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Research Article

Numerical Solutions of Coupled Fractional Diffusion-Reaction Equations Using q-HMTM and MVIM

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Abstract: This paper gives a thorough description of q-Homotopy Mohand Transform Method (q-HMTM) and Mohand Transform Variational Iteration Method in solving nonlinear fractional partial differential equations that characterize an extensive variety of intricate physical and engineering processes. The Caputo-type fractional derivative of these equations provide important memory and hereditary effects. An extensive numerical study is implemented, covering the profiles of the solutions, error tables and two dimensional error plots. Quantitative findings indicate that they both give small absolute errors, with the q-HMTM having a faster convergence and being more stable to measure small variations in the behavior of solutions over different fractional orders. In comparison, Mohand Transform Variational Iteration Method offers accurate approximations at relatively higher error growth with small fractional parameters. The results demonstrate that q-HMTM has high accuracy and computational efficiency, and it is especially applicable when it is needed to simulate systems with nonlocal dynamics.

Keywords: physical equations, coupled system of diffusion-reaction equation, q-Homotopy Mohand Transform Method (q-HMTM), Mohand Variational Iteration Method (MVIM), fractional order differential equation, Caputo operator

MSC: 35R11

1. Introduction

Fractional calculus is a new variant of traditional calculus which allows for generalizing the integer-order differentiation and integration to non-integer orders (i.e., fractional orders). In contrast to the conventional calculus, which represent the systems with local instant responses, the fractional-order calculus brings the memory and hereditary characteristics of processes to the base of mathematics, so that it has a significant advantage in describing those physical phenomena of real world with abnormal diffusion, viscoelasticity, long-range dependence, etc. [1–3]. The classical operators of fractional calculus, the Riemann-Liouville, Caputo and Grunwald-Letnikov derivative operators, have already been applied to several different fields, including control systems, bioengineering, signal processing and financial mathematics [4–6]. For instance, in the context of viscoelastic material modeling, fractional derivative-based models yield a better fit of experimental data than integer order-based models as a result of their capability to describe complex time dependent properties [7, 8]. In addition, fractional-order models have been proven effective to describe the development of infectious

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diseases, control systems and neural networks, in particular, offering an improved flexibility and accuracy than their classical versions [9–11]. While for differential equations fractional analogues of numerical methods were developed and adapted to approximate solutions of Fractional Differential Equations (FDEs), typically without closed-form solutions [12–14] and so on [15–19].

Nonlinear Fractional Partial Differential Equations (NFPDEs) come as an effective model in modeling various complex physical problems in which memory effects, nonlocality phenomena, and anomalous diffusion take place. Favouring them over classical PDEs, NFPDEs involve the derivatives of non-integer order, which provides better approximating of realistic physical phenomena like the viscoelastic deformation, the turbulent flows or the biological systems [1, 3, 7]. These equations are appropriate to describe physical systems in which the rate of change of the system is sensitive not only to the present state of the system, but also on the past history of the process [20–24]. In physical context, the NFPDEs describe the subdiffusive or superdiffusive transport phenomena, which have been widely seen in porous material dyelectric materials and anomalous heat conduction [8, 25]. In kinetic-reaction systems, the non-local transport is represented by means of the fractional derivatives, while the nonlinear reaction terms describe complex interactions, such as chemical kinetics, biological cell dynamics or ecological competition [26–28]. For example, the fractional Fisher-Kolmogorov equation and the fractional reaction-diffusion equation are used for modeling tumor growth, the epidemic spreading process, and catalytic reaction [29, 30].

In the recent years, there was a significant progress in developing efficient numerical schemes for solving NFPDEs, which generalize classical models to make them more compatible with memory and hereditary effects appearing in the diffusion-reaction equations. Spectral methods have been a popular choice for the numerical methods, as they are spectrally accurate and exponentially convergent in smooth problems [31], and finite element formulations have been extended to also include such complicated geometries and boundary conditions as fractional operators [32]. In addition, some adaptive time-stepping schemes were proposed for the efficient numerical solution of the strong singularities and non-local memory effects which often arise in NFPDEs [33] may also be more desirable both in stability and efficiency than uniform discretization. In this respect, the work at hand is different because it considers coupled diffusion-reaction equations, which have been far less studied than single-equation models. This work is novel in the sense that it generalizes q-Homotopy Mohand Transform Method (q-HMTM) and Mohand Variational Iteration Method (MVIM) to coupled systems, and provides a comparative study between both methods since no any comparisons have been present in literature.

The q-HTM, which was initially developed as an extension of the Homotopy Perturbation Method (HPM) and includes a parameter $q \in (0, 1)$ for regulating the deformation process of a simple problem into a complex one [34]. The proposed method enhances the convergence by providing flexibility in the choice of building the homotopy series, especially for nonlinear and fractional systems. Now the q-HTM might be slow among suboptimal or even instable for certain boundary conditions or stiff like systems. To deal with this problem, the q-HMTM is employed by integrating the Mohand Transform (MT) an integral transform similar to Laplace transform which is useful to solve differential equations having non-integer order derivative and the q-Homotopy algebra. The MT renders the initial conditions and fractional derivatives manageable by converting the initial differential equation into an algebraic form which can be easily handled [35]. When combined with q-HTM, the q-HMTM can greatly enhance the convergence characterise and lead to direct and recursive construction of approximate solution, and are also applicable to strongly nonlinear problems.

On the other hand, the MVIM is a kind of extension of the well-known VIM, originally developed by He's [36], and it has been successfully applied to solve the linear and nonlinear problems. The correctional functionals of VIM are built based on Lagrange multipliers, and a semi-analytical iterative strategy is proposed to solve the VIM. MVIM differs in the sense that it incorporates the MT under the framework of the variational iteration method. This hybridization takes the advantage of transform not only to simplify the fractional operators and initial conditions but also it does keep the VIM's strengths that is to obtain a fast convergent series solution in a closed form without any linearization and discretization [37]. q-HMTM and MVIM have attempted to overcome the drawbacks of the previous approaches using the benefits of integral transforms and homotopy/variational techniques. They are especially important for the physical and engineering problems formulated by time-fractional PDEs, which describe the dynamics of diffusion-reaction systems, population dynamics, and electromagnetic theories, and many more [38–48]. Even comparative studies reveal that these methods are impressively successful in achieving high accuracy with simplicity and computational efficiency compared with conventional techniques.

The novelty of this contribution consists in the generalization of two semi-analytic methods, the q-HMTM and MVIM based on the MT, and apply them for the first time to coupled fractional diffusion-reaction systems with such a systematic comparison in terms of accuracy, stability and convergence. MT is the keystone in both methods, which renders fractional operators simpler to analyze in an algebraic sense and makes it easier to deal with nonlocal memory effects associated with Caputo type derivatives . The direct applications of these methods to nonlinear system of fractional diffusion-reactions has never been reported before. The key contributions of the present paper can be summarized as follows: (i) proposing and extending q-HMTM and MVIM to the coupled fractional PDEs under consideration via MT making them available for wide range of problems; (ii) offering a comprehensive comparative discussion along with error tables and figures supported by simulations, which reflects better efficiency and accurate results of q-HMTM; (iii) understanding the physical meaning of fractional parameter that governs memory effects and anomalous diffusion phenomena to connect theoretical aspects with applications for nonlinear science as well as engineering fields.

The workflow of this study is structured in the following way: In Section 2, the mathematical foundation of the Mohand transformation and some basic definition related to the fractional calculus is described. The methodology is outlined in Section 3 and the q-HMTM and the MVIM are developed in detail. Sections 4 and 5 give two examples of the NFPDEs that show the step-by-step implementation of the methods. The discussion of the results is presented in Section 6, and, lastly, in Section 7 the paper closes with the summary of main findings of the work and the benefits of methods suggested in its effort to address the nonlocal dynamics.

2. Concepts of MT

In this section we will first address broadly the MT and associated concepts.

Definition 1 Assuming the function $\psi(\sigma)$ for which [49] indicates the MT as:

$$M[\psi(\sigma)] = R(s) = s^2 \int_0^{\sigma} \psi(\sigma) e^{-s\sigma} d\sigma, \;\; k_1 \le s \le k_2.$$

The inverse MT is denoted as:

$$M^{-1}[R(s)] = \psi(\sigma).$$

Definition 2 The derivative of fractional order in the framework of the MT is given as [50]:

$$M[\psi^{\mu}(\sigma)] = s^{\mu} R(s) - \sum_{k=0}^{n-1} \frac{\psi^k(0)}{s^{k-(\mu+1)}}, \; 0 < \mu \le n.$$

Definition 3 Some of the features of the Mohand transformation is:

1. $M[\psi'(\sigma)] = sR(s) - s^2 R(0)$,
2. $M[\psi''(\sigma)] = s^2 R(s) - s^3 R(0) - s^2 R'(0)$,
3. $M[\psi^n(\sigma)] = s^n R(s) - s^{n+1} R(0) - s^n R'(0) - \cdots - s^n R^{n-1}(0)$.

Lemma 1 Assume that we have an exponentially ordered function $\psi(\beta,\, \sigma)$. Subsequently, the MT is defined as $M[R(s)] = \psi(\beta,\, \sigma)$:

$$M\left[D_\sigma^{r\mu}\psi(\beta,\ \sigma)\right] = s^{r\mu}R(s) - \sum_{j=0}^{r-1} s^{\mu(r-j)-1} D_\sigma^{j\mu}\psi(\beta,\ 0),\ 0 < \mu \leq 1, \tag{1}$$

where $\beta = (\beta_1,\ \beta_2,\ \cdots,\ \beta_\mu) \in \mathbb{R}^\mu$ and $\mu \in \mathbb{N}$ and $D_\sigma^{r\mu} = D_\sigma^\mu . D_\sigma^\mu . \cdots . D_\sigma^\mu$ (r-times).

Proof. We will employ the induction method to demonstrate the validity of Equation (1). By solving Equation (1) with $r = 1$, the resulting outcome is obtained.

$$M\left[D_\sigma^{2\mu}\psi(\beta,\ \sigma)\right] = s^{2\mu}R(s) - s^{2\mu-1}\psi(\beta,\ 0) - s^{\mu-1}D_\sigma^\mu\psi(\beta,\ 0).$$

By Definition 2, Equation (1) proves to be valid for $r = 1$. Equation (1) is modified by replacing $r = 2$ to obtain the following:

$$M\left[D_r^{2\mu}\psi(\beta,\ \sigma)\right] = s^{2\mu}R(s) - s^{2\mu-1}\psi(\beta,\ 0) - s^{\mu-1}D_\sigma^\mu\psi(\beta,\ 0). \tag{2}$$

We will examine the Equation (2) left-hand side.

$$L.H.S = M\left[D_\sigma^{2\mu}\psi(\beta,\ \sigma)\right]. \tag{3}$$

In addition, Equation (3) may be expressed as follows:

$$L.H.S = M\left[D_\sigma^\mu D_\sigma^\mu\psi(\beta,\ \sigma)\right]. \tag{4}$$

Let

$$z(\beta,\ \sigma) = D_\sigma^\mu\psi(\beta,\ \sigma). \tag{5}$$

In order to determine the following, substitute Equation (5) into Equation (4) and solve.

$$L.H.S = M\left[D_\sigma^\mu z(\beta,\ \sigma)\right]. \tag{6}$$

The Caputo derivative induces the subsequent modifications to Equation (6):

$$L.H.S = M\left[J^{1-\mu}z'(\beta,\ \sigma)\right]. \tag{7}$$

In Equation (7), the subsequent modifications are the result of the Riemann-Liouville (RL) integral:

$$L.H.S = \frac{M\left[z'(\beta,\ \sigma)\right]}{s^{1-\mu}}. \tag{8}$$

Equation (8) is employed to acquire these results through the MT derivative feature.

$$L.H.S = s^{\mu} Z(\beta,\ s) - \frac{z(\beta,\ 0)}{s^{1-\mu}}. \tag{9}$$

The subsequent outcome was achieved by employing Equation (5).

$$Z(\beta,\ s) = s^{\mu} R(s) - \frac{\psi(\beta,\ 0)}{s^{1-\mu}}.$$

As, $M[z(\sigma,\ \beta)] = Z(\beta,\ s)$. Accordingly, Equation (9) may also be represented as follows:

$$L.H.S = s^{2\mu} R(s) - \frac{\psi(\beta,\ 0)}{s^{1-2\mu}} - \frac{D_{\sigma}^{\mu}\psi(\beta,\ 0)}{s^{1-\mu}}. \tag{10}$$

Assuming that Equation (1) is true for $r = K$. Equation (1) is modified by substituting $r = K$.

$$M\left[D_{\sigma}^{K\mu}\psi(\beta,\ \sigma)\right] = s^{K\mu} R(s) - \sum_{j=0}^{K-1} s^{\mu(K-j)-1} D_{\sigma}^{j\mu} D_{\sigma}^{j\mu}\psi(\beta,\ 0),\ 0 < \mu \leq 1. \tag{11}$$

Showing that Equation (1) is true for $r = K+1$ is the subsequent step. Substitute $r = K+1$ to solve Equation (1).

$$M\left[D_{\sigma}^{(K+1)\mu}\psi(\beta,\ \sigma)\right] = s^{(K+1)\mu} R(s) - \sum_{j=0}^{K} s^{\mu((K+1)-j)-1} D_{\sigma}^{j\mu}\psi(\beta,\ 0). \tag{12}$$

Equation (12) can be employed to achieve this outcome.

$$L.H.S = M\left[D_{\sigma}^{K\mu}\left(D_{\sigma}^{K\mu}\right)\right]. \tag{13}$$

Let

$$D_{\sigma}^{K\mu} = g(\beta,\ \sigma).$$

The following result is achieved by utilizing Equation (13):

$$L.H.S = M\left[D_{\sigma}^{\mu} g(\beta,\ \sigma)\right]. \tag{14}$$

The RL integral and Caputo's derivative are both implemented to modify Equation (14).

$$L.H.S = s^{\mu} M\left[D_{\sigma}^{K\mu} \psi(\beta,\ \sigma)\right] - \frac{g(\beta,\ 0)}{s^{1-\mu}}. \tag{15}$$

Using Equation (11), Equation (15) becomes:

$$L.H.S = s^{r\mu} R(s) - \sum_{j=0}^{r-1} s^{\mu(r-j)-1} D_{\sigma}^{j\mu} \psi(\beta,\ 0). \tag{16}$$

We can also denote Equation (16) as:

$$L.H.S = M\left[D_{\sigma}^{r\mu} \psi(\beta,\ 0)\right].$$

Hence, Equation (1) is true for $r = K+1$ using induction technique, which implies that Equation (1) for all positive integer is true.

Lemma 2 Let us suppose that the order of the function $\psi(\beta,\ \sigma)$ is exponential. $R(s) = M[\psi(\beta,\ \sigma)]$ represents the MT of $\psi(\beta,\ \sigma)$. The expression of the Multiple Fractional Power Series (MFPS) in relation to the MT is as follows:

$$R(s) = \sum_{r=0}^{\infty} \frac{\hbar_r(\beta)}{s^{r\mu+1}}, \quad s > 0, \tag{17}$$

where, $\beta = (s_1,\ \beta_2,\ \cdots,\ \beta_\mu) \in \mathbb{R}^\mu,\ \mu \in \mathbb{N}$.

Proof. Consider the Taylor series:

$$\psi(\beta,\ \sigma) = \hbar_0(\beta) + \hbar_1(\beta)\frac{\sigma^\mu}{\mu[\mu+1]} + +\hbar_2(\beta)\frac{\sigma^{2\mu}}{\mu[2\mu+1]} + \cdots. \tag{18}$$

The resulting outcome is obtained by applying MT to Equation (18).

$$M\left[\psi(\beta,\ \sigma)\right] = M\left[\hbar_0(\beta)\right] + M\left[\hbar_1(\beta)\frac{\sigma^\mu}{\mu[\mu+1]}\right] + M\left[\hbar_1(\beta)\frac{\sigma^{2\mu}}{\mu[2\mu+1]}\right] + \cdots.$$

Make use of the features of MT to achieve the desired result.

$$M\left[\psi(\beta,\ \sigma)\right] = \hbar_0(\beta)\frac{1}{s} + \hbar_1(\beta)\frac{\mu[\mu+1]}{\mu[\mu+1]}\frac{1}{s^{\mu+1}} + \hbar_2(\beta)\frac{\mu[2\mu+1]}{\mu[2\mu+1]}\frac{1}{s^{2\mu+1}}\cdots.$$

Consequently, this leads to a new Taylor series.

Lemma 3 The modified Taylor series in MFPS has the following form when $M[\psi(\beta,\ \sigma)] = R(s)$ denotes the MT:

$$\hbar_0(\beta) = \lim_{s\to\infty} sR(s) = \psi(\beta,\ 0). \tag{19}$$

Proof. Consider the Taylor series:

$$\hbar_0(\beta) = sR(s) - \frac{\hbar_1(\beta)}{s^{\mu}} - \frac{\hbar_2(\beta)}{s^{2\mu}} - \cdots. \tag{20}$$

Calculating and simplifying the limit given in Equation (19) yields Equation (20).

Theorem 1 Assume the function $M[\psi(\beta,\ \sigma)]$ for which $R(s)$ in MFPS form is given as:

$$R(s) = \sum_{r=0}^{\infty} \frac{\hbar_r(\beta)}{s^{r\mu+1}},\ s > 0,$$

where $\beta = (\beta_1,\ \beta_2,\ \cdots,\ \beta_\mu) \in \mathbb{R}^{\mu}$ and $\mu \in \mathbb{N}$. Then we have

$$\hbar_r(\beta) = D_r^{r\mu}\psi(\beta,\ 0),$$

where, $D_\sigma^{r\mu} = D_\sigma^{\mu}.D_\sigma^{\mu}.\cdots.D_\sigma^{\mu}$ (r-times).

Proof. Assume the Taylor series:

$$\hbar_1(\beta) = s^{\mu+1}R(s) - s^{\mu}\hbar_0(\beta) - \frac{\hbar_2(\beta)}{s^{\mu}} - \frac{\hbar_3(\beta)}{s^{2\mu}} - \cdots. \tag{21}$$

By applying limit to Equation (21), the following result is obtained:

$$\hbar_1(\beta) = \lim_{s\to\infty}(s^{\mu+1}R(s) - s^{\mu}\hbar_0(\beta)) - \lim_{s\to\infty}\frac{\hbar_2(\beta)}{s^{\mu}} - \lim_{s\to\infty}\frac{\hbar_3(\beta)}{s^{2\mu}} - \cdots.$$

Simplify the Equation (21).

$$\hbar_1(\beta) = \lim_{s\to\infty}(s^{\mu+1}R(s) - s^{\mu}\hbar_0(\beta)). \tag{22}$$

We develop the following form of Equation (22) by applying the basic ideas provided in Lemma 1.

$$\hbar_1(\beta) = \lim_{s\to\infty}(sM\left[D_\sigma^{\mu}\psi(\beta,\ \sigma)\right](s)). \tag{23}$$

The foundation for deriving Equation (23) is Lemma 2.

$$\hbar_1(\beta) = D_\sigma^\mu \psi(\beta,\ 0).$$

We have to apply the Taylor series and once more take $\lim_{s\to\infty}$ to get the following result.

$$\hbar_2(\beta) = s^{2\mu+1}R(s) - s^{2\mu}\hbar_0(\beta) - s^\mu \hbar_1(\beta) - \frac{\hbar_3(\beta)}{s^\mu} - \cdots.$$

Lemma 2 provides the next results.

$$\hbar_2(\beta) = \lim_{s\to\infty} s\left(s^{2\mu}R(s) - s^{2\mu-1}\hbar_0(\beta) - s^{\mu-1}\hbar_1(\beta)\right). \tag{24}$$

Lemmas 1 and 3 serve as the foundation for the following modifications to Equation (24):

$$\hbar_2(\beta) = D_\sigma^{2\mu} \psi(\beta,\ 0).$$

Following the same process, we get:

$$\hbar_3(\beta) = \lim_{s\to\infty} s\left(M\left[D_\sigma^{2\mu} \psi(\beta,\ \mu)\right](s)\right).$$

This final result is obtained from Lemma 3.

$$\hbar_3(\beta) = D_\sigma^{3\mu} \psi(\beta,\ 0).$$

Generally

$$\hbar_r(\beta) = D_\sigma^{r\mu} \psi(\beta,\ 0).$$

Proved. □

The following theorem defines and clarifies more effectively the Taylor's series convergence.

Theorem 2 Lemma 2 characterises the MFTS expression, which is represented as follows: $M[\psi(\sigma,\ \beta)] = R(s)$, when $\left|s^a M\left[D_\sigma^{(K+1)\mu} \psi(\beta,\ \sigma)\right]\right| \le T$, for all $s > 0$ and $0 < \mu \le 1$, the residual $H_K(\beta,\ s)$ of the new MFTS and validate the inequality:

$$|H_K(\beta,\ s)| \le \frac{T}{s^{(K+1)\mu+1}},\ s > 0.$$

Proof. Let $M\left[D_{\sigma}^{r\mu}\psi(\beta,\ \sigma)\right](s)$ is defined on $s>0$ for $r=0,\ 1,\ 2,\ \ldots,\ K+1$ and assume $|sM[D_{\sigma^{K+1}}\psi(\beta,\ \sigma)]|\leq T$. The resultant relationship can be found with the help of the new form of Taylor series.

$$H_K(\beta,\ s)=R(s)-\sum_{r=0}^{K}\frac{\hbar_r(\beta)}{s^{r\mu+1}}. \tag{25}$$

We get the following result by using Theorem 1 and Equation (25):

$$H_K(\beta,\ s)=R(s)-\sum_{r=0}^{K}\frac{D_{\sigma}^{r\mu}\psi(\beta,\ 0)}{s^{r\mu+1}}. \tag{26}$$

Multiply $s^{(K+1)a+1}$ with Equation (26):

$$s^{(K+1)\mu+1}H_K(\beta,\ s)=s\left(s^{(K+1)\mu}R(s)-\sum_{r=0}^{K}s^{(K+1-r)\mu-1}D_{\sigma}^{r\mu}\psi(\beta,\ 0)\right). \tag{27}$$

When Lemma 1 is used, Equation (27) takes the following specific form:

$$s^{(K+1)\mu+1}H_K(\beta,\ s)=sM\left[D_{\sigma}^{(K+1)\mu}\psi(\beta,\ \sigma)\right]. \tag{28}$$

Take the absolute:

$$\left|s^{(K+1)\mu+1}H_K(\beta,\ s)\right|=\left|sM\left[D_{\sigma}^{(K+1)\mu}\psi(\beta,\ \sigma)\right]\right|. \tag{29}$$

The condition specifies in Equation (29) must be applied in order to get the desired result.

$$\frac{-T}{s^{(K+1)\mu+1}}\leq H_K(\beta,\ s)\leq\frac{T}{s^{(K+1)\mu+1}}. \tag{30}$$

The result of Equation (30) may also be expressed as follows:

$$|H_K(\beta,\ s)|\leq\frac{T}{s^{(K+1)\mu+1}}.$$

This leads to the establishment of unique conditions for the series' convergence.

3. Methodology

3.1 *q-HMTM*

Assume the general PDE of the time fractional order:

$$D_{\sigma}^{\mu}\psi(\beta,\ \sigma)+\mathscr{R}\psi(\beta,\ \sigma)+\mathscr{N}\psi(\beta,\ \sigma)=\mathbb{H}(\beta,\ \sigma),\ \ n-1<\mu\leq n. \tag{31}$$

The derivative of Caputo is represented by $D_{\sigma}^{\mu}\psi(\beta,\ \sigma)$ and the source term is $\mathbb{H}(\beta,\ \sigma)$. $\mathscr{N}$ represents the nonlinear operator, while $\mathscr{R}$ represents the linear operator.

The transformation of Mohand is applied to Equation (31).

$$\mathscr{M}[\psi(\beta,\ \sigma)]-\frac{1}{s^{\mu}}\sum_{k=0}^{n-1}s^{\mu-k-1}\psi^{k}(\beta,\ 0)+\frac{1}{s^{\mu}}[\mathscr{M}[\mathscr{R}\psi(\beta,\ \sigma)]+\mathscr{M}[\mathscr{N}\psi(\beta,\ \sigma)]-\mathscr{M}[\mathbb{H}(\beta,\ \sigma)]]=0. \tag{32}$$

The non-linear operator defined as follows:

$$\begin{aligned} N[\zeta(\beta,\ \sigma;\ q)]=&\mathscr{M}[\zeta(\beta,\ \sigma;\ q)]-\frac{1}{s^{\mu}}\sum_{k=0}^{n-1}s^{\mu-k-1}\zeta^{k}(\beta,\ \sigma;\ q)(0^{+})+\frac{1}{s^{\mu}}[\mathscr{M}[\mathscr{R}\zeta(\beta,\ \sigma;\ q)] \\ &+\mathscr{M}[\mathscr{N}\zeta(\beta,\ \sigma;\ q)]-\mathscr{M}[\mathbb{H}(\beta,\ \sigma)]]. \end{aligned} \tag{33}$$

With respect to $\beta,\ \sigma$, and $q\in\left[0,\ \frac{1}{n}\right]$, the real-valued function in the present case is $\zeta(\beta,\ \sigma;\ q)$. We describe a homotopy in the following manner:

$$(1-nq)\mathscr{M}[\zeta(\beta,\ \sigma;\ q)-\psi_0(\beta,\ \sigma)]=\hbar q\mathfrak{h}(\beta,\ \sigma)\mathscr{N}[\zeta(\beta,\ \sigma;\ q)]. \tag{34}$$

Within the equation illustrated above, the auxiliary parameter is $\hbar=0$ whereas the initial condition is ψ_0.

The following is applicable for $\frac{1}{n}$ as well as 0.

$$\zeta(\beta,\ \sigma;\ 0)=\psi_0(\beta,\ \sigma),\ \ \zeta\left(\beta,\ \sigma;\ \frac{1}{n}\right)=\psi(\beta,\ \sigma). \tag{35}$$

Due of the intensification of q, the solution $\zeta(\beta,\ \sigma;\ q)$ differs from the initial condition $\psi_0(\beta,\ \sigma)$ to $\psi(\beta,\ \sigma)$. With respect to q, we may use the Taylor theorem on $\zeta(\beta,\ \sigma;\ q)$ to determine the following:

$$\zeta(\beta,\ \sigma;\ q)=\psi_0(\beta,\ \sigma)+\sum_{m=1}^{\infty}\psi_m(\beta,\ \sigma)q^{m}, \tag{36}$$

where

$$\psi_m=\frac{1}{m!}\frac{\partial^{m}\zeta(\beta,\ \sigma;\ q)}{\partial q^{m}}|_{q=0}. \tag{37}$$

When $\psi_0(\beta,\ \sigma)$, n, and $\hbar$ are set appropriately, the series (34) converges at $q = \frac{1}{n}$. Consequently,

$$\zeta(\beta,\ \sigma;\ q) = \psi_0(\beta,\ \sigma) + \sum_{m=1}^{\infty} \psi_m(\beta,\ \sigma) \left(\frac{1}{n}\right)^m. \tag{38}$$

The derivative of Equation (34) with regard to the embedding parameter q may be obtained by setting $q = 0$ and dividing by $m!$.

$$\mathscr{M}[\psi_m(\beta,\ \sigma) - k_m \psi_{m-1}(\beta,\ \sigma)] = \hbar \mathfrak{h}(\beta,\ \sigma)\mathscr{R}_m(\overrightarrow{\psi}_{m-1}). \tag{39}$$

The following is the definition of the vectors and the auxiliary parameter $\hbar \neq 0$:

$$\overrightarrow{\psi}_m = [\psi_0(\beta,\ \sigma),\ \psi_1(\beta,\ \sigma),\ \ldots,\ \psi_m(\beta,\ \sigma)]. \tag{40}$$

Apply the inverse transformation of Mohand on Equation (39).

$$\psi_m(\beta,\ \sigma) = k_m \psi_{m-1}(\beta,\ \sigma) + \hbar \mathscr{M}^{-1}[\mathfrak{h}(\beta,\ \sigma)\mathscr{R}_m(\overrightarrow{\psi}_{m-1})], \tag{41}$$

$$\mathscr{R}_m(\overrightarrow{\mathbb{K}}_{m-1}) = \frac{1}{(m-1)!} \frac{\partial^{m-1} \mathscr{N}[\zeta(\beta,\ \sigma;\ q)]}{\partial q^{m-1}} \Big|_{q=0},$$

$$k_m = \begin{cases} 0 \text{ if } m \leq 1, \\ 1 \text{ if } m > 1. \end{cases} \tag{42}$$

Solving Equation (41) help us to determine the constituents of the q-HMTM solution.

3.2 *MVIM*

Assume the general PDE of the time fractional order:

$$D_{\sigma}^{\mu} \psi(\beta,\ \sigma) = \mathscr{R}\psi(\beta,\ \sigma) + \mathscr{N}\psi(\beta,\ \sigma) + \mathbb{H}(\beta,\ \sigma),\ \ n-1 < \mu \leq n. \tag{43}$$

Initial condition

$$\psi(\beta,\ 0) = \psi_0(\beta). \tag{44}$$

Apply the transformation of Mohand on Equation (43).

$$\mathscr{M}[D_\sigma^\mu \psi(\beta,\ \sigma)] = \mathscr{M}[\mathscr{R}\psi(\beta,\ \sigma) + \mathscr{N}\psi(\beta,\ \sigma) + \mathbb{H}(\beta,\ \sigma)]. \tag{45}$$

We get the following result by means of the iterative characteristic of transform:

$$\mathscr{M}[\psi(\beta,\ \sigma)] - \sum_{k=0}^{m-1} s^{\mu-k-1} \frac{\partial^k \psi(\beta,\ \sigma)}{\partial \sigma^k}|_{\sigma=0} = \mathscr{M}[\mathscr{R}\psi(\beta,\ \sigma) + \mathscr{N}\psi(\beta,\ \sigma) + \mathbb{H}(\beta,\ \sigma)]. \tag{46}$$

The Lagrange multiplier $(-\lambda(s))$ is used with the iterative technique to find the solution.

$$\mathscr{M}[\psi_{n+1}(\beta,\ \sigma)] = \mathscr{M}[\psi_n(\beta,\ \sigma)] - \lambda(s)\left[\mathscr{M}[\psi_n(\beta,\ \sigma)] - \sum_{k=0}^{m-1} s^{\mu-k-1} \frac{\partial^k \psi(\beta,\ 0)}{\partial \sigma^k}\right]. \tag{47}$$

Where $\lambda(s) = -\frac{1}{s^\mu}$ and Equation (47) are substituted in Equation (46), to get

$$\begin{aligned} \mathscr{M}[\psi_{n+1}(\beta,\ \sigma)] =& \mathscr{M}[\psi_n(\beta,\ \sigma)] - \lambda(s)\Bigg[\mathscr{M}[\psi_n(\beta,\ \sigma)] - \sum_{k=0}^{m-1} s^{\mu-k-1} \frac{\partial^k \psi(\beta,\ 0)}{\partial \sigma^k} \\ &+ \mathscr{M}[\mathscr{R}\psi(\beta,\ \sigma) + \mathscr{N}\psi(\beta,\ \sigma) + \mathbb{H}(\beta,\ \sigma)]\Bigg]. \end{aligned} \tag{48}$$

Apply the inverse transformation of Mohand on Equation (48):

$$\psi_{n+1}(\beta,\ \sigma) = \psi_n(\beta,\ \sigma) + \mathscr{M}^{-1}\left[\frac{1}{s^\mu}\sum_{k=0}^{m-1} s^{\mu-k-1} \frac{\partial^k \psi(\beta,\ 0)}{\partial \sigma^k} + \mathscr{M}[\mathscr{R}\psi(\beta,\ \sigma) + \mathscr{N}\psi(\beta,\ \sigma) + \mathbb{H}(\beta,\ \sigma)]\right]. \tag{49}$$

The initial condition is given as:

$$\psi_0(\beta,\ \sigma) = \mathscr{M}^{-1}\left[\frac{1}{s^\mu}\sum_{k=0}^{m-1} s^{\mu-k-1} \frac{\partial^k \psi(\beta,\ 0)}{\partial \sigma^k}\right]. \tag{50}$$

The iterative scheme is given as:

$$\psi_{n+1}(\beta,\ \sigma) = \psi_n(\beta,\ \sigma) + \mathscr{M}^{-1}\left[\frac{1}{s^\mu}\sum_{k=0}^{m-1} s^{\mu-k-1} \frac{\partial^k \psi(\beta,\ 0)}{\partial \sigma^k} + \mathscr{M}[\mathscr{R}\psi(\beta,\ \sigma) + \mathscr{N}\psi(\beta,\ \sigma) + \mathbb{H}(\beta,\ \sigma)]\right]. \tag{51}$$

4. Problem 1

4.1 *Solution via q-HMTM*

Take into consideration the Physical model of time fractional order [51]:

$$D_{\sigma}^{\mu}\psi_1(\beta,\,\sigma)-\frac{\partial^2\psi_1(\beta,\,\sigma)}{\partial\beta^2}+{\psi_1}^3(\beta,\,\sigma)+\psi_1(\beta,\,\sigma)\psi_2(\beta,\,\sigma)-\psi_1(\beta,\,\sigma)=0,$$

$$D_{\sigma}^{\mu}\psi_2(\beta,\,\sigma)-\frac{\partial^2\psi_2(\beta,\,\sigma)}{\partial\beta^2}+{\psi_2}^2(\beta,\,\sigma)+\psi_1(\beta,\,\sigma)\psi_2(\beta,\,\sigma)-\psi_2(\beta,\,\sigma)=0, \tag{52}$$

$$\sigma>0,\ \beta\in R,\ 0<\mu\leq 1.$$

Initial conditions:

$$\psi_1(\beta,\,0)=\frac{e^{k\beta}}{e^{k\beta}+1},$$

$$\psi_2(\beta,\,0)=\frac{\frac{3e^{k\beta}}{4}+1}{\left(e^{k\beta}+1\right)^2}, \tag{53}$$

$$\mathscr{M}[\psi_1(\beta,\,\sigma)]+s\left(\frac{e^{k\beta}}{e^{k\beta}+1}\right)+\frac{1}{s^{\mu}}\mathscr{M}\left[-\frac{\partial^2\psi_1(\beta,\,\sigma)}{\partial\beta^2}+{\psi_1}^3(\beta,\,\sigma)+\psi_1(\beta,\,\sigma)\psi_2(\beta,\,\sigma)-\psi_1(\beta,\,\sigma)\right]=0,$$

$$\mathscr{M}[\psi_2(\beta,\,\sigma)]+s\left(\frac{\frac{3e^{k\beta}}{4}+1}{\left(e^{k\beta}+1\right)^2}\right)+\frac{1}{s^{\mu}}\mathscr{M}\left[-\frac{\partial^2\psi_2(\beta,\,\sigma)}{\partial\beta^2}+{\psi_2}^2(\beta,\,\sigma)+\psi_1(\beta,\,\sigma)\psi_2(\beta,\,\sigma)-\psi_2(\beta,\,\sigma)\right]=0. \tag{54}$$

We characterise the nonlinear operators as follows:

$$\mathscr{N}^2[\zeta_1(\beta,\,\sigma;\,q),\,\zeta_2(\beta,\,\sigma;\,q)]=\mathscr{M}[\zeta_1(\beta,\,\sigma;\,q)]+s\left(\frac{e^{k\beta}}{e^{k\beta}+1}\right)+\frac{1}{s^{\mu}}\mathscr{M}\left[-\frac{\partial^2\zeta_1(\beta,\,\sigma;\,q)}{\partial\beta^2}+{\zeta_1}^3(\beta,\,\sigma;\,q)\right.$$

$$\left.+\zeta_1(\beta,\,\sigma;\,q)\zeta_2(\beta,\,\sigma;\,q)-\zeta_1(\beta,\,\sigma;\,q)\right],$$

$$\mathscr{N}^3[\zeta_1(\beta,\,\sigma;\,q),\,\zeta_2(\beta,\,\sigma;\,q)]=\mathscr{M}[\zeta_2(\beta,\,\sigma;\,q)]+s\left(\frac{\frac{3e^{k\beta}}{4}+1}{\left(e^{k\beta}+1\right)^2}\right)+\frac{1}{s^{\mu}}\mathscr{M}\left[-\frac{\partial^2\zeta_2(\beta,\,\sigma;\,q)}{\partial\beta^2}+{\zeta_2}^2(\beta,\,\sigma;\,q)\right.$$

$$+\zeta_1(\beta,\ \sigma;\ q)\zeta_2(\beta,\ \sigma;\ q)-\zeta_2(\beta,\ \sigma;\ q)\Big]. \tag{55}$$

The Mohand operators are describe as follows:

$$\begin{aligned}
&\mathscr{M}[\psi_{1m}(\beta,\ \sigma)-k_m\psi_{1m-1}(\beta,\ \sigma)]=\hbar\mathfrak{h}(\beta,\ \sigma)\mathscr{R}_{1,\ m}[\overrightarrow{\psi_1}_{m-1},\ \overrightarrow{\psi_2}_{m-1}],\\
&\mathscr{M}[\psi_{2m}(\beta,\ \sigma)-k_m\psi_{2m-1}(\beta,\ \sigma)]=\hbar\mathfrak{h}(\beta,\ \sigma)\mathscr{R}_{2,\ m}[\overrightarrow{\psi_1}_{m-1},\ \overrightarrow{\psi_2}_{m-1}].
\end{aligned} \tag{56}$$

Here,

$$\begin{aligned}
&\mathscr{R}_{1,\ m}[\overrightarrow{\psi_1}_{m-1},\ \overrightarrow{\psi_2}_{m-1}]\\
&=\mathscr{M}[\psi_{1m-1}(\beta,\ \sigma)]+s\left(1-\frac{k_m}{n}\right)\left(\frac{e^{k\beta}}{e^{k\beta}+1}\right)+\frac{1}{s^{\mu}}\mathscr{M}\Bigg[-\frac{\partial^2\psi_{1m-1}(\beta,\ \sigma)}{\partial\beta^2}\\
&\quad+\sum_{r=0}^{m-1}\sum_{i=0}^{m-1-i}\psi_{1r}(\beta,\ \sigma)\psi_{1i}(\beta,\ \sigma)\psi_{1m-1-r-i}(\beta,\ \sigma)+\sum_{i=0}^{m-1}\psi_{1i}(\beta,\ \sigma)\psi_{2m-1-i}(\beta,\ \sigma)-\psi_{1m-1}(\beta,\ \sigma)\Bigg],\\
&\mathscr{R}_{2,\ m}[\overrightarrow{\psi_1}_{m-1},\ \overrightarrow{\psi_2}_{m-1}]\\
&=\mathscr{M}[\psi_{2m-1}(\beta,\ \sigma)]+s\left(1-\frac{k_m}{n}\right)\left(\frac{\frac{3e^{k\beta}}{4}+1}{\left(e^{k\beta}+1\right)^2}\right)+\frac{1}{s^{\mu}}\mathscr{M}\Bigg[-\frac{\partial^2\psi_{2m-1}(\beta,\ \sigma)}{\partial\beta^2}\\
&\quad+\sum_{i=0}^{m-1}\psi_{2i}(\beta,\ \sigma)\psi_{2m-1-i}(\beta,\ \sigma)+\sum_{i=0}^{m-1}\psi_{1i}(\beta,\ \sigma)\psi_{2m-1-i}(\beta,\ \sigma)-\psi_{2m-1}(\beta,\ \sigma)\Bigg],
\end{aligned} \tag{57}$$

$$\begin{aligned}
&\psi_{1m}(\beta,\ \sigma)=k_m\psi_{1m-1}(\beta,\ \sigma)+\hbar\mathscr{M}^{-1}\left[\mathfrak{h}(\beta,\ \sigma)\mathscr{R}_{1,\ m}(\overrightarrow{\psi_1}_{m-1},\ \overrightarrow{\psi_2}_{m-1})\right],\\
&\psi_{2m}(\beta,\ \sigma)=k_m\psi_{2m-1}(\beta,\ \sigma)+\hbar\mathscr{M}^{-1}\left[\mathfrak{h}(\beta,\ \sigma)\mathscr{R}_{2,\ m}(\overrightarrow{\psi_1}_{m-1},\ \overrightarrow{\psi_2}_{m-1})\right].
\end{aligned} \tag{58}$$

We get the following result by using initial conditions (58).

$$\psi_{1_1}(\beta,\ \sigma) = \frac{\hbar\sigma^{\mu}e^{\beta k}\left(\left(4k^2-5\right)e^{\beta k}-4k^2\right)}{4\mu(\mu+1)\left(e^{\beta k}+1\right)^3},$$

$$\psi_{2_1}(\beta,\ \sigma) = -\frac{\hbar\sigma^{\mu}e^{\beta k}\left(12k^2e^{2\beta k}+\left(16k^2+3\right)e^{\beta k}-20k^2+4\right)}{16\mu(\mu+1)\left(e^{\beta k}+1\right)^4}, \tag{59}$$

$$\psi_{1_2}(\beta,\ \sigma) = -\frac{\hbar\sigma^{\mu}e^{\beta k}}{8\mu(\mu+1)\mu(2\mu+1)\left(e^{\beta k}+1\right)^5}\Big(\hbar\sigma^{\mu}\mu(\mu+1)\Big(-8k^4+4\Big(2k^4-5k^2+5\Big)e^{3\beta k}$$

$$+\Big(88k^4-60k^2+2\Big)e^{\beta k}+\Big(-88k^4+104k^2-11\Big)e^{2\beta k}\Big)$$

$$-2n\mu(2\mu+1)\Big(e^{\beta k}+1\Big)^2\Big(\Big(4k^2-5\Big)e^{\beta k}-4k^2\Big)\Big),$$

$$\psi_{2_2}(\beta,\ \sigma) = \frac{1}{32\mu(\mu+1)\mu(2\mu+1)\Big(e^{\beta k}+1\Big)^6}\Big(\hbar\sigma^{\mu}e^{\beta k}\Big(\hbar\sigma^{\mu}\mu(\mu+1)\Big(24k^4e^{4\beta k}-2\Big(56k^4-18k^2+15\Big)e^{3\beta k}$$

$$-\Big(528k^4+8k^2+73\Big)e^{2\beta k}+\Big(528k^4-92k^2-50\Big)e^{\beta k}-8\Big(5k^4-2k^2+1\Big)\Big)-2n\mu(2\mu+1)\Big(e^{\beta k}+1\Big)^2$$

$$\times\Big(12k^2e^{2\beta k}+\Big(16k^2+3\Big)e^{\beta k}-20k^2+4\Big)\Big)\Big), \tag{60}$$

and so on.

In this way, the additional terms in the solution are obtained. The following is the solution to Equation (52) using q-HATM:

$$\psi_1(\beta,\ \sigma) = \psi_{1_0}+\sum_{m=1}^{\infty}\psi_{1_m}\left(\frac{1}{n}\right)^m,$$

$$\psi_2(\beta,\ \sigma) = \psi_{2_0}+\sum_{m=1}^{\infty}\psi_{2_m}\left(\frac{1}{n}\right)^m. \tag{61}$$

$$\psi_1(\beta,\ \sigma) = \frac{e^{k\beta}}{e^{k\beta}+1}+\frac{\hbar\sigma^{\mu}e^{\beta k}\left(\left(4k^2-5\right)e^{\beta k}-4k^2\right)}{4\mu(\mu+1)\left(e^{\beta k}+1\right)^3}-\frac{\hbar\sigma^{\mu}e^{\beta k}}{8\mu(\mu+1)\mu(2\mu+1)\left(e^{\beta k}+1\right)^5}\Big(\hbar\sigma^{\mu}\mu(\mu+1)\Big(-8k^4$$

$$+4\Big(2k^4-5k^2+5\Big)e^{3\beta k}+\Big(88k^4-60k^2+2\Big)e^{\beta k}+\Big(-88k^4+104k^2-11\Big)e^{2\beta k}\Big)$$

$$-2n\mu(2\mu+1)\left(e^{\beta k}+1\right)^2\times\left(\left(4k^2-5\right)e^{\beta k}-4k^2\right)\right)+\cdots,$$

$$\psi_2(\beta,\ \sigma)=\frac{\frac{3e^{k\beta}}{4}+1}{\left(e^{k\beta}+1\right)^2}-\frac{\hbar\sigma^{\mu}e^{\beta k}\left(12k^2e^{2\beta k}+\left(16k^2+3\right)e^{\beta k}-20k^2+4\right)}{16\mu(\mu+1)\left(e^{\beta k}+1\right)^4}+\frac{1}{32\mu(\mu+1)\mu(2\mu+1)\left(e^{\beta k}+1\right)^6}$$

$$\left(\hbar\sigma^{\mu}e^{\beta k}\left(\hbar\sigma^{\mu}\mu(\mu+1)\left(24k^4e^{4\beta k}-2\left(56k^4-18k^2+15\right)e^{3\beta k}-\left(528k^4+8k^2+73\right)e^{2\beta k}\right.\right.\right.$$

$$\left.+\left(528k^4-92k^2-50\right)e^{\beta k}-8\left(5k^4-2k^2+1\right)\right)$$

$$\left.\left.-2n\mu(2\mu+1)\left(e^{\beta k}+1\right)^2\left(12k^2e^{2\beta k}+\left(16k^2+3\right)e^{\beta k}-20k^2+4\right)\right)\right)+\cdots. \tag{62}$$

For $\mu=1$, $\hbar=-1$ and $n=1$ solutions $\sum_{m=1}^{N}\psi_{1m}\left(\frac{1}{n}\right)^m$ and $\sum_{m=1}^{N}\psi_{2m}\left(\frac{1}{n}\right)^m$ converges to the exact solutions as $N\to\infty$.

$$\begin{aligned}\psi_1(\beta,\ \sigma)&=\frac{e^{k(c\sigma+\beta)}}{e^{k(c\sigma+\beta)}+1},\\ \psi_2(\beta,\ \sigma)&=\frac{\frac{3}{4}e^{k(c\sigma+\beta)}+1}{\left(e^{k(c\sigma+\beta)}+1\right)^2}.\end{aligned} \tag{63}$$

4.2 *Solution via MVIM*

Take into consideration the Physical model of time fractional order:

$$\begin{aligned}D_{\sigma}^{\mu}\psi_1(\beta,\ \sigma)&=\frac{\partial^2\psi_1(\beta,\ \sigma)}{\partial\beta^2}-\psi_1{}^3(\beta,\ \sigma)-\psi_1(\beta,\ \sigma)\psi_2(\beta,\ \sigma)+\psi_1(\beta,\ \sigma),\\ D_{\sigma}^{\mu}\psi_2(\beta,\ \sigma)&=\frac{\partial^2\psi_2(\beta,\ \sigma)}{\partial\beta^2}-\psi_2{}^2(\beta,\ \sigma)-\psi_1(\beta,\ \sigma)\psi_2(\beta,\ \sigma)+\psi_2(\beta,\ \sigma),\end{aligned} \tag{64}$$

$$\sigma>0,\ \beta\in R,\ 0<\mu\leq 1.$$

Initial conditions:

$$\psi_1(\beta,\ 0) = \frac{e^{k\beta}}{e^{k\beta}+1},$$
$$\psi_2(\beta,\ 0) = \frac{\frac{3e^{k\beta}}{4}+1}{\left(e^{k\beta}+1\right)^2}. \tag{65}$$

We can get subsequent result by applying the recursive formula given in Equation (51).

$$\psi_{1_{n+1}}(\beta,\ \sigma) = \psi_{1_n}(\beta,\ \sigma) + \mathscr{M}^{-1}\left[\frac{1}{s^\mu}\sum_{k=0}^{m-1} s^{\mu-k-1}\frac{\partial^k \psi_1(\beta,\ 0)}{\partial \sigma^k} + \mathscr{M}\left[\frac{\partial^2 \psi_{1_n}(\beta,\ \sigma)}{\partial \beta^2} - \psi_{1\,n}^3(\beta,\ \sigma)\right.\right.$$
$$\left.\left. - \psi_{1_n}(\beta,\ \sigma)\psi_{2_n}(\beta,\ \sigma) + \psi_{1_n}(\beta,\ \sigma)\right]\right],$$
$$\psi_{2_{n+1}}(\beta,\ \sigma) = \psi_{2_n}(\beta,\ \sigma) + \mathscr{M}^{-1}\left[\frac{1}{s^\mu}\sum_{k=0}^{m-1} s^{\mu-k-1}\frac{\partial^k \psi_2(\beta,\ 0)}{\partial \sigma^k} + \mathscr{M}\left[\frac{\partial^2 \psi_{2_n}(\beta,\ \sigma)}{\partial \beta^2} - \psi_{2\,n}^2(\beta,\ \sigma)\right.\right.$$
$$\left.\left. - \psi_{1_n}(\beta,\ \sigma)\psi_{2_n}(\beta,\ \sigma) + \psi_{2_n}(\beta,\ \sigma)\right]\right]. \tag{66}$$

The second approximation is obtained by putting $n = 0$ into the equation above:

$$\psi_{1_1}(\beta,\ \sigma) = \psi_{1_0}(\beta,\ \sigma) + \mathscr{M}^{-1}\left[\frac{1}{s^\mu}\sum_{k=0}^{m-1} s^{\mu-k-1}\frac{\partial^k \psi_1(\beta,\ 0)}{\partial \sigma^k} + \mathscr{M}\left[\frac{\partial^2 \psi_{1_0}(\beta,\ \sigma)}{\partial \beta^2} - \psi_{1\,0}^3(\beta,\ \sigma)\right.\right.$$
$$\left.\left. - \psi_{1_0}(\beta,\ \sigma)\psi_{2_0}(\beta,\ \sigma) + \psi_{1_0}(\beta,\ \sigma)\right]\right],$$
$$\psi_{2_1}(\beta,\ \sigma) = \psi_{2_0}(\beta,\ \sigma) + \mathscr{M}^{-1}\left[\frac{1}{s^\mu}\sum_{k=0}^{m-1} s^{\mu-k-1}\frac{\partial^k \psi_2(\beta,\ 0)}{\partial \sigma^k} + \mathscr{M}\left[\frac{\partial^2 \psi_{2_0}(\beta,\ \sigma)}{\partial \beta^2} - \psi_{2\,0}^2(\beta,\ \sigma)\right.\right.$$
$$\left.\left. - \psi_{1_0}(\beta,\ \sigma)\psi_{2_0}(\beta,\ \sigma) + \psi_{2_0}(\beta,\ \sigma)\right]\right], \tag{67}$$

by simplification, we get

$$
\psi_{1\,1}(\beta,\ \sigma)=\frac{e^{\beta k}}{e^{\beta k}+1}-\frac{\sigma^{\mu}e^{\beta k}\left(e^{\beta k}\left(4k^{2}+4e^{\beta k}+3\right)-4k^{2}+4\right)}{4\mu(\mu+1)\left(e^{\beta k}+1\right)^{3}},
$$

$$
\psi_{2\,1}(\beta,\ \sigma)=\frac{\frac{3e^{\beta k}}{4}+1}{\left(e^{\beta k}+1\right)^{2}}+\frac{\sigma^{\mu}\left(12\left(k^{2}-1\right)e^{3\beta k}-20\left(k^{2}+2\right)e^{\beta k}+\left(16k^{2}-37\right)e^{2\beta k}-16\right)}{16\mu(\mu+1)\left(e^{\beta k}+1\right)^{4}}. \tag{68}
$$

Substitute $n=1$ in Equation (66), to obtain

$$
\psi_{1\,2}(\beta,\ \sigma)=\psi_{1\,1}(\beta,\ \sigma)+\mathscr{M}^{-1}\left[\frac{1}{s^{\mu}}\sum_{k=0}^{m-1}s^{\mu-k-1}\frac{\partial^{k}\psi_{1}(\beta,\ 0)}{\partial\sigma^{k}}+\mathscr{M}\left[\frac{\partial^{2}\psi_{1\,1}(\beta,\ \sigma)}{\partial\beta^{2}}-\psi_{1\,1}^{3}(\beta,\ \sigma)\right.\right.
$$

$$
\left.\left.-\psi_{1\,1}(\beta,\ \sigma)\psi_{2\,1}(\beta,\ \sigma)+\psi_{1\,1}(\beta,\ \sigma)\right]\right],
$$

$$
\psi_{2\,2}(\beta,\ \sigma)=\psi_{2\,1}(\beta,\ \sigma)+\mathscr{M}^{-1}\left[\frac{1}{s^{\mu}}\sum_{k=0}^{m-1}s^{\mu-k-1}\frac{\partial^{k}\psi_{2}(\beta,\ 0)}{\partial\sigma^{k}}+\mathscr{M}\left[\frac{\partial^{2}\psi_{2\,1}(\beta,\ \sigma)}{\partial\beta^{2}}-\psi_{2\,1}^{2}(\beta,\ \sigma)\right.\right.
$$

$$
\left.\left.-\psi_{1\,1}(\beta,\ \sigma)\psi_{2\,1}(\beta,\ \sigma)+\psi_{2\,1}(\beta,\ \sigma)\right]\right]. \tag{69}
$$

After simplifying the expression, the required solution is obtain.

$$
\psi_{1\,2}(\beta,\ \sigma)=\frac{e^{\beta k}}{64\left(e^{\beta k}+1\right)^{9}}\Bigg(64\left(e^{\beta k}+1\right)^{8}-\left(16\sigma^{\mu}\left(e^{\beta k}+1\right)^{6}\left(e^{\beta k}\left(4k^{2}+4e^{\beta k}+3\right)-4k^{2}+4\right)\right)/\left(\mu(\mu+1)\right)
$$

$$
+\left(8\sigma^{2\mu}\left(e^{\beta k}+1\right)^{4}\left(11\left(8k^{4}-8k^{2}+5\right)e^{2\beta k}+\left(-8k^{4}+36k^{2}+30\right)e^{3\beta k}+\left(-88k^{4}+44k^{2}+32\right)e^{\beta k}\right.\right.
$$

$$
\left.\left.+8\left(k^{4}-2k^{2}+2\right)+24e^{4\beta k}\right)\right)/\left(\mu(2\mu+1)\right)+\left(\sigma^{4\mu}\mu(3\mu+1)e^{2\beta k}\left(e^{\beta k}\left(4k^{2}+4e^{\beta k}+3\right)-4k^{2}\right.\right.
$$

$$
\left.\left.+4\right)^{3}\right)/\left(\mu(\mu+1)^{3}\mu(4\mu+1)\right)+\left(4\sigma^{3\mu}\mu(2\mu+1)e^{4\beta k}\cosh^{2}\left(\frac{\beta k}{2}\right)\right.
$$

$$
\left(-4k^{2}\sinh(\beta k)+4\left(k^{2}-2\right)\cosh(\beta k)-4k^{2}-3\right)\left(8\left(2k^{2}+1\right)\sinh(\beta k)+8\left(7k^{2}+11\right)\cosh(\beta k)\right.
$$

$$-64k^2+32\sinh(2\beta k)+64\cosh(2\beta k)+85\Big)\Big)/\Big(\mu(\mu+1)^2\mu(3\mu+1)\Big)\Big), \tag{70}$$

$$\psi_{2_2}(\beta,\ \sigma)=\frac{1}{256\left(e^{\beta k}+1\right)^8}\left(64\left(e^{\beta k}+1\right)^6\left(3e^{\beta k}+4\right)+\left(16\sigma^{\mu}\left(e^{\beta k}+1\right)^4\left(12\left(k^2-1\right)e^{3\beta k}-20\left(k^2+2\right)e^{\beta k}\right.\right.\right.$$

$$+\left(16k^2-37\right)e^{2\beta k}-16\Big)\Big)/\Big(\mu(\mu+1)\Big)$$

$$+\left(8\sigma^{2\mu}\left(e^{\beta k}+1\right)^2\left(e^{\beta k}\left(-40k^4+\left(411-8k^2\left(66k^2+13\right)\right)e^{2\beta k}\right.\right.\right.$$

$$+96k^2+24\left(k^4-2k^2+2\right)e^{4\beta k}-4\left(28k^4+31k^2-52\right)e^{3\beta k}+4\left(132k^4+k^2+115\right)e^{\beta k}+272\Big)+64\Big)\Big)$$

$$/\Big(\mu(2\mu+1)\Big)+\Big(\sigma^{3\mu}\mu(2\mu+1)e^{2\beta k}\Big(12\left(k^2-1\right)e^{3\beta k}-20\left(k^2+2\right)e^{\beta k}+\left(16k^2-37\right)e^{2\beta k}-16\Big)$$

$$\Big(8\left(k^2+12\right)\cosh(\beta k)-16k^2-16\sinh(\beta k)+32\cosh(2\beta k)+65\Big)\Big)/\Big(\mu(\mu+1)^2\mu(3\mu+1)\Big)\Big). \tag{71}$$

5. Problem 2

5.1 *Solution via q-HMTM*

Assume the time fractional coupled system of diffusion-reaction equation [51]:

$$\begin{aligned} &D_{\sigma}^{\mu}\psi_1(\beta,\ \sigma)-\frac{\partial^2\psi_1(\beta,\ \sigma)}{\partial\beta^2}+\psi_1^2(\beta,\ \sigma)+\psi_1(\beta,\ \sigma)\psi_2(\beta,\ \sigma)-\psi_1(\beta,\ \sigma)=0,\\ &D_{\sigma}^{\mu}\psi_2(\beta,\ \sigma)-\frac{\partial^2\psi_2(\beta,\ \sigma)}{\partial\beta^2}+\psi_1(\beta,\ \sigma)\psi_2(\beta,\ \sigma)=0, \end{aligned} \tag{72}$$

where $0<\mu\leq 1$.

Initial condition:

$$\begin{aligned} \psi_1(\beta,\ 0)&=\frac{e^{k\beta}}{\left(e^{\frac{k\beta}{2}}+1\right)^2},\\ \psi_2(\beta,\ 0)&=\frac{1}{e^{\frac{k\beta}{2}}+1}, \end{aligned} \tag{73}$$

$$\mathscr{M}[\psi_1(\beta,\,\sigma)]+s\left(\frac{e^{k\beta}}{\left(e^{\frac{k\beta}{2}}+1\right)^2}\right)+\frac{1}{s^{\mu}}\mathscr{M}\left[-\frac{\partial^2\psi_1(\beta,\,\sigma)}{\partial\beta^2}+\psi_1^2(\beta,\,\sigma)+\psi_1(\beta,\,\sigma)\psi_2(\beta,\,\sigma)-\psi_1(\beta,\,\sigma)\right]=0,$$

$$\mathscr{M}[\psi_2(\beta,\,\sigma)]+s\left(\frac{1}{e^{\frac{k\beta}{2}}+1}\right)+\frac{1}{s^{\mu}}\mathscr{M}\left[-\frac{\partial^2\psi_2(\beta,\,\sigma)}{\partial\beta^2}+\psi_1(\beta,\,\sigma)\psi_2(\beta,\,\sigma)\right]=0. \tag{74}$$

We characterise the nonlinear operators as follows:

$$\begin{aligned}\mathscr{N}^2[\zeta_1(\beta,\,\sigma;\,q),\zeta_2(\beta,\,\sigma;\,q)]=&\mathscr{M}[\zeta_1(\beta,\,\sigma;\,q)]+s\left(\frac{e^{k\beta}}{\left(e^{\frac{k\beta}{2}}+1\right)^2}\right)+\frac{1}{s^{\mu}}\mathscr{M}\left[-\frac{\partial^2\zeta_1(\beta,\,\sigma;\,q)}{\partial\beta^2}\right.\\&\left.+\zeta_1^2(\beta,\,\sigma;\,q)(\beta,\,\sigma)+\zeta_1(\beta,\,\sigma;\,q)\zeta_2(\beta,\,\sigma;\,q)-\zeta_1(\beta,\,\sigma;\,q)\right],\\\mathscr{N}^3[\zeta_1(\beta,\,\sigma;\,q),\zeta_2(\beta,\,\sigma;\,q)]=&\mathscr{M}[\zeta_2(\beta,\,\sigma;\,q)]+s\left(\frac{1}{e^{\frac{k\beta}{2}}+1}\right)\\&+\frac{1}{s^{\mu}}\mathscr{M}\left[-\frac{\partial^2\zeta_2(\beta,\,\sigma;\,q)}{\partial\beta^2}+\zeta_1(\beta,\,\sigma;\,q)\zeta_2(\beta,\,\sigma;\,q)\right].\end{aligned} \tag{75}$$

The Mohand operators are describe as follows:

$$\begin{aligned}\mathscr{M}[\psi_{1_m}(\beta,\,\sigma)-k_m\psi_{1_{m-1}}(\beta,\,\sigma)]&=\hbar\mathfrak{h}(\beta,\,\sigma)\mathscr{R}_{1,\,m}\left[\overrightarrow{\psi_1}_{m-1},\,\overrightarrow{\psi_2}_{m-1}\right],\\\mathscr{M}[\psi_{2_m}(\beta,\,\sigma)-k_m\psi_{2_{m-1}}(\beta,\,\sigma)]&=\hbar\mathfrak{h}(\beta,\,\sigma)\mathscr{R}_{2,\,m}\left[\overrightarrow{\psi_1}_{m-1},\,\overrightarrow{\psi_2}_{m-1}\right].\end{aligned} \tag{76}$$

Here,

$$\begin{aligned}&\mathscr{R}_{1,\,m}[\overrightarrow{\psi_1}_{m-1},\,\overrightarrow{\psi_2}_{m-1}]\\&=\mathscr{M}[\psi_{1_{m-1}}(\beta,\,\sigma)]+s\left(1-\frac{k_m}{n}\right)\left(\frac{e^{k\beta}}{\left(e^{\frac{k\beta}{2}}+1\right)^2}\right)+\frac{1}{s^{\mu}}\mathscr{M}\left[-\frac{\partial^2\psi_{1_{m-1}}(\beta,\,\sigma)}{\partial\beta^2}\right.\end{aligned}$$

$$+\sum_{i=0}^{m-1}\psi_{1_i}(\beta,\ \sigma)\psi_{1_{m-1-i}}(\beta,\ \sigma)+\sum_{i=0}^{m-1}\psi_{1_i}(\beta,\ \sigma)\psi_{2_{m-1-i}}(\beta,\ \sigma)-\psi_{1_{m-1}}(\beta,\ \sigma)\Bigg],$$

$$\mathscr{R}_{2,\ m}[\overrightarrow{\psi}_{1m-1},\ \overrightarrow{\psi}_{2m-1}]$$

$$=\mathscr{M}[\psi_{2_{m-1}}(\beta,\ \sigma)]+s\left(1-\frac{k_m}{n}\right)\left(\frac{1}{e^{\frac{k\beta}{2}}+1}\right)$$

$$+\frac{1}{s^{\mu}}\mathscr{M}\left[-\frac{\partial^2\psi_{2_{m-1}}(\beta,\ \sigma)}{\partial\beta^2}+\sum_{i=0}^{m-1}\psi_{1_i}(\beta,\ \sigma)\psi_{2_{m-1-i}}(\beta,\ \sigma)\right], \tag{77}$$

$$\psi_{1_m}(\beta,\ \sigma)=k_m\psi_{1_{m-1}}(\beta,\ \sigma)+\hbar\mathscr{M}^{-1}[\mathfrak{h}(\beta,\ \sigma)\mathscr{R}_{1,\ m}(\overrightarrow{\psi}_{1m-1},\ \overrightarrow{\psi}_{2m-1}),$$

$$\psi_{2_m}(\beta,\ \sigma)=k_m\psi_{2_{m-1}}(\beta,\ \sigma)+\hbar\mathscr{M}^{-1}[\mathfrak{h}(\beta,\ \sigma)\mathscr{R}_{2,\ m}(\overrightarrow{\psi}_{1m-1},\ \overrightarrow{\psi}_{2m-1}). \tag{78}$$

We get the following result by using initial conditions (78).

$$\psi_{1_1}(\beta,\ \sigma)=\frac{\hbar\sigma^{\mu}e^{\beta k}\left(\left(k^2-2\right)e^{\frac{\beta k}{2}}-2k^2\right)}{2\mu(\mu+1)\left(e^{\frac{\beta k}{2}}+1\right)^4},$$

$$\psi_{2_1}(\beta,\ \sigma)=\frac{\hbar\sigma^{\mu}e^{\frac{\beta k}{2}}\left(k^2-\left(k^2-4\right)e^{\frac{\beta k}{2}}\right)}{4\mu(\mu+1)\left(e^{\frac{\beta k}{2}}+1\right)^3}, \tag{79}$$

$$\psi_{1_2}(\beta,\ \sigma)=\hbar\sigma^{\mu}e^{\beta k}\left(-\frac{\hbar\sigma^{\mu}\left(-8k^4+\left(k^2-4\right)k^2e^{\frac{3\beta k}{2}}+\left(33k^2-28\right)k^2e^{\frac{\beta k}{2}}-2\left(9k^4-16k^2+8\right)e^{\beta k}\right)}{8\mu(2\mu+1)\left(e^{\frac{\beta k}{2}}+1\right)^6}\right.$$

$$\left.+\frac{n\left(\left(k^2-2\right)e^{\frac{\beta k}{2}}-2k^2\right)}{2\mu(\mu+1)\left(e^{\frac{\beta k}{2}}+1\right)^4}\right),$$

$$\psi_{2_2}(\beta,\ \sigma)=\hbar\sigma^{\mu}e^{\frac{\beta k}{2}}\left(\frac{\hbar\sigma^{\mu}\left(e^{\frac{\beta k}{2}}-1\right)\left(k^4-2\left(5k^2-16\right)k^2e^{\frac{\beta k}{2}}+\left(k^2-4\right)^2e^{\beta k}\right)}{16\mu(2\mu+1)\left(e^{\frac{\beta k}{2}}+1\right)^5}-\frac{n\left(\left(k^2-4\right)e^{\frac{\beta k}{2}}-k^2\right)}{4\mu(\mu+1)\left(e^{\frac{\beta k}{2}}+1\right)^3}\right), \tag{80}$$

and so on.

In this way, the additional terms in the solution are obtained. The following is the solution to Equation (72) using q-HATM:

$$\psi_1(\beta,\ \sigma) = \psi_{1\,0} + \sum_{m=1}^{\infty} \psi_{1\,m}\left(\frac{1}{n}\right)^m,$$

$$\psi_2(\beta,\ \sigma) = \psi_{2\,0} + \sum_{m=1}^{\infty} \psi_{2\,m}\left(\frac{1}{n}\right)^m. \tag{81}$$

$$\psi_1(\beta,\ \sigma) = \frac{e^{k\beta}}{\left(e^{\frac{k\beta}{2}}+1\right)^2} + \frac{\hbar\sigma^\mu e^{\beta k}\left(\left(k^2-2\right)e^{\frac{\beta k}{2}}-2k^2\right)}{2\mu(\mu+1)\left(e^{\frac{\beta k}{2}}+1\right)^4} + \hbar\sigma^\mu e^{\beta k}\left(\frac{n\left(\left(k^2-2\right)e^{\frac{\beta k}{2}}-2k^2\right)}{2\mu(\mu+1)\left(e^{\frac{\beta k}{2}}+1\right)^4}\right.$$
$$\left. - \frac{\hbar\sigma^\mu\left(-8k^4+\left(k^2-4\right)k^2e^{\frac{3\beta k}{2}}+\left(33k^2-28\right)k^2e^{\frac{\beta k}{2}}-2\left(9k^4-16k^2+8\right)e^{\beta k}\right)}{8\mu(2\mu+1)\left(e^{\frac{\beta k}{2}}+1\right)^6}\right),$$

$$\psi_2(\beta,\ \sigma) = \frac{1}{e^{\frac{k\beta}{2}}+1} + \frac{\hbar\sigma^\mu e^{\frac{\beta k}{2}}\left(k^2-\left(k^2-4\right)e^{\frac{\beta k}{2}}\right)}{4\mu(\mu+1)\left(e^{\frac{\beta k}{2}}+1\right)^3} + \hbar\sigma^\mu e^{\frac{\beta k}{2}}\left(-\frac{n\left(\left(k^2-4\right)e^{\frac{\beta k}{2}}-k^2\right)}{4\mu(\mu+1)\left(e^{\frac{\beta k}{2}}+1\right)^3}\right.$$
$$\left. + \frac{\hbar\sigma^\mu\left(e^{\frac{\beta k}{2}}-1\right)\left(k^4-2\left(5k^2-16\right)k^2e^{\frac{\beta k}{2}}+\left(k^2-4\right)^2e^{\beta k}\right)}{16\mu(2\mu+1)\left(e^{\frac{\beta k}{2}}+1\right)^5}\right). \tag{82}$$

For $\mu = 1$, $\hbar = -1$ and $n = 1$ solutions $\sum_{m=1}^{N} \psi_{1\,m}\left(\frac{1}{n}\right)^m$ and $\sum_{m=1}^{N} \psi_{2\,m}\left(\frac{1}{n}\right)^m$ converges to the exact solutions as $N \to \infty$.

$$\psi_1(\beta,\ \sigma) = \frac{e^{k(c\sigma+\beta)}}{\left(e^{\frac{1}{2}k(c\sigma+\beta)}+1\right)^2},$$
$$\psi_2(\beta,\ \sigma) = \frac{1}{e^{\frac{1}{2}k(c\sigma+\beta)}+1}. \tag{83}$$

5.2 *Solution via MVIM*

Assume the time fractional coupled system of diffusion-reaction equation:

$$D_\sigma^\mu \psi_1(\beta,\ \sigma) - \frac{\partial^2\psi_1(\beta,\ \sigma)}{\partial\beta^2} + \psi_1^2(\beta,\ \sigma) + \psi_1(\beta,\ \sigma)\psi_2(\beta,\ \sigma) - \psi_1(\beta,\ \sigma) = 0,$$

$$D_{\sigma}^{\mu}\psi_2(\beta,\ \sigma)-\frac{\partial^2\psi_2(\beta,\ \sigma)}{\partial\beta^2}+\psi_1(\beta,\ \sigma)\psi_2(\beta,\ \sigma)=0,$$

$$\text{where}\quad 0<\mu\leq 1. \tag{84}$$

Initial conditions:

$$\psi_1(\beta,\ 0)=\frac{e^{k\beta}}{\left(e^{\frac{k\beta}{2}}+1\right)^2},$$

$$\psi_2(\beta,\ 0)=\frac{1}{e^{\frac{k\beta}{2}}+1}. \tag{85}$$

We can get subsequent result by applying the recursive formula given in Equation (51).

$$\psi_{1_{n+1}}(\beta,\ \sigma)=\psi_{1_n}(\beta,\ \sigma)+\mathscr{M}^{-1}\left[\frac{1}{s^{\mu}}\sum_{k=0}^{m-1}s^{\mu-k-1}\frac{\partial^k\psi_1(\beta,\ 0)}{\partial\sigma^k}+\mathscr{M}\left[-\frac{\partial^2\psi_{1_n}(\beta,\ \sigma)}{\partial\beta^2}+\psi_{1_n}^2(\beta,\ \sigma)\right.\right.$$

$$\left.\left.+\psi_{1_n}(\beta,\ \sigma)\psi_{2_n}(\beta,\ \sigma)-\psi_{1_n}(\beta,\ \sigma)\right]\right],$$

$$\psi_{2_{n+1}}(\beta,\ \sigma)=\psi_{2_n}(\beta,\ \sigma)+\mathscr{M}^{-1}\left[\frac{1}{s^{\mu}}\sum_{k=0}^{m-1}s^{\mu-k-1}\frac{\partial^k\psi_2(\beta,\ 0)}{\partial\sigma^k}+\mathscr{M}\left[-\frac{\partial^2\psi_{2_n}(\beta,\ \sigma)}{\partial\beta^2}+\psi_{1_n}(\beta,\ \sigma)\psi_{2_n}(\beta,\ \sigma)\right]\right]. \tag{86}$$

The second approximation is obtained by putting $n=0$ into the equation above:

$$\psi_{1_1}(\beta,\ \sigma)=\psi_{1_0}(\beta,\ \sigma)+\mathscr{M}^{-1}\left[\frac{1}{s^{\mu}}\sum_{k=0}^{m-1}s^{\mu-k-1}\frac{\partial^k\psi_1(\beta,\ 0)}{\partial\sigma^k}+\mathscr{M}\left[-\frac{\partial^2\psi_{1_0}(\beta,\ \sigma)}{\partial\beta^2}+\psi_{1_0}^2(\beta,\ \sigma)\right.\right.$$

$$\left.\left.+\psi_{1_0}(\beta,\ \sigma)\psi_{2_0}(\beta,\ \sigma)-\psi_{1_0}(\beta,\ \sigma)\right]\right], \tag{87}$$

$$\psi_{2_1}(\beta,\ \sigma)=\psi_{2_0}(\beta,\ \sigma)+\mathscr{M}^{-1}\left[\frac{1}{s^{\mu}}\sum_{k=0}^{m-1}s^{\mu-k-1}\frac{\partial^k\psi_2(\beta,\ 0)}{\partial\sigma^k}+\mathscr{M}\left[-\frac{\partial^2\psi_{2_0}(\beta,\ \sigma)}{\partial\beta^2}+\psi_{1_0}(\beta,\ \sigma)\psi_{2_0}(\beta,\ \sigma)\right]\right],$$

by simplification, we get

$$\psi_{1_1}(\beta,\ \sigma)=\frac{e^{\beta k}}{\left(e^{\frac{\beta k}{2}}+1\right)^2}-\frac{\sigma^{\mu}e^{\beta k}\left(\left(k^2+2\right)e^{\frac{\beta k}{2}}-2k^2+2e^{\beta k}+2\right)}{2\mu(\mu+1)\left(e^{\frac{\beta k}{2}}+1\right)^4},$$

$$\psi_{2_1}(\beta,\ \sigma)=\frac{1}{e^{\frac{\beta k}{2}}+1}+\frac{\left(k^2-4\right)\sigma^{\mu}e^{\beta k}-k^2\sigma^{\mu}e^{\frac{\beta k}{2}}}{4\mu(\mu+1)\left(e^{\frac{\beta k}{2}}+1\right)^3}, \tag{88}$$

Substitute $n=1$ in Equation (86) to obtain

$$\psi_{1_2}(\beta,\ \sigma)=\psi_{1_1}(\beta,\ \sigma)+\mathscr{M}^{-1}\left[\frac{1}{s^{\mu}}\sum_{k=0}^{m-1}s^{\mu-k-1}\frac{\partial^k\psi_1(\beta,\ 0)}{\partial\sigma^k}+\mathscr{M}\left[-\frac{\partial^2\psi_{1_1}(\beta,\ \sigma)}{\partial\beta^2}+\psi_{1_1}^2(\beta,\ \sigma)\right.\right.$$

$$\left.\left.+\psi_{1_1}(\beta,\ \sigma)\psi_{2_1}(\beta,\ \sigma)-\psi_{1_1}(\beta,\ \sigma)\right]\right], \tag{89}$$

$$\psi_{2_2}(\beta,\ \sigma)=\psi_{2_1}(\beta,\ \sigma)+\mathscr{M}^{-1}\left[\frac{1}{s^{\mu}}\sum_{k=0}^{m-1}s^{\mu-k-1}\frac{\partial^k\psi_2(\beta,\ 0)}{\partial\sigma^k}+\mathscr{M}\left[-\frac{\partial^2\psi_{2_1}(\beta,\ \sigma)}{\partial\beta^2}+\psi_{1_1}(\beta,\ \sigma)\psi_{2_1}(\beta,\ \sigma)\right]\right].$$

After simplifying the expression, the required solution is obtain.

$$\psi_{1_2}(\beta,\ \sigma)$$

$$=\frac{e^{\beta k}}{\left(e^{\frac{\beta k}{2}}+1\right)^2}-\frac{\sigma^{\mu}e^{\beta k}\left(\left(k^2+2\right)e^{\frac{\beta k}{2}}-2k^2+2e^{\beta k}+2\right)}{2\mu(\mu+1)\left(e^{\frac{\beta k}{2}}+1\right)^4}$$

$$-\frac{\sigma^{3\mu}\mu(2\mu+1)e^{\frac{3\beta k}{2}}\left(\left(k^2+2\right)e^{\frac{\beta k}{2}}-2k^2+2e^{\beta k}+2\right)\left(-4\left(k^2-2\right)e^{\frac{\beta k}{2}}+\left(k^2+8\right)e^{\beta k}+k^2+4e^{\frac{3\beta k}{2}}\right)}{8\mu(\mu+1)^2\mu(3\mu+1)\left(e^{\frac{\beta k}{2}}+1\right)^8}$$

$$+\frac{\sigma^{2\mu}e^{\beta k}\left(8\left(k^2-1\right)^2+\left(-k^4+12k^2+32\right)e^{\frac{3\beta k}{2}}+2\left(9k^4-16k^2+20\right)e^{\beta k}+\left(-33k^4+4k^2+16\right)e^{\frac{\beta k}{2}}+16e^{2\beta k}\right)}{8\mu(2\mu+1)\left(e^{\frac{\beta k}{2}}+1\right)^6},$$

$$\psi_{2_2}(\beta,\ \sigma)$$

$$
=\frac{1}{e^{\frac{\beta k}{2}}+1}+\frac{\left(k^2-4\right)\sigma^{\mu}e^{\beta k}-k^2\sigma^{\mu}e^{\frac{\beta k}{2}}}{4\mu(\mu+1)\left(e^{\frac{\beta k}{2}}+1\right)^3}
$$

$$
+\frac{\sigma^{2\mu}e^{\frac{\beta k}{2}}\left(-k^4+\left(k^4-8k^2+32\right)e^{\frac{3\beta k}{2}}+\left(11k^4-32k^2+16\right)e^{\frac{\beta k}{2}}+\left(-11k^4+40k^2+16\right)e^{\beta k}\right)}{16\mu(2\mu+1)\left(e^{\frac{\beta k}{2}}+1\right)^5}
$$

$$
+\frac{\sigma^{3\mu}\mu(2\mu+1)e^{\frac{3\beta k}{2}}\left(\left(k^2-4\right)e^{\frac{\beta k}{2}}-k^2\right)\left(\left(k^2+2\right)e^{\frac{\beta k}{2}}-2k^2+2e^{\beta k}+2\right)}{8\mu(\mu+1)^2\mu(3\mu+1)\left(e^{\frac{\beta k}{2}}+1\right)^7}. \tag{90}
$$

6. Numerical results and discussion

The systems of PDEs solved in this work using q-HMTM and MVIM are the Caputo fractional-order diffusion-reaction models: the time-fractional derivative of order $0<\mu\leq 1$ brings memory and hereditary into dynamics. From the phenomenological point of view, these models capture the dynamics resulting from diffusion mechanisms on a homogeneous background (that accounts for the spatial spreading between the quantities ψ_1 and ψ_2) which interacts with nonlinearities representing self-interaction and cross-interaction terms among two different components. It is essential because the empirical anomalous diffusion behavior, wherein a system moves in ways that contradict classical Brownian motion and instead give way to sub-diffusive or history-dependent dynamics found in porous media, viscoelastic materials, and biological systems, must be described by the fractional operator. The nonlinear terms ψ_1^2, ψ_2^2 and coupling product $\psi_1\psi_2$ correspond to cooperative or competitive interaction, ubiquitous in chemical kinetics, population dynamics and pattern formation of nonlinear science. The selected initial conditions are localized configurations, which may mimic the concentration profiles or population densities in the beginning of the process. Hence, these fractional coupled diffusion-reaction equations offer a physically sensible approach for describing real-world dynamics in applications occurring in nonlinear science; where both the diffusion and the reaction are affected by memory which results with richer and more realistic behavior than classical integer-order models.

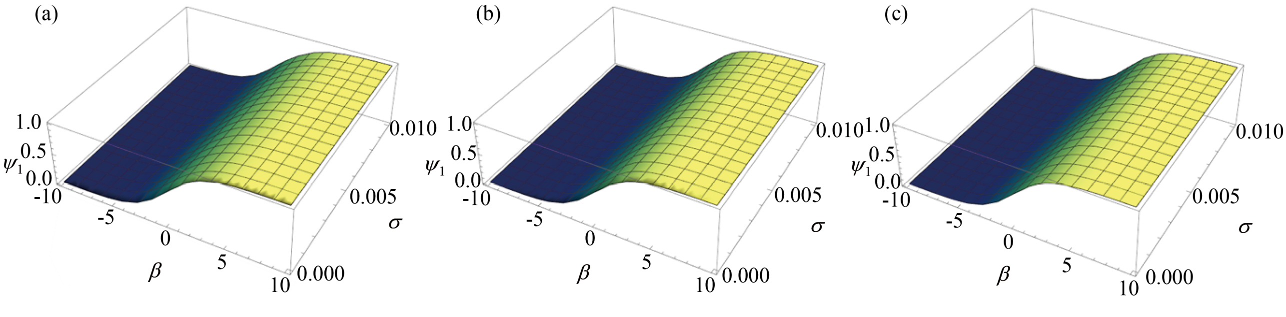

Figure 1. Graphical analysis of fractional order σ for the solution $\psi_1(\beta,\ \sigma)$. The subfigures (a) show the effect of $\mu=0.4$, (b) show the effect of $\mu=0.6$, and (c) show the effect of $\mu=0.8$ of problem 1 for $\sigma=0.01$, $k=2/3$, and $c=1$

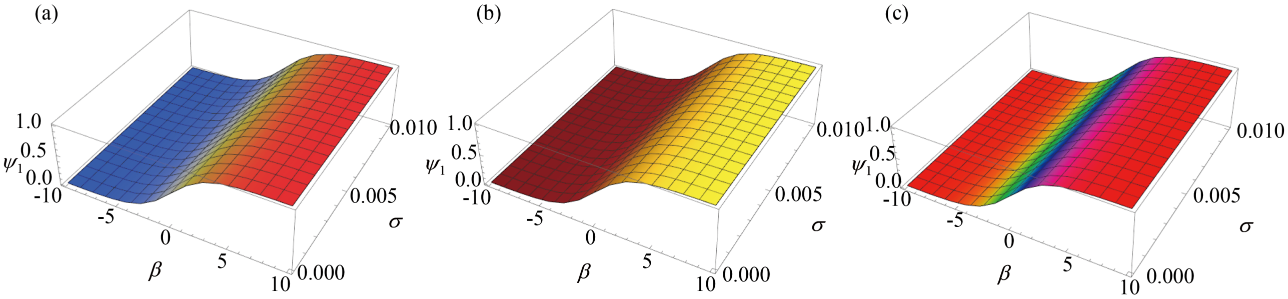

Figure 2. Graphical comparison of the q-HMTM solution (a), MVIM solution (b), and exact solution (c) of problem 1 for $\mu = 1.0$, $\sigma = 0.01$, $k = 2/3$, and $c = 1$

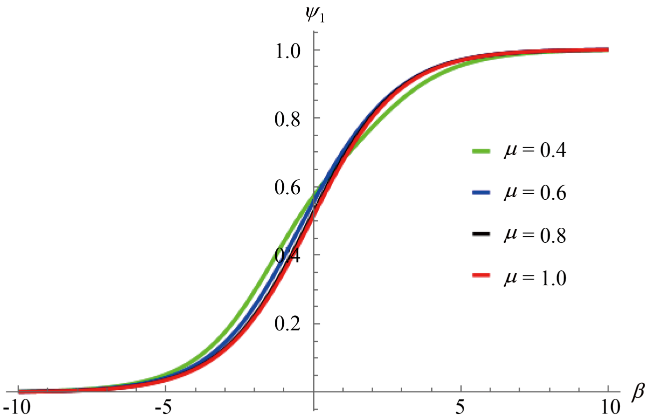

Figure 3. Graphical comparison of fractional order μ for the solution $\psi_1(\beta, \sigma)$ of problem 1 for $\sigma = 0.01$, $k = 2/3$, and $c = 1$

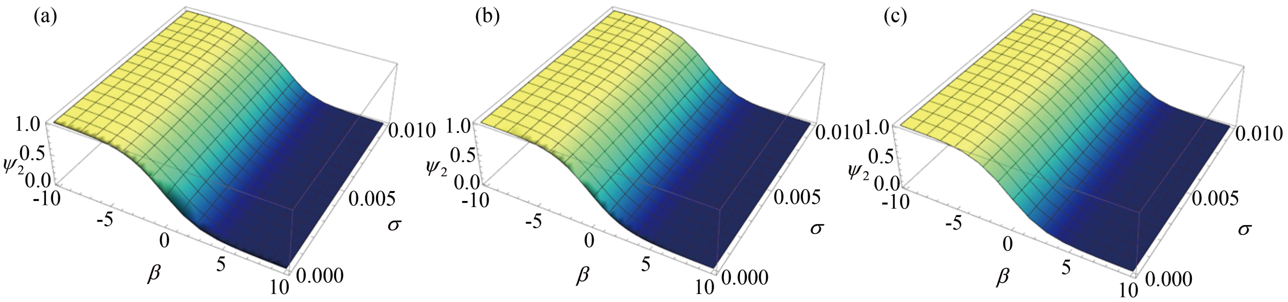

Figure 4. Graphical analysis of fractional order σ for the solution $\psi_2(\beta, \sigma)$. The subfigures (a) show the effect of $\mu = 0.4$, (b) show the effect of $\mu = 0.6$, and (c) show the effect of $\mu = 0.8$ of problem 1 for $\sigma = 0.01$, $k = 2/3$, and $c = 1$

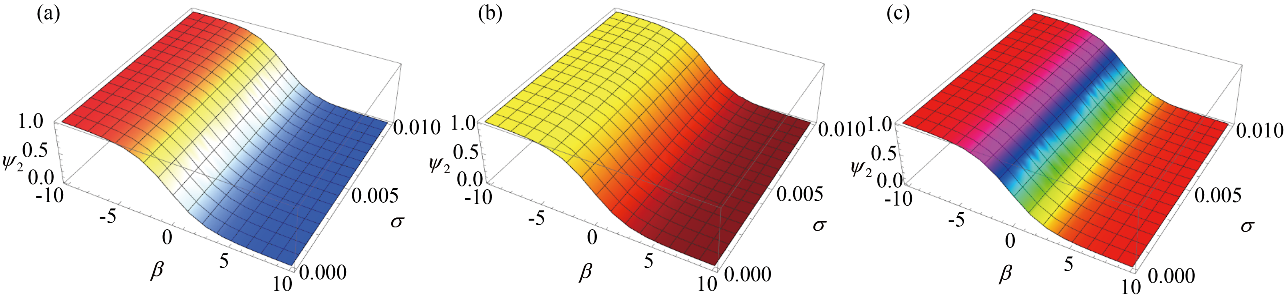

Figure 5. Graphical comparison of the q-HMTM solution (a), MVIM solution (b), and exact solution (c) of problem 1 for $\mu = 1.0$, $\sigma = 0.01$, $k = 2/3$, and $c = 1$

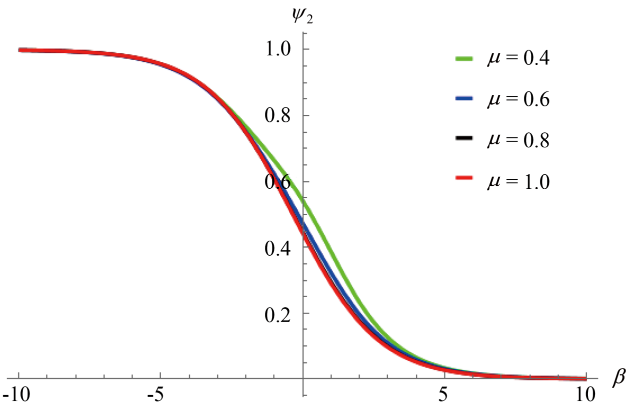

Figure 6. Graphical comparison of fractional order μ for the solution $\psi_2(\beta,\ \sigma)$ of problem 1 for $\sigma = 0.01$, $k = 2/3$, and $c = 1$

In Figures 1-6, we have shown the numerical solutions of $\psi_1(\beta,\ \sigma)$ and $\psi_2(\beta,\ \sigma)$ which corresponds to the Problem 1 for different values of fractional parameter μ and time σ, along with exact solution. These figures confirm the effectiveness of the proposed techniques and indicate the physical importance for varying fractional order. In particular, the influence of the parameter μ on the dynamics of $\psi_1(\beta,\ \sigma)$ at a fixed time $\sigma = 0.01$, is shown in Figure 1. For smaller fractional value $\mu = 0.4$, the solution displays considerable transitions and large memory effects, implying the predominance of nonlocality and weak diffusion. For $\mu = 0.6, 0.8$, the profiles become smoother and more diffusive as we increase μ indicating a transition between anomalous to classical diffusion regime.

The solution $\psi_1(\beta,\ \sigma)$ obtained through q-HMTM and MVIM are presented as compared with exact solution for $\mu = 1.0$ and $\sigma = 0.01$ in Figure 2. The excellent agreement that we obtain from the proposed methods demonstrate that, they reliably capture the essential dynamics. The fractional orders comparison are shown in Figure 3, and it is observed that as $\mu \to 1$, the solution approaches to classical model, this confirms the validity of fractional model comparing with integer-order models.

A similar analysis for $\psi_2(\beta,\ \sigma)$ is presented in Figures 4-6. Similar patterns are observed: the smaller fractional orders lead to more oscillatory profiles because of more pronounced memory effects and larger values of the order provide smoother solutions as an evidence of less diffusive carriers. These results emphasize the role of fractional operators for modeling real-world systems with anomalous diffusion rates that cannot be fully represented by classical PDEs.

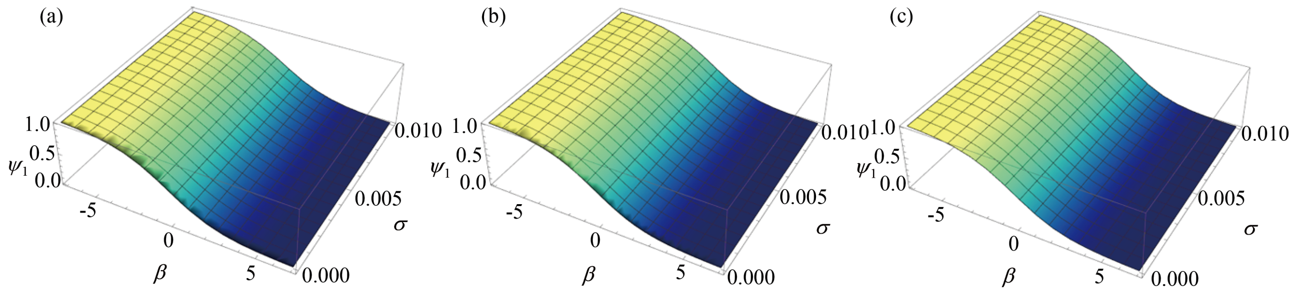

Figure 7. Graphical analysis of fractional order σ for the solution $\psi_1(\beta,\ \sigma)$. The subfigures (a) show the effect of $\mu = 0.4$, (b) show the effect of $\mu = 0.6$, and (c) show the effect of $\mu = 0.8$ of problem 2 for $\sigma = 0.01$, $k = 2/3$, and $c = 1$

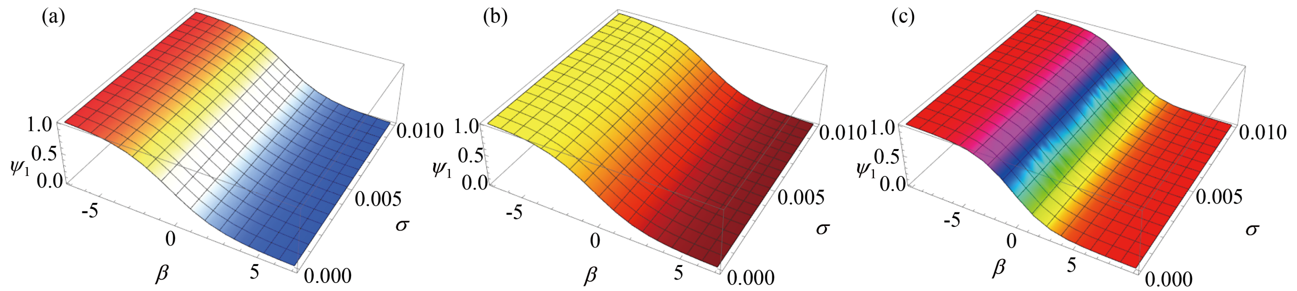

Figure 8. Graphical comparison of the q-HMTM solution (a), MVIM solution (b), and exact solution (c) of problem 2 for $\mu = 1.0$, $\sigma = 0.01$, $k = 2/3$, and $c = 1$

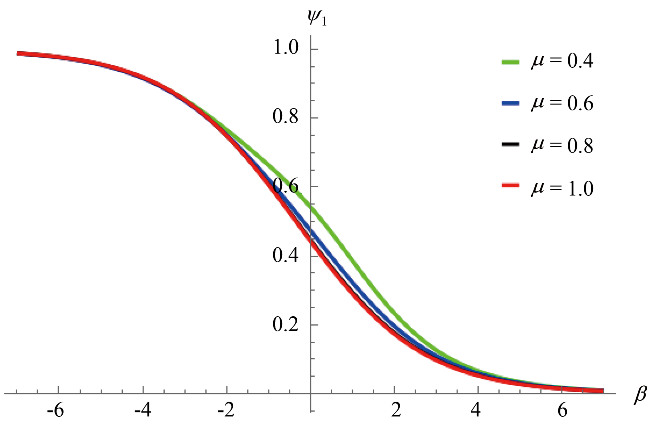

Figure 9. Graphical comparison of fractional order μ for the solution $\psi_1(\beta,\ \sigma)$ of problem 2 for $\sigma = 0.01$, $k = 2/3$, and $c = 1$

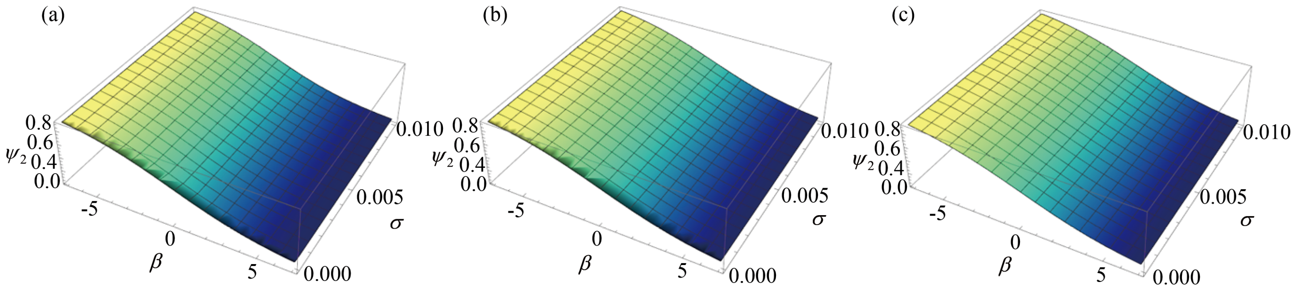

Figure 10. Graphical analysis of fractional order σ for the solution $\psi_2(\beta, \sigma)$. The subfigures (a) show the effect of $\mu = 0.4$, (b) show the effect of $\mu = 0.6$, and (c) show the effect of $\mu = 0.8$ of problem 2 for $\sigma = 0.01$, $k = 2/3$, and $c = 1$

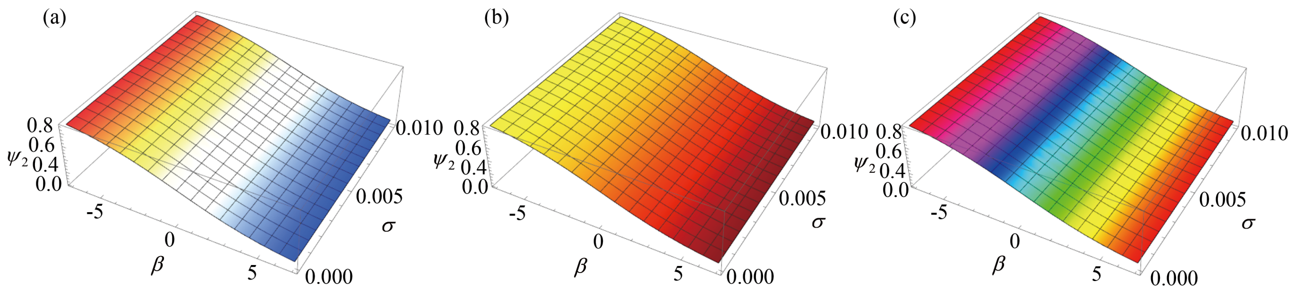

Figure 11. Graphical comparison of the q-HMTM solution (a), MVIM solution (b), and exact solution (c) of problem 2 for $\mu = 1.0$, $\sigma = 0.01$, $k = 2/3$, and $c = 1$

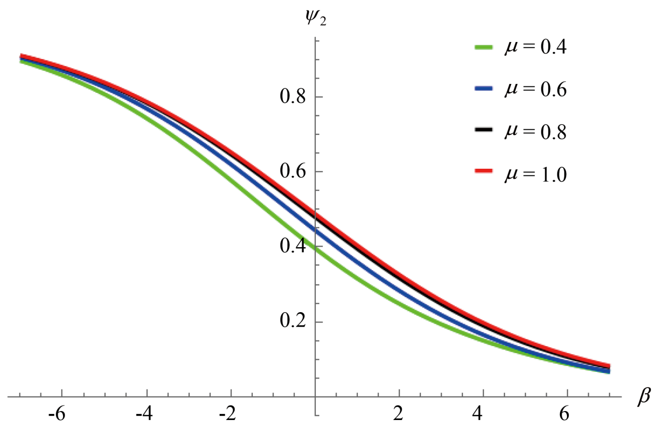

Figure 12. Graphical comparison of fractional order μ for the solution $\psi_2(\beta, \sigma)$ of problem 2 for $\sigma = 0.01$, $k = 2/3$, and $c = 1$

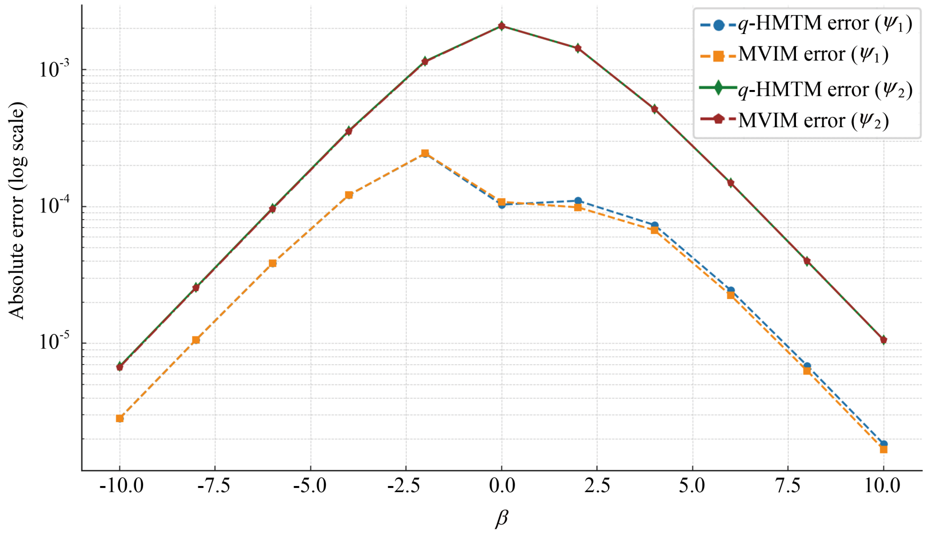

Figure 13. Absolute error comparison of problem 1

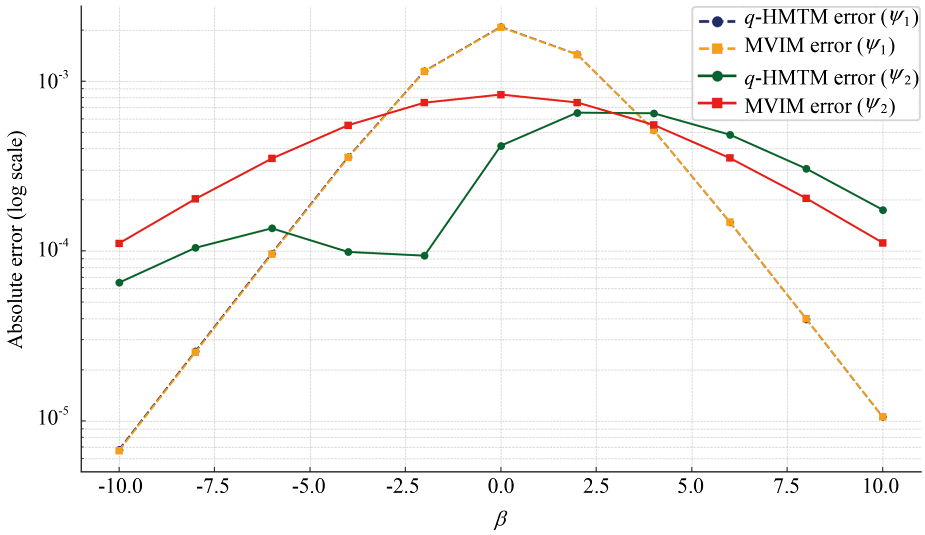

Figure 14. Absolute error comparison of problem 2

Tables 1 and 2 for Problem 1 confirm quantitatively that both methods give good approximations of ψ_1 and ψ_2, with the absolute errors mostly being small. In Figures 7-12 and Tables 3 and 4 for Problem 2, the trends outlined are similar but ψ_2 has slightly large errors for MVIM at lower β values. However, both approaches maintain stable error levels as shown in Figures 13 and 14.

Table 1. q-HMTM and MVIM solution error analysis of $\psi_1(\beta, \sigma)$ for k = 2/3, and c = 1 of problem 1

σ	β	q-HMTM$_{\mu=1}$	MVIM$_{\mu=1}$	Exact	q-HMTM error$_{\mu=1.00}$	MVIM error$_{\mu=1.00}$
	-10	0.001276	0.001276	0.001279	2.830701×10^{-6}	2.830701×10^{-6}
	-8	0.004826	0.004826	0.004836	1.060080×10^{-5}	1.060090×10^{-5}
	-6	0.018066	0.018066	0.018104	3.830350×10^{-5}	3.830390×10^{-5}
	-4	0.065254	0.065254	0.065375	1.212830×10^{-4}	1.213010×10^{-4}
	-2	0.209468	0.209468	0.209711	2.430450×10^{-4}	2.436050×10^{-4}
0.01	0	0.501564	0.501558	0.501667	1.025350×10^{-4}	1.083920×10^{-4}
	2	0.792600	0.792588	0.792490	1.097680×10^{-4}	9.844470×10^{-5}
	4	0.935508	0.935502	0.935435	7.321050×10^{-5}	6.698020×10^{-5}
	6	0.982156	0.982154	0.982131	2.444280×10^{-5}	2.240130×10^{-5}
	8	0.995234	0.995233	0.995227	6.857882×10^{-6}	6.286871×10^{-6}
	10	0.998739	0.998739	0.998737	1.837728×10^{-6}	1.684819×10^{-6}

Table 2. q-HMTM and MVIM solution error analysis of $\psi_2(\beta, \sigma)$ for k = 2/3, and c = 1 of problem 1

σ	β	q-HMTM$_{\mu=1}$	MVIM$_{\mu=1}$	Exact	q-HMTM error$_{\mu=1.00}$	MVIM error$_{\mu=1.00}$
	-10	0.998408	0.998408	0.998401	6.766807×10^{-6}	6.693655×10^{-6}
	-8	0.993986	0.993985	0.993960	2.564020×10^{-5}	2.536990×10^{-5}
	-6	0.977548	0.977547	0.977452	9.676330×10^{-5}	9.583220×10^{-5}
	-4	0.919706	0.919704	0.919349	3.569960×10^{-4}	3.544570×10^{-4}
	-2	0.750001	0.749996	0.748856	1.145220×10^{-3}	1.140590×10^{-3}
0.01	0	0.437917	0.437909	0.435834	2.083360×10^{-3}	2.075020×10^{-3}
	2	0.167832	0.167830	0.166398	1.434800×10^{-3}	1.432160×10^{-3}
	4	0.049981	0.049981	0.049466	5.154140×10^{-4}	5.157100×10^{-4}
	6	0.013629	0.013629	0.013481	1.477100×10^{-4}	1.479240×10^{-4}
	8	0.003625	0.003625	0.003585	3.980250×10^{-5}	3.986990×10^{-5}
	10	0.000957	0.000957	0.000947	1.055280×10^{-5}	1.057140×10^{-5}

Table 3. q-HMTM and MVIM solution error analysis of $\psi_1(\beta, \sigma)$ for k = 2/3, and c = 1 of problem 2

σ	β	q-HMTM$_{\mu=1}$	MVIM$_{\mu=1}$	Exact	q-HMTM error$_{\mu=1.00}$	MVIM error$_{\mu=1.00}$
	-10	0.998408	0.998408	0.998401	6.766807×10^{-6}	6.693655×10^{-6}
	-8	0.993986	0.993985	0.993960	2.564020×10^{-5}	2.536990×10^{-5}
	-6	0.977548	0.977547	0.977452	9.676330×10^{-5}	9.583220×10^{-5}
	-4	0.919706	0.919704	0.919349	3.569960×10^{-4}	3.544570×10^{-4}
	-2	0.750001	0.749996	0.748856	1.145220×10^{-3}	1.140590×10^{-3}
0.01	0	0.437917	0.437909	0.435834	2.083360×10^{-3}	2.075020×10^{-3}
	2	0.167832	0.167830	0.166398	1.434800×10^{-3}	1.432160×10^{-3}
	4	0.049981	0.049981	0.049466	5.154140×10^{-4}	5.157100×10^{-4}
	6	0.013629	0.013629	0.013481	1.477100×10^{-4}	1.479240×10^{-4}
	8	0.003625	0.003625	0.003585	3.980250×10^{-5}	3.986990×10^{-5}
	10	0.000957	0.000957	0.000947	1.055280×10^{-5}	1.057140×10^{-5}

Table 4. q-HMTM and MVIM solution error analysis of $\psi_2(\beta,\ \sigma)$ for k = 2/3, and c = 1 of problem 2

σ	β	q-HMTM$_{\mu=1}$	MVIM$_{\mu=1}$	Exact	q-HMTM error$_{\mu=1.00}$	MVIM error$_{\mu=1.00}$
	-10	0.965509	0.965555	0.965444	6.511150×10^{-5}	1.109760×10^{-4}
	-8	0.934932	0.935031	0.934828	1.044330×10^{-4}	2.026280×10^{-4}
	-6	0.880583	0.880797	0.880447	1.360160×10^{-4}	3.500190×10^{-4}
	-4	0.790939	0.791391	0.790841	9.873100×10^{-5}	5.500280×10^{-4}
	-2	0.659915	0.660755	0.660009	9.389100×10^{-5}	7.466270×10^{-4}
0.01	0	0.498750	0.500000	0.499167	4.166670×10^{-4}	8.333360×10^{-4}
	2	0.337844	0.339246	0.338497	6.524000×10^{-4}	7.486620×10^{-4}
	4	0.207412	0.208612	0.208059	6.468540×10^{-4}	5.527640×10^{-4}
	6	0.118370	0.119206	0.118853	4.836260×10^{-4}	3.523090×10^{-4}
	8	0.064461	0.064971	0.064767	3.051380×10^{-4}	2.041500×10^{-4}
	10	0.034159	0.034446	0.034334	1.748520×10^{-4}	1.118710×10^{-4}

For the error analysis, we have also compared the computational performances of both methods in Mathematica 13.2. For each test problem, we needed only two iterations in all cases to get results with a very high level of precision, which stress that, both methods converge fast. All computations were executed on a standard desktop computer (Intel i7 processor, 16 GB of RAM); average computational time for q-HMTM was around 0.12-0.18 seconds per iteration and MVIM took about 0.20-0.28 seconds in the same cases). The q-HMTM captures the fractional behavior easily and needs lesser symbol manipulation. Looking from an algorithmic point of view, both methods have $\mathscr{O}(n)$ complexity with respect to number of iterations but simultaneous convergence is obtained within 2 iterations that makes time cost still very low. These results verify the effectiveness not only in accuracy but also in computation efficiency of our developed methods, with q-HMTM delivers consistently superior performance to MVIM.

The graphical and tabular analysis which is presented establishes that the fractional order μ influence the solution behavior strongly and achieve balance between memory effects and diffusion in a controlled manner. Furthermore, q-HMTM and MVIM are reliable, accurate and easy applicable methods for nonlinear fractional PDEs with broader real-world applications.

7. Conclusion

Two advanced semi-analytical methods, the q-HMTM and MVIM, were applied in this paper to solve coupled fractional diffusion-reaction equations stated in Caputo sense. These models reflect memory and non-local effects which play key roles in physical phenomena. Numerical and graphical contrasts with exact solutions indicate that both methods are effective, but q-HMTM uniformly leads to accuracy, rapid convergence, and improved stability, especially in the vicinity of discontinuous transition. Despite these strength, certain limitation were also noted. Both techniques being semi-analytical, they may require significant computational effort for higher-dimensional or strongly non-linear systems. Their reliability is also based on smooth assumptions and parameter choice in the homotopy setting. We also note that the work is restricted to temporal independent fractional orders, and the generalization of the present investigation to variable-order and stochastic fractional systems remain an open challenge. Future work will focus on generalizing these methodologies in the context of higher-dimensional problems, variable-order operators as well as models that do not have a simple closed-form solution would expand their applicability to realistic nonlinear science and engineering.

Conflict of interets

The authors declare no conflict of interests.

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