enumeration problems with multiple constraints. For approximation theory, the presented formulas offer promising approaches to develop new approximation methods for special functions, analyze convergence properties of series expansions involving binomial coefficients, and create improved numerical techniques

with tighter error bounds, potentially benefiting statistical modeling and probability distribution approximations in data science applications.

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Conflict of interest

The authors declare no competing financial interest.

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Appendix

Proof of Theorem 1 By linearity, (2)–(4) will hold for some T coefficients if they hold for any $\{P_j\}$. We now prove that they hold for $P_j = D_j$ of (5). Note that $U_{N,\,r}(D_j)$ is not altered by changing r by m. So, we may assume that $0 \le r < m$. If N is not a positive integer then the sum is infinite and we assume that $|t| < 1$.

First consider the case $j = 0$. For ℓ an integer,

$$\left(1 + w_m^{\ell} t\right)^N = \sum_{k=0}^{\infty} \binom{N}{k} \left(w_m^{\ell} t\right)^k = \sum_{r=0}^{m-1} w_m^{-\ell r} U_{N,\,r,\,0} \tag{A1}$$

since we may replace any sum

$$\sum_{k=-\infty}^{\infty} f(k)$$

by

$$\sum_{r=0}^{m-1} \sum_{k=-\infty}^{\infty} f(mk - r).$$

The $m \times m$ Fourier matrix

$$\mathbf{F} = m^{-\frac{1}{2}} \left(w_m^{-\ell r} : 0 \le \ell,\ r < m\right)$$

has inverse $\overline{\mathbf{F}}$, its conjugate. See p.271–274 of [16] for more on $\mathbf{F}$. Taking $\ell = 0, \ldots, m-1$ we can write this system as

$$\alpha_\ell = m^{\frac{1}{2}} \sum_{r=0}^{m-1} F_{\ell,\,r} U_{N,\,r,\,0},$$

where α_ℓ is the left hand side of (A1), with inverse

$$U_{N,\,r,\,0} = m^{-\frac{1}{2}} \sum_{\ell=0}^{m-1} \overline{F}_{\ell,\,r} \alpha_\ell = m^{-1} \sum_{\ell=0}^{m} w_m^{\ell r} \alpha_\ell.$$

This proves (2). Since

$$(n)_i \binom{N}{n} = (N)_i \binom{N-i}{n-i},$$

it follows that

$$U_{N,\,r}(D_j) = (N)_j t^j U_{N-j,\,r+j,\,0}. \tag{A2}$$

So, from (2), (2)–(4) hold for $P_j = D_j$ with T^D of (6). The proof is complete. □

Proof of Theorem 2 By (8),

$$(x+r)^a = \sum_{i=0}^{a} (x)_i T_{r,\,a,\,i},$$

so that

$$(mk)^a = \sum_{i=0}^{a} (mk-r)_i T_{a,\,i,\,r}, \tag{A3}$$

$$P_j(k) = \sum_{a=0}^{j} c_a m^{-a} \sum_{i=0}^{a} (mk-r)_i T_{r,\,a,\,i}, \tag{A4}$$

$$U_{N,\,r,\,j} = \sum_{a=0}^{j} c_a m^{-a} \sum_{i=0}^{a} i! T_{r,\,a,\,i} U^{\#}_{N,\,r,\,i},$$

where $U^{\#}_{N,\,r,\,i}$ is $U_{N,\,r,\,i}(D_j)$ of (A2). □

Proof of Theorem 3 For all $\boldsymbol{\ell}$ in $\mathcal{N}^p$

$$\left(1 + \sum_{a=1}^{p} w_{m_a}^{\ell_a} t_a\right)^N = \sum_{\mathbf{k}=-\boldsymbol{\infty}}^{\boldsymbol{\infty}} \binom{N}{\mathbf{k}} \prod_{a=1}^{p} \left(w_{m_a}^{\ell_a} t_a\right)^{k_a} = \sum_{\mathbf{0}\le\mathbf{r}<\mathbf{m}} \left(\prod_{a=1}^{p} w_{m_a}^{-\ell_a r_a}\right) U_{N,\,\mathbf{r},\,\mathbf{0}}, \tag{A5}$$

replacing

$$\sum_{\mathbf{k}=-\boldsymbol{\infty}}^{\boldsymbol{\infty}} f(\mathbf{k})$$

by

$$\sum_{\mathbf{0}\le\mathbf{r}<\mathbf{m}} \sum_{\mathbf{k}=-\boldsymbol{\infty}}^{\boldsymbol{\infty}} f(\mathbf{mk}-\mathbf{r}).$$

Now restrict $\boldsymbol{\ell}$ to $\mathbf{0} \le \boldsymbol{\ell} < \mathbf{m}$ and invert (A5) as for $p = 1$. □

Proof of Lemma 1 First suppose that $p = 1$. Let F be the forward operator

$$FA_{\mathbf{s}} = A_{\mathbf{s}+\mathbf{1}},\ FB_{\mathbf{n}} = B_{\mathbf{n}+\mathbf{1}}.$$

We wish to prove that $A_{\mathbf{s}} \equiv (F-1)^{\mathbf{s}} B_{\mathbf{0}}$ is the solution of $(F+1)^{\mathbf{n}} A_{\mathbf{0}} \equiv B_{\mathbf{n}}$. This follows since

$$\sum_{\mathbf{s}=\mathbf{0}}^{\mathbf{n}} \binom{\mathbf{n}}{\mathbf{s}} (F-1)^{\mathbf{s}} B_{\mathbf{0}} = (1+F-1)^{\mathbf{n}} B_{\mathbf{0}} = B_{\mathbf{n}}.$$

For $p > 1$, just apply this result p times. □

Proof of Theorem 4 Note that (15) follows from (13) and

$$\binom{N}{\mathbf{n}}\binom{\mathbf{n}}{\mathbf{s}} = \binom{N}{\mathbf{s}}\binom{N-|\mathbf{s}|}{\mathbf{n}-\mathbf{s}}. \tag{A6}$$

Now use (14) to obtain (16). □

Proof of Theorem 5 For $\boldsymbol{\ell}$ in $\mathscr{N}^p$,

$$\left(\sum_{a=1}^{p} \mathbf{w}_{\mathbf{m}}^{\ell_a} t_a\right)^N = \sum_{\mathbf{k}=-\boldsymbol{\infty}}^{\boldsymbol{\infty}} \left\{ \binom{N}{\mathbf{k}} \prod_{a=1}^{p} \left(w_{m_a}^{\ell_a} t_a\right)^{k_a} : |\mathbf{k}| = N \right\} = \sum_{\mathbf{0} \le \mathbf{r} < \mathbf{m}} \left(\prod_{a=1}^{p} w_{m_a}^{-\ell_a r_a}\right) U'_{N,\,\mathbf{r},\,\mathbf{0}}.$$

Now restrict $\boldsymbol{\ell}$ to $\mathbf{0} \le \boldsymbol{\ell} < \mathbf{m}$ and invert. □