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Research Article

Exploring Adjacency Recognizable Colorings in Graphs: A New Variant of Adjacency Codes

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Abstract: Let G be a nontrivial connected graph and $c: V(G) \to \{1, 2, \ldots, k\}$ be a coloring of G, where adjacent vertices may be colored the same. For any vertex v of G, the adjacency code $\mathrm{ad}_c(v)$ of v with respect to c is defined as the ordered k-tuple $(a_1, a_2, \ldots, a_k)$, where a_i is the number of vertices adjacent to v that are colored i for $1 \le i \le k$. The coloring c is called adjacency recognizable if distinct vertices have distinct adjacency codes, and the adjacency recognition number of G, denoted $\mathrm{an}(G)$, is the minimum positive integer k for which G admits an adjacency recognizable k-coloring. This study examines the relationships between the recognition number and the adjacency recognition number of a graph, with the novelty that adjacency recognizable colorings are based only on the colors of neighboring vertices, excluding the vertex itself, and do not require proper colorings, leading to new behaviors and existence conditions. Our findings indicate that adjacency recognition numbers are not defined for certain graphs. In addition, the bounds of $\mathrm{an}(G)$ are presented in terms of the order of G, depending on its diameter.

***Keywords*:** adjacency recognizable coloring, adjacency code, adjacency recognition number

MSC: 05C15, 05C78

1. Introduction

Graph coloring plays a crucial role in distinguishing vertices within a graph, forming a foundational aspect of graph theory. The development and analysis of new coloring frameworks not only advance theoretical understanding but also support practical applications in areas such as complex network analysis, materials modeling, and computational methods. Previous research has demonstrated that vertex-distinguishing and neighborhood-based colorings can be applied to real-world problems such as wireless channel assignment, where transmitters need different frequencies to avoid interference, as shown by Huo et al. [1] and by Ma and Yang [2]; more recently, spectrum graph coloring has been used effectively for Wi-Fi channel assignment in real deployments [3]. In addition, Ramar and Swaminathan [4, 5] studied neighborhood distinguishing colorings in relation to network identification, coding, and fault detection. In this broader setting, the chromatic polynomial of a graph provides a fundamental tool for counting the number of proper colorings with a specified number of colors, ensuring that adjacent vertices are assigned different colors [6].

One of the central challenges in graph theory is to distinguish vertices of a connected graph in a meaningful way. To this end, various coloring strategies have been proposed. Among them, proper edge coloring stands out for its ability to

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distinguish vertices based on the colors of their incident edges. Hornák and Šoták [7] studied the notion of proper edge coloring, formally defined as a function $c: E(G) \to \{1, 2, \ldots, k\}$ for a graph G and integer $k > 0$, with the requirement that incident edges obtain distinct colors. They proposed that for any two distinct vertices u and v, the sets of colors assigned to the edges incident to u and v are distinct. This type of coloring, later known as a *vertex distinguishing proper edge coloring* by Burris and Schelp [8], has been further studied in [1, 9–12]. These investigations have deepened our understanding of how edge colorings can be employed to uniquely identify graph vertices, leading to new theoretical insights and potential algorithmic applications.

In related work, Esperet et al. [13] examined the notion of proper vertex coloring, formally given by a function $c: V(G) \to \{1, 2, \ldots, k\}$ where k is a positive integer, in which adjacent vertices receive different colors. They showed that for any pair of adjacent vertices u and v, the sets of colors assigned to the closed neighborhoods of u and v differ whenever these neighborhoods are distinct. This coloring is referred to as a *locally identifying coloring* of a graph G. This concept has been further explored in several studies, including [14–16].

According to Chartrand et al. [17], vertices in a graph can be distinguished using unique color codes, which are determined not only by the color of the vertex itself but also by the number of neighboring vertices in each color class. This approach represented a significant contribution to the investigation of vertex colorings where neighboring vertices are allowed to receive identical colors.

Let G be a connected graph with at least two vertices. Suppose $c: V(G) \to \{1, 2, \ldots, k\}$ is a vertex coloring of G for some integer $k > 0$, where adjacent vertices are allowed to have the same color. Unless otherwise specified, the term "coloring" in this paper refers to a vertex coloring whenever the meaning is unambiguous. For a given coloring c, the *color code* of a vertex v, denoted by $\text{code}_c(v)$, is defined as the ordered $(k+1)$-tuple $\text{code}_c(v) = (a_0, a_1, a_2, \ldots, a_k)$, where a_0 is the color assigned to v (that is, $c(v) = a_0$) and for $1 \le i \le k$, a_i is the number of vertices adjacent to v that are colored i. Note that $\sum_{i=1}^{k} a_i = \deg_G(v)$. A coloring c is *recognizable* provided that no two distinct vertices share the same color code. For a graph G, the *recognition number* $\text{rn}(G)$ is the least positive integer k for which G has a recognizable k-coloring. Whenever a connected graph is colored so that every vertex has a different color, the coloring is recognizable, and consequently, the recognition number is well defined. The investigation of recognizable colorings was further extended to cycles and trees in the work of Dorfling et al. [18].

Building upon previous studies on recognizable colorings and recognition numbers of graphs, this study adapts the traditional notion of a vertex color code in a graph G by omitting the color of the vertex v itself (represented by a_0). We refer to this as the *adjacency code* of v. For a coloring $c: V(G) \to \{1, 2, \ldots, k\}$ of a graph G, where adjacent vertices may share the same color, the *adjacency code* associated with vertex v under coloring c, denoted $\text{ad}_c(v)$, is defined as the ordered k-tuple $\text{ad}_c(v) = (a_1, a_2, \ldots, a_k)$, in which a_i represents the number of vertices adjacent to v that receive color i, for $1 \le i \le k$. In some instances, we denote $\text{ad}_c(v) = (a_1^v, a_2^v, \ldots, a_k^v)$ or simply $\text{ad}(v) = (a_1^v, a_2^v, \ldots, a_k^v)$.

A coloring c is considered *adjacency recognizable* provided that no two distinct vertices share the same adjacency code. A graph G is said to have *adjacency recognition number* $\text{an}(G)$ given by the minimum k for which G can be assigned an adjacency recognizable k-coloring. Under the framework of proper vertex coloring—where adjacent vertices are given distinct colors—Ramar and Swaminathan [4, 5] examined neighborhood distinguishing colorings in graphs. These works focused on distinguishing vertices based on the k-tuples of colors in their neighborhoods, offering a different, yet related perspective on adjacency recognizable k-colorings of graphs.

The motivation and importance of studying adjacency recognizable colorings arise from the challenge of distinguishing vertices using only neighborhood information. Adjacency recognizable colorings distinguish vertices through adjacency codes without requiring proper colorings, which makes them different from earlier approaches. This leads to new behaviors and existence conditions, and introduces a new invariant that enriches graph coloring theory while also offering potential applications in network identification, frequency assignment, and error detection.

To illustrate the concept of adjacency recognizable k-colorings, consider the graph G on four vertices, where $V(G) = \{u_1, u_2, u_3, u_4\}$. Define the coloring $c_1: V(G) \to \{1, 2, 3\}$ by $c_1(u_1) = 1$, $c_1(u_2) = 1$, $c_1(u_3) = 2$, and $c_1(u_4) = 3$, as shown in Figure 1. Under the coloring c_1, the vertices of G have adjacency codes $\text{ad}_{c_1}(u_1) = (1, 1, 0)$, $\text{ad}_{c_1}(u_2) = (1, 1, 0)$, $\text{ad}_{c_1}(u_3) = (2, 0, 1)$, and $\text{ad}_{c_1}(u_4) = (0, 1, 0)$. Since $\text{ad}_{c_1}(u_1) = \text{ad}_{c_1}(u_2)$, c_1 is not an adjacency recognizable 3-coloring of G.

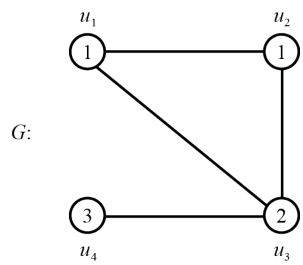

Figure 1. The graph G having order 4 with coloring c_1

As an alternative, define $c_2 : V(G) \to \{1, 2, 3, 4\}$ such that $c_2(u_i) = i$ for all $1 \le i \le 4$, as presented in Figure 2. The adjacency codes of the vertices of G are $\text{ad}_{c_2}(u_1) = (0, 1, 1, 0)$, $\text{ad}_{c_2}(u_2) = (1, 0, 1, 0)$, $\text{ad}_{c_2}(u_3) = (1, 1, 0, 1)$, and $\text{ad}_{c_2}(u_4) = (0, 0, 1, 0)$. Since no two vertices have the same adjacency code, c_2 is an adjacency recognizable 4-coloring of G. Consequently, $\text{an}(G) \le 4$.

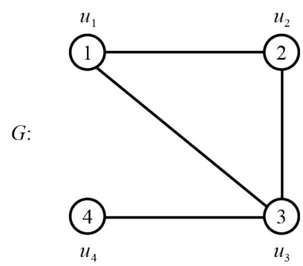

Figure 2. The graph G having order 4 with coloring c_2

However, it turns out that there is an adjacency recognizable 2-coloring c defined by $c(u_1) = 1$, $c(u_2) = 2$, $c(u_3) = 1$, and $c(u_4) = 1$, as shown in Figure 3. The adjacency codes of the vertices of G are $\text{ad}_c(u_1) = (1, 1)$, $\text{ad}_c(u_2) = (2, 0)$, $\text{ad}_c(u_3) = (2, 1)$, and $\text{ad}_c(u_4) = (1, 0)$. Thus $\text{an}(G) \le 2$. Moreover, there is no adjacency recognizable 1-coloring of G. Therefore, $\text{an}(G) = 2$.

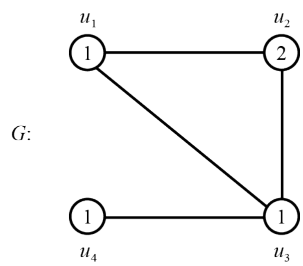

Figure 3. The graph G having order 4 with coloring c

The following result establishes the relationship between an adjacency recognizable coloring and a recognizable coloring of a graph.

Theorem 1 If c is an adjacency recognizable coloring of G, then c is also a recognizable coloring of G.

However, the converse is not true, as illustrated by the complete bipartite graph $K_{r,s}$, where $1 \le r \le s$, $s \ge 2$, and $\text{rn}(K_{r,s})$ is defined (see [17]). We claim that the adjacency recognition number of $K_{r,s}$ is not defined. In order to demonstrate this, assume that c serves as an adjacency recognizable coloring of $K_{r,s}$. Let $U = \{u_1, u_2, \ldots, u_r\}$ and $W = \{w_1, w_2, \ldots, w_s\}$ be the two partite sets of $K_{r,s}$. For all pairs of $1 \leq i < j \leq r$, since the neighborhoods of u_i and u_j are the same, specifically $N(u_i) = N(u_j) = \{w_1, w_2, \ldots, w_s\}$, it follows that the adjacency codes of u_i and u_j are also identical, which is impossible. Furthermore, from Theorem 1, the next corollary is immediate.

Corollary 1 If G is a connected graph for which $\mathrm{an}(G)$ is defined, then $\mathrm{rn}(G) \leq \mathrm{an}(G)$.

2. Definitions and notations

For completeness and better understanding, we summarize the fundamental ideas of graph theory and introduce the definitions and notations employed in this work. Common graphs such as complete graphs, cycles, cubic graphs, and complete bipartite graphs are included, together with specialized notions related to recognizable colorings and adjacency recognizable colorings. These preliminaries will be used often, so it is important to keep them in mind as we go. For graph-theoretical notation and terminology not explicitly defined in this paper, we refer the reader to the book [19].

Definition 1 (Graph) A *graph* is denoted by $G = (V, E)$, where $V(G)$ is the finite nonempty set of objects called *vertices* and $E(G)$ is the set of unordered pairs of distinct vertices called *edges*.

Definition 2 (Vertex Coloring) A *vertex coloring* of a graph G refers to a function

$$c : V(G) \to \{1, 2, \ldots, k\},$$

without the restriction that adjacent vertices must be colored differently.

Definition 3 (Color Code) For a vertex $v \in V(G)$, the *color code* with respect to coloring c is the ordered $(k+1)$-tuple

$$\mathrm{code}_c(v) = (a_0, a_1, \ldots, a_k),$$

where $a_0 = c(v)$ and a_i is the number of neighbors of v colored i for $1 \leq i \leq k$.

Definition 4 (Recognizable Coloring and Recognition Number) A coloring c is said to be a *recognizable coloring* when every pair of distinct vertices in G receives different color codes. The *recognition number* of G, written as $\mathrm{rn}(G)$, is the least integer k for which G possesses a recognizable k-coloring.

Definition 5 (Adjacency Code) For a vertex $v \in V(G)$, the *adjacency code* with respect to coloring c is the ordered k-tuple

$$\mathrm{ad}_c(v) = (a_1, a_2, \ldots, a_k),$$

where a_i is the number of neighbors of v colored i for $1 \leq i \leq k$.

Definition 6 (Adjacency Recognizable Coloring and Adjacency Recognition Number) A coloring c is said to be an *adjacency recognizable coloring* whenever every pair of distinct vertices of G yields different adjacency codes. The *adjacency recognition number* of G, denoted $\mathrm{an}(G)$, is the least integer k for which G possesses an adjacency recognizable k-coloring.

Definition 7 (Neighborhoods and Twins [20]) For $u \in V(G)$, the *neighborhood* of u is $N(u) = \{v \in V(G) : uv \in E(G)\}$ and the *closed neighborhood* is $N[u] = N(u) \cup \{u\}$. Two vertices u and v are *false twins* if $N(u) = N(v)$, and *true twins* if $N[u] = N[v]$.

Definition 8 (Complete Graph) The *complete graph* on n vertices, denoted K_n, is the graph in which every pair of distinct vertices is adjacent.

Definition 9 (Cycle) The *cycle* on n vertices, denoted C_n, is the graph in which vertices can be arranged in a cycle such that each vertex is adjacent to exactly two others.

Definition 10 (Cubic Graph [19]) A *cubic graph* is a graph in which every vertex has degree three.

Definition 11 (Bipartite Graph) A graph G is called *bipartite* if its vertex set $V(G)$ can be divided into two disjoint subsets U and W such that every edge of G connects a vertex from U to a vertex from W.

Definition 12 (Complete Bipartite Graph) The *complete bipartite graph* with partite sets of size r and s, denoted $K_{r,\,s}$, is the bipartite graph in which every vertex of the first partite set is adjacent to every vertex of the second partite set.

Notation 1

$V(G)$: The vertex set of a graph G.

$E(G)$: The edge set of a graph G.

$\deg_G(v)$: The degree of a vertex v in G.

$N(u) = \{v \in V(G) : uv \in E(G)\}$: The open neighborhood of vertex u.

$N[u] = N(u) \cup \{u\}$: The closed neighborhood of vertex u.

$\text{code}_c(v)$: The color code of vertex v with respect to a coloring c.

$\text{rn}(G)$: The recognition number of G.

$\text{ad}_c(v)$: The adjacency code of vertex v with respect to a coloring c.

$\text{an}(G)$: The adjacency recognition number of G.

K_n: The complete graph on n vertices.

C_n: The cycle on n vertices.

$K_{r,\,s}$: The complete bipartite graph with partite sets of size r and s.

3. The existence of adjacency recognizable colorings of graphs

In this section, we investigate conditions for the existence of adjacency recognizable colorings of graphs. The following results will be useful in our analysis.

Lemma 1 For any coloring c of a graph G, if u and v are two vertices of G with $\deg_G(u) \neq \deg_G(v)$ then

$$\text{ad}_c(u) \neq \text{ad}_c(v).$$

Proof. Let c be a k-coloring of a graph G. By contrapositive, we assume that u and v are two vertices such that $\text{ad}_c(u) = \text{ad}_c(v)$. This implies that $(a_1^u, a_2^u, \ldots, a_k^u) = (a_1^v, a_2^v, \ldots, a_k^v)$ and so $a_i^u = a_i^v$ for all i where $1 \leq i \leq k$. Consequently, $\sum_{i=1}^k a_i^u = \sum_{i=1}^k a_i^v$. Since for any vertex w of G, $\sum_{i=1}^k a_i^w = \deg_G(w)$, the result $\deg_G(u) = \deg_G(v)$ follows. □

Lemma 1 asserts that in order to show a coloring c of a graph G is adjacency recognizable, it suffices to check that every pair of vertices of equal degree has distinct adjacency codes.

Theorem 2 For any adjacency recognizable coloring c of a graph G, if u and v are true twins in G, then

$$c(u) \neq c(v).$$

Proof. Let $c : V(G) \to \{1, 2, \ldots, \ell\}$ be an adjacency recognizable ℓ-coloring of G. Suppose that $u, v \in V(G)$ are true twins, meaning that their closed neighborhoods are identical, i.e., $N[u] = N[v]$. By the definition of adjacency code, for each color i with $1 \leq i \leq \ell$, we record the number of neighbors of a vertex that receive color i. Denote these by

$$\text{ad}(u) = (a_1^u, a_2^u, \ldots, a_\ell^u), \quad \text{ad}(v) = (a_1^v, a_2^v, \ldots, a_\ell^v).$$

Assume, to the contrary, that the coloring c gives the same color to both vertices u and v. Given that u and v possess exactly the same neighbors and both are colored alike, it holds that for each color i, the number of u's neighbors colored i is the same as the number of v's neighbors colored i. Thus, we obtain

$$a_i^u = a_i^v \quad \text{for all } 1 \le i \le \ell,$$

which implies that $\mathrm{ad}(u) = \mathrm{ad}(v)$. This means that u and v cannot be distinguished by their adjacency codes, contradicting the assumption that c is adjacency recognizable. Therefore, our initial assumption is false, and we must have that vertex u and vertex v are assigned different colors. □

In particular, for any two vertices u and v of the complete graph K_n with $n \ge 2$, they are true twins. By Theorem 2, any adjacency recognizable coloring of K_n must assign a different color to each vertex. Thus, $\mathrm{an}(K_n) = n$ for $n \ge 2$.

The following result gives a characterization, in terms of necessary and sufficient conditions, for the existence of the adjacency recognition number $\mathrm{an}(G)$, relating it to the absence of false twins.

Theorem 3 Let G be a connected graph. Then the adjacency recognition number $\mathrm{an}(G)$ exists if and only if G contains no false twins.

Proof. First, suppose that the adjacency recognition number of G exists and $\mathrm{an}(G) = \ell$. For contradiction, assume that in the graph G there exist vertices u and v with the same neighborhood, namely $N(u) = N(v)$. Let $c : V(G) \to \{1, 2, \ldots, \ell\}$ be an adjacency recognizable ℓ-coloring of G. For $\mathrm{ad}(u) = (a_1^u, a_2^u, \ldots, a_\ell^u)$ and $\mathrm{ad}(v) = (a_1^v, a_2^v, \ldots, a_\ell^v)$, since u and v are not adjacent and $N(u) = N(v)$, the number of vertices colored i for $1 \le i \le \ell$ that are adjacent to u and v is the same. Thus, $a_i^u = a_i^v$ for all $1 \le i \le \ell$, implying that $\mathrm{ad}(u) = \mathrm{ad}(v)$. Consequently, G cannot have false twins.

For the converse, suppose that G contains no false twins, with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$. We define a coloring $c : V(G) \to \{1, 2, \ldots, n\}$ by $c(v_i) = i$ such that each vertex v_i is assigned color i for every $1 \le i \le n$.

We claim that c is an adjacency recognizable n-coloring of G. Let v_i and v_j be two distinct vertices of G such that $1 \le i < j \le n$.

If v_i and v_j are neighbors in G, then

$$\mathrm{ad}(v_i) = (\ldots, a_i^{v_i} = 0, \ldots, a_j^{v_i} = 1, \ldots),$$

$$\mathrm{ad}(v_j) = (\ldots, a_i^{v_j} = 1, \ldots, a_j^{v_j} = 0, \ldots).$$

So $\mathrm{ad}(v_i) \ne \mathrm{ad}(v_j)$.

In contrast, if v_i and v_j are not neighbors in G, then, without loss of generality, there exists a vertex v_k that is adjacent to v_i but not to v_j. Thus, $a_k^{v_i} = 1 \ne 0 = a_k^{v_j}$, implying that $\mathrm{ad}(v_i) \ne \mathrm{ad}(v_j)$. Hence, as claimed, c is an adjacency recognizable n-coloring of G. Thus, the adjacency recognition number of G exists and $\mathrm{an}(G) \le n$. □

4. The number of unique adjacency codes

Understanding the number of unique adjacency codes for a vertex based on its degree is fundamental in analyzing adjacency recognizable colorings of graphs. The following theorem determines this number, providing a combinatorial foundation for studying adjacency codes in graphs.

Theorem 4 Let c be a k-coloring of a nontrivial connected graph G. The number of different possible adjacency codes of a vertex of degree r in G is

$$\binom{r+k-1}{r}.$$

Proof. Let c be a k-coloring of the vertices of G. For a vertex v of degree r in G, $\text{ad}_c(v) = (a_1, a_2, \ldots, a_k)$ where a_i is the number of vertices adjacent to v that are colored i for $1 \le i \le k$. We observe that $\sum_{i=1}^{k} a_i = r$. Then the number of different possible adjacency codes $(a_1, a_2, \ldots, a_k)$ is $\binom{r+k-1}{r}$. □

The following corollary is an immediate consequence of Theorem 4.

Corollary 2 For a nontrivial connected graph G, if c is an adjacency recognizable k-coloring, then the number of vertices of degree r is bounded above by

$$\binom{r+k-1}{r}.$$

The following results describe the relationship between the order and the adjacency recognition number for both connected cubic graphs and cycles, as derived from Corollary 2.

Proposition 1 For a connected cubic graph G of order n, if the adjacency recognition number of G is k, then it follows that

$$n \le \frac{k^3 + 3k^2 + 2k}{6}.$$

Proof. Let G denote a connected cubic graph having n vertices and $\text{an}(G) = k$. Consider a vertex $v \in V(G)$, the degree of v is 3. According to Corollary 2, we obtain $n \le \binom{3+k-1}{3}$ or $n \le \frac{k^3+3k^2+2k}{6}$. □

In the contrapositive form of Proposition 1, one considers any connected cubic graph G of order n such that $n \ge \frac{k^3+3k^2+2k}{6} + 1$. Then $\text{an}(G) \ge k+1$, which is useful for determining the adjacency recognition number of a connected cubic graph. However, the converse does not necessarily hold. For instance, the complete graph K_4, which is cubic of order 4, satisfies $4 = \frac{2^3+3(2^2)+2(2)}{6}$, yet $\text{an}(K_4) = 4 > 2$.

Next, we examine the relationship between the order and the adjacency recognition number of a cycle C_n, where $n \ge 3$.

Proposition 2 For a cycle C_n, if $\text{an}(C_n) = k$, then

$$n \le \frac{k^2+k}{2}.$$

The contrapositive of Proposition 2 is likewise useful for estimating the adjacency recognition number of a cycle.

5. Bounds on the adjacency recognition number

As established earlier, whenever the adjacency recognition number $\text{an}(G)$ exists, a connected graph G with all vertices colored distinctly always produces an adjacency recognizable coloring. Hence, $\text{an}(G) \le n$.

Moreover, it is a classical result that any nontrivial graph must contain at least two vertices of equal degree. If the same color is assigned to all vertices, then any two vertices with the same degree will have identical adjacency codes. Hence, for a nontrivial connected graph G of order n, if its adjacency recognition number $\mathrm{an}(G)$ is defined, then

$$2 \leq \mathrm{an}(G) \leq n.$$

Theorem 5 Suppose G is a connected graph of order n that contains no false twins and $\mathrm{diam}(G) \geq 5$. Then

$$\mathrm{an}(G) \leq n-1.$$

Proof. Let G be a connected graph of order n such that G contains no false twins and $\mathrm{diam}(G) = \ell \geq 5$. Suppose that $V(G) = \{u_0, u_1, u_2, \ldots, u_{n-1}\}$. Without loss of generality, let $P = (u_0, u_1, u_2, \ldots, u_{\ell-1}, u_\ell)$ be a $u_0 - u_\ell$ geodesic of length ℓ. Define an $(n-1)$-coloring c by

$$c(u_i) = \begin{cases} \ell, & \text{if } i = 0, \\ i, & \text{if } 1 \leq i \leq n-1. \end{cases}$$

We claim that c is an adjacency recognizable $(n-1)$-coloring of G. Assume, to the contrary, that there are two vertices x and y such that

$$\mathrm{ad}(x) = (a_1^x, a_2^x, \ldots, a_{n-1}^x) = (a_1^y, a_2^y, \ldots, a_{n-1}^y) = \mathrm{ad}(y).$$

The proof proceeds by analyzing two cases.

Case 1. Vertices x and y are adjacent in G.

Then $\{x, y\} \neq \{u_0, u_\ell\}$ and $c(x) \neq c(y)$. Let $x = u_i$ and $y = u_j$ where $0 \leq i < j \leq n-1$ and $\{i, j\} \neq \{0, \ell\}$. Thus $a_j^x = 0 \neq 1 = a_j^y$, which is a contradiction.

Case 2. Vertices x and y are not adjacent in G. We examine three subcases.

Subcase 2.1. $a_\ell^x = a_\ell^y = 2$.

Then both x and y are adjacent to u_0 and u_ℓ, which implies that G contains a $u_0 - u_\ell$ path (u_0, x, u_ℓ) of length 2, contradicting $\mathrm{diam}(G) = \ell \geq 5$.

Subcase 2.2. $a_\ell^x = a_\ell^y = 0$.

Then neither x nor y is adjacent to u_0 and u_ℓ. Since $\mathrm{ad}(x) = \mathrm{ad}(y)$, it implies that $N(x) = N(y)$. Thus x and y are false twins, which is not possible.

Subcase 2.3. $a_\ell^x = a_\ell^y = 1$.

If both x and y are adjacent to u_0 or both to u_ℓ, then $N(x) = N(y)$, contradicting the no false twins condition. Without loss of generality, suppose $xu_0, yu_\ell \in E(G)$ and $xu_\ell, yu_0 \notin E(G)$. Then $N(x) - \{u_0\} = N(y) - \{u_\ell\}$.

If $N(x) - \{u_0\} = N(y) - \{u_\ell\} \neq \emptyset$, then there is u_k, where $1 \leq k \leq n-1$ and $k \neq \ell$ such that $xu_k, yu_k \in E(G)$. As a consequence, the graph G contains a $u_0 - u_\ell$ path (u_0, x, u_k, y, u_ℓ) of length 4, contradicting $\mathrm{diam}(G) = \ell \geq 5$. On the other hand, if $N(x) - \{u_0\} = N(y) - \{u_\ell\} = \emptyset$, then x and y are end vertices of G. Thus there is an $x - y$ geodesic $(x, u_0, u_1, u_2, \ldots, u_\ell, y)$ of length $d(x, y) = \ell + 2$, which again contradicts the definition of $\mathrm{diam}(G)$.

Since all possible cases lead to a contradiction, c must be an adjacency recognizable $(n-1)$-coloring of G. Therefore, $\mathrm{an}(G) \leq n-1$. □

However, there exists a connected graph G with $\mathrm{diam}(G) < 5$ such that $\mathrm{an}(G) = n-1$ as we show below.

Proposition 3 For an odd integer n where $n \geq 5$,

$$\mathrm{an}\left(\left\lfloor \frac{n}{2} \right\rfloor K_2 + K_1\right) = n-1.$$

Proof. Let $G = \left\lfloor \frac{n}{2} \right\rfloor K_2 + K_1$, where the vertex set of $\left\lfloor \frac{n}{2} \right\rfloor K_2$ is $\{v_1, v_2, \ldots, v_{n-1}\}$ and $v_i v_{i+1} \in E\left(\left\lfloor \frac{n}{2} \right\rfloor K_2\right)$ for every odd integer i, $1 \leq i \leq n-2$, and the vertex set of K_1 is $\{u\}$. Additionally, the vertex u in K_1 is adjacent to all vertices in $V\left(\left\lfloor \frac{n}{2} \right\rfloor K_2\right)$.

First, we demonstrate that $\mathrm{an}(G)$ does not exceed $n-1$. Let c be an $(n-1)$-coloring which assigns color 1 to vertex u, and for each $1 \leq j \leq n-1$, assign color j to vertex v_j. Thus

$$\mathrm{ad}(u) = (a_1^u = 1,\ a_2^u = 1,\ a_3^u = 1,\ \ldots, 1, 1)$$

$$\mathrm{ad}(v_1) = (a_1^{v_1} = 1,\ a_2^{v_1} = 1,\ a_3^{v_1} = 0,\ \ldots, 0, 0)$$

$$\mathrm{ad}(v_2) = (a_1^{v_2} = 2,\ a_2^{v_2} = 0,\ a_3^{v_2} = 0,\ \ldots, 0, 0)$$

For an odd integer i where $3 \leq i \leq n-2$,

$$\mathrm{ad}(v_i) = (1, 0, \ldots, 0,\ a_i^{v_i} = 0,\ a_{i+1}^{v_i} = 1,\ 0, \ldots, 0)$$

$$\mathrm{ad}(v_{i+1}) = (1, 0, \ldots, 0,\ a_i^{v_{i+1}} = 1,\ a_{i+1}^{v_{i+1}} = 0,\ 0, \ldots, 0)$$

Given that no two vertices share the same adjacency code, one concludes that c is an adjacency recognizable $(n-1)$-coloring of G. Thus $\mathrm{an}(G) \leq n-1$.

Next, we show that $\mathrm{an}(G) \geq n-1$. Suppose, to the contrary, that there exists an adjacency recognizable ℓ-coloring c' of G where $l \leq n-2$. As a result, there are two vertices v_x and v_y, distinct from each other, such that $c'(v_x) = c'(v_y)$ for some $1 \leq x < y \leq n-1$. We claim that $v_x v_y \notin E(G)$. Assume, to the contrary, that this statement does not hold. Then v_x and v_y are true twins and so by Theorem 2, $c'(v_x) \neq c'(v_y)$, which is a contradiction. Therefore there are v_w and v_z such that $v_x v_w \in E(G)$ and $v_y v_z \in E(G)$ where $w \neq z$. Hence $N(v_w) = \{u, v_x\}$ and $N(v_z) = \{u, v_y\}$. Without loss of generality, suppose that $c'(u) = 1$. If $c'(u) = c'(v_x) = c'(v_y)$, then

$$\mathrm{ad}(v_w) = (a_1^{v_w} = 2, a_2^{v_w} = 0, a_3^{v_w} = 0, \ldots, a_\ell^{v_w} = 0)$$

$$\mathrm{ad}(v_z) = (a_1^{v_z} = 2, a_2^{v_z} = 0, a_3^{v_z} = 0, \ldots, a_\ell^{v_z} = 0)$$

Thus $\mathrm{ad}(v_w) = \mathrm{ad}(v_z)$, which is a contradiction.

Now, if $c'(u) \neq c'(v_x) = c'(v_y)$, then

$$\mathrm{ad}(v_w) = (a_1^{v_w} = 1, 0, \ldots, 0, a_{c'(v_x)}^{v_w} = 1, 0, \ldots, 0)$$

$$\mathrm{ad}(v_z) = (a_1^{v_z} = 1, 0, \ldots, 0, a_{c'(v_x)}^{v_z} = 1, 0, \ldots, 0)$$

Thus $\mathrm{ad}(v_w) = \mathrm{ad}(v_z)$, which is a contradiction. As a consequence, c' is not an adjacency recognizable ℓ-coloring of G. Hence $\mathrm{an}(G) \geq n-1$. Therefore, $\mathrm{an}(G) = n-1$. $\square$

6. Open question

We have argued that if $G = K_n$, then $\mathrm{an}(G) = n$. Furthermore, if G is a nontrivial connected graph without false twins and $G \neq K_n$, then $\mathrm{diam}(G) \geq 2$. According to Theorem 5, if $\mathrm{diam}(G) \geq 5$, then $\mathrm{an}(G) \leq n-1$. Extending this result to graphs where $2 \leq \mathrm{diam}(G) \leq 4$ could conclusively prove that $\mathrm{an}(G) = n$ if and only if $G = K_n$.

7. Conclusion

In this paper, we introduce and investigate the concept of adjacency recognizable colorings in graphs, which extends the traditional framework of recognizable colorings by considering adjacency codes that exclude the color of a vertex itself. We provide necessary and sufficient conditions for the existence of the adjacency recognition number. In particular, we established that for connected graphs without false twins and with sufficiently large diameter, the adjacency recognition number is strictly less than the order of the graph. In addition, explicit examples were constructed to demonstrate that this bound does not always hold for graphs with smaller diameters. Adjacency recognizable colorings thus provide both new theoretical insights and practical tools for applications in communication networks, including frequency assignment, network identification, and error detection. Future directions include characterizing graph classes where the adjacency recognition number equals the graph order, as well as exploring algorithmic implications and applications within theoretical computer science.

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Conflict of interest

The authors declare no competing financial interest.

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