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Research Article

On Spectrum of the Weakly Zero-Divisor Graph

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Abstract: Let us consider the finite commutative ring R, whose unity is $1 \neq 0$. The weakly zero-divisor graph, denoted by $W\Gamma(R)$, is an undirected graph whose distinct vertices c_1 and c_2 are adjacent if and only if, there exist $r \in \text{ann}(c_1)$ and $s \in \text{ann}(c_2)$ that satisfy the condition $rs = 0$. This article finds the Seidel Laplacian and Seidel signless Laplacian spectrum for the graph $W\Gamma(Z_n)$ for various values of n.

***Keywords*:** ring of integers modulo n, weakly zero-divisor graph, spectrum of graph, Seidel Laplacian and Seidel signless Laplacian spectrum

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1. Introduction

In this article, a commutative ring having identity $1 \neq 0$ shall be denoted by R. When an element c_2, different from zero ($0 \neq c_2 \in R$), exists such that $c_1c_2 = 0$, then the nonzero element c_1 is called a zero-divisor of R. $Z(R)$ is the collection of those zero-divisors in the ring R and $Z(R)^* = Z(R) \setminus \{0\}$.

The graph $G = (V, E)$ has been defined, where V denotes the set of vertices and E denotes the set of edges of G. When two distinct vertices of graph G, c_1 and c_2 are adjacent to each other in graph G, the notation $c_1 \sim c_2$ represents this. In a graph G, the set of vertices adjacent to a vertex c is called its *neighborhood*; this neighborhood is represented by the notation $N_G(c)$. K_m refers to the complete graph with m vertices, $deg(c)$, the degree of vertex c, represents the number of edges incident with $c \in V$. For every vertex $c \in G$, G is k-regular if $deg(c) = k$. Let $A_{k\times k}$ be any square matrix and let $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_k$ be its different eigenvalues with multiplicities of $f_1, f_2, f_3, \ldots, f_k$ respectively. The *spectrum* of A is then denoted by $\sigma(A)$, which is defined by

$$\sigma(A) = \begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \cdots & \lambda_k \\ f_1 & f_2 & f_3 & \cdots & f_k \end{Bmatrix}. \tag{1}$$

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Van Lint and Seidel [1] introduced the Seidel matrix of G, defined as $S(G) = [s_{ij}]$ where,

$$(s_{ij}) = \begin{cases} -1, & c_i \sim c_j \\ 1, & c_i \nsim c_j \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

The Seidel spectrum of G is denoted by $spec^S(G)$. Let $D_s(G) = diag(n-2d_1-1,\, n-2d_2-1,\, \ldots,\, n-2d_n-1)$ be the diagonal matrix where d_i is the degree of the vertex c_i. The Seidel Laplacian matrix [2] of a graph is defined as

$$SL(G) = D_s(G) - S(G). \tag{3}$$

And the Seidel signless Laplacian matrix of a graph is defined as

$$SL^+(G) = D_s(G) + S(G). \tag{4}$$

Nikmeher et al. [3] introduced the idea of a weakly zero-divisor graph of ring R. The weakly zero-divisor graph of ring R is represented by the symbol $W\Gamma(R)$. This undirected simple graph $W\Gamma(R)$ has a vertex set as set of non-zero zero-divisors of R. The two distinct vertices, c_1 and c_2, are adjacent if and only if $r \in \text{ann}(c_1)$ and $s \in \text{ann}(c_2)$ exist, satisfying the condition that $rs = 0$. The weakly zero-divisor graph's spanning sub-graph is easily observed to be the zero-divisor graph of a ring.

The Seidel Laplacian and Seidel signless Laplacian spectrum of the weakly zero-divisor graph Z_n of is found in this paper for various values of n. More information about spectrum of graphs based on different structure can be found in [4–8]. The definitions, lemmas, and theorems that used to support the main results are presented in Section 2. In section 3, we calculate the Seidel Laplacian spectrum of $W\Gamma(Z_n)$. In section 4, we find the Seidel signless Laplacian spectrum of the weakly zero-divisor graph $W\Gamma(Z_n)$, when n is the product of primes and their powers and also for $n = \Psi_1\Psi_2\ldots\Psi_t\eta_1^{d_1}\eta_2^{d_2}\ldots\eta_s^{d_s}(d_s \geq 2,\, t \geq 1,\, s \geq 0)$ where Ψ_i's and η_i''s are distinct primes.

2. Preliminaries

Definition 1 "Let $G(V, E)$ be a graph of order m having vertex set $\{c_1, c_2, \ldots, c_m\}$ and $F_k(V_k, E_k)$ be disjoint graphs of order m_k, $1 \leq k \leq m$. The graph $F_1, F_2, \ldots, F_m$ formed the generalized join graph $G[F_1, F_2, \ldots, F_m]$ and whenever c_k and c_l are adjacent in G, joined each vertex of F_k to every vertex of F_l, $1 \leq l, k \leq m$."

$\tau(j_1)$ indicates the number of positive divisors of a positive integer j_1. For j_2 to not divide j_1, we write $j_2 \nmid j_1$. The greatest common divisor of j_1 and j_2 is shown by (j_1, j_2). The number of positive integers smaller than or equal to j_1 that are relatively prime to j_1 is indicated by Eulur's phi function $\phi(j_1)$. If $j_1 = \Psi_1^{h_1}\Psi_2^{h_2}\ldots\Psi_k^{h_k}$, where $h_1, h_2, \ldots, h_k$ are positive integers and $\Psi_1, \Psi_2, \ldots, \Psi_k$ are distinct primes, then j_1 is in *prime decoposition*.

Let $j_1, j_2, \ldots, j_k$ be the proper divisors of n. For $1 \leq i \leq k$, consider the following sets

$$A_{j_i} = \{x \in \mathbb{Z}_n \colon (x, n) = j_i\}. \tag{5}$$

Moreover, observe that for $i \neq s$, $A_{j_i} \cap A_{j_s} = \emptyset$. As a result, the vertex set of $W\Gamma(Z_n)$ has a partition formed by the sets $A_{j_1}, A_{j_2}, \ldots, A_{j_k}$ i.e. $V(W\Gamma(Z_n)) = A_{j_1} \cup A_{j_2} \cup \cdots \cup A_{j_k}$, as a result. The following lemma provides information about the cardinality of each A_{j_i}.

Lemma 1 [9, Lemma 2.1] "Let j_i be the proper divisor of n then $|A_{j_i}| = \phi(\frac{n}{j_i})$ for $1 \leq i \leq k$".

Lemma 2 [8] Let n be represented as $n = l_1 l_2 \ldots l_m w_1{}^{s_1} w_2{}^{s_2} \ldots w_i{}^{s_i}$ where $l_i's$, $w_i's$ are distinct primes and $i \geq 0$, $s_i \geq 2$ and $m \geq 1$. Suppose, the set of divisors of n are $\{j_1, j_2, \ldots, j_k\}$. If $j_r \in \{l_1, l_2, \ldots, l_m\}$ then the induced subgraph of $W\Gamma(Z_n)$ by A_{j_r} is $\overline{K}_{\phi(\frac{n}{j_r})}$.

Corollary 1 [8] Let j_t be the proper divisor of positive integer n. The following assertions are true:

1. For $t \in \{1, 2, \ldots, k\}$, the induced subgraph $W\Gamma(A_{j_t})$ of $W\Gamma(Z_n)$, formed by the vertices in the set A_{j_t} is take two forms: either $\overline{K}_{\phi(\frac{n}{j_t})}$ or $K_{\phi(\frac{n}{j_t})}$.

2. For $t, q \in \{1, 2, \ldots, k\}$ and $t \neq q$, a vertex within A_{j_t} is connected to either all or none of the vertices in A_{j_q} in the graph $W\Gamma(Z_n)$.

The sub-graphs $W\Gamma(A_{j_t})$ created within the structure of $W\Gamma(Z_n)$ can be classified as either complete graphs or empty graphs, as shown by the previously noted Corollary 2.1. The graph $\delta_n^\star$ is created as a complete graph by utilizing the set of all proper divisors of n, represented by the notation $\{j_1, j_2, \ldots, j_s\}$.

Lemma 3 [8] $W\Gamma(Z_n) = \delta_n^*[W\Gamma(A_{j_1}), W\Gamma(A_{j_2}), \ldots, W\Gamma(A_{j_s})]$ where $j_1, j_2, \ldots, j_s$ are all the proper divisors of n.

3. Seidel Laplacian spectrum of the weakly zero-divisor graph

In this section, we will highlight the primary results of Seidel Laplacian spectrum of the weakly zero-divisor graph. For $r \in \{1, 2, \ldots, k\}$ the induced subgraph $W\Gamma(A_{j_r})$ of $W\Gamma(Z_n)$, formed by the vertices in the set A_{j_r} is either $\overline{K}_{\phi(\frac{n}{j_r})}$ or $K_{\phi(\frac{n}{j_r})}$. The Seidel Laplacian spectrum of complete graph K_l and its complement graph $\overline{K_l}$ on l vertices is given by

$$spec^{SL}(K_l) = \left\{ \begin{matrix} 0 & -l \\ 1 & l-1 \end{matrix} \right\} \text{ and } spec^{SL}(\overline{K_l}) = \left\{ \begin{matrix} 0 & l \\ 1 & l-1 \end{matrix} \right\} \text{respectively.} \tag{6}$$

The following theorem provides the generalized join graph's Seidel spectrum of regular graphs.

Theorem 1 [2] Consider $G[L_1, L_2, \ldots, L_k]$ where G is simple connected graph with vertices labeled as $1, 2, \ldots, k$ and $S = [s_{ij}]_{k\times k}$ is the Seidel matrix of G and L_j is r_j- regular and $|V(L_j)| = n_j$, for every $j = 1, 2, \ldots, k$. Let $\{\sigma_{j1}^{SL} = 0, \sigma_{j2}^{SL}, \ldots, \sigma_{jn_j}^{SL}\}$ be the Seidel Laplacian eigenvalues of L_j, for $j = 1, 2, \ldots, k$. Then, the Seidel Laplacian spectrum of the G-join of the graph $L_1, L_2, \ldots, L_k$ is given by,

$$spec^{SL}(G[L_1, L_2, \ldots, L_k]) = \left(\bigcup_{j=1}^{k} \bigcup_{i=2}^{n_j} (\sigma_{ji}^{SL} + \tau_j) \right) \bigcup spec(T_{SL}(G)), \tag{7}$$

where $\tau_j = \sum_{i=1}^{k} s_{ij} n_i$ and

$$T_{SL}(G) = \begin{bmatrix} \tau_1 & -s_{1,2}n_2 & \ldots & -s_{1,k}n_k \\ -s_{1,2}n_1 & \tau_2 & \ldots & -s_{2,k}n_k \\ \vdots & \vdots & \ddots & \vdots \\ -s_{1,k}n_1 & -s_{2,k}n_2 & \ldots & \tau_k \end{bmatrix}. \tag{8}$$

Lemma 4 Let n be the product of two different primes Ψ_1 and Ψ_2. The Seidel Laplacian spectrum of $W\Gamma(Z_n)$ is given by,

$$\begin{Bmatrix} \Psi_2-\Psi_1 & \Psi_1-\Psi_2 \\ \Psi_2-2 & \Psi_1-2 \end{Bmatrix}. \tag{9}$$

The remaining two Seidel Laplacian eigenvalues of the graph $W\Gamma(Z_n)$ are the eigenvalues of the matrix,

$$\begin{bmatrix} 1-\Psi_1 & \Psi_1-1 \\ \Psi_2-1 & 1-\Psi_2 \end{bmatrix}. \tag{10}$$

Proof. The proper divisors of n are Ψ_1, Ψ_2 and $\Psi_1<\Psi_2$. Also, by the definition of δ_n^*; $\Psi_1\sim\Psi_2$. Now by Lemma 3, we have $W\Gamma(Z_{\Psi_1\Psi_2})=\delta^*_{\Psi_1\Psi_2}[W\Gamma(A_{\Psi_1}),\, W\Gamma(A_{\Psi_2})]$. Therefore, by Lemma 1 and Corollary 1, we have $W\Gamma(A_{\Psi_1})=\overline{K}_{\phi(\Psi_2)}$ and $W\Gamma(A_{\Psi_2})=\overline{K}_{\phi(\Psi_1)}$. Consequently, in order of proper divisor sequence we have $n_1=\phi(\Psi_2)=\Psi_2-1,\ n_2=\phi(\Psi_1)=\Psi_1-1$, value of $\tau_1=1-\Psi_1$ and $\tau_2=1-\Psi_2$. Therefore, the Seidel Laplacian spectrum of the graph $W\Gamma(Z_{\Psi_1\Psi_2})$ is

$$\begin{Bmatrix} \Psi_2-\Psi_1 & \Psi_1-\Psi_2 \\ \Psi_2-2 & \Psi_1-2 \end{Bmatrix}. \tag{11}$$

And the characteristic polynomial of the matrix provided below, can be used to determine the remaining eigenvalues,

$$\begin{bmatrix} 1-\Psi_1 & \Psi_1-1 \\ \Psi_2-1 & 1-\Psi_2 \end{bmatrix}. \tag{12}$$

The matrix (12) has a characteristic polynomial $\lambda^2-\lambda(2-\Psi_1-\Psi_2)$. □

Example 1 The Seidel Laplacian spectrum of the graph $W\Gamma(Z_6)$ is given by,

$$\begin{Bmatrix} 1 & -1 \\ 1 & 0 \end{Bmatrix}. \tag{13}$$

The remaining two Seidel Laplacian eigenvalues of the graph $W\Gamma(Z_6)$ are the eigenvalues of the matrix,

$$\begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix}. \tag{14}$$

Proof. The proper divisors of 6 are 2, 3 and $2<3$. Also, by the definition of δ_6^*; $2\sim 3$. Now by Lemma 3, we have $W\Gamma(Z_6)=\delta_6^*[W\Gamma(A_2),\, W\Gamma(A_3)]$. Therefore, by Lemma 1 and Corollary 1, we have $W\Gamma(A_2)=\overline{K}_2$ and $W\Gamma(A_3)=\overline{K}_1$. Consequently, in order of proper divisor sequence we have $n_1=2,\ n_2=1$, value of $\tau_1=-1$ and $\tau_2=-2$. Therefore, the Seidel Laplacian spectrum of the graph $W\Gamma(Z_6)$ is

$$\left\{\begin{matrix} 1 & -1 \\ 1 & 0 \end{matrix}\right\}. \tag{15}$$

And the characteristic polynomial of the matrix provided below, can be used to determine the remaining eigenvalues,

$$\begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix}. \tag{16}$$

The matrix (16) has a characteristic polynomial $\lambda^2 + 3\lambda$. □

Lemma 5 For distinct primes Ψ_1 and Ψ_2, if $n = {\Psi_1}^2\Psi_2$ then, the Seidel Laplacian spectrum of $W\Gamma(Z_n)$ is given by

$$\left\{\begin{matrix} -E & 2(\Psi_1^2 - \Psi_1) - E \\ \Psi_1\Psi_2 - 4 & \Psi_1^2 - \Psi_1 - 1 \end{matrix}\right\}. \tag{17}$$

Where $E = \phi(\Psi_1\Psi_2) + \phi(\Psi_1^2) + \phi(\Psi_2) + \phi(\Psi_1)$. The remaining four Seidel Laplacian eigenvalues of the graph $W\Gamma(Z_n)$ are the eigenvalues of the matrix (19).

Proof. Let $n = {\Psi_1}^2\Psi_2$, where $\Psi_1 < \Psi_2$, note that $\delta^*_{{\Psi_1}^2\Psi_2}$ is complete graph on vertices $\{\Psi_1, \Psi_2, {\Psi_1}^2, \Psi_1\Psi_2\}$. By Lemma 3, we have $W\Gamma(Z_{{\Psi_1}^2\Psi_2}) = \delta^*_{{\Psi_1}^2\Psi_2}[W\Gamma(A_{\Psi_1}), W\Gamma(A_{\Psi_2}), W\Gamma(A_{{\Psi_1}^2}), W\Gamma(A_{\Psi_1\Psi_2})]$. Therefore, by Lemma 1 and Corollary 1, we have $W\Gamma(A_{\Psi_1}) = K_{\phi(\Psi_1\Psi_2)}$, $W\Gamma(A_{\Psi_2}) = \overline{K}_{\phi({\Psi_1}^2)}$, $W\Gamma(A_{{\Psi_1}^2}) = K_{\phi(\Psi_2)}$ and $W\Gamma(A_{\Psi_1\Psi_2}) = K_{\phi(\Psi_1)}$. Also $n_1 = \phi(\Psi_1\Psi_2)$, $n_2 = \phi({\Psi_1}^2)$, $n_3 = \phi(\Psi_2)$, $n_4 = \phi(\Psi_1)$. By using Theorem 1, the value of $\tau_i = n_i - E$ for $1 \leq i \leq 4$ where $E = \phi(\Psi_1\Psi_2) + \phi(\Psi_1^2) + \phi(\Psi_2) + \phi(\Psi_1)$. Therefore, by Theorem 1, the Seidel Laplacian spectrum of the graph $W\Gamma(Z_{{\Psi_1}^2\Psi_2})$ is

$$\left\{\begin{matrix} -E & 2(\Psi_1^2 - \Psi_1) - E \\ \Psi_1\Psi_2 - 4 & \Psi_1^2 - \Psi_1 - 1 \end{matrix}\right\}. \tag{18}$$

$E = \phi(\Psi_1\Psi_2) + \phi(\Psi_1^2) + \phi(\Psi_2) + \phi(\Psi_1)$. And the matrix given in (19), can be used to determine the remaining four eigenvalues.

$$\begin{bmatrix} \phi(\Psi_1\Psi_2) - E & \phi(\Psi_1^2) & \phi(\Psi_2) & \phi(\Psi_1) \\ \phi(\Psi_1\Psi_2) & \phi(\Psi_1^2) - E & \phi(\Psi_2) & \phi(\Psi_1) \\ \phi(\Psi_1\Psi_2) & \phi(\Psi_1^2) & \phi(\Psi_2) - E & \phi(\Psi_1) \\ \phi(\Psi_1\Psi_2) & \phi(\Psi_1^2) & \phi(\Psi_2) & \phi(\Psi_1) - E \end{bmatrix}. \tag{19}$$

□

Example 2 The Seidel Laplacian spectrum of the graph $W\Gamma(Z_{28})$, shown in Figure 1, is given by

$$\left\{\begin{matrix} -15 & -11 \\ 10 & 1 \end{matrix}\right\}. \tag{20}$$

The remaining four Seidel Laplacian eigenvalues of the graph $W\Gamma(Z_{28})$ are the eigenvalues of the matrix (22).

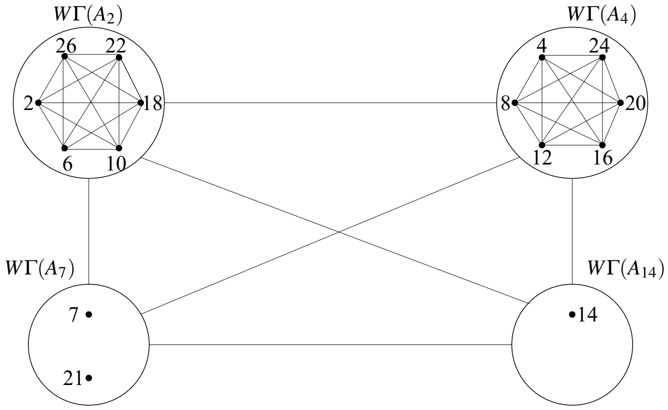

Figure 1. Weakly zero-divisor graph $W\Gamma(\mathbb{Z}_{28})$

Proof. Let $n = 28$, note that δ^*_{28} is complete graph on vertices $\{2, 7, 4, 14\}$. By Lemma 3, we have $W\Gamma(Z_{28}) = \delta^*_{28}[W\Gamma(A_2), W\Gamma(A_7), W\Gamma(A_4), W\Gamma(A_{14})]$. Therefore, by Lemma 1 and Corollary 1, we have $W\Gamma(A_2) = K_6$, $W\Gamma(A_7) = \overline{K}_2$, $W\Gamma(A_4) = K_6$ and $W\Gamma(A_{14}) = K_1$. Also $n_1 = 6$, $n_2 = 2$, $n_3 = 6$, $n_4 = 1$ and $E = 6+2+6+1 = 15$. By using Theorem 1, the value of $\tau_1 = -9$, $\tau_2 = -13$, $\tau_3 = -9$ and $\tau_4 = -14$. Therefore the Seidel Laplacian spectrum of the graph $W\Gamma(Z_{28})$ is given by

$$\left\{ \begin{array}{cc} -15 & -11 \\ 10 & 1 \end{array} \right\}. \tag{21}$$

Characteristic polynomial and the eigenvalues of the matrix (22) are respectively, $\lambda(\lambda^3 + 45\lambda^2 + 675\lambda + 3{,}375) = \lambda(\lambda + 15)^3$, and $\{-15, -15, -15, 0\}$.

$$M = \begin{bmatrix} -9 & 2 & 6 & 1 \\ 6 & -13 & 6 & 1 \\ 6 & 2 & -9 & 1 \\ 6 & 2 & 6 & -14 \end{bmatrix}. \tag{22}$$

□

Lemma 6 For distinct prime $\Psi_1, \Psi_2, \Psi_3,$ if $n = \Psi_1\Psi_2\Psi_3$ then, the Seidel Laplacian spectrum of the graph $W\Gamma(Z_n)$ is given by,

$$\left\{ \begin{array}{cccc} 2\phi(\Psi_2\Psi_3) - E & 2\phi(\Psi_1\Psi_3) - E & 2\phi(\Psi_1\Psi_2) - E & -E \\ \phi(\Psi_2\Psi_3) - 1 & \phi(\Psi_1\Psi_3) - 1 & \phi(\Psi_1\Psi_2) - 1 & \phi(\Psi_1) + \phi(\Psi_2) + \phi(\Psi_3) - 3 \end{array} \right\}, \tag{23}$$

where $E = \phi(\Psi_1) + \phi(\Psi_2) + \phi(\Psi_3) + \phi(\Psi_1\Psi_2) + \phi(\Psi_1\Psi_3) + \phi(\Psi_2\Psi_3)$. The remaining Seidel Laplacian eigenvalues of the graph $W\Gamma(Z_n)$ are the eigenvalues of the matrix (25).

Proof. Let $n = \Psi_1\Psi_2\Psi_3$, where $\Psi_1 < \Psi_2 < \Psi_3$, note that $\delta^*_{\Psi_1\Psi_2\Psi_3}$ is complete graph on vertices $\{\Psi_1, \Psi_2, \Psi_3, \Psi_1\Psi_2, \Psi_1\Psi_3, \Psi_2\Psi_3\}$. Now, by Lemma 3, we have, $W\Gamma(Z_{\Psi_1\Psi_2\Psi_3}) = \delta^*_{\Psi_1\Psi_2\Psi_3}[W\Gamma(A_{\Psi_1}), W\Gamma(A_{\Psi_2}), W\Gamma(A_{\Psi_3}), W\Gamma(A_{\Psi_1\Psi_2}), W\Gamma(A_{\Psi_1\Psi_3}), W\Gamma(A_{\Psi_2\Psi_3})]$. Therefore, by Lemma 1 and Corollary 1, we have $W\Gamma(A_{\Psi_1}) = \overline{K}_{\phi(\Psi_2\Psi_3)}$, $W\Gamma(A_{\Psi_2}) = \overline{K}_{\phi(\Psi_1\Psi_3)}$, $W\Gamma(A_{\Psi_3}) = \overline{K}_{\phi(\Psi_1\Psi_2)}$, $W\Gamma(A_{\Psi_1\Psi_2}) = K_{\phi(\Psi_3)}$, $W\Gamma(A_{\Psi_1\Psi_3}) = K_{\phi(\Psi_2)}$ and $W\Gamma(A_{\Psi_2\Psi_3}) = K_{\phi(\Psi_1)}$. And $n_1 = \phi(\Psi_2\Psi_3)$, $n_2 = \phi(\Psi_1\Psi_3)$, $n_3 = \phi(\Psi_1\Psi_2)$, $n_4 = \phi(\Psi_3)$, $n_5 = \phi(\Psi_2)$ and $n_6 = \phi(\Psi_1)$. It follows that from Theorem 1, $\tau_i = n_i - E$ for $1 \le i \le 6$ where $E = \phi(\Psi_1) + \phi(\Psi_2) + \phi(\Psi_3) + \phi(\Psi_1\Psi_2) + \phi(\Psi_1\Psi_3) + \phi(\Psi_2\Psi_3)$. Therefore, by Theorem 1, the Seidel Laplacian spectrum of the graph $W\Gamma(Z_{\Psi_1\Psi_2\Psi_3})$ is

$$\left\{ \begin{matrix} 2\phi(\Psi_2\Psi_3) - E & 2\phi(\Psi_1\Psi_3) - E & 2\phi(\Psi_1\Psi_2) - E & -E \\ \phi(\Psi_2\Psi_3) - 1 & \phi(\Psi_1\Psi_3) - 1 & \phi(\Psi_1\Psi_2) - 1 & \phi(\Psi_1) + \phi(\Psi_2) + \phi(\Psi_3) - 3 \end{matrix} \right\}, \tag{24}$$

where $E = \phi(\Psi_1) + \phi(\Psi_2) + \phi(\Psi_3) + \phi(\Psi_1\Psi_2) + \phi(\Psi_1\Psi_3) + \phi(\Psi_2\Psi_3)$. And the matrix given in (25), can be used to determine the remaining six eigenvalues,

$$M = \begin{bmatrix} n_1 - E & \phi(\Psi_1\Psi_3) & \phi(\Psi_1\Psi_2) & \phi(\Psi_3) & \phi(\Psi_2) & \phi(\Psi_1) \\ \phi(\Psi_2\Psi_3) & n_2 - E & \phi(\Psi_1\Psi_2) & \phi(\Psi_3) & \phi(\Psi_2) & \phi(\Psi_1) \\ \phi(\Psi_2\Psi_3) & \phi(\Psi_1\Psi_3) & n_3 - E & \phi(\Psi_3) & \phi(\Psi_2) & \phi(\Psi_1) \\ \phi(\Psi_2\Psi_3) & \phi(\Psi_1\Psi_3) & \phi(\Psi_1\Psi_2) & n_4 - E & \phi(\Psi_2) & \phi(\Psi_1) \\ \phi(\Psi_2\Psi_3) & \phi(\Psi_1\Psi_3) & \phi(\Psi_1\Psi_2) & \phi(\Psi_3) & n_5 - E & \phi(\Psi_1) \\ \phi(\Psi_2\Psi_3) & \phi(\Psi_1\Psi_3) & \phi(\Psi_1\Psi_2) & \phi(\Psi_3) & \phi(\Psi_2) & n_6 - E \end{bmatrix}. \tag{25}$$

□

Theorem 2 Let $n = \Psi_1^K$ where $K = 2j$, Ψ_1 is a prime and $j \ge 3$ is a positive integer. Then Seidel Laplacian spectrum of the graph $W\Gamma(Z_{\Psi_1{}^{2j}})$ is consists of the eigenvalue $-E$ with multiplicities $\Psi_1^{2j-1} - 2j$ for $i = 1, 2, 3, \dots, j-1, j, j+1, j+2, \dots, 2j-1$. Where $E = \sum_{i=1}^{2j-1} \phi(\Psi_1^i)$, the remaining Seidel Laplacian eigenvalues of the graph $W\Gamma(Z_{\Psi_1{}^{2j}})$ are eigenvalues of the matrix (27).

Proof. For $n = \Psi_1^{2j}$, where j is a positive integer and Ψ_1 is a prime, the proper divisors of Ψ_1^{2j} are $\Psi_1, \Psi_1^2, \Psi_1^3, \dots, \Psi_1^{j-1}, \Psi_1^j, \Psi_1^{j+1}, \dots, \Psi_1^{2j-2}, \Psi_1^{2j-1}$. By Lemma 3, we have $W\Gamma(Z_{\Psi_1{}^{2j}}) = \delta^*_{\Psi_1{}^{2j}}[W\Gamma(A_{\Psi_1}), W\Gamma(A_{\Psi_1{}^2}), \dots, W\Gamma(A_{\Psi_1{}^{j-1}}), W\Gamma(A_{\Psi_1{}^j}), \dots, W\Gamma(A_{\Psi_1{}^{2j-2}}), W\Gamma(A_{\Psi_1{}^{2j-1}})]$. Therefore, by Lemma 1 and Corollary 1, we get

$$W\Gamma(Z_{\Psi_1{}^{2j}}) = \delta^*_{\Psi_1{}^{2j}}[K_{\phi(\Psi_1{}^{2j-1})}, K_{\phi(\Psi_1{}^{2j-2})}, \dots, K_{\phi(\Psi_1{}^{j+1})}, K_{\phi(\Psi_1{}^{j})}, \dots, K_{\phi(\Psi_1{}^{2})}, K_{\phi(\Psi_1)}]. \tag{26}$$

And $n_i = \phi(\Psi_1{}^{2j-i})$ for $i = 1, 2, 3, \dots, j-1, j, j+1, j+2, \dots, 2j-1$. It follows that from Theorem 1, $\tau_i = n_i - E$ for $i = 1, 2, 3, \dots, j-1, j, j+1, j+2, \dots, 2j-1$. Where $E = \sum_{i=1}^{2j-1} \phi(\Psi_1^i)$. Therefore, by Theorem 1, the Seidel Laplacian spectrum of the graph $W\Gamma(Z_{\Psi_1{}^{2j}})$ is consists of the eigenvalue $-E$ with multiplicities $\Psi_1^{2j-1} - 2j$ for $i = 1, 2, 3, \dots, j-1, j, j+1, j+2, \dots, 2j-1$. The roots of the characteristic polynomial of the matrix (27) can be used to determine the remaining eigenvalues,

$$\begin{bmatrix} \phi({\Psi_1}^{2j-1})-E & \phi({\Psi_1}^{2j-2}) & \dots & \phi({\Psi_1}^{j+1}) & \phi({\Psi_1}^{j}) & \dots & \phi({\Psi_1}^{2}) & \phi(\Psi_1) \\ \phi({\Psi_1}^{2j-1}) & \phi({\Psi_1}^{2j-2})-E & \dots & \phi({\Psi_1}^{j+1}) & \phi({\Psi_1}^{j}) & \dots & \phi({\Psi_1}^{2}) & \phi(\Psi_1) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \dots & \vdots & \vdots \\ \phi({\Psi_1}^{2j-1}) & \phi({\Psi_1}^{2j-2}) & \dots & \phi({\Psi_1}^{j+1})-E & \phi({\Psi_1}^{j}) & \dots & \phi({\Psi_1}^{2}) & \phi(\Psi_1) \\ \phi({\Psi_1}^{2j-1}) & \phi({\Psi_1}^{2j-2}) & \dots & \phi({\Psi_1}^{j+1}) & \phi({\Psi_1}^{j})-E & \dots & \phi({\Psi_1}^{2}) & \phi(\Psi_1) \\ \vdots & \vdots & \dots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \phi({\Psi_1}^{2j-1}) & \phi({\Psi_1}^{2j-2}) & \dots & \phi({\Psi_1}^{j+1}) & \phi({\Psi_1}^{j}) & \dots & \phi({\Psi_1}^{2})-E & \phi(\Psi_1) \\ \phi({\Psi_1}^{2j-1}) & \phi({\Psi_1}^{2j-2}) & \dots & \phi({\Psi_1}^{j+1}) & \phi({\Psi_1}^{j}) & \dots & \phi({\Psi_1}^{2}) & \phi(\Psi_1)-E \end{bmatrix}. \quad (27)$$

□

Example 3 The Seidel Laplacian spectrum of the graph $W\Gamma(Z_{81})$ is consists of the eigenvalue -26 with multiplicity 23. The remaining Seidel Laplacian eigenvalues of the graph $W\Gamma(Z_{81})$ are eigenvalues of the matrix (28).

Proof. For $n = 81$, the proper divisors of 81 are 3, 9, 27. By Lemma 3, we have $W\Gamma(Z_{81}) = \delta^*_{81}[W\Gamma(A_3), W\Gamma(A_9), W\Gamma(A_{27})]$. Therefore, by Lemma 1 and Corollary 1, we get $W\Gamma(Z_{81}) = \delta^*_{81}[K_{18}, K_6, K_2]$ and $n_1 = 18$, $n_2 = 6$, $n_3 = 2$. It follows that from Theorem 1, $\tau_1 = -8$, $\tau_2 = -20$, $\tau_3 = -24$ and $E = 18+6+2 = 26$. Therefore, by Theorem 1, the Seidel Laplacian spectrum of the graph $W\Gamma(Z_{81})$ is consists of the eigenvalue -26 with multiplicity 23. Characteristic polynomial and the eigenvalues of the matrix (28) are respectively, $-\lambda^3 - 52\lambda^2 - 676\lambda = -\lambda(\lambda^2 + 52\lambda + 676) = -\lambda(\lambda + 26)^2$, and $\{0, -26, -26\}$.

$$M = \begin{bmatrix} -8 & 6 & 2 \\ 18 & -20 & 2 \\ 18 & 6 & -24 \end{bmatrix}. \quad (28)$$

□

Theorem 3 Let $n = \Psi_1^K$ where $K = 2j+1$, Ψ_1 is a prime and $j \geq 3$ is a positive integer. Then Seidel Laplacian spectrum of the graph $W\Gamma(Z_{{\Psi_1}^{2j+1}})$ is consists of the eigenvalue $-E$ with multiplicities $\Psi_1^{2j} - (2j+1)$ for $i = 1, 2, 3, \dots, j-1, j, j+1, j+2, \dots, 2j-1, 2j$. Where $E = \sum_{i=1}^{2j} \phi(\Psi_1^i)$, the remaining Seidel Laplacian eigenvalues of the graph $W\Gamma(Z_{{\Psi_1}^{2j+1}})$ are eigenvalues of the matrix (29).

Proof. Similarly as above Theorem 3, we can prove that the Seidel Laplacian spectrum of the graph $W\Gamma(Z_{{\Psi_1}^{2j+1}})$ is consists of the eigenvalue $-E$ with multiplicities $\Psi_1^{2j} - (2j+1)$ for $i = 1, 2, 3, \dots, j-1, j, j+1, j+2, \dots, 2j-1, 2j$. The remaining, Seidel Laplacian eigenvalues of the graph $W\Gamma(Z_{{\Psi_1}^{2j+1}})$ are eigenvalues of the matrix's (29).

$$\begin{bmatrix} \phi({\Psi_1}^{2j})-E & \phi({\Psi_1}^{2j-1}) & \dots & \phi({\Psi_1}^{j+1}) & \phi({\Psi_1}^{j}) & \dots & \phi({\Psi_1}^{2}) & \phi(\Psi_1) \\ \phi({\Psi_1}^{2j}) & \phi({\Psi_1}^{2j-1})-E & \dots & \phi({\Psi_1}^{j+1}) & \phi({\Psi_1}^{j}) & \dots & \phi({\Psi_1}^{2}) & \phi(\Psi_1) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \dots & \vdots & \vdots \\ \phi({\Psi_1}^{2j}) & \phi({\Psi_1}^{2j-1}) & \dots & \phi({\Psi_1}^{j+1})-E & \phi({\Psi_1}^{j}) & \dots & \phi({\Psi_1}^{2}) & \phi(\Psi_1) \\ \phi({\Psi_1}^{2j}) & \phi({\Psi_1}^{2j-1}) & \dots & \phi({\Psi_1}^{j+1}) & \phi({\Psi_1}^{j})-E & \dots & \phi({\Psi_1}^{2}) & \phi(\Psi_1) \\ \vdots & \vdots & \dots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \phi({\Psi_1}^{2j}) & \phi({\Psi_1}^{2j-1}) & \dots & \phi({\Psi_1}^{j+1}) & \phi({\Psi_1}^{j}) & \dots & \phi({\Psi_1}^{2})-E & \phi(\Psi_1) \\ \phi({\Psi_1}^{2j}) & \phi({\Psi_1}^{2j-1}) & \dots & \phi({\Psi_1}^{j+1}) & \phi({\Psi_1}^{j}) & \dots & \phi({\Psi_1}^{2}) & \phi(\Psi_1)-E \end{bmatrix}. \quad (29)$$

□

Example 4 The Seidel Laplacian spectrum of the graph $W\Gamma(Z_{32})$ is consists of the eigenvalue -15 with multiplicity 11. The remaining Seidel Laplacian eigenvalues of the graph $W\Gamma(Z_{32})$ are eigenvalues of the matrix (30).

Proof. For $n = 32$ the proper divisors of 32 are 2, 4, 8, 16. By Lemma 3, we have $W\Gamma(Z_{32}) = \delta^*_{32}[W\Gamma(A_2), W\Gamma(A_4), W\Gamma(A_8), W\Gamma(A_{16})]$. Therefore, by Lemma 1 and Corollary 1, we get $W\Gamma(Z_{32}) = \delta^*_{32}[K_8, K_4, K_2, K_1]$ and $n_1 = 8$, $n_2 = 4$, $n_3 = 2$, $n_4 = 1$. It follows that from Theorem 1, $E = 8+4+2+1 = 15$ and $\tau_1 = -7$, $\tau_2 = -11$, $\tau_3 = -13$, $\tau_4 = -14$. Therefore, by Theorem 1, the Seidel Laplacian spectrum of the graph $W\Gamma(Z_{32})$ is consists of the eigenvalue -15 with multiplicity 11. Characteristic polynomial and the eigenvalues of the matrix (30) are respectively, $\lambda(\lambda^3+45\lambda^2+675\lambda+3{,}375) = \lambda(\lambda+15)^3$, and $\{0, -15, -15, -15\}$.

$$M = \begin{bmatrix} -7 & 4 & 2 & 1 \\ 8 & -11 & 2 & 1 \\ 8 & 4 & -13 & 1 \\ 8 & 4 & 2 & -14 \end{bmatrix}. \tag{30}$$

□

Theorem 4 For distinct primes Ψ_1, Ψ_2 and where t is positive integer, if $n = {\Psi_1}^t\Psi_2$ then the Seidel Laplacian spectrum of the graph $W\Gamma(Z_n)$ is

$$\left\{ \begin{matrix} -E & 2(\Psi_1^t - \Psi_1^{t-1}) - E \\ \Psi_1^{t-1}\Psi_2 - 2t & \Psi_1^t - \Psi_1^{t-1} - 1 \end{matrix} \right\}. \tag{31}$$

Where $E = \sum_{k=1}^{t} \phi(\Psi_1^k) + \sum_{k=0}^{t-1} \phi({\Psi_1}^k\Psi_2)$ and the matrix (35) provides the remaining eigenvalues.

Proof. Let $n = {\Psi_1}^t\Psi_2$, where $\Psi_1 < \Psi_2$, note that $\delta^*_{{\Psi_1}^t\Psi_2}$ is complete graph on vertices $\{\Psi_1, \Psi_1^2, \ldots, {\Psi_1}^t, \Psi_2, \Psi_1\Psi_2, {\Psi_1}^2\Psi_2, \ldots, {\Psi_1}^{t-1}\Psi_2\}$. By lemma 3, we have

$$W\Gamma(Z_{{\Psi_1}^t\Psi_2}) = \delta^*_{{\Psi_1}^t\Psi_2}[W\Gamma(A_{\Psi_1}), W\Gamma(A_{{\Psi_1}^2}), \ldots, W\Gamma(A_{{\Psi_1}^t}), W\Gamma(A_{\Psi_2}), W\Gamma(A_{\Psi_1\Psi_2}), \ldots, W\Gamma(A_{{\Psi_1}^{t-1}\Psi_2})]. \tag{32}$$

Therefore, by Lemma 1 and Corollary 1, we get

$$W\Gamma(Z_{{\Psi_1}^t\Psi_2}) = \delta^*_{{\Psi_1}^t\Psi_2}[K_{\phi({\Psi_1}^{t-1}\Psi_2)}, K_{\phi({\Psi_1}^{t-2}\Psi_2)}, \ldots, K_{\phi(\Psi_2)}, \overline{K}_{\phi({\Psi_1}^t)}, K_{\phi({\Psi_1}^{t-1})}, \ldots, K_{\phi(\Psi_1)}]. \tag{33}$$

And $n_{\Psi_1} = \phi({\Psi_1}^{t-1}\Psi_2), n_{{\Psi_1}^2} = \phi({\Psi_1}^{t-2}\Psi_2), \ldots, n_{{\Psi_1}^t} = \phi(\Psi_2), n_{\Psi_1\Psi_2} = \phi({\Psi_1}^{t-1}), \ldots, n_{{\Psi_1}^r\Psi_2} = \phi({\Psi_1}^{t-r}), \ldots, n_{{\Psi_1}^{t-1}\Psi_2} = \phi(\Psi_1)$ and $n_{\Psi_2} = \phi({\Psi_1}^t)$. It follows that from Theorem 1, $\tau_{d_j} = n_{d_j} - E$ where $d_j \in \{\Psi_1, \Psi_1^2, \ldots, {\Psi_1}^t, \Psi_1\Psi_2, {\Psi_1}^2\Psi_2, \ldots, {\Psi_1}^{t-1}\Psi_2\}$ and $E = \sum_{k=1}^{t} \phi(\Psi_1^k) + \sum_{k=0}^{t-1} \phi({\Psi_1}^k\Psi_2)$. By Theorem 1, the Seidel Laplacian spectrum of the graph $W\Gamma(Z_n)$ is

$$\left\{ \begin{matrix} -E & 2(\Psi_1^t - \Psi_1^{t-1}) - E \\ \Psi_1^{t-1}\Psi_2 - 2t & \Psi_1^t - \Psi_1^{t-1} - 1 \end{matrix} \right\}. \tag{34}$$

The roots of the characteristic polynomial of the matrix (35) can be used to determine the remaining eigenvalues

$$\begin{bmatrix} n_{\Psi_1}-E & n_{\Psi_1^2} & \cdots & n_{\Psi_1^t} & n_{\Psi_2} & n_{\Psi_1\Psi_2} & \cdots & n_{\Psi_1^{t-1}\Psi_2} \\ n_{\Psi_1} & n_{\Psi_1^2}-E & \cdots & n_{\Psi_1^t} & n_{\Psi_2} & n_{\Psi_1\Psi_2} & \cdots & n_{\Psi_1^{t-1}\Psi_2} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ n_{\Psi_1} & n_{\Psi_1^2} & \cdots & n_{\Psi_1^t}-E & n_{\Psi_2} & n_{\Psi_1\Psi_2} & \cdots & n_{\Psi_1^{t-1}\Psi_2} \\ n_{\Psi_1} & n_{\Psi_1^2} & \cdots & n_{\Psi_1^t} & n_{\Psi_2}-E & n_{\Psi_1\Psi_2} & \cdots & n_{\Psi_1^{t-1}\Psi_2} \\ n_{\Psi_1} & n_{\Psi_1^2} & \cdots & n_{\Psi_1^t} & n_{\Psi_2} & n_{\Psi_1\Psi_2}-E & \cdots & n_{\Psi_1^{t-1}\Psi_2} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ n_{\Psi_1} & n_{\Psi_1^2} & \cdots & n_{\Psi_1^t} & n_{\Psi_2} & n_{\Psi_1\Psi_2} & \cdots & n_{\Psi_1^{t-1}\Psi_2}-E \end{bmatrix}. \tag{35}$$

□

Theorem 5 Let $n = \Psi_1\Psi_2\ldots\Psi_t\eta_1^{d_1}\eta_2^{d_2}\ldots\eta_s^{d_s}(d_i \geq 2, t \geq 1, s \geq 0)$ where Ψ_i's and η_i's are the distinct primes. Let $\beta = \{\Psi_1, \Psi_2, \ldots, \Psi_t\}$ and $\{c_1, c_2, \ldots, c_{\tau(n)-2}\}$ represents the collection of all proper divisors of n. Then, the Seidel Laplacian spectrum $W\Gamma(Z_n)$ consists of eigenvalues $-E$ with multiplicity $\phi(\frac{n}{c_i})-1$ when $c_i \notin \beta$ and $2\phi(\frac{n}{c_j})-E$ with multiplicity $\phi(\frac{n}{c_j})-1$ when $c_j \in \beta$ for $1 \leq i, j \leq \tau(n)-2$, where $E = \sum_{i=1}^{\tau(n)-2}\phi(\frac{n}{c_i})$. The characteristic polynomial of the matrix (36) provides the remaining eigenvalues.

Proof. Suppose that $n = \Psi_1\Psi_2\ldots\Psi_t\eta_1^{d_1}\eta_2^{d_2}\ldots\eta_s^{d_s}(d_i \geq 2, t \geq 1, s \geq 0)$ where Ψ_i's and η_i's are the distinct primes. Let $\beta = \{\Psi_1, \Psi_2, \ldots, \Psi_t\}$ and $\{c_1, c_2, \ldots, c_{\tau(n)-2}\}$ represents the collection of all proper divisors of n. Now by Lemma 2, the following conclusions can be drawn: for each $c_i \in \beta$, we have $W\Gamma(A_{c_i}) = \overline{K}_{\phi(\frac{n}{c_i})}$ and for $c_j \notin \beta$ we have $W\Gamma(A_{c_i}) = K_{\phi(\frac{n}{c_i})}$ for $1 \leq i, j \leq \tau(n)-2$. Also, $n_{c_i} = \phi(\frac{n}{c_i}), n_{c_j} = \phi(\frac{n}{c_j})$ for all $c_i \in \beta$ and $c_j \notin \beta$. It follows that from Theorem 1, $\tau_{c_i} = n_{c_i} - E$ where $E = \sum_{i=1}^{\tau(n)-2}\phi(\frac{n}{c_i})$. By Theorem 1, Seidel Laplacian eigenvalues of the graph $W\Gamma(Z_n)$ are respectively, $-E$ with multiplicity $\phi(\frac{n}{c_i})-1$ when $c_i \notin \beta$ and $2\phi(\frac{n}{c_j})-E$ with multiplicity $\phi(\frac{n}{c_j})-1$ when $c_j \in \beta$ for $1 \leq i, j \leq \tau(n)-2$. The roots of the characteristic polynomial of the matrix (36) can be used to determine the remaining eigenvalues.

$$Y = \begin{bmatrix} n_{c_1}-E & \cdots & n_{c_t} & n_{c_{t+1}} & \cdots & n_{\tau(n)-2} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ n_{c_1} & \cdots & n_{c_t}-E & n_{c_{t+1}} & \cdots & n_{\tau(n)-2} \\ n_{c_1} & \cdots & n_{c_t} & n_{c_{t+1}}-E & \cdots & n_{\tau(n)-2} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ n_{c_1} & \cdots & n_{c_t} & n_{c_{t+1}} & \cdots & n_{\tau(n)-2}-E \end{bmatrix}. \tag{36}$$

□

Example 5 The Seidel Laplacian spectrum of the graph $W\Gamma(Z_{12})$ is given by

$$\begin{Bmatrix} -7 & -3 \\ 2 & 1 \end{Bmatrix}. \tag{37}$$

The remaining four Seidel Laplacian eigenvalues of the graph $W\Gamma(Z_{12})$ are the eigenvalues of the matrix (39).

Proof. Let $n = 2^2 \cdot 3 = 12$, where $2 < 3$, note that δ_{12}^* is complete graph. By theorem 5 proper divisor of 12 are $\{c_1, c_2, c_3, c_4\} = \{2, 3, 4, 6\}$ and $\beta = \{3\}$. So we have $W\Gamma(A_3) = \overline{K}_2$, $W\Gamma(A_2) = K_2$, $W\Gamma(A_4) = K_2$, $W\Gamma(A_6) = K_1$. Also

$n_1 = 2,\ n_2 = 2,\ n_3 = 2,\ n_4 = 1$ and $E = 2+2+2+1 = 7$. By using Theorem 5, the value of $\tau_1 = \tau_2 = \tau_3 = -5$ and $\tau_4 = -6$. Therefore the Seidel Laplacian spectrum of the graph $W\Gamma(Z_{12})$ is given by

$$\left\{ \begin{array}{cc} -7 & -3 \\ 2 & 1 \end{array} \right\}. \tag{38}$$

Characteristic polynomial and the eigenvalues of the matrix (39) are respectively, $\lambda^4 + 21\lambda^3 + 147\lambda^2 + 343\lambda = \lambda(\lambda^3 + 21\lambda^2 + 147\lambda + 343) = \lambda(\lambda+7)^3$, and $\{0, -7, -7, -7\}$.

$$M = \begin{bmatrix} -5 & 2 & 2 & 1 \\ 2 & -5 & 2 & 1 \\ 2 & 2 & -5 & 1 \\ 2 & 2 & 2 & -6 \end{bmatrix}. \tag{39}$$

□

4. Seidel signless Laplacian spectrum of the weakly zero-divisor graph

In this section, we will highlight some results of Seidel signless Laplacian spectrum of the weakly zero-divisor graph. The Seidel signless Laplacian spectrum of complete graph K_l and its complement graph $\overline{K_l}$ on l vertices is given by

$$spec^{SL^+}(K_l) = \left\{ \begin{array}{cc} 2(n_i - 2r_i - 1) & -1 \\ 1 & l-1 \end{array} \right\} \text{ and } spec^{SL^+}(\overline{K_l}) = \left\{ \begin{array}{cc} 2(n_i - 2r_i - 1) & 1 \\ 1 & l-1 \end{array} \right\} \text{ respectively.} \tag{40}$$

The following theorem provides the generalized join graph's Seidel signless Laplacian spectrum in terms of the spectrum of regular graphs.

Theorem 6 [2] Consider $G[L_1, L_2, \ldots, L_k]$ where G is simple connected graph with vertices labeled as $1, 2, \ldots, k$ and $S = [s_{ij}]_{k\times k}$ is the Seidel matrix of G and L_j is r_j- regular and $|V(L_j)| = n_j$, for every $j = 1, 2, \ldots, k$. Let $\{\sigma_{j1}^{SL^+} = 2(n_j - 2r_j - 1), \sigma_{j2}^{SL^+}, \ldots, \sigma_{jn_j}^{SL^+}\}$ be the Seidel signless Laplacian eigenvalues of L_j, for $j = 1, 2, \ldots, k$. Then, the Seidel signless Laplacian spectrum of the G-join of the graph $L_1, L_2, \ldots L_k$ is given by,

$$spec^{SL^+}(G[L_1, L_2, \ldots, L_k]) = \left(\bigcup_{j=1}^{k}\bigcup_{i=2}^{n_j}(\sigma_{ji}^{SL^+} + \tau_j)\right)\bigcup spec(T_{SL^+}(G)). \tag{41}$$

Where $\tau_j = \sum_{i=1}^{k} s_{ij} n_i$ and

$$T_{SL^+}(G) = \begin{bmatrix} 2(n_1 - 2r_1 - 1) + \tau_1 & s_{1,2}n_2 & \ldots & s_{1,k}n_k \\ s_{1,2}n_1 & 2(n_2 - 2r_2 - 1) + \tau_2 & \ldots & s_{2,k}n_k \\ \vdots & \vdots & \ddots & \vdots \\ s_{1,k}n_1 & s_{2,k}n_2 & \ldots & 2(n_k - 2r_k - 1) + \tau_k \end{bmatrix}. \tag{42}$$

Lemma 7 Let n be the product of two different primes Ψ_1 and Ψ_2. Then, the Seidel signless Laplacian spectrum of $W\Gamma(Z_n)$ is given by,

$$\left\{ \begin{array}{cc} 2-\Psi_1 & 2-\Psi_2 \\ \Psi_2-2 & \Psi_1-2 \end{array} \right\}. \tag{43}$$

The remaining two Seidel signless Laplacian eigenvalues of the graph $W\Gamma(Z_n)$ are the eigenvalues of the matrix,

$$\begin{bmatrix} 2\Psi_2-\Psi_1-3 & 1-\Psi_1 \\ 1-\Psi_2 & 2\Psi_1-\Psi_2-3 \end{bmatrix}. \tag{44}$$

Proof. The proper divisors of n are Ψ_1 and Ψ_2 and $\Psi_1 < \Psi_2$. Also, by the definition of δ_n^*; $\Psi_1 \sim \Psi_2$. Now by Lemma 3, we have $W\Gamma(Z_{\Psi_1\Psi_2}) = \delta^*_{\Psi_1\Psi_2}[W\Gamma(A_{\Psi_1}), W\Gamma(A_{\Psi_2})]$. Therefore, by Lemma 1 and Corollary 1, we have $W\Gamma(A_{\Psi_1}) = \overline{K}_{\phi(\Psi_2)}$ and $W\Gamma(A_{\Psi_2}) = \overline{K}_{\phi(\Psi_1)}$. Consequently, in order of proper divisor sequence we have $n_1 = \phi(\Psi_2) = \Psi_2 - 1,\ n_2 = \phi(\Psi_1) = \Psi_1 - 1$, value of $\tau_1 = 1-\Psi_1$, $\tau_2 = 1-\Psi_2$, $r_1 = r_2 = 0$ and $E = \phi(\Psi_1) + \phi(\Psi_2)$. Therefore, the Seidel signless Laplacian spectrum of the graph $W\Gamma(Z_{\Psi_1\Psi_2})$ is

$$\left\{ \begin{array}{cc} 2-\Psi_1 & 2-\Psi_2 \\ \Psi_2-2 & \Psi_1-2 \end{array} \right\}. \tag{45}$$

And the characteristic polynomial of the matrix provided below, can be used to determine the remaining eigenvalues,

$$\begin{bmatrix} 2\Psi_2-\Psi_1-3 & 1-\Psi_1 \\ 1-\Psi_2 & 2\Psi_1-\Psi_2-3 \end{bmatrix}. \tag{46}$$

The matrix (46) has a characteristic polynomial $\lambda^2 - \lambda(\Psi_1 + \Psi_2 - 6) + 2(2\Psi_1\Psi_2 - \Psi_1^2 - \Psi_2^2 - \Psi_1 - \Psi_2 + 4)$. □

Example 6 The Seidel signless Laplacian spectrum of the graph $W\Gamma(Z_{10})$ is given by,

$$\left\{ \begin{array}{cc} 0 & -3 \\ 3 & 0 \end{array} \right\}. \tag{47}$$

The remaining two Seidel signless Laplacian eigenvalues of the graph $W\Gamma(Z_{10})$ are the eigenvalues of the matrix,

$$\begin{bmatrix} 5 & -1 \\ -4 & -4 \end{bmatrix}. \tag{48}$$

Proof. The proper divisors of 10 are 2, 5 and $2 < 5$. Also, by the definition of δ_{10}^*; $2 \sim 5$. Now by Lemma 3, we have $W\Gamma(Z_{10}) = \delta^*_{10}[W\Gamma(A_2), W\Gamma(A_5)]$. Therefore, by Lemma 1 and Corollary 1, we have $W\Gamma(A_2) = \overline{K}_4$ and $W\Gamma(A_5) = \overline{K}_1$. Consequently, in order of proper divisor sequence we have $n_1 = 4,\ n_2 = 1$, value of $\tau_1 = -1$, $\tau_2 = -4$, $r_1 = r_2 = 0$ and $E = 4 + 1 = 5$. Therefore, the Seidel signless Laplacian spectrum of the graph $W\Gamma(Z_{10})$ is

$$\left\{\begin{matrix} 0 & -3 \\ 3 & 0 \end{matrix}\right\}. \tag{49}$$

And the characteristic polynomial of the matrix provided below, can be used to determine the remaining eigenvalues,

$$\begin{bmatrix} 5 & -1 \\ -4 & -4 \end{bmatrix}. \tag{50}$$

The matrix (50) has a characteristic polynomial $\lambda^2 - \lambda - 24$. □

Lemma 8 For distinct primes Ψ_1 and Ψ_2, if $n = \Psi_1{}^2\Psi_2$ then, the Seidel signless Laplacian spectrum of $W\Gamma(Z_n)$ is given by

$$\left\{\begin{matrix} 1-\Psi_1^2-\Psi_2 & 2-\Psi_1\Psi_2 & -1-(\Psi_1^2+\Psi_1\Psi_2-\Psi_1-\Psi_2) & 2\Psi_1-\Psi_1\Psi_2-\Psi_1^2-1 \\ \Psi_1\Psi_2-\Psi_1-\Psi_2 & \Psi_1^2-\Psi_1-1 & \Psi_2-2 & \Psi_1-2 \end{matrix}\right\}. \tag{51}$$

Where $E = \phi(\Psi_1\Psi_2) + \phi(\Psi_1^2) + \phi(\Psi_2) + \phi(\Psi_1)$. The remaining four Seidel signless Laplacian eigenvalues of the graph $W\Gamma(Z_n)$ are the eigenvalues of the matrix (53).

Proof. Let $n = \Psi_1{}^2\Psi_2$, where $\Psi_1 < \Psi_2$, note that $\delta^*_{\Psi_1{}^2\Psi_2}$ is complete graph on vertices $\{\Psi_1,\ \Psi_2,\ \Psi_1{}^2,\ \Psi_1\Psi_2\}$. By Lemma 3, we have $W\Gamma(Z_{\Psi_1{}^2\Psi_2}) = \delta^*_{\Psi_1{}^2\Psi_2}[W\Gamma(A_{\Psi_1}),\ W\Gamma(A_{\Psi_2}),\ W\Gamma(A_{\Psi_1{}^2}),\ W\Gamma(A_{\Psi_1\Psi_2})]$. Therefore, by Lemma 1 and Corollary 1, we have $W\Gamma(A_{\Psi_1}) = K_{\phi(\Psi_1\Psi_2)}$, $W\Gamma(A_{\Psi_2}) = \overline{K}_{\phi(\Psi_1{}^2)}$, $W\Gamma(A_{\Psi_1{}^2}) = K_{\phi(\Psi_2)}$ and $W\Gamma(A_{\Psi_1\Psi_2}) = K_{\phi(\Psi_1)}$. Also $n_1 = \phi(\Psi_1\Psi_2)$, $n_2 = \phi(\Psi_1{}^2)$, $n_3 = \phi(\Psi_2)$, $n_4 = \phi(\Psi_1)$. By using Theorem 6, the value of τ_i are, $\tau_i = n_i - E$, for $1 \le i \le 4$ and $r_1 = \phi(\Psi_1\Psi_2) - 1$, $r_2 = 0$, $r_3 = \phi(\Psi_2) - 1$ and $r_4 = \phi(\Psi_1) - 1$. where $E = \phi(\Psi_1\Psi_2) + \phi(\Psi_1^2) + \phi(\Psi_2) + \phi(\Psi_1)$. Therefore, by Theorem 6, the Seidel signless Laplacian spectrum of the graph $W\Gamma(Z_{\Psi_1{}^2\Psi_2})$ is

$$\left\{\begin{matrix} 1-\Psi_1^2-\Psi_2 & 2-\Psi_1\Psi_2 & -1-(\Psi_1^2+\Psi_1\Psi_2-\Psi_1-\Psi_2) & 2\Psi_1-\Psi_1\Psi_2-\Psi_1^2-1 \\ \Psi_1\Psi_2-\Psi_1-\Psi_2 & \Psi_1^2-\Psi_1-1 & \Psi_2-2 & \Psi_1-2 \end{matrix}\right\}. \tag{52}$$

And the matrix given in (53), can be used to determine the remaining four eigenvalues.

$$\begin{bmatrix} 2-\phi(\Psi_1\Psi_2)-E & -\phi(\Psi_1^2) & -\phi(\Psi_2) & -\phi(\Psi_1) \\ -\phi(\Psi_1\Psi_2) & 3\phi(\Psi_1^2)-2-E & -\phi(\Psi_2) & -\phi(\Psi_1) \\ -\phi(\Psi_1\Psi_2) & -\phi(\Psi_1^2) & 2-\phi(\Psi_2)-E & -\phi(\Psi_1) \\ -\phi(\Psi_1\Psi_2) & -\phi(\Psi_1^2) & -\phi(\Psi_2) & 2-\phi(\Psi_1)-E \end{bmatrix}. \tag{53}$$

□

Lemma 9 For distinct prime Ψ_1, Ψ_2, Ψ_3, if $n = \Psi_1\Psi_2\Psi_3$ then, the Seidel signless Laplacian spectrum of the graph $W\Gamma(Z_n)$ is

$$\left\{\begin{matrix} 1+\phi(\Psi_2\Psi_3)-E & 1+\phi(\Psi_1\Psi_3)-E & 1+\phi(\Psi_1\Psi_2)-E & \phi(\Psi_3)-(1+E) & \phi(\Psi_2)-(1+E) & J' \\ \phi(\Psi_2\Psi_3)-1 & \phi(\Psi_1\Psi_3)-1 & \phi(\Psi_1\Psi_2)-1 & \phi(\Psi_3)-1 & \phi(\Psi_2)-1 & J'' \end{matrix}\right\}. \tag{54}$$

Where $J' = \phi(\Psi_1) - (1+E)$, $J'' = \phi(\Psi_1) - 1$ and $E = \phi(\Psi_1) + \phi(\Psi_2) + \phi(\Psi_3) + \phi(\Psi_1\Psi_2) + \phi(\Psi_1\Psi_3) + \phi(\Psi_2\Psi_3)$, The remaining Seidel signless Laplacian eigenvalues of the graph $W\Gamma(Z_n)$ are the eigenvalues of the matrix (56).

Proof. Let $n = \Psi_1\Psi_2\Psi_3$, where $\Psi_1 < \Psi_2 < \Psi_3$, note that $\delta^*_{\Psi_1\Psi_2\Psi_3}$ is complete graph on vertices $\{\Psi_1, \Psi_2, \Psi_3, \Psi_1\Psi_2, \Psi_1\Psi_3, \Psi_2\Psi_3\}$. Now, by Lemma 3, we have, $W\Gamma(Z_{\Psi_1\Psi_2\Psi_3}) = \delta^*_{\Psi_1\Psi_2\Psi_3}[W\Gamma(A_{\Psi_1}), W\Gamma(A_{\Psi_2}), W\Gamma(A_{\Psi_3}), W\Gamma(A_{\Psi_1\Psi_2}), W\Gamma(A_{\Psi_1\Psi_3}), W\Gamma(A_{\Psi_2\Psi_3})]$. Therefore, by Lemma 1 and Corollary 1, we have $W\Gamma(A_{\Psi_1}) = \overline{K}_{\phi(\Psi_2\Psi_3)}$, $W\Gamma(A_{\Psi_2}) = \overline{K}_{\phi(\Psi_1\Psi_3)}$, $W\Gamma(A_{\Psi_3}) = \overline{K}_{\phi(\Psi_1\Psi_2)}$, $W\Gamma(A_{\Psi_1\Psi_2}) = K_{\phi(\Psi_3)}$, $W\Gamma(A_{\Psi_1\Psi_3}) = K_{\phi(\Psi_2)}$ and $W\Gamma(A_{\Psi_2\Psi_3}) = K_{\phi(\Psi_1)}$. And $n_1 = \phi(\Psi_2\Psi_3)$, $n_2 = \phi(\Psi_1\Psi_3)$, $n_3 = \phi(\Psi_1\Psi_2)$, $n_4 = \phi(\Psi_3)$, $n_5 = \phi(\Psi_2)$ and $n_6 = \phi(\Psi_1)$. It follows that from Theorem 6, $\tau_i = n_i - E$ for $1 \le i \le 6$ and $r_1 = r_2 = r_3 = 0$, $r_4 = \phi(\Psi_3) - 1$, $r_5 = \phi(\Psi_2) - 1$ and $r_6 = \phi(\Psi_1) - 1$ where $E = \phi(\Psi_1) + \phi(\Psi_2) + \phi(\Psi_3) + \phi(\Psi_1\Psi_2) + \phi(\Psi_1\Psi_3) + \phi(\Psi_2\Psi_3)$. Therefore, by Theorem 6, the Seidel signless Laplacian spectrum of the graph $W\Gamma(Z_{\Psi_1\Psi_2\Psi_3})$ is

$$\left\{ \begin{array}{cccccc} 1+\phi(\Psi_2\Psi_3)-E & 1+\phi(\Psi_1\Psi_3)-E & 1+\phi(\Psi_1\Psi_2)-E & \phi(\Psi_3)-(1+E) & \phi(\Psi_2)-(1+E) & J' \\ \phi(\Psi_2\Psi_3)-1 & \phi(\Psi_1\Psi_3)-1 & \phi(\Psi_1\Psi_2)-1 & \phi(\Psi_3)-1 & \phi(\Psi_2)-1 & J'' \end{array} \right\}. \tag{55}$$

Where $J' = \phi(\Psi_1) - (1+E)$, $J'' = \phi(\Psi_1) - 1$ and $E = \phi(\Psi_1) + \phi(\Psi_2) + \phi(\Psi_3) + \phi(\Psi_1\Psi_2) + \phi(\Psi_1\Psi_3) + \phi(\Psi_2\Psi_3)$, And the matrix given in (56), can be used to determine the remaining six eigenvalues,

$$M = \begin{bmatrix} A & -\phi(\Psi_1\Psi_3) & -\phi(\Psi_1\Psi_2) & -\phi(\Psi_3) & -\phi(\Psi_2) & -\phi(\Psi_1) \\ -\phi(\Psi_2\Psi_3) & B & -\phi(\Psi_1\Psi_2) & -\phi(\Psi_3) & -\phi(\Psi_2) & -\phi(\Psi_1) \\ -\phi(\Psi_2\Psi_3) & -\phi(\Psi_1\Psi_3) & C & -\phi(\Psi_3) & -\phi(\Psi_2) & -\phi(\Psi_1) \\ -\phi(\Psi_2\Psi_3) & -\phi(\Psi_1\Psi_3) & -\phi(\Psi_1\Psi_2) & D & -\phi(\Psi_2) & -\phi(\Psi_1) \\ -\phi(\Psi_2\Psi_3) & -\phi(\Psi_1\Psi_3) & -\phi(\Psi_1\Psi_2) & -\phi(\Psi_3) & E' & -\phi(\Psi_1) \\ -\phi(\Psi_2\Psi_3) & -\phi(\Psi_1\Psi_3) & -\phi(\Psi_1\Psi_2) & -\phi(\Psi_3) & -\phi(\Psi_2) & F \end{bmatrix}, \tag{56}$$

where $A = 3\phi(\Psi_2\Psi_3) - 2 - E$, $B = 3\phi(\Psi_1\Psi_3) - 2 - E$, $C = 3\phi(\Psi_1\Psi_2) - 2 - E$, $D = 2 - \phi(\Psi_3) - E$, $E' = 2 - \phi(\Psi_2) - E$ and $F = 2 - \phi(\Psi_1) - E$. □

Example 7 For distinct prime 2, 3, 5, if $n = 2.3.5 = 30$ then, the Seidel signless Laplacian spectrum of the graph $W\Gamma(Z_{30})$ is

$$\left\{ \begin{array}{cccccc} -12 & -16 & -18 & -18 & -20 & -21 \\ 7 & 3 & 1 & 3 & 1 & 0 \end{array} \right\}. \tag{57}$$

The remaining Seidel signless Laplacian eigenvalues of the graph $W\Gamma(Z_{30})$ are the eigenvalues of the matrix (59).

Proof. Let $n = 2 \cdot 3 \cdot 5 = 30$, where $2 < 3 < 5$, note that δ^*_{30} is complete graph on vertices $\{2, 3, 5, 6, 10, 15\}$. Now, by Lemma 3, we have, $W\Gamma(Z_{30}) = \delta^*_{30}[W\Gamma(A_2), W\Gamma(A_3), W\Gamma(A_5), W\Gamma(A_6), W\Gamma(A_{10}), W\Gamma(A_{15})]$. Therefore, by Lemma 1 and Corollary 1, we have $W\Gamma(A_2) = \overline{K}_{\phi(15)}$, $W\Gamma(A_3) = \overline{K}_{\phi(10)}$, $W\Gamma(A_5) = \overline{K}_{\phi(6)}$, $W\Gamma(A_6) = K_{\phi(5)}$, $W\Gamma(A_{10}) = K_{\phi(3)}$ and $W\Gamma(A_{15}) = K_{\phi(2)}$. Also $n_1 = 8$, $n_2 = 4$, $n_3 = 2$, $n_4 = 4$, $n_5 = 2$, $n_6 = 1$ and $E = 8+4+2+4+2+1 = 21$. It follows that from Theorem 6, $\tau_1 = -13$, $\tau_2 = -17$, $\tau_3 = -19$, $\tau_4 = -17$, $\tau_5 = -19$, $\tau_6 = -20$, and $r_1 = r_2 = r_3 = 0$, $r_4 = 3$, $r_5 = 1$ and $r_6 = 0$. Therefore, by Theorem 6, the Seidel signless Laplacian spectrum of the graph $W\Gamma(Z_{30})$ is

$$\left\{ \begin{array}{cccccc} -12 & -16 & -18 & -18 & -20 & -21 \\ 7 & 3 & 1 & 3 & 1 & 0 \end{array} \right\}. \tag{58}$$

Characteristic polynomial and the eigenvalues of the matrix (59) are respectively, $\lambda^6+91\lambda^5+3{,}082\lambda^4+46{,}534\lambda^3+254{,}293\lambda^2-614{,}897\lambda-7{,}676{,}304=(\lambda+19)(\lambda^5+72\lambda^4+1{,}714\lambda^3+13{,}968\lambda^2-11{,}099\lambda-404{,}016)=(\lambda+19)(\lambda+19)(\lambda^4+53\lambda^3+707\lambda^2+535\lambda-21{,}264)$ and $\{-19, -19, -32.19, -15.82, -9.41, 4.43\}$

$$M=\begin{bmatrix} 1 & -4 & -2 & -4 & -2 & -1 \\ -8 & -11 & -2 & -4 & -2 & -1 \\ -8 & -4 & -17 & -4 & -2 & -1 \\ -8 & -4 & -2 & -23 & -2 & -1 \\ -8 & -4 & -2 & -4 & -21 & -1 \\ -8 & -4 & -2 & -4 & -2 & -20 \end{bmatrix}. \tag{59}$$

□

Theorem 7 Let $n=\Psi_1^K$, where $K=2j$, Ψ_1 is a prime and $j\geq 3$ is a positive integer. Then Seidel signless Laplacian spectrum of the graph $W\Gamma(Z_{\Psi_1{}^{2j}})$ is consists of the eigenvalue $\phi(\Psi_1^{2j-i})-(1+E)$ with multiplicities $\phi(\Psi_1^{2j-i})-1$ for $i=1,2,3,\ldots,j-1,j,j+1,j+2,\ldots,2j-1$, where $E=\sum_{i=1}^{2j-1}\phi(\Psi_1^i)$, the remaining Seidel signless Laplacian eigenvalues of the graph $W\Gamma(Z_{\Psi_1{}^{2j}})$ are eigenvalues of the matrix (61).

Proof. For $n=\Psi_1^{2j}$, where j is a positive integer and Ψ_1 is a prime, the proper divisors of Ψ_1^{2j} are $\Psi_1, \Psi_1^2, \Psi_1^3, \ldots, \Psi_1^{j-1}, \Psi_1^j, \Psi_1^{j+1}, \ldots, \Psi_1^{2j-2}, \Psi_1^{2j-1}$. By Lemma 3, we have $W\Gamma(Z_{\Psi_1{}^{2j}})=\delta^*_{\Psi_1{}^{2j}}[W\Gamma(A_{\Psi_1}), W\Gamma(A_{\Psi_1{}^2}), \ldots, W\Gamma(A_{\Psi_1{}^{j-1}}), W\Gamma(A_{\Psi_1{}^j}), \ldots, W\Gamma(A_{\Psi_1{}^{2j-2}}), W\Gamma(A_{\Psi_1{}^{2j-1}})]$. Therefore, by Lemma 1 and Corollary 1, we get

$$W\Gamma(Z_{\Psi_1{}^{2j}})=\delta^*_{\Psi_1{}^{2j}}[K_{\phi(\Psi_1{}^{2j-1})}, K_{\phi(\Psi_1{}^{2j-2})}, \ldots, K_{\phi(\Psi_1{}^{j+1})}, K_{\phi(\Psi_1{}^{j})}, \ldots, K_{\phi(\Psi_1{}^{2})}, K_{\phi(\Psi_1)}]. \tag{60}$$

And $n_i=\phi({\Psi_1}^{2j-i})$ for $i=1, 2, 3, \ldots, j-1, j, j+1, j+2, \ldots, 2j-1$. It follows that from Theorem 6, $\tau_i=n_i-E$ and also $r_i=\phi({\Psi_1}^{2j-i})-1$ for $i=1, 2, 3, \ldots, j-1, j, j+1, j+2, \ldots, 2j-1$ and where $E=\sum_{i=1}^{2j-1}\phi(\Psi_1^i)$. Therefore, by Theorem 6, the Seidel signless Laplacian spectrum of the graph $W\Gamma(Z_{\Psi_1{}^{2j}})$ is consists of the eigenvalue $\phi(\Psi_1^{2j-i})-(1+E)$ with multiplicities $\phi(\Psi_1^{2j-i})-1$ for $i=1, 2, 3, \ldots, j-1, j, j+1, j+2, \ldots, 2j-1$. The roots of the characteristic polynomial of the matrix (61) can be used to determine the remaining eigenvalues,

$$\begin{bmatrix} K & -\phi({\Psi_1}^{2j-2}) & \ldots & -\phi({\Psi_1}^{j+1}) & -\phi({\Psi_1}^{j}) & \ldots & -\phi({\Psi_1}^{2}) & -\phi(\Psi_1) \\ -\phi({\Psi_1}^{2j-1}) & L & \ldots & -\phi({\Psi_1}^{j+1}) & -\phi({\Psi_1}^{j}) & \ldots & -\phi({\Psi_1}^{2}) & -\phi(\Psi_1) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ldots & \vdots & \vdots \\ -\phi({\Psi_1}^{2j-1}) & -\phi({\Psi_1}^{2j-2}) & \ldots & M & -\phi({\Psi_1}^{j}) & \ldots & -\phi({\Psi_1}^{2}) & -\phi(\Psi_1) \\ -\phi({\Psi_1}^{2j-1}) & -\phi({\Psi_1}^{2j-2}) & \ldots & -\phi({\Psi_1}^{j+1}) & P & \ldots & -\phi({\Psi_1}^{2}) & -\phi(\Psi_1) \\ \vdots & \vdots & \ldots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -\phi({\Psi_1}^{2j-1}) & -\phi({\Psi_1}^{2j-2}) & \ldots & -\phi({\Psi_1}^{j+1}) & -\phi({\Psi_1}^{j}) & \ldots & Q & -\phi(\Psi_1) \\ -\phi({\Psi_1}^{2j-1}) & -\phi({\Psi_1}^{2j-2}) & \ldots & -\phi({\Psi_1}^{j+1}) & -\phi({\Psi_1}^{j}) & \ldots & -\phi({\Psi_1}^{2}) & R \end{bmatrix}, \tag{61}$$

where $K=2-\phi({\Psi_1}^{2j-1})-E$, $L=2-\phi({\Psi_1}^{2j-2})-E$, $M=2-\phi({\Psi_1}^{j+1})-E$, $P=2-\phi({\Psi_1}^{j})-E$, $Q=2-\phi({\Psi_1}^{2})-E$, $R=2-\phi(\Psi_1)-E$. □

Theorem 8 Let $n=\Psi_1^K$, where $K=2j+1$, Ψ_1 is a prime and $j\geq 3$ is a positive integer. Then the Seidel signless Laplacian spectrum of the graph $W\Gamma(Z_{\Psi_1{}^{2j+1}})$ is consists of the eigenvalue $\phi(\Psi_1^{2j-i+1})-(1+E)$ with multiplicities

$\phi(\Psi_1^{2j-i+1})-1$ for $i=1, 2, 3, \ldots, j-1, j, j+1, j+2, \ldots, 2j-1, 2j$, where $E=\sum_{i=1}^{2j}\phi(\Psi_1^i)$, the remaining Seidel signless Laplacian eigenvalues of the graph $W\Gamma(Z_{\Psi_1{}^{2j+1}})$ are eigenvalues of the matrix (62).

Proof. Similarly as above Theorem 7, we can prove that the Seidel signless Laplacian spectrum of the graph $W\Gamma(Z_{\Psi_1{}^{2j+1}})$ is consists of the eigenvalue $\phi(\Psi_1^{2j-i+1})-(1+E)$ with multiplicities $\phi(\Psi_1^{2j-i+1})-1$ for $i=1, 2, 3, \ldots, j-1, j, j+1, j+2, \ldots, 2j-1, 2j$. Where $E=\sum_{i=1}^{2j}\phi(\Psi_1^i)$, The remaining, Seidel signless Laplacian eigenvalues of the graph $W\Gamma(Z_{\Psi_1{}^{2j+1}})$ are eigenvalues of the matrix (62).

$$\begin{bmatrix} K & -\phi({\Psi_1}^{2j-1}) & \ldots & -\phi({\Psi_1}^{j+1}) & -\phi({\Psi_1}^{j}) & \ldots & -\phi({\Psi_1}^{2}) & -\phi(\Psi_1) \\ -\phi({\Psi_1}^{2j}) & L & \ldots & -\phi({\Psi_1}^{j+1}) & -\phi({\Psi_1}^{j}) & \ldots & -\phi({\Psi_1}^{2}) & -\phi(\Psi_1) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ldots & \vdots & \vdots \\ -\phi({\Psi_1}^{2j}) & -\phi({\Psi_1}^{2j-1}) & \ldots & M & -\phi({\Psi_1}^{j}) & \ldots & -\phi({\Psi_1}^{2}) & -\phi(\Psi_1) \\ -\phi({\Psi_1}^{2j}) & -\phi({\Psi_1}^{2j-1}) & \ldots & -\phi({\Psi_1}^{j+1}) & P & \ldots & -\phi({\Psi_1}^{2}) & -\phi(\Psi_1) \\ \vdots & \vdots & \ldots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -\phi({\Psi_1}^{2j}) & -\phi({\Psi_1}^{2j-1}) & \ldots & -\phi({\Psi_1}^{j+1}) & -\phi({\Psi_1}^{j}) & \ldots & Q & -\phi(\Psi_1) \\ -\phi({\Psi_1}^{2j}) & -\phi({\Psi_1}^{2j-1}) & \ldots & -\phi({\Psi_1}^{j+1}) & -\phi({\Psi_1}^{j}) & \ldots & -\phi({\Psi_1}^{2}) & R \end{bmatrix}, \tag{62}$$

where $K=2-\phi({\Psi_1}^{2j})-E$, $L=2-\phi({\Psi_1}^{2j-1})-E$, $M=2-\phi({\Psi_1}^{j+1})-E$, $P=2-\phi({\Psi_1}^{j})-E$, $Q=2-\phi({\Psi_1}^{2})-E$, $R=2-\phi(\Psi_1)-E$. □

Theorem 9 For distinct primes Ψ_1, Ψ_2 where t is positive integer, if $n={\Psi_1}^t\Psi_2$ then the Seidel signless Laplacian spectrum of the graph $W\Gamma(Z_n)$ is

$$\left\{\begin{matrix} \phi(\frac{n}{\Psi_1})-(1+E) & \phi(\frac{n}{\Psi_1^2})-(1+E) & \ldots & \phi(\frac{n}{\Psi_1^t})-(1+E) & 1+\phi(\frac{n}{\Psi_2})-E & \phi(\frac{n}{\Psi_1\Psi_2})-(1+E) & \ldots \\ \phi(\frac{n}{\Psi_1})-1 & \phi(\frac{n}{\Psi_1^2})-1 & \ldots & \phi(\frac{n}{\Psi_1^t})-1 & \phi(\frac{n}{\Psi_2})-1 & \phi(\frac{n}{\Psi_1\Psi_2})-1 & \ldots \end{matrix}\right.$$

$$\left.\begin{matrix} \phi(\frac{n}{\Psi_1^{t-2}\Psi_2})-(1+E) & \phi(\frac{n}{\Psi_1^{t-1}\Psi_2})-(1+E) \\ \phi(\frac{n}{\Psi_1^{t-2}\Psi_2})-1 & \phi(\frac{n}{\Psi_1^{t-1}\Psi_2})-1 \end{matrix}\right\}.$$

Where $E=\sum_{k=1}^{t}\phi(\Psi_1^k)+\sum_{k=0}^{t-1}\phi({\Psi_1}^k\Psi_2)$, and the matrix (61) provides the remaining eigenvalues.

Proof. Let $n={\Psi_1}^t\Psi_2$, where $\Psi_1<\Psi_2$, note that $\delta^*_{{\Psi_1}^t\Psi_2}$ is complete graph on vertices $\{\Psi_1, \Psi_1^2, \ldots, {\Psi_1}^t, \Psi_2, \Psi_1\Psi_2, {\Psi_1}^2\Psi_2, \ldots, {\Psi_1}^{t-1}\Psi_2\}$. By lemma 3, we have

$$W\Gamma(Z_{{\Psi_1}^t\Psi_2})=\delta^*_{{\Psi_1}^t\Psi_2}[W\Gamma(A_{\Psi_1}), W\Gamma(A_{{\Psi_1}^2}), \ldots, W\Gamma(A_{{\Psi_1}^t}), W\Gamma(A_{\Psi_2}), W\Gamma(A_{\Psi_1\Psi_2}), , \ldots, W\Gamma(A_{{\Psi_1}^{t-1}\Psi_2})]. \tag{63}$$

Therefore, by Lemma 1 and Corollary 1, we get

$$W\Gamma(Z_{{\Psi_1}^t\Psi_2})=\delta^*_{{\Psi_1}^t\Psi_2}[K_{\phi({\Psi_1}^{t-1}\Psi_2)}, K_{\phi({\Psi_1}^{t-2}\Psi_2)}, \ldots, K_{\phi(\Psi_2)}, \overline{K}_{\phi({\Psi_1}^t)}, K_{\phi({\Psi_1}^{t-1})}, \ldots, K_{\phi(\Psi_1)}]. \tag{64}$$

And $n_{\Psi_1} = \phi({\Psi_1}^{t-1}\Psi_2), n_{{\Psi_1}^2} = \phi({\Psi_1}^{t-2}\Psi_2), \ldots, n_{{\Psi_1}^t} = \phi(\Psi_2), n_{\Psi_1\Psi_2} = \phi({\Psi_1}^{t-1}), \ldots, n_{{\Psi_1}^r\Psi_2} = \phi({\Psi_1}^{t-r}), \ldots, n_{{\Psi_1}^{t-1}\Psi_2}$ $= \phi(\Psi_1)$ and $n_{\Psi_2} = \phi({\Psi_1}^t)$. It follows that from Theorem 6, $\tau_{d_j} = n_{d_j} - E$ also $r_{\Psi_1} = \phi({\Psi_1}^{t-1}\Psi_2) - 1, r_{{\Psi_1}^2} = \phi({\Psi_1}^{t-2}\Psi_2) -$ $1, \ldots, r_{{\Psi_1}^t} = \phi(\Psi_2) - 1, r_{\Psi_1\Psi_2} = \phi({\Psi_1}^{t-1}) - 1, \ldots, r_{{\Psi_1}^r\Psi_2} = \phi({\Psi_1}^{t-r}) - 1, \ldots, r_{{\Psi_1}^{t-1}\Psi_2} = \phi(\Psi_1) - 1$ and $r_{\Psi_2} = 0$. Where $d_j \in \{\Psi_1, \Psi_1^2, \ldots, {\Psi_1}^t, \Psi_1\Psi_2, {\Psi_1}^2\Psi_2, \ldots, {\Psi_1}^{t-1}\Psi_2\}$ and $E = \sum_{k=1}^{t} \phi(\Psi_1^k) + \sum_{k=0}^{t-1} \phi({\Psi_1}^k\Psi_2)$. By Theorem 6, the Seidel signless Laplacian spectrum of the graph $W\Gamma(Z_{{\Psi_1}^t\Psi_2})$ is

$$\left\{ \begin{array}{ccccccc} \phi(\frac{n}{\Psi_1}) - (1+E) & \phi(\frac{n}{\Psi_1^2}) - (1+E) & \ldots & \phi(\frac{n}{\Psi_1^t}) - (1+E) & 1 + \phi(\frac{n}{\Psi_2}) - E & \phi(\frac{n}{\Psi_1\Psi_2}) - (1+E) & \ldots \\ \phi(\frac{n}{\Psi_1}) - 1 & \phi(\frac{n}{\Psi_1^2}) - 1 & \ldots & \phi(\frac{n}{\Psi_1^t}) - 1 & \phi(\frac{n}{\Psi_2}) - 1 & \phi(\frac{n}{\Psi_1\Psi_2}) - 1 & \ldots \end{array} \right.$$

$$\left. \begin{array}{cc} \phi(\frac{n}{\Psi_1^{t-2}\Psi_2}) - (1+E) & \phi(\frac{n}{\Psi_1^{t-1}\Psi_2}) - (1+E) \\ \phi(\frac{n}{\Psi_1^{t-2}\Psi_2}) - 1 & \phi(\frac{n}{\Psi_1^{t-1}\Psi_2}) - 1 \end{array} \right\}.$$

Where $E = \sum_{k=1}^{t} \phi(\Psi_1^k) + \sum_{k=0}^{t-1} \phi({\Psi_1}^k\Psi_2)$, the roots of the characteristic polynomial of the matrix (65) can be used to determine the remaining eigenvalues

$$\begin{bmatrix} 2-(n_{\Psi_1}+E) & -n_{\Psi_1^2} & \ldots & -n_{\Psi_1^t} & -n_{\Psi_2} & -n_{\Psi_1\Psi_2} & \ldots & -n_{{\Psi_1}^{t-1}\Psi_2} \\ -n_{\Psi_1} & 2-(n_{\Psi_1^2}+E) & \ldots & -n_{\Psi_1^t} & -n_{\Psi_2} & -n_{\Psi_1\Psi_2} & \ldots & -n_{{\Psi_1}^{t-1}\Psi_2} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -n_{\Psi_1} & -n_{\Psi_1^2} & \ldots & 2-(n_{\Psi_1^t}+E) & -n_{\Psi_2} & -n_{\Psi_1\Psi_2} & \ldots & -n_{{\Psi_1}^{t-1}\Psi_2} \\ -n_{\Psi_1} & n_{\Psi_1^2} & \ldots & -n_{\Psi_1^t} & 3n_{\Psi_2}-(2+E) & -n_{\Psi_1\Psi_2} & \ldots & -n_{{\Psi_1}^{t-1}\Psi_2} \\ -n_{\Psi_1} & -n_{\Psi_1^2} & \ldots & -n_{\Psi_1^t} & -n_{\Psi_2} & B' & \ldots & -n_{{\Psi_1}^{t-1}\Psi_2} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -n_{\Psi_1} & -n_{\Psi_1^2} & \ldots & -n_{\Psi_1^t} & -n_{\Psi_2} & -n_{\Psi_1\Psi_2} & \ldots & A' \end{bmatrix}, \tag{65}$$

where $A' = 2 - (n_{{\Psi_1}^{t-1}\Psi_2} + E), B' = 2 - (n_{\Psi_1\Psi_2} + E)$. □

Theorem 10 Let $n = \Psi_1\Psi_2 \ldots \Psi_t \eta_1^{d_1} \eta_2^{d_2} \ldots \eta_s^{d_s} (d_i \geq 2, t \geq 1, s \geq 0)$ where Ψ_i's and η_i's are the distinct primes. Let $\beta = \{\Psi_1, \Psi_2, \ldots, \Psi_t\}$ and $\{c_1, c_2, \ldots, c_{\tau(n)-2}\}$ represents the collection of all proper divisors of n. Then, the Seidel signless Laplacian spectrum $W\Gamma(Z_n)$ consists of eigenvalues $\phi(\frac{n}{c_j}) - (1+E)$ with multiplicity $\phi(\frac{n}{c_j}) - 1$ when $c_j \notin \beta$ and $1 + \phi(\frac{n}{c_i}) - E$ with multiplicity $\phi(\frac{n}{c_i}) - 1$ when $c_i \in \beta$ for $1 \leq i, j \leq \tau(n) - 2$, where $E = \sum_{i=1}^{\tau(n)-2} \phi(\frac{n}{c_i})$. And the characteristic polynomial of the matrix (66) provides the remaining, eigenvalues.

Proof. Suppose that $n = \Psi_1\Psi_2 \ldots \Psi_t \eta_1^{d_1} \eta_2^{d_2} \ldots \eta_s^{d_s} (d_i \geq 2, t \geq 1, s \geq 0)$ where Ψ_i's and η_i's are the distinct primes. Let $\beta = \{\Psi_1, \Psi_2, \ldots, \Psi_t\}$ and $\{c_1, c_2, \ldots, c_{\tau(n)-2}\}$ represents the collection of all proper divisors of n. Now by Lemma 2, the following conclusions can be drawn: for each $c_i \in \beta$, we have $W\Gamma(A_{c_i}) = \overline{K}_{\phi(\frac{n}{c_i})}$ and for $c_j \notin \beta$ we have $W\Gamma(A_{c_j}) = K_{\phi(\frac{n}{c_j})}$.

$$Y=\begin{bmatrix} 2(n_1-2r_1-1)+n_1-E & \cdots & -n_{c_t} & -n_{c_{t+1}} & \cdots & -n_{\tau(n)-2} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ -n_{c_1} & \cdots & 2(n_{c_t}-2r_{c_t}-1)+n_{c_t}-E & -n_{c_{t+1}} & \cdots & -n_{\tau(n)-2} \\ -n_{c_1} & \cdots & -n_{c_t} & G' & \cdots & -n_{\tau(n)-2} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -n_{c_1} & \cdots & -n_{c_t} & -n_{c_{t+1}} & \cdots & F' \end{bmatrix}, \tag{66}$$

where $G'=2(n_{c_{t+1}}-2r_{c_{t+1}}-1)+n_{c_{t+1}}-E$ and $F'=2(n_{\tau(n)-2}-2r_{\tau(n)-2}-1)+n_{\tau(n)-2}-E$.

For $1\leq i,\ j\leq \tau(n)-2$, we have, $n_{c_i}=\phi(\frac{n}{c_i})$, $n_{c_j}=\phi(\frac{n}{c_j})$ for all $c_i\in\beta$ and $c_j\notin\beta$. It follows that from Theorem 6, $\tau_{c_i}=n_{c_i}-E$. Also, $r_{c_i}=0$ for $c_i\in\beta$ and $r_{c_j}=\phi(\frac{n}{c_j})-1$ for $c_j\notin\beta$ where $E=\sum_{i=1}^{\tau(n)-2}\phi(\frac{n}{c_i})$. By Theorem 6, the Seidel signless Laplacian spectrum $W\Gamma(Z_n)$ consists of eigenvalues $\phi(\frac{n}{c_j})-(1+E)$ with multiplicity $\phi(\frac{n}{c_j})-1$ when $c_j\notin\beta$ and $1+\phi(\frac{n}{c_i})-E$ with multiplicity $\phi(\frac{n}{c_i})-1$ when $c_i\in\beta$ for $1\leq i,\ j\leq\tau(n)-2$. The roots of the characteristic polynomial of the matrix (66) can be used to determine the remaining eigenvalues. □

Example 8 Seidel signless Laplacian spectrum of the graph $W\Gamma(Z_{66})$ is shown in Figure 2, is

$$\begin{Bmatrix} -24 & -34 & -42 & -36 & -44 & -45 \\ 19 & 9 & 1 & 9 & 1 & 0 \end{Bmatrix}. \tag{67}$$

The remaining six Seidel signless Laplacian eigenvalues of the graph $W\Gamma(Z_{66})$ are the eigenvalues of the matrix (69).

The proper divisors of 66 are 2, 3, 11, 6, 22 and 33. Note that δ^*_{66} is complete graph on vertices 2, 3, 11, 6, 22 and 33 Now by Lemma 3, we have $W\Gamma(Z_{66})=\delta^*_{66}[W\Gamma(A_2), W\Gamma(A_3), W\Gamma(A_{11}), W\Gamma(A_6), W\Gamma(A_{22}), W\Gamma(A_{33})]$. Therefore, by Lemma 1 and Corollary 1, we have $W\Gamma(A_2)=\overline{K_{20}}$, $W\Gamma(A_3)=\overline{K}_{10}$, $W\Gamma(A_{11})=\overline{K}_2$, $W\Gamma(A_6)=K_{10}$, $W\Gamma(A_{22})=K_2$ and $W\Gamma(A_{33})=K_1$. Now according the proper divisor sequence, we have, $n_1=20$, $n_2=10$, $n_3=2$, $n_4=10$, $n_5=2$, $n_6=1$. Further, we have $r_1=r_2=r_3=0$, $r_4=9$, $r_5=1$ and $r_6=0$. And by using Theorem 6, the value of $\tau_i=n_i-E$ for $1\leq i\leq 6$, where $E=45$.

Consequently, Seidel signless Laplacian spectrum of the graph $W\Gamma(Z_{66})$ is given by Theorem 6,

$$\begin{Bmatrix} -24 & -34 & -42 & -36 & -44 & -45 \\ 19 & 9 & 1 & 9 & 1 & 0 \end{Bmatrix}. \tag{68}$$

And eigenvalues of the matrix (69) are respectively $\{-43, -43, -66.340, -39.486, -15.688, 20.514\}$.

$$M=\begin{bmatrix} 13 & -10 & -2 & -10 & -2 & -1 \\ -20 & -17 & -2 & -10 & -2 & -1 \\ -20 & -10 & -41 & -10 & -2 & -1 \\ -20 & -10 & -2 & -53 & -2 & -1 \\ -20 & -10 & -2 & -10 & -45 & -1 \\ -20 & -10 & -2 & -10 & -2 & -44 \end{bmatrix}. \tag{69}$$

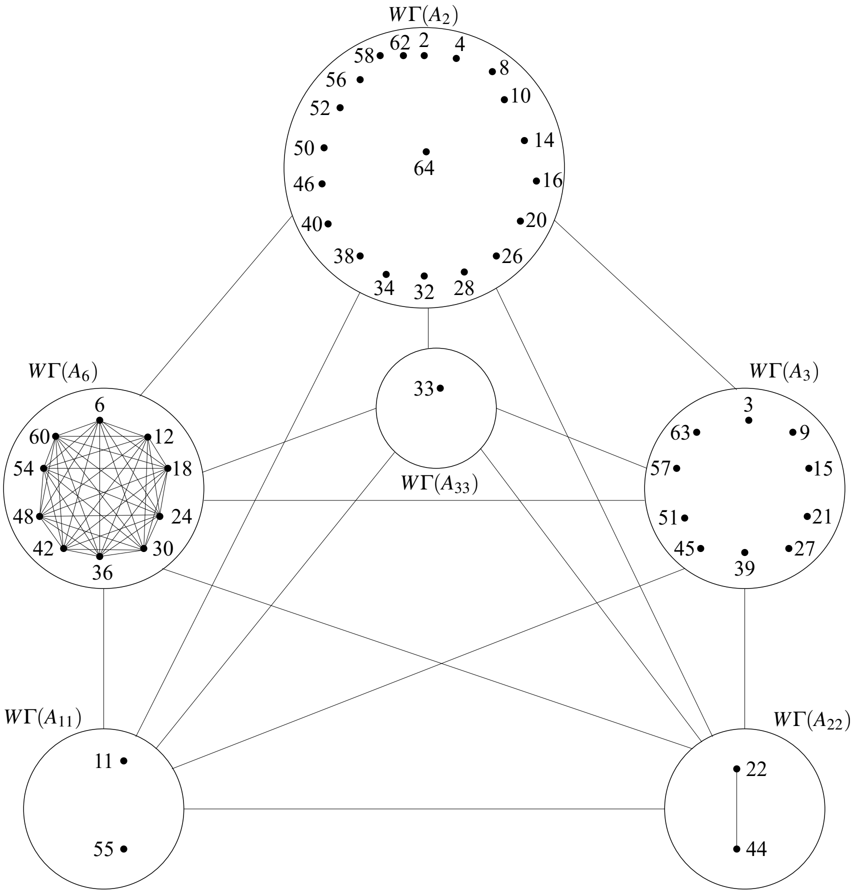

Figure 2. Weakly zero-divisor graph $W\Gamma(\mathbb{Z}_{66})$

5. Conclusion

Our result gives Seidel Laplacian and Seidel signless Laplacian spectrum of the weakly zero-divisor graph of integer modulo n by using the generalized join graph of induced subgraphs.

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Conflict of interest

The authors declare no competing financial interest.

References

[1] Van Lint JH, Seidel JJ. Equilateral point sets in elliptic geometry. *Indagationes Mathematicae*. 1966; 28: 335-348.

[2] Magi PM, Magie JS, Anjaly K. Seidel Laplacian and signless Laplacian spectrum of the zero-divisor graph on the ring of integers modulo n. *Mathematics and Statistics*. 2021; 9(6): 917-926. Available from: http://doi.org/10.13189/ms.2021.090607.

[3] Nikmehr MJ, Azadi A, Nikandish R. The weakly zero-divisor graph of a commutative ring. *Journal of the Argentine Mathematical Union*. 2021; 62(1): 105-116. Available from: https://doi.org/10.33044/revuma.1677.

[4] Anwar M, Mozumder MR, Rashid M, Raza MA. Somber index and somber spectrum of cozero-divisor graph of $\mathbb{Z}_n$. *Results in Mathematics*. 2024; 79(4): 146. Available from: https://doi.org/10.1007/s00025-024-02174-8.

[5] Rehman N, Nazim, Alghamdi AM, Almotairi ES. Randić spectrum of the weakly zero-divisor graph of the ring $\mathbb{Z}_\mathrm{n}$. *AKCE International Journal of Graphs and Combinatorics*. 2024; 21(3): 302-309. Available from: https://doi.org/10.1080/09728600.2024.2358360.

[6] Rashid M, Alali AS, Mozumder MR, Ahmed W. Spectrum of the cozero-divisor graph associated to ring $\mathbb{Z}_n$. *Axioms*. 2023; 12(10): 957. Available from: https://doi.org/10.3390/axioms12100957.

[7] Rashid M, Mozumder MR, Anwar M. Signless Laplacian spectrum of the cozero-divisor graph of the commutative ring $\mathbb{Z}_n$. *Georgian Mathematical Journal*. 2024; 31(4): 687-699. Available from: https://doi.org/10.1515/gmj-2023-2098.

[8] Shariq M, Mathil P, Kumar J. Laplacian spectrum of weakly zero-divisor graph of the ring $\mathbb{Z}_n$. *arXiv:230712757*. 2023. Available from: https://doi.org/10.48550/arXiv.2307.12757.

[9] Young M. Adjacency matrices of zero-divisor graphs of integer modulo n. *Involve, a Journal of Mathematics*. 2015; 8: 753-761. Available from: https://doi.org/10.2140/involve.2015.8.753.