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Research Article

Attractive Solutions for Hilfer-Katugampola Fuzzy Fractional Neutral Differential Equations

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Abstract: This study investigates the existence and local asymptotic stability of solutions for fuzzy fractional neutral differential equations involving the Hilfer-Katugampola fractional derivative. The existence of mild solutions is established by employing techniques from fractional calculus, semigroup theory, the Laplace transform, and Krasnoselskii's fixed point theorem. Furthermore, the local asymptotic stability of the attractive solution is analyzed. An illustrative example is provided to demonstrate the applicability and effectiveness of the theoretical results.

Keywords: Hilfer-Katugampola fractional derivative, fixed point theory, attractivity, asymptotic stability

MSC: 34A08, 26A33, 93B05

1. Introduction

In recent years, fractional calculus and fractional differential equations have become central topics in mathematics. The framework of fractional calculus has proven applicable across diverse domains, including social sciences, physics, signal and image processing, biology, control theory, and engineering. Fractional differential equations provide effective tools for modeling a wide range of phenomena and often offer more accurate representations than classical integer-order models. Furthermore, the generalization of differential equations through differential inclusions, closely related to control theory and it is valuable for analyzing dynamical systems where the evolution is not solely determined by the current state. Numerous studies have explored the existence of solutions for fractional differential inclusions and systems. Various works highlighted the theoretical foundations and practical applications of fractional calculus [1–4], emphasizing the ability of fractional differential equations to capture complex dynamics in real-world systems. Maya-Franco et al. [5] investigated fractional differential equations within the fractional Schrödinger framework to model quantum systems with memory effects, while Santana-Carrillo et al. [6] used them to analyze entropy measures in quantum potentials, illustrating the impact of fractional-order dynamics. Additional studies support both the theory and applications of fractional calculus [7–11].

Katugampola [12] introduced a class of fractional operators known as the Katugampola fractional integral and derivative, which are characterized by an additional parameter $\rho > 0$. As ρ approaches 0_+, these operators tend toward the Hadamard fractional operators, while setting $\rho = 0$ recovers the Riemann-Liouville fractional operators. This parameter-dependent structure offers a unified framework, allowing results established for the Katugampola operators, which can be

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apply for both Riemann-Liouville and Hadamard cases simultaneously. Owing to their versatility, these operators have gained attention and it has been widely studied in recent research.

Hilfer [13] introduced the Hilfer fractional derivative, which bridges the Riemann-Liouville and Caputo derivatives. Gu and Trujillo [14] examined the existence of mild solutions for evolution equations involving the Hilfer fractional derivative, contributing to the advancement of fractional calculus and its relevance in modeling key physical and engineering phenomena. Nandhaprasadh and Udhayakumar [15] studied the controllability results for Hilfer fractional Sobolev-type stochastic differential systems with infinite delay. Balachandran et al. [16] introduced a general type of new Hadamard fractional integrals and derivatives with respect to another function and investigated some of their properties. In a related development, Oliveira and De Oliveira [17] introduced a new fractional derivative that integrates the features of both the Hilfer and Katugampola derivatives.

Fuzzy fractional differential equations combine fractional calculus and fuzzy set theory to model systems with memory effects and uncertainties. They have been widely applied in engineering, physics, biology, and control systems. Recent studies on fuzzy fractional differential equations have focused on modeling and controlling such systems under uncertainty. For instance, Boulkroune et al. [18] investigated output-feedback control for projective lag-synchronization of uncertain chaotic systems, while Zouari et al. [19] studied robust neural adaptive control for uncertain nonlinear multivariable systems. These studies demonstrate the effectiveness of advanced control strategies in managing complex dynamics under uncertain conditions.

Bede and Stefanini [20] explored the concept of differentiability for functions that can take fuzzy values. The authors extended the classical notions of differentiability to accommodate fuzzy-valued functions, providing a comprehensive framework for their analysis. Fuzzy numbers, first proposed by Chang and Zadeh [21], serve as an essential tool for representing uncertainty in mathematical models. Agarwal et al. [22] were the first to introduce the concept of fuzzy fractional differential equations, which has been a subject of extensive exploration and practical applications, covering theories and solution methodologies.

Arshad [23] focused on the properties of fuzzy fractional differential equations and analyzed their existence and uniqueness through fuzzy integral equivalent equations. Allahviranloo et al. [24] expanded this field by exploring fuzzy fractional differential equations via the Caputo fractional gH-derivative, focusing on their existence and uniqueness. Recently, Zhang et al. [25] investigated and established the controllability of Sobolev-type fuzzy Hilfer fractional integro-differential inclusions with Clarke subdifferential type.

Abbas and Benchohra [26] investigated the existence and attractivity of solutions for fractional-order integral equations within the framework of Fréchet spaces. Abbas et al. [27] focused on establishing existence and attractivity results for solutions to Hilfer fractional differential equations.

Chen et al. [28] investigated the Hilfer-Katugampola fuzzy fractional differential equation with nonlocal conditions. Hariharan and Udhayakumar [29] explored the existence of mild solutions for fuzzy fractional differential equations via the Hilfer-Katugampola fractional derivative with the following initial condition.

Based on the discussion above, this paper investigates the existence of attractive solutions for fuzzy fractional neutral differential equations involving the Hilfer-Katugampola fractional derivative, subject to the following initial condition:

$$\begin{cases} {}^{\rho}D_{0^+}^{\alpha,\,\beta}\left[x(t)-Q(t,\,x(t))\right]=Ax(t)+f(t,\,x(t)), & t\in[0,\,\infty)=U \\ {}^{\rho}I_{0^+}^{1-\gamma}x(0)=x_0, \qquad \gamma=\alpha+\beta(1-\alpha), & \end{cases} \tag{1}$$

where ${}^{\rho}D_{0_+}^{\alpha,\,\beta}$ denotes the Hilfer-Katugampola fractional derivative of order $\alpha\in(0,\,1)$ and type $\beta\in[0,\,1]$, with an additional parameter $\rho>0$. The state variable $x(\cdot)$ takes values in the space of all fuzzy numbers F, where a fuzzy number is defined as a fuzzy set $x:\mathbb{R}\to[0,\,1]$. The operator A is the infinitesimal generator of a compact operator semigroup $J(t)t\geq 0$, and $f:U\times F\to F$ is a fuzzy-valued function. Moreover, ${}^{\rho}I0^{+^{1-\gamma}}$ denotes the Hilfer-Katugampola fractional integral of order $1-\gamma$.

Additionally, the remainder of the paper is organized as follows: Section 2 presents the core concepts relevant to the present investigation. Section 3 discusses the existence and attractive results, while Section 4 provides an illustrative example to enhance understanding. Finally, Section 5 concludes the paper.

2. Preliminaries

Definition 2.1 [30] A fuzzy number is a fuzzy set on $\mathbb{R}$ that satisfies properties such as convexity, normality, and upper semicontinuity. A common way to define a distance between fuzzy numbers $\widetilde{\mathfrak{A}}$ and $\widetilde{\mathfrak{B}}$ is using a norm-based metric, such as:

$$d(\widetilde{\mathfrak{A}}, \widetilde{\mathfrak{B}}) = \|\widetilde{\mathfrak{A}} - \widetilde{\mathfrak{B}}\|,$$

where $\|\cdot\|$ is a norm that measures differences between fuzzy numbers.

Definition 2.2 [25] In a fuzzy number space, the Hausdorff distance can be defined similarly to classical sets:

$$d_H(\widetilde{\mathfrak{A}}, \widetilde{\mathfrak{B}}) = \max\left\{\sup_{\mathfrak{g}\in\widetilde{\mathfrak{A}}}\inf_{\varsigma\in\widetilde{\mathfrak{B}}}\|\mathfrak{g}-\varsigma\|,\ \sup_{\varsigma\in\widetilde{\mathfrak{B}}}\inf_{\mathfrak{g}\in\widetilde{\mathfrak{A}}}\|\varsigma-\mathfrak{g}\|\right\}.$$

where $\mathfrak{g}$, ς represent elements in the support of the fuzzy numbers, and $\|\varsigma - \mathfrak{g}\|$ is the norm-based distance between them.

Let us consider the function $g : N \to F$ (where $N = [0, \mathfrak{g}]$ and $\mathfrak{g} > 0$) with the supremum norm

$$\|g\|_\infty = \sup_{t\in\mathfrak{g}} |g(t)|.$$

A space is defined by

$$AC^1(N) = \left\{x : N \to F : \frac{dx}{dt} \in AC(N)\right\},$$

where $AC^1(N)$ denotes the set of absolutely continuous functions from N into F. A mapping $g : N \to F$ with a given norm, and $\mathscr{L}^1$ denotes the space of Lebesgue integrable functions, defined as

$$\|g\|_1 = \int_0^{\mathfrak{g}} |g(t)|dt.$$

We define the weighted spaces of continuous functions $\mathbb{C}_\beta(N)$ and $\mathbb{C}^1_\beta(N)$ as follows:

$$\mathbb{C}_\beta(N) = \left\{x : (0, \mathfrak{g}] \to F\colon \left(\frac{t^\rho}{\rho}\right)^{1-\gamma} x(t) \in \mathbb{C}\right\}$$

with the norm

$$\|x\|_{\mathbb{C}_\beta} = \sup_{t\in N}\left|\left(\frac{t^\rho}{\rho}\right)^{1-\gamma} x(t)\right|$$

and weighted space we define as

$$\mathbb{C}^1_\beta(N) = \left\{x \in \mathbb{C} : \frac{dx}{dt} \in \mathbb{C}_\beta\right\}$$

with the norm x

$$\|x\|_{\mathbb{C}^1_\beta} = \|x\|_\infty + \|\mathfrak{h}'\|_{\mathbb{C}_\beta}$$

$$FC_\beta = FC_\beta(U),$$

where FC denotes the space of all continuous and bounded fuzzy-valued functions from U into F, and the weighted space FC_β is defined by

$$FC_\beta = \left\{x : (0, +\infty) \to F : \left(\frac{t^\rho}{\rho}\right)^{1-\gamma} x(t) \in FC\right\}$$

with the norm

$$\|x\|_{FC_\beta} = \sup_{t\in\mathfrak{g}}\left|\left(\frac{t^\rho}{\rho}\right)^{1-\gamma} x(t)\right|.$$

Definition 2.3 [31] The katugampola fractional integral of order m with the lower limit u of $x \in FC_\beta$ for $-\infty < 0 < t < \infty$ is defined by

$$^\rho I^\alpha_{0^+} x(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^t \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{1-\alpha}} x(s)ds,$$

where $t > b$, $\rho > 0$ and $\alpha > 0$.

The Katugampola fractional integral is defined with respect to an additional parameter $\rho > 0$. These operators have special properties based on the value of ρ.

Note 2.1 Specifically, as $\rho \to 0^+$, the Katugampola fractional integral converge to the Hadamard fractional integral,

$$\lim_{\rho\to 0} I^{\alpha,\rho}_{0^+} x(t) = \frac{1}{\Gamma(\alpha)}\int_0^t \left(\log\frac{t}{s}\right)^{\alpha-1} x(s)\frac{ds}{s},$$

when the parameter $\rho = 1$, they coincide with the Riemann-Liouville fractional integral,

$$I_{0^+}^{\alpha,\,1}x(t)=\frac{1}{\Gamma(\alpha)}\int_0^t\frac{x(s)}{(t-s)^{1-\alpha}}ds.$$

Definition 2.4 [13] Let the order α and type β satisfy $\alpha\in(0,\,1]$ and $\beta\in[0,\,1]$. The fuzzy Hilfer-Katugampola fractional derivative with respect to t, for $\rho>0$, of a function $t\in\mathscr{C}_{1-\gamma,\,\rho}[0,\,u]$, is defined by

$$^{\rho}D_{0^+}^{\alpha,\,\beta}x(t)=\left(^{\rho}I_{0^+}^{\beta(1-\alpha)}s^{\rho-1}\frac{d}{ds}\,^{\rho}I_{0^+}^{(1-\alpha)(1-\beta)}\right)x(t)$$

$$=\left(^{\rho}I_{0^+}^{\beta(1-\alpha)}\gamma_{\rho}\,^{\rho}I_{0^+}^{(1-\alpha)(1-\beta)}\right)x(t).$$

Definition 2.5 [32] A function $f:U\times F\to F$ is said to satisfy the Carathéodory conditions if the following properties hold:

• The map $t\to f(t,\,x)$ is measurable for $x\in FC_{\beta}$;

• The map $x\to f(t,\,x)$ is continuous for each $t\in U$.

Lemma 2.1 [10] Assume that the linear operator A acts as the infinitesimal generator of a semigroup of class C_0 if and only if

• The set A has the property of being closed and $D(A)=\mathbb{B}$.

• The resolvent set $p(A)$ of A includes positive real numbers, $\forall\,\beta>0$,

$$\|R(\mathfrak{g},\,A)\|\leq\frac{1}{\mathfrak{g}},$$

where

$$R(\mathfrak{g},\,A)=(\mathfrak{g}^{q}I-A)^{-1}s=\int_0^{\infty}e^{-\mathfrak{g}^{q}t}J(t)sdt.$$

Lemma 2.2 The Hilfer-Katugampola fuzzy fractional differential equation (1) can be equivalently expressed in the form of the following integral equation:

$$x(t)=\frac{x_0-Q(0,\,x(0))}{\Gamma(\gamma)}\left(\frac{t^{\rho}}{\rho}\right)^{\gamma-1}+\frac{1}{\Gamma(\beta)}\int_0^t\left(\frac{t^{\rho}-s^{\rho}}{\rho}\right)^{\beta-1}s^{\rho-1}f(s,\,x(s))ds+Q(t,\,x(t))$$

$$+\frac{1}{\Gamma(\beta)}\int_0^t\left(\frac{t^{\rho}-s^{\rho}}{\rho}\right)^{\beta-1}s^{\rho-1}AQ(s,\,x(s))ds. \tag{2}$$

Note 2.2 [33] The wright function $\mathscr{W}_x(\delta)$ is defind by,

$$\mathscr{W}_{\beta}(\delta)=\sum_{\nu=1}^{\infty}\frac{(-\delta)^{\nu-1}}{(\nu-1)!\,\beta(1-\beta\nu)},$$

where $\beta \in (0, 1)$, $\delta \in [0, +\infty)$. Which satisfies the equality given below

$$\int_0^{\infty} \delta^s \mathscr{W}_\beta(\delta) d\delta = \frac{\Gamma(1+s)}{(1+\beta s)}, \quad s \geq 0,$$

where $\mathscr{W}_\beta(\delta)$ is a probability density function and is given as follows:

$$\mathscr{W}_\beta(\delta) = \frac{1}{\beta} \delta^{-1-\frac{1}{\beta}} \mathfrak{g}_\beta \left(\delta^{-\frac{1}{\beta}}\right) \leq 0,$$

for $0 < \beta < 1$, $\delta \in [0, +\infty)$.

Definition 2.6 A mild solution of the Hilfer-Katugampola fuzzy fractional differential equation (1) is defined as a function $x \in FC_\beta$.

$$x(t) = S_{\alpha,\beta}\left(\frac{t^\rho}{\rho}\right)[x_0 - Q(0, x(0))] + \int_0^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\beta-1} s^{\rho-1} K_\beta \left(\frac{t^\rho - s^\rho}{\rho}\right) f(s, x(s)) ds + Q(t, x(t))$$

$$+ \int_0^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\beta-1} s^{\rho-1} K_\beta \left(\frac{t^\rho - s^\rho}{\rho}\right) AQ(s, x(s)) ds. \tag{3}$$

where

$$S_{\alpha,\beta}\left(\frac{t^\rho}{\rho}\right) = I_{0_+}^{1+\rho(\gamma-1)-\beta} \left(\frac{t^\rho}{\rho}\right)^{\beta-1} K_\beta \left(\frac{t^\rho}{\rho}\right), \quad K_\beta \left(\frac{t^\rho}{\rho}\right) = \int_0^{\infty} \beta \delta \mathscr{W}_\beta(\delta) J\left(\left(\frac{t^\rho}{\rho}\right)^\beta \delta\right) d\delta.$$

Definition 2.7 [34] As for operator $S_{\alpha,\beta}\left(\frac{t^\rho}{\rho}\right)$ and $K_\beta\left(\frac{t^\rho}{\rho}\right)$.

- $\left\{S_{\alpha,\beta}\left(\frac{t^\rho}{\rho}\right)\right\}_{t>0}$ and $\left\{K_\beta\left(\frac{t^\rho}{\rho}\right)\right\}_{t>0}$ are linear, bounded and compact operator's, thus we have

$$\left|S_{\alpha,\beta}\left(\frac{t^\rho}{\rho}\right)x\right| \leq \frac{Lt^{\rho(\gamma-1)}\rho^{1-\gamma}\Gamma(\rho(\gamma-1)+1)}{(\Gamma(\gamma))^2}|x|, \quad \left|K_\beta\left(\frac{t^\rho}{\rho}\right)x\right| \leq \frac{L}{\Gamma(\beta)}|x|, \quad \forall\, \beta \in \mathbb{B}.$$

- $\left\{S_{\alpha,\beta}\left(\frac{t^\rho}{\rho}\right)\right\}_{t>0}$ and $\left\{K_\beta\left(\frac{t^\rho}{\rho}\right)\right\}_{t>0}$ are strongly continuous operators, i.e., $\forall t_1, t_2 \in U$, we have

$$\left|S_{\alpha,\beta}\left(\frac{t_2^\rho}{\rho}\right)x - S_{\alpha,\beta}\left(\frac{t_1^\rho}{\rho}\right)x\right| \to 0, \quad \left|K_\beta\left(\frac{t_2^\rho}{\rho}\right)x - K_\beta\left(\frac{t_1^\rho}{\rho}\right)x\right| \to 0, \text{ as } t_2^\rho \to t_1^\rho.$$

Let $\Xi : \Psi \to \Psi$ and $\Psi \subset FC_\beta$ (where, Ψ is non-empty). Let the solution of the equation be

$$(\Xi x)(t) = x(t). \tag{4}$$

We initiate the following concepts of attractivity of the solutions for the equation (4).

Definition 2.8 [27] The equation (4) are Locally Asysmptotically Stable (LAS) or Locally Attractive (LA) in the space FC_β, there exist a ball $\mathscr{B}(x_0, i)$ in FC_β such that solution of the equation is $x = x(t)$ and $\mathfrak{h} = \mathfrak{h}(t)$ of the equation (4) belongs to $\mathscr{B}(x_0, i) \cup \Psi$ then the equation is

$$\lim_{t\to\infty}(x(t) - \mathfrak{h}(t)) = 0, \tag{5}$$

equation (5) is uniform with respect to the ball $\mathscr{B}(x_0, i) \cup \Psi$. Then the solution of the equation (4) is said to be uniformly LAS or uniformly LA.

Proposition 2.1 [35] Let $0 < \beta \leq 1$ and $\forall x \in D(A)$, $\exists\, G_\beta > 0$ such that

$$\left\| A^m K_\beta\left(\frac{t^\rho}{\rho}\right)x\right\| \leq \frac{\rho^{m\beta}\beta G_m \Gamma(2-m)}{(t^\rho)^{m\beta}\Gamma(1+\beta(1-m))}\|x\|, \quad 0 < t \leq b.$$

Theorem 2.1 [32] Consider the Banach space $\mathbb{B}$. If $M, N : D \to \mathbb{B}$, then D is a closed, bounded, and convex subset of a Banach space $\mathbb{B}$ such that

(i) $Mh + Ng \in \mathbb{B}\ \forall$ pair of $h, g \in D$,

(ii) M is contraction mapping,

(iii) N is compact and continuous,

then $M(h) + N(h) = h$ has a solution in D.

3. Existence and attractive results

The analysis that follows is based on the following hypotheses:

$(H1)$ The function $f : U \times F \to F$ satisfy Caratheodory conditions.

$(H2)$ Then there exists a mapping $\Phi_f : U \to U$, $x \in F$ such that:

$$\left|f(t, x(t))\right| \leq \Phi_f(t),\ \ \forall t \in U,$$

and

$$\lim_{t\to\infty}\left(\frac{t^\rho}{\rho}\right)^{1-\gamma}(I_0^\alpha \Phi_f)(t) = 0.$$

We define

$$\Delta = \left(\frac{t^\rho}{\rho}\right)^\beta \frac{\mathbb{B}(1, \beta)}{\Gamma(\beta)}, \quad \text{and} \quad \Phi_f^* = \sup_{t\in U}\left(\frac{t^\rho}{\rho}\right)^{1-\gamma}(\Phi_f)(t).$$

$(H3)$ Let $Q : U \times F \to F$ be a continuous function. Suppose there exist constants $m \in (0, 1)$ with $m\beta > \frac{1}{2}$ and $L_Q > 0$ such that $Q \in D(A^m)$ and, for all $x,\ \nu \in F$ and $t \in U$, the following condition holds:

$$|A^m Q(t,\ x(t))| \le L_Q\left[1 + \left(\frac{t^\rho}{\rho}\right)^{1-\gamma}|x(t)|\right], \quad (t,\ x) \in U \times F,$$

and

$$|A^m Q(t,\ x(t)) - A^m Q(t,\ \nu(t))| \le L_Q\left(\frac{t^\rho}{\rho}\right)^{1-\gamma}|x-\nu|, \quad (t,\ x),\ (t,\ \nu) \in U \times F.$$

For our convenience, let us consider

$$|A^{-m}| = L_A.$$

Theorem 3.1 Suppose that hypotheses $(H1)$-$(H3)$ are satisfied. Then, the Hilfer-Katugampola fuzzy fractional differential equation (1) admits an attractive solution on U.

Proof. We define $B_q = B(0,\ q) = \left\{x \in FC_\beta : \|x\|_{FC_\beta} = \sup_{t\in U}\left(\frac{t^\rho}{\rho}\right)^{1-\gamma}|x(t)| \le q\right\}$. It is clear that B_q is a closed, bounded, and convex subset of FC_β for all $q > 0$. We define the operator $\Xi : FC_\beta \to FC_\beta$ by

$$x(t) = S_{\alpha,\beta}\left(\frac{t^\rho}{\rho}\right)[x_0 - Q(0,\ x(0))] + \int_0^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\beta-1} s^{\rho-1}K_\beta\left(\frac{t^\rho - s^\rho}{\rho}\right) f(s,\ x(s))ds + Q(t,\ x(t))$$

$$+ \int_0^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\beta-1} s^{\rho-1}K_\beta\left(\frac{t^\rho - s^\rho}{\rho}\right) AQ(s,\ x(s))ds.$$

We split the operator Ξ into two operators, Ξ_1 and Ξ_2, on B_q, where

$$(\Xi_1 x)(t) = S_{\alpha,\beta}\left(\frac{t^\rho}{\rho}\right)[x_0 - Q(0,\ x(0))] + Q(t,\ x(t)) + \int_0^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\beta-1} s^{\rho-1}K_\beta\left(\frac{t^\rho - s^\rho}{\rho}\right) AQ(s,\ x(s))ds,$$

$$(\Xi_2 x)(t) = \int_0^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\beta-1} s^{\rho-1}K_\beta\left(\frac{t^\rho - s^\rho}{\rho}\right) f(s,\ x(s))ds, \quad t \in U.$$

To apply Theorem 2.1, we provide the following proof.

Case 1. The operator Ξ maps the set B_q into itself. Let $q > 0$. Then, for all $t \in U$, there exists

$$\left\|(\Xi x)(t)\right\|_{FC_\beta} > q, \tag{6}$$

we have

$$\left\|(\Xi x)(t)\right\|_{FC_\beta} = \sup_{t\in U}\left(\frac{t^\rho}{\rho}\right)^{1-\gamma}\Bigg|S_{\alpha,\beta}\left(\frac{t^\rho}{\rho}\right)[x_0 - Q(0, x(0))] + \int_0^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\beta-1} s^{\rho-1} K_\beta\left(\frac{t^\rho - s^\rho}{\rho}\right) f(s, x(s))ds$$

$$+ Q(t, x(t)) + \int_0^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\beta-1} s^{\rho-1} K_\beta\left(\frac{t^\rho - s^\rho}{\rho}\right) AQ(s, x(s))ds\Bigg|$$

$$\leq \frac{L\Gamma(\rho(\gamma-1)+1)}{(\Gamma(\gamma))^2}[x_0 - L_Q L_A] + L_Q L_A(1+q) + \frac{L\Phi_f^*}{\Gamma(\beta)}\left(\frac{t^\rho}{\rho}\right)^\beta \frac{\mathbb{B}(1, \beta)}{\Gamma(\beta)}$$

$$+ \left(\frac{t^\rho}{\rho}\right)^{1-\gamma+m\beta-\beta} \frac{\beta G_{1-m}\Gamma(1+m)}{\Gamma(1+m\beta)} L_Q[1+q]\frac{\mathbb{B}(1, \beta)}{\Gamma(\beta)}$$

$$\leq \frac{L\Gamma(\rho(\gamma-1)+1)}{(\Gamma(\gamma))^2}[x_0 - L_Q L_A] + L_Q L_A(1+q) + \frac{L\Phi_f^*}{\Gamma(\beta)}\Delta$$

$$+ \left(\frac{t^\rho}{\rho}\right)^{1-\gamma+m\beta-2\beta} \frac{\beta G_{1-m}\Gamma(1+m)}{\Gamma(1+m\beta)} \Delta L_Q[1+q].$$

Next, dividing both sides by q and taking the limit as $q \to \infty$, we arrive at a contradiction to our assumption (6). Therefore, Ξx is bounded.

Case 2. Ξ_1 is a contraction mapping. For all $x, \nu \in FC_\beta$, we have

$$\left\|\Xi_1 x(t) - \Xi_1 \nu(t)\right\|_{FC_\beta}$$

$$\leq \left|\left(\frac{t^\rho}{\rho}\right)^{1-\gamma} S_{\alpha,\beta}\left(\frac{t^\rho}{\rho}\right)[Q(0, x(0)) - Q(0, \nu(0))]\right| + \left|\left(\frac{t^\rho}{\rho}\right)^{1-\gamma}[Q(t, x(t)) - Q(t, \nu(t))]\right|$$

$$+ \left|\int_0^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\beta-1} s^{\rho-1} K_\beta\left(\frac{t^\rho - s^\rho}{\rho}\right) A\left[Q(s, x(s)) - Q(s, \nu(s))\right]ds\right|$$

$$\leq \frac{LL_A L_Q\Gamma(\rho(\gamma-1)+1)}{(\Gamma(\gamma))^2}|x-\nu| + \left(\frac{t^\rho}{\rho}\right)^{2-2\gamma} L_A L_Q|x-\nu| + \left(\frac{t^\rho}{\rho}\right)^{2-2\gamma+m\beta-2\beta} \frac{\beta G_{1-m}\Gamma(1+m)}{\Gamma(1+m\beta)}\Delta L_Q|x-\nu|$$

$$\leq \left[\frac{LL_A L_Q\Gamma(\rho(\gamma-1)+1)}{(\Gamma(\gamma))^2} + \left(\frac{t^\rho}{\rho}\right)^{2-2\gamma+m\beta-2\beta} \frac{\beta G_{1-m}\Gamma(1+m)}{\Gamma(1+m\beta)}\Delta L_Q + \left(\frac{t^\rho}{\rho}\right)^{2-2\gamma} L_A L_Q\right]|x-\nu|$$

$$\leq \mathbb{T}|x-\nu|.$$

Since $\mathbb{T} < 1$, it follows that Ξ_1 is a contraction mapping.

Case 3. We show that Ξ_2 is continuous. Let $\{x_n\}$ be a sequence in B_q satisfying $\lim_{n\to\infty} \|x_n - x\| = 0$. Using $(H1)$, it follows that

$$\left\|\Xi_2 x_n - \Xi_2 x\right\|_{FC_\beta} = \sup_{t\in U} \left(\frac{t^\rho}{\rho}\right)^{1-\gamma} \left| \int_0^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\beta-1} s^{\rho-1} K_\beta \left(\frac{t^\rho - s^\rho}{\rho}\right) f(s, x_n(s))ds \right.$$

$$\left. - \int_0^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\beta-1} s^{\rho-1} K_\beta \left(\frac{t^\rho - s^\rho}{\rho}\right) f(s, x(s))ds \right|$$

$$\leq \frac{L}{\Gamma(\beta)} \left(\frac{t^\rho}{\rho}\right)^{1-\gamma} \int_0^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\beta-1} s^{\rho-1} |f(s, x_n(s)) - f(s, x(s))| ds$$

$$\leq \frac{L\Delta}{\Gamma(\beta)} \left(\frac{t^\rho}{\rho}\right)^{1-\gamma} |f(\cdot, x_n(\cdot)) - f(\cdot, x(\cdot))|.$$

By the Lebesgue dominated convergence theorem, we obtain

$$\left\|\Xi_2 x_n - \Xi_2 x\right\|_{FC_\beta} \to 0 \text{ as } n \to \infty.$$

Thus, we have

$$\lim_{n\to\infty} \left\|\Xi_2 x_n - \Xi_2 x\right\|_{FC_\beta} = 0.$$

Therefore, Ξ_2 is continuous.

Case 4. We need to show that $\Xi_2(B_q)$ is equicontinuous. Let $t_1, t_2 \in U$ with $t_1 < t_2$. For each $x \in B_q$, we have

$$\|(\Xi_2 x)(t_2) - (\Xi_2 x)(t_1)\|_{FC_\beta}$$

$$= \sup_{t\in U} \left| \left(\frac{t_2^\rho}{\rho}\right)^{1-\gamma} \int_0^{t_2} \left(\frac{t_2^\rho - s^\rho}{\rho}\right)^{\beta-1} s^{\rho-1} K_\beta \left(\frac{t_2^\rho - s^\rho}{\rho}\right) f(s, x(s))ds \right.$$

$$\left. - \left(\frac{t_1^\rho}{\rho}\right)^{1-\gamma} \int_0^{t_1} \left(\frac{t_1^\rho - s^\rho}{\rho}\right)^{\beta-1} s^{\rho-1} K_\beta \left(\frac{t_1^\rho - s^\rho}{\rho}\right) f(s, x(s))ds \right|$$

$$\leq \left| \frac{L}{\Gamma(\beta)} \left[\left(\frac{t_2^\rho}{\rho}\right)^{1-\gamma} - \left(\frac{t_1^\rho}{\rho}\right)^{1-\gamma} \right] \int_0^{t_2} \left(\frac{t_2^\rho - s^\rho}{\rho}\right)^{\beta-1} s^{\rho-1} f(s, x(s))ds \right|$$

$$+\left|\frac{L}{\Gamma(\beta)}\left(\frac{t_1^\rho}{\rho}\right)^{1-\gamma}\int_{t_1}^{t_2}\left(\frac{t_2^\rho-s^\rho}{\rho}\right)^{\beta-1}s^{\rho-1}f(s,x(s))ds\right|$$

$$+\left|\frac{L}{\Gamma(\beta)}\left(\frac{t_1^\rho}{\rho}\right)^{1-\gamma}\int_0^{t_1}\left[\left(\frac{t_2^\rho-s^\rho}{\rho}\right)^{\beta-1}-\left(\frac{t_1^\rho-s^\rho}{\rho}\right)^{\beta-1}\right]s^{\rho-1}f(s,x(s))ds\right|$$

$$=I_1+I_2+I_3,$$

where

$$I_1=\left|\frac{L}{\Gamma(\beta)}\left[\left(\frac{t_2^\rho}{\rho}\right)^{1-\gamma}-\left(\frac{t_1^\rho}{\rho}\right)^{1-\gamma}\right]\int_0^{t_2}\left(\frac{t_2^\rho-s^\rho}{\rho}\right)^{\beta-1}s^{\rho-1}f(s,x(s))ds\right|,$$

$$I_2=\left|\frac{L}{\Gamma(\beta)}\left(\frac{t_1^\rho}{\rho}\right)^{1-\gamma}\int_{t_1}^{t_2}\left(\frac{t_2^\rho-s^\rho}{\rho}\right)^{\beta-1}s^{\rho-1}f(s,x(s))ds\right|,$$

$$I_3=\left|\frac{L}{\Gamma(\beta)}\left(\frac{t_1^\rho}{\rho}\right)^{1-\gamma}\int_0^{t_1}\left[\left(\frac{t_2^\rho-s^\rho}{\rho}\right)^{\beta-1}-\left(\frac{t_1^\rho-s^\rho}{\rho}\right)^{\beta-1}\right]s^{\rho-1}f(s,x(s))ds\right|.$$

By the hypothese $(H2)$, we have

$$I_1\leq\frac{L}{\Gamma(\beta)}\left[\left(\frac{t_2^\rho}{\rho}\right)^{1-\gamma}-\left(\frac{t_1^\rho}{\rho}\right)^{1-\gamma}\right]\int_0^{t_2}\left(\frac{t_2^\rho-s^\rho}{\rho}\right)^{\beta-1}s^{\rho-1}|f(s,x(s))|ds$$

$$\leq\frac{L\Upsilon^*r}{\Gamma(\beta)}\left[\left(\frac{t_2^\rho}{\rho}\right)^{1-\gamma}-\left(\frac{t_1^\rho}{\rho}\right)^{1-\gamma}\right]\int_0^{t_2}\left(\frac{t_2^\rho-s^\rho}{\rho}\right)^{\beta-1}s^{\rho-1}ds$$

$$\leq\frac{L\Upsilon^*r}{\Gamma(\beta)}\Delta\left[\left(\frac{t_2^\rho}{\rho}\right)^{1-\gamma}-\left(\frac{t_1^\rho}{\rho}\right)^{1-\gamma}\right]\to 0\quad\text{as}\quad t_2\to t_1$$

$$I_2\leq\frac{L}{\Gamma(\beta)}\left(\frac{t_1^\rho}{\rho}\right)^{1-\gamma}\int_{t_1}^{t_2}\left(\frac{t_2^\rho-s^\rho}{\rho}\right)^{\beta-1}s^{\rho-1}|f(s,x(s))|ds$$

$$\leq\frac{L\Upsilon^*r}{\Gamma(\beta)}\left(\frac{t_1^\rho}{\rho}\right)^{1-\gamma}\int_{t_1}^{t_2}\left(\frac{t_2^\rho-s^\rho}{\rho}\right)^{\beta-1}s^{\rho-1}ds$$

$$\leq\frac{L\Upsilon^*r\mathbf{B}(1,\beta)}{[\Gamma(\beta)]^2}\left(\frac{t_1^\rho}{\rho}\right)^{1-\gamma}\left(\frac{t_2^\rho-t_1^\rho}{\rho}\right)^{\beta}\to 0\quad\text{as}\quad t_2\to t_1$$

$$I_3 \leq \frac{L}{\Gamma(\beta)}\left(\frac{t_1^\rho}{\rho}\right)^{1-\gamma}\int_0^{t_1}\left[\left(\frac{t_2^\rho - s^\rho}{\rho}\right)^{\beta-1} - \left(\frac{t_1^\rho - s^\rho}{\rho}\right)^{\beta-1}\right]s^{\rho-1}|f(s, x(s))|ds$$

$$\leq \frac{L\Upsilon^* r}{\Gamma(\beta)}\left(\frac{t_1^\rho}{\rho}\right)^{1-\gamma}\int_0^{t_1}\left[\left(\frac{t_2^\rho - s^\rho}{\rho}\right)^{\beta-1} - \left(\frac{t_1^\rho - s^\rho}{\rho}\right)^{\beta-1}\right]s^{\rho-1}ds$$

$$\leq \frac{L\Upsilon^* r\mathbf{B}(1, \beta)}{[\Gamma(\beta)]^2}\left(\frac{t_1^\rho}{\rho}\right)^{1-\gamma}\left[\left(\frac{t_2^{\beta\rho} - t_1^{\beta\rho}}{\rho}\right) - \left(\frac{t_2^\rho - t_1^\rho}{\rho}\right)^{\beta}\right] \to 0 \quad \text{as } t_2 \to t_1$$

Combining the estimates of I_1, I_2, and I_3, we deduce that

$$\|(\Xi_2 x)(t_2) - (\Xi_2 x)(t_1)\|_{FC_\beta} \to 0 \quad \text{as } t_2 \to t_1.$$

Hence, Ξ_2 is equicontinuous.

Case 5. We need to show that, for any $t \in U$,

$$\mathscr{V}(t) = \{(\Xi_2 \widehat{x})(t) : \widehat{x} \in B_q\}$$

is relatively compact in FC_β. We define the operator $\Xi_2^{\mathfrak{h}, \mathfrak{g}}$ for all $\mathfrak{h}, \mathfrak{g} > 0$ on B_q by

$$(\Xi_2^{\mathfrak{h}, \mathfrak{g}}\widehat{x})(t) = \int_0^{t-\mathfrak{h}}\int_{\mathfrak{g}}^{\infty}\left(\frac{t^\rho - s^\rho}{\rho}\right)^{\beta-1}s^{\rho-1}\beta\delta\mathscr{W}_\beta(\delta)J\left(\left(\frac{t^\rho - s^\rho}{\rho}\right)^{\beta}\delta\right)f(s, \widehat{x}(s))d\delta ds$$

$$= \beta J\left[\left(\frac{\mathfrak{h}^\rho}{\rho}\right)^{\beta}\mathfrak{g}\right]\int_0^{t-\mathfrak{h}}\int_{\mathfrak{g}}^{\infty}\left(\frac{t^\rho - s^\rho}{\rho}\right)^{\beta-1}s^{\rho-1}\delta\mathscr{W}_\beta(\delta)J\left[\left(\frac{t^\rho - s^\rho}{\rho}\right)^{\beta}\delta - \left(\frac{\mathfrak{h}^\rho}{\rho}\right)^{\beta}\mathfrak{g}\right]f(s, \widehat{x}(s))d\delta ds.$$

Since $J\left[\left(\frac{\mathfrak{h}^\rho}{\rho}\right)^{\beta}\mathfrak{g}\right]$ is compact for $\left(\frac{\mathfrak{h}^\rho}{\rho}\right)^{\beta}\mathfrak{g} > 0$, then the set $(\mathscr{V}^{\mathfrak{h}, \mathfrak{g}}\widehat{x})(t) = \{(\Xi_2^{\mathfrak{h}, \mathfrak{g}}\widehat{x})(t) : \widehat{x} \in B_q\}$ is relatively compact in FC_β. For every $\mathfrak{h} \in (0, t)$, $\mathfrak{g} > 0$, and we get that

$$\left\|(\Xi_2\widehat{x})(t) - (\Xi_2^{\mathfrak{h}, \mathfrak{g}}\widehat{x})(t)\right\|_{FC_\beta}$$

$$\leq \sup_{t\in U}\left(\frac{t^\rho}{\rho}\right)^{1-\gamma}\left[\beta L_E\int_0^{t-\mathfrak{h}}\int_0^{\mathfrak{g}}\left(\frac{t^\rho - s^\rho}{\rho}\right)^{\beta-1}s^{\rho-1}\delta\mathscr{W}_\beta(\delta)\left|f(s, \widehat{x}(s))\right|d\delta ds\right.$$

$$\left. + \beta L_E\int_{t-\mathfrak{h}}^{t}\int_0^{\infty}\left(\frac{t^\rho - s^\rho}{\rho}\right)^{\beta-1}s^{\rho-1}\delta\mathscr{W}_\beta(\delta)\left|f(s, \widehat{x}(s))\right|d\delta ds\right]$$

$$\leq \sup_{t\in U}\left(\frac{t^{\rho}}{\rho}\right)^{1-\gamma}\frac{\beta L_E t^{\rho+\alpha\rho(\beta-1)}\Phi_f(t)}{t^{1+\alpha(\beta-1)}}\mathbb{B}_{(\frac{t-\mathfrak{g}}{t})^{\rho}}(1,\,\beta)\left[\frac{1}{\Gamma(1+\beta)}+\int_0^{\mathfrak{g}}\delta\mathscr{W}_\beta(\delta)d\delta\right]$$

$$\leq \frac{\beta L_E t^{\rho+\alpha\rho(\beta-1)}\Phi_f^*}{t^{1+\alpha(\beta-1)}}\mathbb{B}_{(\frac{t-\mathfrak{g}}{t})^{\rho}}(1,\,\beta)\left[\frac{1}{\Gamma(1+\beta)}+\int_0^{\mathfrak{g}}\delta\mathscr{W}_\beta(\delta)d\delta\right].$$

From the above boundedness condition, it follows that the right-hand side of the equation tends to 0. That is,

$$\left\|(\Xi_2\widehat{x})(t)-(\Xi_2^{\mathfrak{h},\,\mathfrak{g}}\widehat{x})(t)\right\|_{FC_\beta}\to 0$$

as $\mathfrak{h}$, $\mathfrak{g}\to 0$. Moreover, by Note 2.2, we obtain

$$\int_0^{\infty}\delta\mathscr{W}_\beta(\delta)\,d\delta=\frac{1}{\Gamma(1+\beta)}.$$

Thus, by the Arzelà-Ascoli theorem, we deduce that Ξ_2 is compact. Hence, $\Xi=\Xi_1+\Xi_2$ is relatively compact in F. By applying Krasnoselskii's fixed point Theorem 2.4, the operator Ξ has a fixed point x, which is a mild solution of the system (1) on U. □

Lemma 3.1 We prove local asymptotic stability. Assume that x_0 is a solution of the system (1) under the conditions specified in this theorem. By considering $x\to\mathscr{B}(x_0,\,2\mathbb{T}^*)$, we obtain

$$\begin{aligned}\|(\Xi x)(t)-(x_0)(t)\|_{FC_{\mathfrak{h}}}\leq&\left|\left(\frac{t^{\rho}}{\rho}\right)^{1-\gamma}S_{\alpha,\,\beta}\left(\frac{t^{\rho}}{\rho}\right)[Q(0,\,x(0))-Q(0,\,x_0(0))]\right|\\&+\left|\left(\frac{t^{\rho}}{\rho}\right)^{1-\gamma}[Q(t,\,x(t))-Q(t,\,x_0(t))]\right|\\&+\left|\left(\frac{t^{\rho}}{\rho}\right)^{1-\gamma}\int_0^t\left(\frac{t^{\rho}-s^{\rho}}{\rho}\right)^{\beta-1}s^{\rho-1}K_\beta\left(\frac{t^{\rho}-s^{\rho}}{\rho}\right)A\left[Q(s,\,x(s))-Q(s,\,x_0(s))\right]ds\right|\\&+\frac{L}{\Gamma(\beta)}\left(\frac{t^{\rho}}{\rho}\right)^{1-\gamma}\int_0^t\left(\frac{t^{\rho}-s^{\rho}}{\rho}\right)^{\beta-1}s^{\rho-1}|f(s,\,x(s))-f(s,\,x_0(s))|ds\\\leq&\frac{2LL_AL_Q\Gamma(\rho(\gamma-1)+1)|x|}{(\Gamma(\gamma))^2}+\left(\frac{t^{\rho}}{\rho}\right)^{2-2\gamma}2L_AL_Q|x|\\&+\frac{2L\Delta}{\Gamma(\beta)}\Phi_f^*+\left(\frac{t^{\rho}}{\rho}\right)^{2-2\gamma+m\beta-2\beta}\frac{\beta G_{1-m}2\Gamma(1+m)}{\Gamma(1+m\beta)}\Delta L_Q|x|\leq 2\mathbb{T}^*\end{aligned}$$

We get,

$$\|\Xi(x) - x_0\|_{FC_{\mathfrak{h}}} \leq 2\mathbb{T}^*.$$

We determine that Ξ is a continuous mapping such that

$$\Xi(\mathscr{B}(x_0, 2\mathbb{T}^*)) \subset \mathscr{B}(x_0, 2\mathbb{T}^*).$$

If x is a solution of equation (1), then

$$\begin{aligned}
|x(t) - x_0(t)| &= |(\Xi x)(t) - (\Xi x_0)(t)| \\
&\leq \left| S_{\alpha,\beta}\left(\frac{t^\rho}{\rho}\right)[Q(0, x(0)) - Q(0, x_0(0))] \right| \\
&\quad + \left| [Q(t, x(t)) - Q(t, x_0(t))] \right| \\
&\quad + \left| \int_0^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\beta-1} s^{\rho-1} K_\beta\left(\frac{t^\rho - s^\rho}{\rho}\right) A\left[Q(s, x(s)) - Q(s, x_0(s))\right] ds \right| \\
&\quad + \frac{L}{\Gamma(\beta)} \int_0^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\beta-1} s^{\rho-1} |f(s, x(s)) - f(s, x_0(s))| ds \\
&\leq \frac{L\rho^{1-\gamma}\Gamma(\rho(\gamma-1)+1)}{t^{\rho(1-\gamma)}(\Gamma(\gamma))^2} |Q(0, x(0)) - Q(0, x_0(0))| \\
&\quad + \left(\frac{\rho}{t^\rho}\right)^{\gamma-1} \frac{\left| Q(t, x(t)) - Q(t, x_0(t)) \right|}{\left(\frac{\rho}{t^\rho}\right)^{\gamma-1}} \\
&\quad + \left| \int_0^t \left(\frac{\rho}{t^\rho - s^\rho}\right)^{1-\beta} s^{\rho-1} K_\beta\left(\frac{t^\rho - s^\rho}{\rho}\right) A\left[Q(s, x(s)) - Q(s, x_0(s))\right] ds \right| \\
&\quad + \frac{L}{\Gamma(\beta)} \int_0^t \left(\frac{\rho}{t^\rho - s^\rho}\right)^{1-\beta} s^{\rho-1} |f(s, x(s)) - f(s, x_0(s))| ds.
\end{aligned}$$

Next, taking $\lim\limits_{t\to\infty}$ on both sides, we obtain

$$\lim_{t\to\infty}|x(t)-x_0(t)|=0.$$

Hence, the system (1) is uniformly locally attractive.

4. Example

Examine the following problem

$$\begin{cases} {}^{\rho}D_{0+}^{\frac{4}{5},\frac{1}{3}}x(t, y)\left[x(t, y)-\int_{-\infty}^{0}\mathfrak{h}(\theta, t)x(t, \theta)d\theta\right]=\Delta x(t, y)+\frac{1}{3}e^{-t}x(t, y), \quad t\in[0, \infty)=U_1 \\ x(t, 0)=x(t, \pi)=0, \\ {}^{\rho}I_{0+}^{\frac{2}{15}}x(0, y)=x_0(y), \quad x\in[0, \pi]. \end{cases} \tag{7}$$

Here, ${}^{\rho}D_{0+}^{\frac{4}{5},\frac{1}{3}}$ denotes the Hilfer-Katugampola fractional derivative of order $\alpha=\frac{4}{5}$ and type $\beta=\frac{1}{3}$, with $t\in(0, 1]=U_1$ and $\rho>0$. The function $x(t)(y)=x(t, y)$. The function $f:U_1\times F\to F$ is continuous, given by $f(t, x(t))$, and the function $Q:U\times FC_{\beta}\to F$ is defined as

$$f(t, x(t))(y)=\frac{1}{3}e^{-t}x(t, y),$$

$$Q(t, x(t))=\int_{-\infty}^{0}\mathfrak{h}(\theta, t)x(t, \theta)d\theta.$$

The linear operators $\Delta:\mathrm{Dom}(\Delta)\to\mathbb{B}$ is given by

$$D(\Delta)=\left\{x\in F:\frac{\partial^2 x}{\partial y^2}\in\mathbb{B} \text{ and } x(0, 0)=x(0, 1)=0, \text{ where } \Delta=\frac{\partial^2}{\partial y^2}\right\}.$$

Then, we get

$$\overline{D(\Delta)}=\{x\in F:x(t, 0)=x(t, 1)=0\}.$$

Therefore, Δ is bounded and generates a compact C_0-semigroup $\{T(t)\}_{t\geq 0}$ on $\overline{D(\Delta)}$.

Consequently, system (7) can be rewritten in the form of system (1). The function f satisfies the assumptions $(H1)$-$(H3)$, which implies that Theorem 3.1 is fully applicable. Therefore, the Hilfer-Katugampola fuzzy fractional differential equation (1) admits a mild solution on U_1.

5. Conclusion

This work studied the existence of attractive solutions for fuzzy fractional neutral differential equations involving the Hilfer-Katugampola fractional derivative. The existence of such solutions was established using Krasnoselskii's fixed point theorem. As a result, the system is shown to be locally asymptotically stable. To demonstrate the applicability of the theoretical findings, a detailed illustrative example was provided.

Authors contribution

Ramaraj Hariharan and Ramalingam Udhayakumar contributed equally to the research, conceptualization, and preparation of the article.

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Conflict of interest

The authors declare no competing financial interest.

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