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Research Article

Novel Analytical Approaches for the Fractional Kundu-Mukherjee-Naskar Model in Nonlinear Optics

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Abstract: This study investigates exact traveling wave solutions for the time-fractional Kundu-Mukherjee-Naskar (KMN) equation, which plays a significant role in modeling nonlinear pulse propagation in optical fibers. Two powerful analytical techniquesthe Riccati Modified Extended Simple Equation Method (RMESEM) and the Generalized Projective Riccati equations Method (GPRM) are employed to derive novel soliton and periodic wave results. Consider the fractional derivative in the conformable sense to maintain analytical tractability. Using these methods, we obtain a variety of solutions, including bright solitons, dark solitons, singular solitons, and periodic waves, expressed in terms of hyperbolic, trigonometric, and rational functions. A comparative analysis reveals the strengths and limitations of each method in handling the fractional KMN equation. The results demonstrate that both approaches are effective but yield different forms of solutions, enriching the understanding of nonlinear wave dynamics in optical media. The findings may have potential applications in nonlinear optics and photonics.

Keywords: fractional Kundu-Mukherjee-Naskar equation, modified extended simple equation method, generalized projective Riccati equations, optical solitons, exact solutions

MSC: 34G20, 35A20, 35A22, 35R11

1. Introduction

The Nonlinear Fractional Partial Differential Equations (NFPDEs) have gained a lot of significance in different scientific and engineering domains in the past with the ability to describe various complicated systems. The equations are particularly appropriate in describing memory and hereditary behaviours in dynamic processes. NFPDEs have special significance in modelling anomalous diffusion and they find significant application in fields like physics, chemistry, biology, engineering, finance and economics [1–6]. As such, the study on NFPDEs plays a crucial role in the study and forecasting the behavior of complex systems in many sectors. These have uniqueness in its nature since it is non-local and nonlinear to an extent of causing special strains on it being solved and even interpreted. Traditional analytical and numerical methods are frequently insufficient and new methods and computer codes have to be created. Nevertheless, continuous work on the development of NFPDEs has resulted in significant advances in many fields of science and engineering [7–9]. New analytical and numerical procedures that can be used to address the NFPDEs [10–12] have opened

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up research opportunities and contributed to an improved understanding of more complex phenomena. A wide variety of powerful techniques have been developed for obtaining exact solutions of nonlinear fractional problems, among these are the sine-Gordon expansion method [13], generalized projective Riccati equation method [14], Lie symmetry approach [15], the auxiliary ordinary differential equation method and generalized Riccati method [16], modified Kudryashov method [17], first integral method [18], modified auxiliary equation method [19], extended $\exp(\phi(\xi))$-expansion method [20], unified method [21], Khater methods [22, 23], $\frac{G'}{G}$-expansion methods [24–26], Riccati-Bernoulli Sub-Ordinary Differential Equation (ODE) [27], Poincaré-Lighthill-Kuo method [28], fractional Sin-Gordon method [29], exp-function method [30], tanh-method [31], sub-equation method [32], extended direct algebraic method [33–36], Sardar sub-equation method [37], exponential rational function method [38], and so on [39–44].

Regardless of this variety of methods, expansion-based approaches like the Riccati Modified Extended Simple Equation Method (RMESEM) and the Generalized Projective Riccati equation Method (GPRM) are of particular interest. They offer a systematic framework that not only supports conformable fractional derivatives, but also gives a rich set of explicit traveling wave solutions with bright, dark and singular solitons and periodic waveforms. It is this flexibility that makes them unlike most traditional methods, which tend to be confined to a type of solution and makes them particularly appropriate to explore the fractional Kundu-Mukherjee-Naskar model.

The rest of this paper is structured as follows. Section 2 presents the working mechanism of the RMESEM. Section 3 provides RMESEM solutions to the Kundu Mukherjee Naskar (KMN) equation. Section 4 introduces the GPREs method and Section 5 applies the method to the KMN equation. The results and discussion of the soliton and periodic solutions obtained are presented in Section 6 containing graphical figures. Section 7 provides a comparison of our results with past literature to describe the novelty and relevance. Lastly, Section 8 is the conclusion of the work, which also includes some final remarks and possible future research directions.

1.1 *Kundu-Mukherjee-Naskar equation*

Kundu et al. [45] first proposed the Kundu-Mukherjee-Naskar Equation (KMNE) in 2014 and obtained optical soliton solutions for it. Since then, the KMNE has attracted considerable attention in the research community. The wave dynamics described by this equation are essential for effectively modeling pulse propagation in high-speed communication networks, optical fiber systems, and the formation of rogue waves [46–48] in ocean currents. A typical use of optical fibers is to transmit light from one end to the other. Due to their ability to support faster and longer-distance data transmission than traditional wiring, they are widely utilized in optical communication systems. The primary objective of this study is to develop and examine optical soliton solutions for the generalized Fractional Kundu-Mukherjee-Naskar Equation (gFKMNE), which represents a fractional extension of the original KMNE and is given in the following dimensionless form [49]:

$$iD_t^{\sigma}Z + aD_y^{\beta}(D_x^{\alpha}Z) + ibZ(ZD_x^{\alpha}Z^* - Z^*D_x^{\alpha}Z) = 0. \tag{1}$$

In this context, $0 < \alpha,\ \beta,\ \sigma \leq 1$, $Z = Z(t,\ x,\ y)$ denotes a complex wave envelope, with Z^* being its complex conjugate. The parameters a and b correspond to the dispersion and non linearity terms, respectively. The first term in equation (1) captures the wave's temporal memory, followed by the dispersive component $aD_y^{\beta}(D_x^{\alpha}Z)$. The nonlinear part is characterized by the term involving b, which deviates from the traditional Kerr-type non linearity. It is an equation that describes pulse in optical fibers and communication systems. The generalized Conformable Fractional Derivatives (gCFDs) are denoted as operators α, β, and σ, respectively, by $D_x^{\alpha}(\odot)$, $D_y^{\beta}(\odot)$, and $D_x^{\sigma}(\odot)$.

Equation (1) has been widely used in many areas of different spite in dumping systems, modeling in the river and tidal flows, meteorology and prediction of tsunami. It has enjoyed wide application and it has caught attention of many researchers who want to read more about it and solve it. By way of example, Guneshan et al. [50] used one of improved direct algebraic methods to find the new exact solutions of (1). Rizvi et al. [51] used methods, such as csch method, extended tanh-coth method, extended rational sinh-cosh method and derived several types of soliton solutions—singular soliton, dark soliton and a kind of mixture of both.

The solution of the optical solitons that were obtained by Talarposhti et al. [52] was obtained using the exp-function method to solve the KMNE. In a similar manner, Onder et al. [53] put forward optical solitons by putting together Sardar sub-equation approach and Kudryashov approach. Zafar et al. [54] used exp-function and ext. Jacobi elliptic expansion methods to achieve new soliton structures. Kumar et al. [55] have employed a generalized form of a Kudryashov in this case with a new technique of the auxiliary equation to expose the dark, the bright, periodic U-shaped, and singular solitons.

Additional growth in the solution of (1) has been made by a wide range of analysis tools, such as the semi-inverse procedure [56], and expansion patterns on the base of sine-Gordon and sinh-Gordon forms [57]. Some other remarkable methods are Hamiltonian-based algorithm [58], Laplace-Adomian decomposition method [59] and various solution strategies available at [60–64].

2. The operational procedure of RMESEM

This section outlines the mechanism of operation of the RMESEM. Inspecting the generalized NPDE obtained:

$$A(\mathfrak{f}, \mathfrak{f}_t, \mathfrak{f}_{\rho_1}, \mathfrak{f}_{\rho_2}, \mathfrak{f}\mathfrak{f}_{\rho_1}, \ldots) = 0, \tag{2}$$

where $\mathfrak{f} = \mathfrak{f}(t, \rho_1, \rho_2, \rho_3, \ldots, \rho_r)$.

All steps listed below will be taken in order to solve (2):

1. First $\mathfrak{f}(t, \rho_1, \rho_2, \rho_3, \ldots, \rho_r) = \mathfrak{F}(\sigma)$ is applied as an operation of transformation in the form of variable. In the case of σ there are multiple representations. This procedure transforms Equation (2) as follows to give the following Nonlinear Ordinary Differential Equation (NODE):

$$B(\mathfrak{F}, \mathfrak{F}'\mathfrak{F}, \mathfrak{F}', \ldots) = 0, \tag{3}$$

in which $\mathfrak{F}' = \dfrac{d\mathfrak{F}}{d\sigma}$. The integrating equation is sometimes applied to make the NODE obey the homogeneous balance condition (3).

2. Second, on the basis of the solution of the extended Riccati-NODE, the proposed solution in the form of the finite series of the NODE in (3) is guestimatched based on the solution of extended Riccati-NODE:

$$\mathfrak{F}(\sigma) = \sum_{n=0}^{M} k_n \left(\frac{P'(\sigma)}{P(\sigma)}\right)^n + \sum_{n=0}^{M-1} s_n \left(\frac{P'(\sigma)}{P(\sigma)}\right)^n \cdot \left(\frac{1}{P(\sigma)}\right). \tag{4}$$

Here, $P(\sigma)$ indicates the solution of the ensuing extended Riccati-NODE, and the variables k_n $(n = 0, \ldots, M)$ and $s_n(n = 0, \ldots, M-1)$ stand for the unidentified constants that need to be found later.

$$P'(\sigma) = \lambda + \mu P(\sigma) + \nu (P(\sigma))^2, \tag{5}$$

where λ, μ, and ν are constants.

Table 1. Solutions of RMESEM of equation (5)

S. No.	Cluster	Constraint(s)	$P(\sigma)$	$\left(\frac{P'(\sigma)}{P(\sigma)}\right)$
1	Trigonometric solutions	$B<0,\ \nu\neq 0$	$-\frac{\mu}{2\nu}+\frac{\sqrt{-B}\tan\left(\frac{1}{2}\sqrt{-B}\sigma\right)}{2\nu}$	$-\frac{1}{2}\frac{B\left(1+\left(\tan\left(\frac{1}{2}\sqrt{-B}\sigma\right)\right)^2\right)}{-\mu+\sqrt{-B}\tan\left(\frac{1}{2}\sqrt{-B}\sigma\right)},$
			$-\frac{\mu}{2\nu}-\frac{\sqrt{-B}\cot\left(\frac{1}{2}\sqrt{-B}\sigma\right)}{2\nu},$	$\frac{1}{2}\frac{\left(1+\left(\cot\left(\frac{1}{2}\sqrt{-B}\sigma\right)\right)^2\right)B}{\mu+\sqrt{-B}\cot\left(\frac{1}{2}\sqrt{-B}\sigma\right)},$
			$-\frac{\mu}{2\nu}+\frac{\sqrt{-B}\left(\tan\left(\sqrt{-B}\sigma\right)+\left(\sec\left(\sqrt{-B}\sigma\right)\right)\right)}{2\nu},$	$\frac{-B\left(1+\sin\left(\sqrt{-B}\sigma\right)\right)}{\cos\left(\sqrt{-B}\sigma\right)\left(-\mu\cos\left(\sqrt{-B}\sigma\right)+\sqrt{-B}\sin\left(\sqrt{-B}\sigma\right)+\sqrt{-B}\right)},$
			$-\frac{\mu}{2\nu}+\frac{\sqrt{-B}\left(\tan\left(\sqrt{-B}\sigma\right)-\left(\sec\left(\sqrt{-B}\sigma\right)\right)\right)}{2\nu}.$	$\frac{B\left(\sin\left(\sqrt{-B}\sigma\right)-1\right)}{\cos\left(\sqrt{-B}\sigma\right)\left(-\mu\cos\left(\sqrt{-B}\sigma\right)+\sqrt{-B}\sin\left(\sqrt{-B}\sigma\right)-\sqrt{-B}\right)}.$
2	Hyperbolic solutions	$B>0,\ \nu\neq 0$	$-\frac{\mu}{2\nu}-\frac{\sqrt{B}\tanh\left(\frac{1}{2}\sqrt{B}\sigma\right)}{2\nu},$	$-\frac{1}{2}\frac{\left(-1+\left(\tanh\left(\frac{1}{2}\sqrt{B}\sigma\right)\right)^2\right)B}{\mu+\sqrt{B}\tanh\left(\frac{1}{2}\sqrt{B}\sigma\right)},$
			$-\frac{\mu}{2\nu}-\frac{\sqrt{B}\left(\tanh\left(\sqrt{B}\sigma\right)+i\left(\operatorname{sech}\left(\sqrt{B}\sigma\right)\right)\right)}{2\nu},$	$-\frac{B\left(-1+i\sinh\left(\sqrt{B}\sigma\right)\right)}{\cosh\left(\sqrt{B}\sigma\right)\left(\mu\cosh\left(\sqrt{B}\sigma\right)+\sqrt{B}\sinh\left(\sqrt{B}\sigma\right)+i\sqrt{B}\right)},$
			$-\frac{\mu}{2\nu}-\frac{\sqrt{B}\left(\tanh\left(\sqrt{B}\sigma\right)-i\left(\operatorname{sech}\left(\sqrt{B}\sigma\right)\right)\right)}{2\nu},$	$-\frac{B\left(1+i\sinh\left(\sqrt{B}\sigma\right)\right)}{\cosh\left(\sqrt{B}\sigma\right)\left(-\mu\cosh\left(\sqrt{B}\sigma\right)-\sqrt{B}\sinh\left(\sqrt{B}\sigma\right)+i\sqrt{B}\right)},$
			$-\frac{\mu}{2\nu}-\frac{\sqrt{B}\left(\coth\left(\sqrt{B}\sigma\right)+\left(\operatorname{csch}\left(\sqrt{B}\sigma\right)\right)\right)}{2\nu}.$	$-\frac{1}{4}\frac{B\left(2\left(\cosh\left(\frac{1}{4}\sqrt{B}\sigma\right)\right)^2-1\right)}{\theta\left(-2\mu\theta+\sqrt{B}\right)}.$

Table 1. (cont.)

S. No.	Cluster	Constraint(s)	$P(\sigma)$	$\left(\frac{P'(\sigma)}{P(\sigma)}\right)$
3	Rational solutions	$B=0$	$-2\frac{\lambda(\mu\sigma+2)}{\mu^2\sigma}$,	$-2\frac{1}{\sigma(\mu\sigma+2)}$,
		$B=0$, & $\mu=\nu=0$	$\sigma\lambda$,	$\frac{1}{\sigma}$,
		$B=0$, & $\mu=\lambda=0$	$-\frac{1}{\sigma\nu}$.	$-\frac{1}{\sigma}$.
4	Exponential solutions	$\nu=0$, & $\mu=\varpi$, $\lambda=b\varpi$	$e^{\varpi\sigma}-b$,	$\frac{\varpi e^{\varpi\sigma}}{e^{\varpi\sigma}-b}$,
		$\lambda=0$, & $\mu=\varpi$, $\nu=b\varpi$	$\frac{e^{\varpi\sigma}}{1-be^{\varpi\sigma}}$.	$-\frac{\varpi}{-1+be^{\varpi\sigma}}$.
5	Rational-Hyperbolic solutions	$\lambda=0$, & $\mu\neq 0$, $\nu\neq 0$	$-\frac{\mu a_1}{\nu(\cosh(\mu\sigma)-\sinh(\mu\sigma)+a_2)}$,	$\frac{\mu(\sinh(\mu\sigma)-\cosh(\mu\sigma))}{-\cosh(\mu\sigma)+\sinh(\mu\sigma)-a_2}$,
			$-\frac{\mu(\cosh(\mu\sigma)+\sinh(\mu\sigma))}{\nu(\cosh(\mu\sigma)+\sinh(\mu\sigma)+a_2)}$.	$\frac{\mu a_2}{\cosh(\mu\sigma)+\sinh(\mu\sigma)+a_2}$.

3. We can obtain the positive integer M which we require in Equation (4) by homogeneously balancing the highest degree nonlinear term and the highest-order derivative in Equation (3).

4. The next step briefly is the incorporation of all the variables of the $P(\sigma)$ into an equivalent sequencing upon the insertion of (4) into (3) or the equation that results after combining together (3). Carrying out this treatment gives an equation expressed in terms of $P(\sigma)$ Notation. The variables k_n $(n = 0, \ldots, M)$ and s_n $(n = 0, \ldots, M-1)$ are defined as an algebraic system of equations and variables with other related parameters in case of assuming the coefficients in the equation are zero.

5. In Maple, the nonlinear algebraic system is solved analytically.

6. In order to obtain analytical soliton solutions to (2) we will enter the unknown quantities together with $P(\sigma)$ (solution of equation (5)) into (4) and calculate. Using the general solution form in (5) we could perhaps get a number of clusters of solitonic solution in Table 1. Table 1 is how these clusters are presented.

Where $a_1, a_2 \in R$, $B = \mu^2 - 4\nu\lambda$ and $\theta = \cosh\left(\frac{1}{4}\sqrt{B}\sigma\right)\sinh\left(\frac{1}{4}\sqrt{B}\sigma\right)$.

3. Establishing solutions for Kundu-Mukherjee-Naskar equation

In this section, we apply proposed techniques to build families of results for (1). Applying the wave modification as a component of the procedure, we can (1):

$$
\begin{aligned}
&z = Z(x, y, t) = \exp^{i\xi} Z(\phi),\\
&\zeta = \zeta(t, x, y),\\
&\phi = \phi(t, x, y),\\
&\phi = \frac{p_1\Gamma(n-\alpha+1)x^{\alpha}}{\alpha\Gamma(n)} + \frac{p_2\Gamma(n-\beta+1)y^{\beta}}{\beta\Gamma(n)} + \frac{\rho\Gamma(n-\sigma+1)t^{\sigma}}{\sigma\Gamma(n)},\\
&\zeta = \frac{q_1\Gamma(n-\alpha+1)x^{\alpha}}{\alpha\Gamma(n)} + \frac{q_2\Gamma(n-\beta+1)y^{\beta}}{\beta\Gamma(n)} + \frac{\omega\Gamma(n-\sigma+1)t^{\sigma}}{\sigma\Gamma(n)} + \Theta_0
\end{aligned}
\tag{6}
$$

we will get ODE in the form of;

$$
-ap_1p_2F'' - (\omega + aq_1q_2)F + 2bq_1F^3 = 0. \tag{7}
$$

By proving that F^3 and F'' are homogeneously balanced in (7), we obtain $M + 2 = 3M$, which yields $M = 1$. The supposed solution for (7) that results from inserting $M = 1$ in (4) is as follows:

$$
F(\sigma) = \sum_{n=0}^{1} k_n \left(\frac{P'(\sigma)}{P(\sigma)}\right)^n + \sum_{n=0}^{1-1} s_n \left(\frac{P'(\sigma)}{P(\sigma)}\right)^n \cdot \left(\frac{1}{P(\sigma)}\right). \tag{8}
$$

We generate a set of algebraic equations by substituting (8) in (7). By addressing the resulting system, numerous solutions can be obtained, which are presented as follow:

Case 1:

$$k_0 = -0.0005000000000\,\mu,\ k_1 = 0.001,\ s_0 = -0.001\,\lambda,\ p_1 = \frac{{k_1}^2}{ap_2},\ p_2 = 0.02,\ q_1 = 0.04,\ q_2 = 0.75,\ b = {q_1}^{-1} \tag{9}$$

Case 2:

$$k_0 = 0.01,\ k_1 = 0,\ s_0 = 0.02\,\frac{\lambda}{\mu},\ p_1 = 0.23,\ p_2 = 0.4,\ a = 0.006,\ q_1 = 0.65,\ q_2 = 0.075,\ b = 0.005 \tag{10}$$

Assuming Case 1, for the Kundu-Mukherjee-Naskar Equation given in (1), we build the subsequent sets of solutions:
Set. 1.1: When $B < 0$, $\nu \neq 0$,

$$f_{1,1}(x, y, t) = -0.0005000000000\,\mu - 0.0005000000000\,\frac{B\left(1+\left(\tan\left(1/2\sqrt{-B}\zeta\right)\right)^2\right)}{-\mu+\sqrt{-B}\tan\left(1/2\sqrt{-B}\zeta\right)}$$

$$-0.001\,\lambda\left(-1/2\,\frac{\mu}{\nu}+1/2\,\frac{\sqrt{-B}\tan\left(1/2\sqrt{-B}\zeta\right)}{\nu}\right), \tag{11}$$

$$f_{1,2}(x, y, t) = -0.0005000000000\,\mu + 0.0005000000000\,\frac{B\left(1+\left(\cot\left(1/2\sqrt{-B}\zeta\right)\right)^2\right)}{\mu+\sqrt{-B}\cot\left(1/2\sqrt{-B}\zeta\right)}$$

$$-0.001\,\lambda\left(-1/2\,\frac{\mu}{\nu}-1/2\,\frac{\sqrt{-B}\cot\left(1/2\sqrt{-B}\zeta\right)}{\nu}\right), \tag{12}$$

$$f_{1,3}(x, y, t) = -0.0005000000000\,\mu - 0.001\,\frac{B\left(1+\sin\left(\sqrt{-B}\zeta\right)\right)}{\cos\left(\sqrt{-B}\zeta\right)\left(-\mu\cos\left(\sqrt{-B}\zeta\right)+\sqrt{-B}\sin\left(\sqrt{-B}\zeta\right)+\sqrt{-B}\right)}$$

$$-0.001\,\lambda\left(-1/2\,\frac{\mu}{\nu}+1/2\,\frac{\sqrt{-B}\left(\tan\left(\sqrt{-B}\zeta\right)+\sec\left(\sqrt{-B}\zeta\right)\right)}{\nu}\right), \tag{13}$$

and

$$f_{1,4}(x, y, t) = -0.0005000000000\,\mu + 0.001\,\frac{B\left(\sin\left(\sqrt{-B}\zeta\right)-1\right)}{\cos\left(\sqrt{-B}\zeta\right)\left(-\mu\cos\left(\sqrt{-B}\zeta\right)+\sqrt{-B}\sin\left(\sqrt{-B}\zeta\right)-\sqrt{-B}\right)}$$

$$-0.001\,\lambda\left(-1/2\,\frac{\mu}{\nu}+1/2\,\frac{\sqrt{-B}\left(\tan\left(\sqrt{-B}\zeta\right)-\sec\left(\sqrt{-B}\zeta\right)\right)}{\nu}\right). \tag{14}$$

Set. 1.2: When $B > 0$, $\nu \neq 0$,

$$f_{1,5}(x, y, t) = -0.0005000000000\,\mu - 0.0005000000000\,\frac{B\left(-1+\left(\tanh\left(1/2\sqrt{B}\zeta\right)\right)^2\right)}{\mu+\sqrt{B}\tanh\left(1/2\sqrt{B}\zeta\right)}$$

$$-0.001\,\lambda\left(-1/2\,\frac{\mu}{\nu} - 1/2\,\frac{\sqrt{B}\tanh\left(1/2\sqrt{B}\zeta\right)}{\nu}\right), \tag{15}$$

$$f_{1,6}(x, y, t) = -0.0005000000000\,\mu - 0.001\,\frac{B\left(-1+i\sinh\left(\sqrt{B}\zeta\right)\right)}{\cosh\left(\sqrt{B}\zeta\right)\left(\mu\cosh\left(\sqrt{B}\zeta\right)+\sqrt{B}\sinh\left(\sqrt{B}\zeta\right)+i\sqrt{B}\right)}$$

$$-0.001\,\lambda\left(-1/2\,\frac{\mu}{\nu} - 1/2\,\frac{\sqrt{B}\left(\tanh\left(\sqrt{B}\zeta\right)+i\,\mathrm{sech}\left(\sqrt{B}\zeta\right)\right)}{\nu}\right), \tag{16}$$

$$f_{1,7}(x, y, t) = -0.0005000000000\,\mu - 0.001\,\frac{B\left(1+i\sinh\left(\sqrt{B}\zeta\right)\right)}{\cosh\left(\sqrt{B}\zeta\right)\left(-\mu\cosh\left(\sqrt{B}\zeta\right)-\sqrt{B}\sinh\left(\sqrt{B}\zeta\right)+i\sqrt{B}\right)}$$

$$-0.001\,\lambda\left(-1/2\,\frac{\mu}{\nu} - 1/2\,\frac{\sqrt{B}\left(\tanh\left(\sqrt{B}\zeta\right)-i\,\mathrm{sech}\left(\sqrt{B}\zeta\right)\right)}{\nu}\right), \tag{17}$$

and

$$f_{1,8}(x, y, t) = -0.0005000000000\,\mu$$

$$-0.0002500000000\,\frac{B\left(2\left(\cosh\left(1/4\sqrt{B}\zeta\right)\right)^2-1\right)}{\cosh\left(1/4\sqrt{B}\zeta\right)\sinh\left(1/4\sqrt{B}\zeta\right)\left(-2\mu\cosh\left(1/4\sqrt{B}\zeta\right)\sinh\left(1/4\sqrt{B}\zeta\right)+\sqrt{B}\right)}$$

$$-0.001\,\lambda\left(-1/2\,\frac{\mu}{\nu} - 1/4\,\frac{\sqrt{B}\left(\tanh\left(1/4\sqrt{B}\zeta\right)-\coth\left(1/4\sqrt{B}\zeta\right)\right)}{\nu}\right). \tag{18}$$

Set. 1.3: When $\mu = \varpi$, $\lambda = b\varpi (b \neq 0)$, and $\nu = 0$,

$$f_{1,9}(x, y, t) = -0.0005000000000\,\mu - 0.002\,\frac{1}{\zeta\left(\mu\zeta+2\right)} + 0.002\,\frac{\lambda^2\left(\mu\zeta+2\right)}{\mu^2\zeta}, \tag{19}$$

$$f_{1,10}(x, y, t) = -0.0005000000000\,\mu + 0.001\,\zeta^{-1} - 0.001\,\lambda^2\zeta, \tag{20}$$

$$f_{1,11}(x,y,t) = -0.0005000000000\mu - 0.001\,\zeta^{-1} + 0.001\,\frac{\lambda}{\nu\,\zeta}, \tag{21}$$

$$f_{1,12}(x,y,t) = -0.0005000000000\mu + 0.001\,\frac{\varpi\,\mathrm{e}^{\varpi\,\zeta}}{\mathrm{e}^{\varpi\,\zeta} - b} - 0.001\,\lambda\,\left(\mathrm{e}^{\varpi\,\zeta} - b\right), \tag{22}$$

$$f_{1,13}(x,y,t) = -0.0005000000000\mu - 0.001\,\frac{\varpi}{-1 + b\mathrm{e}^{\varpi\,\zeta}} - 0.001\,\frac{\lambda\,\mathrm{e}^{\varpi\,\zeta}}{1 - b\mathrm{e}^{\varpi\,\zeta}}, \tag{23}$$

$$f_{1,14}(x,y,t) = -0.0005000000000\mu + 0.001\,\frac{\mu\,(\sinh(\mu\,\zeta) - \cosh(\mu\,\zeta))}{-\cosh(\mu\,\zeta) + \sinh(\mu\,\zeta) - a_2}$$

$$+0.001\,\frac{\lambda\,a_1\mu}{\nu\,(\cosh(\mu\,\zeta) - \sinh(\mu\,\zeta) + a_2)}, \tag{24}$$

$$f_{1,15}(x,y,t) = -0.0005000000000\mu + 0.001\,\frac{\mu\,a_2}{\cosh(\mu\,\zeta) + \sinh(\mu\,\zeta) + a_2}$$

$$+0.001\,\frac{\lambda\,\mu\,(\cosh(\mu\,\zeta) + \sinh(\mu\,\zeta))}{\nu\,(\cosh(\mu\,\zeta) + \sinh(\mu\,\zeta) + a_2)}. \tag{25}$$

Assuming Case 2, for the Kundu-Mukherjee-Naskar Equation given in (1), we build the subsequent sets of solutions:
Set. 2.1: When $B < 0$, $\nu \neq 0$,

$$f_{2,1}(x,y,t) = 0.01 + 0.02\,\lambda\,\left(-1/2\,\frac{\mu}{\nu} + 1/2\,\frac{\sqrt{-B}\tan\left(1/2\sqrt{-B}\zeta\right)}{\nu}\right)\mu^{-1}, \tag{26}$$

$$f_{2,2}(x,y,t) = 0.01 + 0.02\,\lambda\,\left(-1/2\,\frac{\mu}{\nu} - 1/2\,\frac{\sqrt{-B}\cot\left(1/2\sqrt{-B}\zeta\right)}{\nu}\right)\mu^{-1}, \tag{27}$$

$$f_{2,3}(x,y,t) = 0.01 + 0.02\,\lambda\,\left(-1/2\,\frac{\mu}{\nu} + 1/2\,\frac{\sqrt{-B}\left(\tan\left(\sqrt{-B}\zeta\right) + \sec\left(\sqrt{-B}\zeta\right)\right)}{\nu}\right)\mu^{-1}, \tag{28}$$

and

$$f_{2,4}(x,y,t) = 0.01 + 0.02\,\lambda\,\left(-1/2\,\frac{\mu}{\nu} + 1/2\,\frac{\sqrt{-B}\left(\tan\left(\sqrt{-B}\zeta\right) - \sec\left(\sqrt{-B}\zeta\right)\right)}{\nu}\right)\mu^{-1}. \tag{29}$$

Set. 3.2: When $B > 0$, $\nu \neq 0$,

$$f_{2,5}(x, y, t) = 0.01 + 0.02\lambda \left(-1/2\frac{\mu}{\nu} - 1/2\frac{\sqrt{B}\tanh\left(1/2\sqrt{B}\zeta\right)}{\nu}\right)\mu^{-1}, \tag{30}$$

$$f_{2,6}(x, y, t) = 0.01 + 0.02\lambda \left(-1/2\frac{\mu}{\nu} - 1/2\frac{\sqrt{B}\left(\tanh\left(\sqrt{B}\zeta\right) + i\,\text{sech}\left(\sqrt{B}\zeta\right)\right)}{\nu}\right)\mu^{-1}, \tag{31}$$

$$f_{2,7}(x, y, t) = 0.01 + 0.02\lambda \left(-1/2\frac{\mu}{\nu} - 1/2\frac{\sqrt{B}\left(\tanh\left(\sqrt{B}\zeta\right) - i\,\text{sech}\left(\sqrt{B}\zeta\right)\right)}{\nu}\right)\mu^{-1}, \tag{32}$$

and

$$f_{2,8}(x, y, t) = 0.01 + 0.02\lambda \left(-1/2\frac{\mu}{\nu} - 1/4\frac{\sqrt{B}\left(\tanh\left(1/4\sqrt{B}\zeta\right) - \coth\left(1/4\sqrt{B}\zeta\right)\right)}{\nu}\right)\mu^{-1}. \tag{33}$$

Set. 3.3: When $\mu = \varpi$, $\lambda = b\varpi (b \neq 0)$, and $\nu = 0$,

$$f_{2,9}(x, y, t) = 0.01 - 0.04\frac{\lambda^2(\mu\zeta + 2)}{\mu^3\zeta}. \tag{34}$$

Set. 3.4: When $\mu = \varpi$, $\nu = b\varpi (b \neq 0)$, and $\lambda = 0$,

$$f_{2,10}(x, y, t) = 0.01 + 0.02\frac{\lambda^2\zeta}{\mu}. \tag{35}$$

Set. 3.5: When $\lambda = 0$, $\nu \neq 0$, and $\mu \neq 0$,

$$f_{2,11}(x, y, t) = 0.01 - 0.02\frac{\lambda}{\mu\nu\zeta}, \tag{36}$$

$$f_{2,12}(x, y, t) = 0.01 + 0.02\frac{\lambda\left(e^{\varpi\zeta} - b\right)}{\mu}, \tag{37}$$

$$f_{2,13}(x, y, t) = 0.01 + 0.02\frac{\lambda e^{\varpi\zeta}}{\mu\left(1 - be^{\varpi\zeta}\right)}, \tag{38}$$

$$f_{2,14}(x, y, t) = 0.01 - 0.02\frac{\lambda a_1}{\nu\left(\cosh(\mu\zeta) - \sinh(\mu\zeta) + a_2\right)}, \tag{39}$$

and

$$f_{2,\,15}(x,\,y,\,t)=0.01-0.02\,\frac{\lambda\,(\cosh\left(\mu\,\zeta\right)+\sinh\left(\mu\,\zeta\right))}{\nu\,(\cosh\left(\mu\,\zeta\right)+\sinh\left(\mu\,\zeta\right)+a_2)}. \tag{40}$$

4. Generalized projective Riccati equations method

When dealing with nonlinear phenomena, one has to have solutions of the following form of $NPDEs$:

$$N(\mathfrak{F},\,\mathfrak{F}_t,\,\mathfrak{F}_z,\,\mathfrak{F}_{tt},\,\mathfrak{F}_{zz},\,\mathfrak{F}_{zt},\,\ldots)=0, \tag{41}$$

N is an expression of a polynomial in the $N(z,\,t)$ and partial derivatives and contains higher order derivatives and nonlinear terms. In this section we give the simplified procedure of the $GPDEs$ method.

1. To begin with, we will need to implement the wave transformation:

$$\mathfrak{F}(z,\,t)=\mathfrak{F}(\phi),\;\;\phi=\vartheta z+\psi t \tag{42}$$

by setting ϑ, ψ to be constants, then the $NPDEs$ (41) can be made to yield the following ODE:

$$H(\mathfrak{F},\,\mathfrak{F}',\,\mathfrak{F}'',\,\ldots)=0, \tag{43}$$

with H a polynomial in $\mathfrak{F}(\phi)$ and any derivatives there of all total, such as $\mathfrak{F}'(\phi)=\dfrac{d\mathfrak{F}}{d\phi}$.

2. Next, we suppose that a formal solution of the ODE (43) exists.

$$\mathfrak{F}(n)=s_0+\sum_{i=1}^{M}\left(G\left(n\right)\right)^{i-1}\left(s_iG\left(n\right)+w_iH\left(n\right)\right). \tag{44}$$

where s_0, s_i, and b_i are constants that still needed to be solved. The functions of $G(n)$ and $H(n)$ are subject to the ODEs.

$$G'(n)=\varepsilon\,G\left(n\right)H\left(n\right), \tag{45}$$

and

$$H'(n)=\sigma+\varepsilon\,\left(H\left(n\right)\right)^2-\mu\,G\left(n\right), \tag{46}$$

where ε, σ, and μ are constants.

Which are referred to each other, by the following primitive integral

$$H(n)^2 = -\varepsilon\left(\frac{(\mu^2-1)G(n)^2}{\sigma} + \varepsilon - 2\mu G(n)^2\right), \tag{47}$$

in the most general instance where σ, μ are non zero constants. In case $\sigma = \mu = 0$, (43) can be formally solved.

$$\mathfrak{F}(n) = \sum_{i=1}^{m} s_i H(n)^i \tag{48}$$

and $H(n)$ satisfy the ODE

$$H'(n) = H(n)^2 \tag{49}$$

3. We can ensure the positive integer M required in Equation (44), by making the following homogeneous balancing of the largest nonlinear term in Equation (43) and the highest order derivative. There could be the case that, following the balance procedure there could be a non-positive for a particular nonlinear equation. The integer number m. When $m = \frac{p}{q}$, then the transformation will be the following

$$\mathfrak{F}(n) = \Theta(n)^{\frac{p}{q}} \tag{50}$$

and when the substitution will be carried out in the (50), the (43) will be reduced to an ODE the case when the balance procedure gives a positive integer number M, see say [40].

4. The replacement of the (44) or (48) with (45) and (46), (47) or (49) in (43), permits the us to bring together all terms of equal power of the order of $H(n)^i G(n)^j$ $(j = 0, 1, ..., i = 0, 1)$. Then we shall equate each of these coefficients before the functions $H(n)^i G(n)^j$ to zero. Through this process, an algebraic set of equations is produced and they can be solved to obtain the values of s_0, s_i, w_i, σ, μ, and ε.

5. The Eq. (45) and (46), has the following solutions (see for example [40]).

(i). $\varepsilon = -1$ and $\sigma \neq 0$

$$G_1 = \frac{\sigma \operatorname{sech}\left(\sqrt{\sigma}\zeta\right)}{\mu \operatorname{sech}\left(\sqrt{\sigma}\zeta\right) + 1}, \tag{51}$$

$$H_1 = \frac{\sqrt{\sigma}\tanh\left(\sqrt{\sigma}\zeta\right)}{\mu \operatorname{sech}\left(\sqrt{\sigma}\zeta\right) + 1}, \tag{52}$$

$$G_2 = \frac{\sigma \operatorname{csch}\left(\sqrt{\sigma}\zeta\right)}{\mu \operatorname{csch}\left(\sqrt{\sigma}\zeta\right) + 1}, \tag{53}$$

$$H_2 = \frac{\sqrt{\sigma}\coth\left(\sqrt{\sigma}\zeta\right)}{\mu \operatorname{csch}\left(\sqrt{\sigma}\zeta\right) + 1}. \tag{54}$$

(ii). If $\varepsilon = 1$ and $\sigma \neq 0$

$$G_3 = \frac{\sigma \csc\left(\sqrt{-\sigma}\zeta\right)}{\mu \csc\left(\sqrt{-\sigma}\zeta\right) + 1}, \tag{55}$$

$$H_3 = \frac{\sqrt{-\sigma} \cot\left(\sqrt{-\sigma}\zeta\right)}{\mu \csc\left(\sqrt{-\sigma}\zeta\right) + 1}, \tag{56}$$

$$G_4 = \frac{\sigma \sec\left(\sqrt{-\sigma}\zeta\right)}{\mu \sec\left(\sqrt{-\sigma}\zeta\right) + 1}, \tag{57}$$

$$H_4 = -\frac{\sqrt{-\sigma} \tan\left(\sqrt{-\sigma}\zeta\right)}{\mu \sec\left(\sqrt{-\sigma}\zeta\right) + 1}. \tag{58}$$

(iii). $\mu = 0$ and $\sigma = 0$

$$G_5 = \frac{K}{\zeta}, \tag{59}$$

$$H_5 = -\frac{1}{\varepsilon \zeta}. \tag{60}$$

6. The substitution of the coefficients s_0, s_i, and w_i, ϑ, ψ, μ, ε, and σ and the functions (51), (53), (63) into (43) or (48), results into the exact solutions of (43).

5. GPREs method for Kundu-Mukherjee-Naskar equation

By proving that F^3 and F'' are homogeneously balanced in (7), we obtain $M+2 = 3M$, which yields $M = 1$. The supposed solution for (7) that results from inserting $M = 1$ in (44) is as follows:

$$\mathfrak{F}(n) = s_0 + \sum_{i=1}^{1} \left(G(n)\right)^{i-1} \left(s_i G(n) + w_i H(n)\right). \tag{61}$$

Therefore, we have

$$\mathfrak{F}(n) = s_0 + s_1 G(n) + w_1 H(n), \tag{62}$$

substituting (62) into (7) and considering the equations (45) and (46) the left-hand-side of (7) becomes a polynomial in terms of $G(n)$ and $H(n)$.

$$-2ap_1p_2s_1\varepsilon^2 G(n)(H(n))^2 - ap_1p_2s_1\varepsilon G(n)\sigma + ap_1p_2s_1\varepsilon (G(n))^2\mu$$

$$-2ap_1p_2w_1\varepsilon H(n)\sigma - 2ap_1p_2w_1\varepsilon^2 (H(n))^3 + 3ap_1p_2w_1\mu\varepsilon G(n)H(n) - aq_1q_2s_0$$

$$-aq_1q_2s_1G(n) - aq_1q_2w_1H(n) + 2bq_1{s_0}^3 + 6bq_1{s_0}^2s_1G(n) + 6bq_1{s_0}^2w_1H(n)$$

$$+6bq_1s_0{s_1}^2(G(n))^2 + 12bq_1s_0s_1G(n)w_1H(n) + 6bq_1s_0{w_1}^2(H(n))^2 + 2bq_1{s_1}^3(G(n))^3$$

$$+6bq_1{s_1}^2(G(n))^2w_1H(n) + 6bq_1s_1G(n){w_1}^2(H(n))^2 + 2bq_1{w_1}^3(H(n))^3 = 0, \quad (63)$$

the algebraic Equation (63) of different powers of in $G(n)$ and $H(n)$ can also be reduced to a simplified algebraic equation depending upon the independent functions $G(n)^j$ and $(G(n))(H(n)^\rho)$.

$$+2bq_1{s_1}^3(G(n))^3 + \left(ap_1p_2s_1\varepsilon\mu + 6bq_1s_0{s_1}^2 + 6bq_1{s_1}^2w_1H(n)\right)(G(n))^2$$

$$+\Big(\left(6bq_1s_1{w_1}^2 - 2ap_1p_2s_1\varepsilon^2\right)(H(n))^2 + (12bq_1s_0s_1w_1 + 3ap_1p_2w_1\mu\varepsilon)H(n)$$

$$-aq_1q_2s_1 - ap_1p_2s_1\varepsilon\sigma + 6bq_1{s_0}^2s_1\Big)G(n) + \left(-2ap_1p_2w_1\varepsilon^2 + 2bq_1{w_1}^3\right)(H(n))^3 + 6bq_1s_0{w_1}^2(H(n))^2$$

$$+\left(6bq_1{s_0}^2w_1 - aq_1q_2w_1 - 2ap_1p_2w_1\varepsilon\sigma\right)H(n) - aq_1q_2s_0 + 2bq_1{s_0}^3 = 0, \quad (64)$$

putting every coefficient of the independent functions of this polynomial $G(n)^j$ and $(G(n))(H(n)^\rho)$ equal to zero, we obtain a system of algebraic equations

$$-aq_1q_2s_0 + 2bq_1{s_0}^3 = 0,$$

$$+2bq_1{s_1}^3 = 0,$$

$$+ap_1p_2s_1\varepsilon\mu + 6bq_1s_0{s_1}^2 = 0,$$

$$+6bq_1{s_1}^2w_1 = 0,$$

$$+6bq_1s_1{w_1}^2 - 2ap_1p_2s_1\varepsilon^2 = 0,$$

$$+12bq_1s_0s_1w_1 + 3ap_1p_2w_1\mu\varepsilon = 0,$$

$$- aq_1q_2s_1 - ap_1p_2s_1\varepsilon\sigma + 6bq_1{s_0}^2s_1 = 0,$$

$$- 2ap_1p_2w_1\varepsilon^2 + 2bq_1{w_1}^3 = 0,$$

$$+ 6bq_1s_0{w_1}^2 = 0,$$

$$+ 6bq_1{s_0}^2w_1 - aq_1q_2w_1 - 2ap_1p_2w_1\varepsilon\sigma = 0. \tag{65}$$

The above equations solution are given as

$$s_0 = 0.01,\ s_1 = 0.02,\ w_1 = 0.0001,\ p_1 = 0.002,\ p_2 = 0.11,\ a = 0.001,\ q_1 = 0.76,\ q_2 = 0.003,\ b = 0. \tag{66}$$

The solution of (51) and (53), when $\varepsilon = 1$ and $\sigma \neq 0$,

$$U_{1,\,1}(x,\, y,\, t) = 0.01 + 0.02\,\frac{\sigma\,\mathrm{sech}\left(\sqrt{\sigma}\zeta\right)}{\mu\,\mathrm{sech}\left(\sqrt{\sigma}\zeta\right) + 1} + 0.0001\,\frac{\sqrt{\sigma}\tanh\left(\sqrt{\sigma}\zeta\right)}{\mu\,\mathrm{sech}\left(\sqrt{\sigma}\zeta\right) + 1}, \tag{67}$$

and

$$U_{1,\,2}(x,\, y,\, t) = 0.01 + 0.02\,\frac{\sigma\,\mathrm{csch}\left(\sqrt{\sigma}\zeta\right)}{\mu\,\mathrm{csch}\left(\sqrt{\sigma}\zeta\right) + 1} + 0.0001\,\frac{\sqrt{\sigma}\coth\left(\sqrt{\sigma}\zeta\right)}{\mu\,\mathrm{csch}\left(\sqrt{\sigma}\zeta\right) + 1}, \tag{68}$$

by considering the second pair of solutions of the Equations (55) and (57), when $\varepsilon = -1$ and $\sigma \neq 0$,

$$U_{1,\,3}(x,\, y,\, t) = 0.01 + 0.02\,\frac{\sigma\,\csc\left(\sqrt{-\sigma}\zeta\right)}{\mu\,\csc\left(\sqrt{-\sigma}\zeta\right) + 1} + 0.0001\,\frac{\sqrt{-\sigma}\cot\left(\sqrt{-\sigma}\zeta\right)}{\mu\,\csc\left(\sqrt{-\sigma}\zeta\right) + 1}, \tag{69}$$

and

$$U_{1,\,4}(x,\, y,\, t) = 0.01 + 0.02\,\frac{\sigma\,\sec\left(\sqrt{-\sigma}\zeta\right)}{\mu\,\sec\left(\sqrt{-\sigma}\zeta\right) + 1} - 0.0001\,\frac{\sqrt{-\sigma}\tan\left(\sqrt{-\sigma}\zeta\right)}{\mu\,\sec\left(\sqrt{-\sigma}\zeta\right) + 1}, \tag{70}$$

now by considering the solutions of the Equation (63), when $\mu = 0$ and $\sigma \neq 0$,

$$U_{1,\,5}(x,\, y,\, t) = 0.01 + 0.02\,\frac{K}{\zeta} - 0.0001\,\frac{1}{\varepsilon\,\zeta}. \tag{71}$$

6. Results and discussion

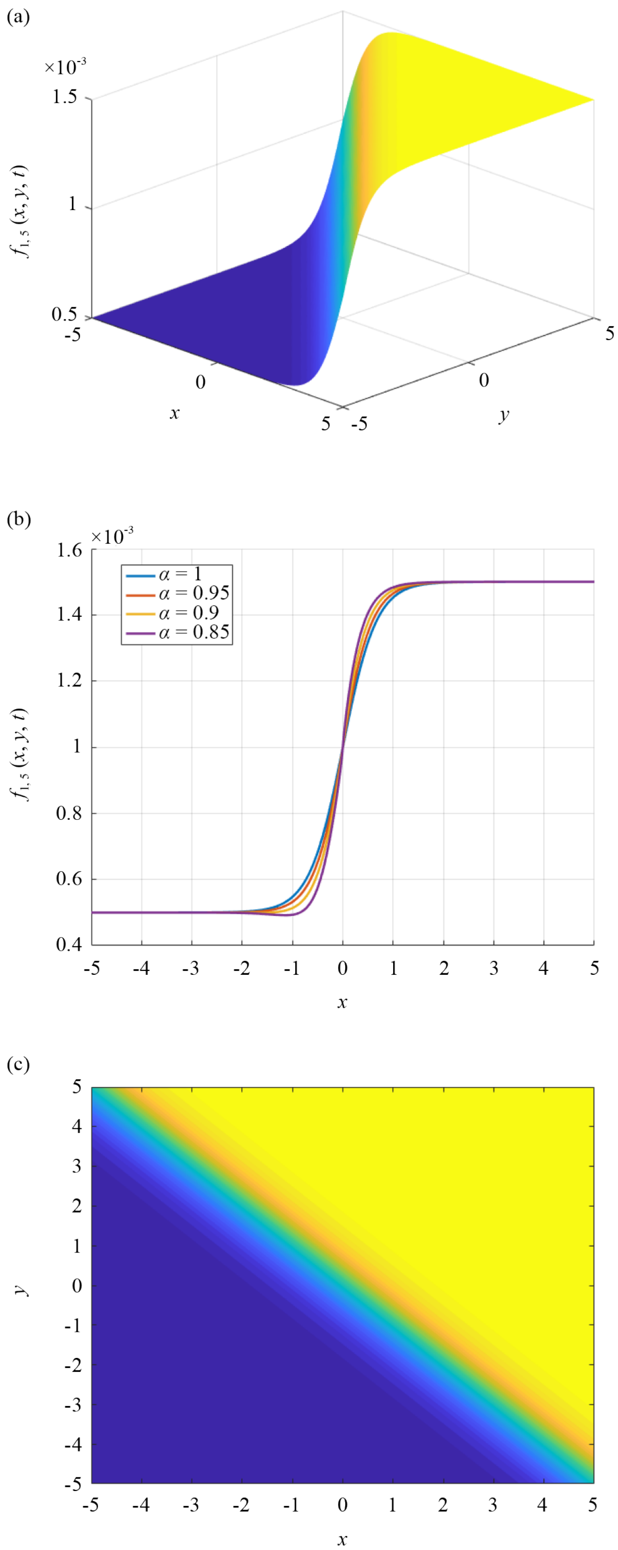

Figure 1. 3D, 2D, and contour plots of the kink solitary wave solution corresponding to Eq. (15) for the function $f_{1,\,5}(x,\,t)$. This figure illustrates how the fractional parameter influences the geometry of the solution in the fractional KMN model

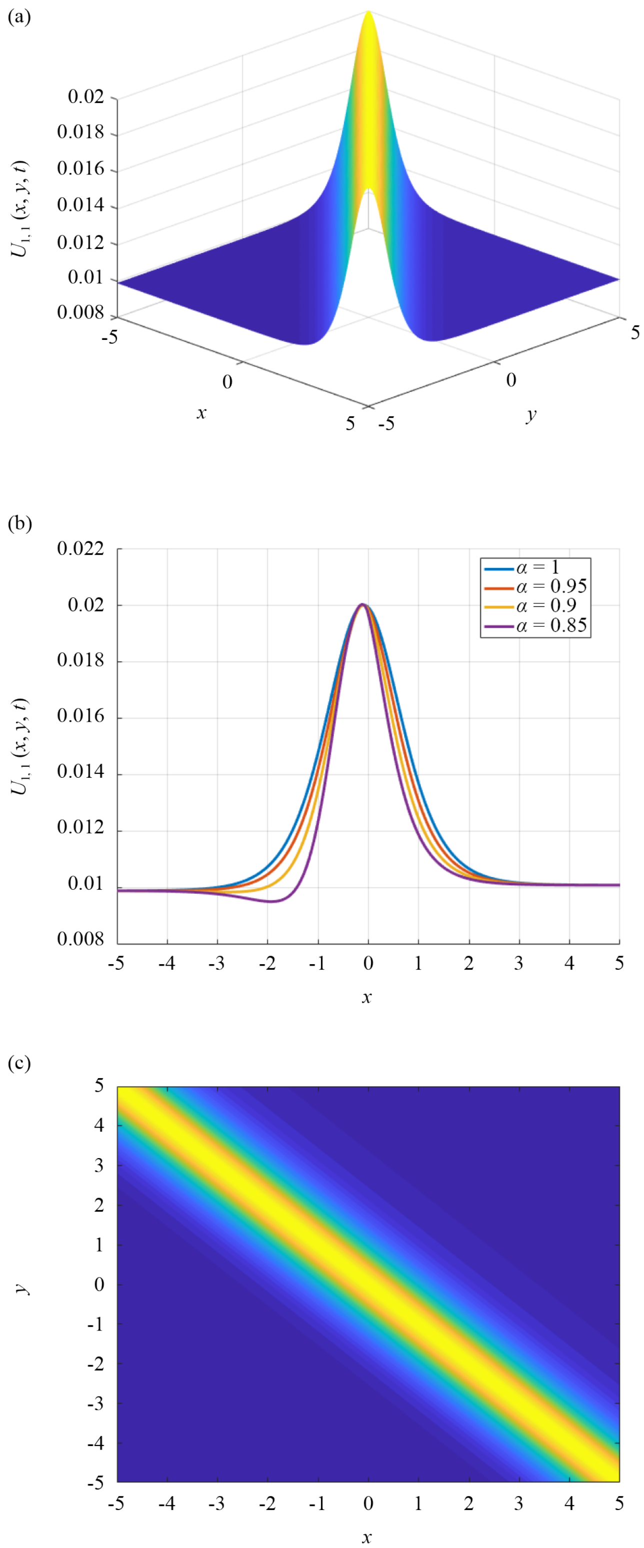

Figure 2. 3D, 2D, and contour representations of the bright kink solution of Eq. (62). These plots highlight the effect of the fractional parameter on the structure of the soliton in the KMN equation

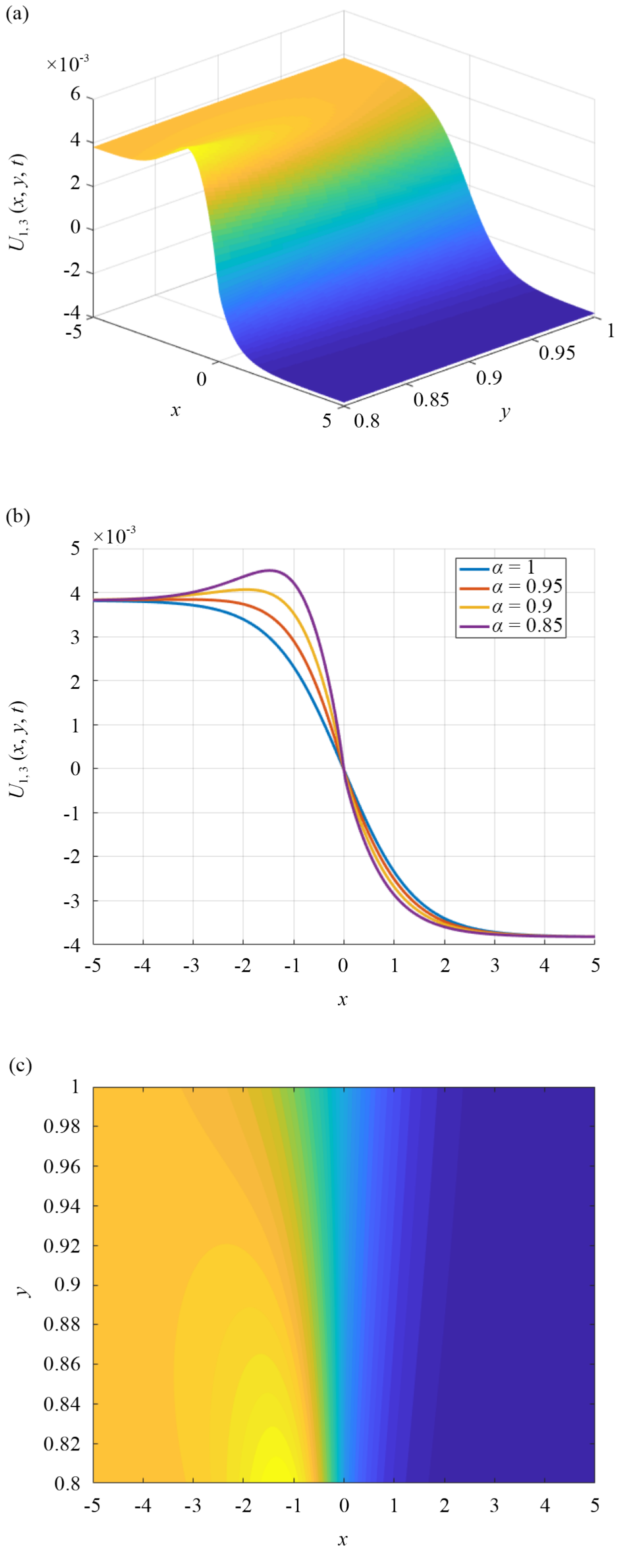

Figure 3. 3D, 2D, and contour plots of the periodic wave soliton solution corresponding to Eq. (64). The results show the anti-kink of the solution and its modulation under fractional variations

Graphical interpretations of the analytical solutions give further understanding of the nonlinear wave dynamics represented by the time-fractional Kundu-Mukherjee-Naskar (KMN) equation. All figures point to different categories of wave structures, which do not only verify the validity of the obtained solutions, but also demonstrate the physical applicability of nonlinear optics and photonics.

Figure 1 shows the kink soliton profile in 3D and 2D profiles and the contour plot. Kink solitons define steep changes of the amplitude of a wave between two opposite-polarity asymptotic states. When applied to nonlinear optics these structures have been connected to switching in optical fibers, where the field intensity is abruptly changed in sign, and can be used in optical signal processing, domain-wall propagation and bistable transmission systems. The contour plot also reveals the localization of the energy in space as well as its sharp interface which renders the anti-kink solution especially useful in all-optical switching and logic devices.

Figure 2 shows a bright kink soliton, which is a localized pulse of high intensity contained in a continuous background. Of particular concern in optical communication systems are such solitons, which are the localized energy packets that can be long-range (undistorted) propagators due to the compromise between dispersion and nonlinearity. Bright solitons have been popular in the modelling of ultrashort optical pulses in fiber lasers, and their resilience to perturbation makes them of interest as high-capacity data carriers, optical memories, and pulse compression systems. The 2D and contour images highlight sharp localization and stability of bright kink solution under conformable fractional dynamics.

An anti-kink wave soliton structure is shown in Figure 3. The solutions are of particular relevance to fiber Bragg gratings, nonlinear photonic crystals and optical lattices in Bose-Einstein condensates where periodic changes of refractive index create controlled light propagation. Anti-kink solitons play an important role in the design of optical modulators, pulse trains, and frequency combs in photonic applications. The related contour plot indicates the repetition of the intensity profile, which indicated the perpetual periodicity of the wave profile, could be used in the frequency-locked optical systems and laser cavity configurations.

Moreover, the impact of the fractional order parameter (α) is evident in all these figures. A range of fractional parameter (α) causes changes in the sharpness, amplitude, and speed of propagation of the solitons, illustrating how a tunable mechanism can be used to manipulate the groups of nonlinear waves with fractional calculus. The property can be applied in the design of reconfigurable photonic systems, wherein it is desired to optimize signal stability, to control dispersion effects, or to design custom pulse profiles in a specific optical device by varying the fractional order.

On the whole, these figures not only represent the precise analytical solutions, but also emphasize their physical significance in the modeling of various nonlinear optical processes. The current findings add to an enhanced comprehension of the dynamics of waves in optical fibers and associated photonic media by connecting the mathematical constructs to real-world applications.

7. Comparison with previous literature

The Kundu-Mukherjee-Naskar (KMN) equation has been previously studied applying various methods of analysis. Indicatively, the exp-function technique was used on the KMN equation in $(2 + 1)D$ in [52] to obtain optical solutions and illustrate the effectiveness of the technique when dealing with highly nonlinear terms. The exp-function method is effective, but the number of classes of exact solutions is usually very small, and the method only gives results in terms of a limited number of functional forms. The current work, in contrast, solves the time-fractional KMN equation in (1 + 1) dimensions and, using the RMESEM approach and the GPRM approach, finds a broader range of solutions, such as bright and singular solitons, and periodic waveforms represented by trigonometric, hyperbolic and rational functions. Correspondingly, in [53], the KMN equation was addressed through the sub-equation approach of Sardar and the new Kudryashov method to yield optical soliton solutions, including topological, anti-topological, periodic and singular solitons, which are graphically represented to describe their dynamics. Yet, contrary to [53], we generalize our findings to the fractional-order case, and we put an emphasis on the impact of the fractional parameter over the wave propagation not represented previously. In this way, the novelty of the current research is not only in the usage of expansion-based techniques that generate richer and more generalized solution families but also in the fact that it is extended to the fractional framework, thus providing new insights into the nonlinear dynamics of the KMN model in optical media.

8. Conclusion

In this study, we successfully derived new exact traveling wave solutions for the time-fractional KMN equation using two advanced analytical techniques: the RMESEM and the GPRM. By employing the conformable fractional derivative, we maintained mathematical consistency while exploring a variety of soliton and periodic wave solutions, including dark solitons, bright solitons, periodic wave structures, and singular solitons, expressed in terms of trigonometric, hyperbolic, and rational functions. A key observation from this work is that both methods are highly effective but yield different forms of solutions, with RMESEM providing a broader class of solutions under certain parameter constraints, while GPRM offers more structured soliton profiles. The results highlight the rich nonlinear dynamics embedded in the fractional KMN equation, which is crucial for understanding optical pulse propagation in dispersive media.

The findings of this study not only contribute to the theoretical understanding of fractional nonlinear models but also have potential implications for optical communications, photonics, and ultra fast laser systems, where such soliton dynamics play a vital role. Future research could explore higher-dimensional extensions, different fractional operators, or stochastic perturbations of the KMN equation to further expand its applicability in nonlinear science.

Conflict of interest

The authors declare no competing financial interest.

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