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Research Article

New Stirling-Type Approximations for the Gamma Function

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Abstract: This paper presents a set of new and precise Stirling-type formulas that approximate the Gamma function and provide associated sharp bounds. The proposed approximations are constructed using ratios of monic polynomials of even degree. In conclusion, an open question is raised regarding the general structure of such Gamma function approximations.

***Keywords*:** gamma function, Stirling's formula, rate of convergence, inequalities

MSC: 33B15, 41A60, 41A21, 26D15

1. Introduction

The gamma function:

$$\Gamma(x+1) = \int_0^\infty t^x e^{-t} dt\ , \quad x > 0$$

is a powerful mathematical tool that generalizes the factorial function to real and complex numbers. Although it originates from pure mathematics, its applications extend far beyond, playing a significant role in various fields of science. However, the gamma function is not straightforward to compute for arbitrary inputs, as it lacks a simple closed-form expression. This complexity has led to the development of bounding techniques and approximations that make it more accessible for scientists.

Undoubtedly, one of the best-known and most widely used formulas for estimating the factorial function and its extension, the gamma function, is the Stirling's formula:

$$\Gamma(n+1) \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n, \quad \text{as } n \to \infty.$$

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It was first introduced in the early 18th century by the Scottish mathematician James Stirling, who published it in his 1730 work Methodus Differentialis. This approximation laid the groundwork for many developments in analysis and probability theory and remains a cornerstone in the study of special functions, including the gamma function. For example, the Beta function can be expressed as a combination of Gamma functions, $B(x, y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$, and the digamma and polygamma functions are derived from the logarithm of the Gamma function. In addition, functions such as hypergeometric or Bessel-type functions frequently involve Gamma function ratios in their definitions or asymptotic expansions. Therefore, obtaining accurate Stirling-type approximations for the Gamma function is crucial, not only for theoretical analysis but also for improving the precision of related functions. Recent works [1–5] highlight the usefulness of such approximations in both analytical and numerical applications.

2. The results

Since the discovery of Stirling's formula, numerous refinements have been made by researchers. We start our study with a recent result of Mahmoud et al. [6], who added a factor to improve Stirling's formula:

$$\Gamma(n+1) \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(\frac{n^2+\frac{7}{60}}{n^2-\frac{1}{20}}\right)^{\frac{n}{2}}, \quad n \in \mathbb{N}. \tag{1}$$

In the beginning, we consider the following family of approximations in this paper, for $a, b, c \in \mathbb{R}$:

$$\Gamma(n+1) \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(\frac{n^4+an^2+b}{n^4+c}\right)^{\frac{n}{4}}. \tag{2}$$

We will find the parameters a, b, and c that give the most accurate approximation (2). We also prove that approximation is better than the Mahmoud approximation (1).

Usually, to an approximation formula of the form

$$f(n) \approx g(n), \quad \text{as } n \to \infty, \tag{3}$$

(in the sense that $\lim_{n\to\infty} f(n)/g(n) = 1$) is associated the sequence

$$\omega_n = \ln \frac{f(n)}{g(n)}.$$

Evidently, the greater the rate of convergence of the sequence ω_n, the more precise the resulting approximation (3).

A main tool for measuring the rate of convergence of a sequence ω_n is the following result, first stated in this form by Mortici [7].

Lemma 1 If $(\omega_n)_{n\geq 1}$ is convergent to zero and

$$\lim_{n\to\infty} n^k(\omega_n - \omega_{n+1}) = l, \quad 0 \neq l \in [-\infty, \infty]$$

for some $k > 1$, then

$$\lim_{n \to \infty} n^{k-1} \omega_n = \frac{l}{k-1}. \tag{4}$$

This lemma has been proved to be useful in the problem of finding some approximation formulas, or to accelerate some convergences. See, e.g., [1, 2, 8, 9] and all reference therein. If the limit (4) holds true with $l \neq 0$, then the sequence ω_n converges to zero like $n^{-(k-1)}$, and we say that the approximation formula (3) is of order $n^{-(k-1)}$. Clearly, the higher k, the more accurate approximation formula (3).

As an example, for Mansour et al. formula (1), we have:

$$\omega_n = \ln \frac{\Gamma(n+1)}{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(\frac{n^2+\frac{7}{60}}{n^2-\frac{1}{20}}\right)^{\frac{n}{2}}},$$

or

$$\omega_n = \ln \Gamma(n+1) - \frac{1}{2} \ln 2\pi n - n \ln n + n - \frac{n}{2} \ln \frac{n^2 + \frac{7}{60}}{n^2 - \frac{1}{20}}.$$

Thus

$$\omega_n - \omega_{n+1} = \frac{461}{181\,440 n^6} - \frac{461}{60\,480 n^7} + O\left(\frac{1}{n^8}\right)$$

and

$$\lim_{n \to \infty} n^6 (\omega_n - \omega_{n+1}) = \frac{461}{181\,440}.$$

According to Lemma 1, $\lim_{n \to \infty} n^5 \omega_n = \frac{461}{907\,200}$, and consequently, the approximation formula (1) is of order n^{-5}.

Remark. The selection of the form (2) is motivated by its natural extension of previous quadratic refinements (see [1, 2]) and by the fact that the exponent $n/4$ allows the systematic cancellation of low-order terms in the expansion of $w_n - w_{n+1}$, as guaranteed by Lemma 1. This choice thus balances simplicity with increased accuracy.

Coming back to the family (2), we define the sequence

$$w_n = \ln \frac{\Gamma(n+1)}{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(\frac{n^4+an^2+b}{n^4+c}\right)^{\frac{n}{4}}}.$$

We have

$$w_n = \ln\Gamma(n+1) - \frac{1}{2}\ln 2\pi n - n\ln n + n - \frac{n}{4}\ln\left(\frac{n^4+an^2+b}{n^4+c}\right),$$

with

$$w_n - w_{n+1} = -\ln(n+1) - \frac{1}{2}\ln\frac{n}{n+1} - n\ln n + (n+1)\ln(n+1) - 1$$

$$-\frac{n}{4}\ln\frac{n^4+an^2+b}{n^4+c} + \frac{n+1}{4}\ln\frac{(n+1)^4+a(n+1)^2+b}{(n+1)^4+c}.$$

To use Lemma 1, we write the difference $w_n - w_{n+1}$ as a power series in n^{-1}:

$$w_n - w_{n+1} = -\left(\frac{1}{4}a - \frac{1}{12}\right)\frac{1}{n^2} + \left(\frac{1}{4}a - \frac{1}{12}\right)\frac{1}{n^3}$$

$$+\left(\frac{3}{8}a^2 - \frac{1}{4}a - \frac{3}{4}b + \frac{3}{4}c + \frac{3}{40}\right)\frac{1}{n^4}$$

$$-\left(\frac{3}{4}a^2 - \frac{1}{4}a - \frac{3}{2}b + \frac{3}{2}c + \frac{1}{15}\right)\frac{1}{n^5}$$

$$-\left(\frac{1}{4}a + \frac{5}{2}b - \frac{5}{2}c - \frac{5}{4}ab - \frac{5}{4}a^2 + \frac{5}{12}a^3 - \frac{5}{84}\right)\frac{1}{n^6}$$

$$-\left(\frac{15}{4}c - \frac{15}{4}b - \frac{1}{4}a + \frac{15}{4}ab + \frac{15}{8}a^2 - \frac{5}{4}a^3 + \frac{3}{56}\right)\frac{1}{n^7}$$

$$-\left(-\frac{7}{16}a^4 + \frac{35}{12}a^3 + \frac{7}{4}a^2b - \frac{21}{8}a^2 - \frac{35}{4}ab + \frac{1}{4}a - \frac{7}{8}b^2\right.$$

$$\left.+\frac{21}{4}b + \frac{7}{8}c^2 - \frac{21}{4}c - \frac{7}{144}\right)\frac{1}{n^8} + O\left(\frac{1}{n^9}\right).$$

Now we are in a position to give the following:

Theorem 1 a) For all $a, b, c \in \mathbb{R},\ a \neq 1/3,$ the approximation formula (2) is of order $n^{-1},$ since

$$\lim_{n\to\infty} n^2(w_n - w_{n+1}) = -\frac{1}{4}a + \frac{1}{12} \quad\text{and}\quad \lim_{n\to\infty} nw_n = -\frac{1}{4}a + \frac{1}{12} \neq 0.$$

b) For all $b, c \in \mathbb{R},\ c \neq b - \frac{2}{45},$ the approximation formula

$$\Gamma(n+1) \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(\frac{n^4 + \frac{1}{3}n^2 + b}{n^4 + c}\right)^{\frac{n}{4}}$$

is of order n^{-3}, since

$$\lim_{n\to\infty} n^4 (w_n - w_{n+1}) = \frac{3}{4}c - \frac{3}{4}b + \frac{1}{30} \text{ and } \lim_{n\to\infty} n^3 w_n = \frac{1}{3}\left(\frac{3}{4}c - \frac{3}{4}b + \frac{1}{30}\right) \neq 0.$$

c) For all $b \in \mathbb{R}$, $b \neq \frac{26}{945}$, the approximation formula

$$\Gamma(n+1) \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(\frac{n^4 + \frac{1}{3}n^2 + b}{n^4 + b - \frac{2}{45}}\right)^{\frac{n}{4}}$$

is of order n^{-5}, since

$$\lim_{n\to\infty} n^6 (w_n - w_{n+1}) = \frac{5}{12}b - \frac{13}{1134} \quad \text{and} \quad \lim_{n\to\infty} n^3 w_n = \frac{1}{5}\left(\frac{5}{12}b - \frac{13}{1134}\right) \neq 0.$$

d) The approximation formula

$$\Gamma(n+1) \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(\frac{n^4 + \frac{1}{3}n^2 + \frac{26}{945}}{n^4 - \frac{16}{945}}\right)^{\frac{n}{4}}$$

(obtained from (2) for $a = 1/3$, $b = 26/945$, $c = -16/945$) is of order n^{-7}, since

$$\lim_{n\to\infty} n^8 (w_n - w_{n+1}) = -\frac{1}{270} \quad \text{and} \quad \lim_{n\to\infty} n^7 w_n = -\frac{1}{1890}.$$

Note that the best constants

$$a = \frac{1}{3},\; b = \frac{26}{945},\; c = -\frac{16}{945}$$

in case d), are obtained from the condition that as many as possible of the first coefficients in the series $w_n - w_{n+1}$ are canceled. More precisely, they are the solution of the system:

$$\begin{cases} \frac{1}{4}a-\frac{1}{12}=0 \\ \frac{3}{8}a^2-\frac{1}{4}a-\frac{3}{4}b+\frac{3}{4}c+\frac{3}{40}=0 \\ \frac{3}{4}a^2-\frac{1}{4}a-\frac{3}{2}b+\frac{3}{2}c+\frac{1}{15}=0 \\ \frac{1}{4}a+\frac{5}{2}b-\frac{5}{2}c-\frac{5}{4}ab-\frac{5}{4}a^2+\frac{5}{12}a^3-\frac{5}{84}=0 \end{cases}.$$

The other cases a)-c) follow by successively vanishing the first three coefficients in the series of $w_n - w_{n+1}$.

3. Further refinements

We also considered, for all parameters α, β, and δ, the following family of approximations, as $n \to \infty$:

$$\Gamma(n+1) \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(\frac{n^6+\alpha n^4+\beta n^2}{n^6+\delta}\right)^{\frac{n}{6}} \tag{5}$$

$$\Gamma(n+1) \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(\frac{n^8+\alpha n^6+\beta n^4}{n^8+\delta n^2}\right)^{\frac{n}{8}}. \tag{6}$$

By using the same method, we obtained the following approximation formulas:

$$\Gamma(n+1) \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(\frac{n^6+\frac{1}{2}n^4+\frac{13}{120}n^2}{n^6-\frac{29}{1680}}\right)^{\frac{n}{6}} \tag{7}$$

$$\Gamma(n+1) \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(\frac{n^8+\frac{2}{3}n^6+\frac{1}{5}n^4}{n^8-\frac{116}{2835}n^2}\right)^{\frac{n}{8}}, \tag{8}$$

which are the best possible through the corresponding family (5)-(6).

Remark. Equations (5), (6), and (8) arise by expanding the logarithmic terms in powers of $1/n$ and applying Lemma 1 to ensure the successive cancellation of coefficients. The resulting polynomial forms are minimal in complexity while providing systematic accuracy improvements, which explains their particular selection among many possible candidates.

Note that after a simplification by n^2, the formula (8) can be written in the form:

$$\Gamma(n+1) \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(\frac{n^6+\frac{2}{3}n^4+\frac{1}{5}n^2}{n^6-\frac{116}{2835}}\right)^{\frac{n}{8}}.$$

We propose the following open question: for integers $k \geq 1$, find the approximation of the form:

$$\Gamma(n+1) \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(\frac{P_{2k}(n)}{Q_{2k}(n)}\right)^{\frac{n}{2k}}, \quad \text{as } n \to \infty, \tag{9}$$

where P_{2k} and Q_{2k} are monic polynomials of $2k$-th degree. The requested polynomials are supposed to be the best possible, in the sense that the corresponding approximation (9) is the most accurate.

Note that we solved here the case $k = 2, 3, 4$ of (9), while previously, Mansour et al. [6] found the approximation formula (1) in case $k = 1$. The approximation formulas (9) are of order n^{-5} in case $k = 1,$ and of order n^{-7} in other cases $k = 2, 3, 4$.

We mention here another idea for using the previous method and Lemma 1. More precisely, we considered the following family of approximations, as $n \to \infty$:

$$\Gamma(n+1) \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(\frac{n^2+a}{n^2+b}\right)^{\frac{n}{2}+\frac{c}{n}}, \tag{10}$$

where $a, b, c \in \mathbb{R}$. Note that Mansour et al.'s approximation formula (1) is obtained as a particular case of (10) for

$$a = \frac{7}{60},\ b = -\frac{1}{20},\ c = 0.$$

Private computations lead us to the best constants

$$a = \frac{151}{504}, b = \frac{67}{504}, c = \frac{461}{5040}$$

that give the most accurate approximation formula (10):

$$\Gamma(n+1) \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(\frac{n^2+\frac{151}{504}}{n^2+\frac{67}{504}}\right)^{\frac{n}{2}+\frac{461}{5040n}}, \quad n \to \infty.$$

This is an approximation of order $n^{-7},$ since the associated sequence

$$W_n = \ln \frac{\Gamma(n+1)}{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(\frac{n^2+\frac{151}{504}}{n^2+\frac{67}{504}}\right)^{\frac{n}{2}+\frac{461}{5040n}}}$$

satisfies:

$$W_n - W_{n+1} = -\frac{34\,333}{13\,063\,680 n^8} + O\left(\frac{1}{n^9}\right), \quad \text{as } n \to \infty.$$

We omit the proofs in this section, for the sake of simplicity.

We give the following:

Theorem 2 The following inequality holds

$$c_1 \leq \frac{\Gamma(x+1)}{\sqrt{2\pi x}\left(\frac{x}{e}\right)^x \left(\frac{x^2+\frac{151}{504}}{x^2+\frac{67}{504}}\right)^{\frac{x}{2}+\frac{461}{5040x}}} < c_2, \quad x \geq 1 \tag{11}$$

with sharp constants $c_1 = \frac{\left(\frac{571}{655}\right)^{2981/5040} e}{\sqrt{2\pi}} \approx 0.999885$ and $c_2 = 1$.

Proof. Consider the function

$$K_1(x) = \ln\left(\frac{\Gamma(x+1)}{\sqrt{2\pi x}\left(\frac{x}{e}\right)^x \left(\frac{x^2+\frac{151}{504}}{x^2+\frac{67}{504}}\right)^{\frac{x}{2}+\frac{461}{5040x}}}\right)$$

and hence

$$K_1'(x) = \frac{-1270080x^4+423360x^3-549360x^2+77448x-50585}{10x\left(504x^2+67\right)\left(504x^2+151\right)}$$

$$+\frac{\left(461-2520x^2\right)\ln\left(\frac{x^2+\frac{151}{504}}{x^2+\frac{67}{504}}\right)}{5040x^2}+\psi(x)-\ln x.$$

Now, consider the function

$$K_2(x) = \frac{5x(7x(8430x-17)+32682)}{30x(x(14x(1405x-237)+11447)-1580)+6241} - \ln x + \psi(x)$$

which satisfies

$$\frac{d}{dx}[K_2(x+2)-K_2(x+1)] = \frac{K_3(x)}{K_4(x)} > 0, \quad x \geq 0$$

where

$$K_3(x) = 1.69847\times 10^{22}x^9 + 2.08731\times 10^{23}x^8 + 1.13303\times 10^{24}x^7$$

$$+3.56209\times 10^{24}x^6 + 7.14298\times 10^{24}x^5 + 9.4691\times 10^{24}x^4$$

$$+8.2897\times10^{24}x^3+4.61086\times10^{24}x^2+1.47263\times10^{24}x+2.04522\times10^{23}$$

and

$$\begin{aligned}K_4(x)=&1.21256\times10^{23}x^{19}+3.31335\times10^{24}x^{18}+4.28782\times10^{25}x^{17}\\&+3.49292\times10^{26}x^{16}+2.00803\times10^{27}x^{15}+8.65624\times10^{27}x^{14}\\&+2.90203\times10^{28}x^{13}+7.74376\times10^{28}x^{12}+1.66925\times10^{29}x^{11}\\&+2.93288\times10^{29}x^{10}+4.2182\times10^{29}x^{9}+4.96684\times10^{29}x^{8}\\&+4.76884\times10^{29}x^{7}+3.7012\times10^{29}x^{6}+2.28822\times10^{29}x^{5}\\&+1.10116\times10^{29}x^{4}+3.97696\times10^{28}x^{3}+1.01437\times10^{28}x^{2}\\&+1.62938\times10^{27}x+1.23965\times10^{26}.\end{aligned}$$

Using the asymptotic expansion of $\psi(x)$, we get

$$K_2(x+2)-K_2(x+1)=\frac{-964337}{61960500x^9}+\frac{332605549}{1579220000x^{10}}+O(x^{-11}),\quad x\to\infty$$

and hence $\lim_{x\to\infty}[K_2(x+2)-K_2(x+1)]=0$. Then

$$K_2(x+2)-K_2(x+1)<0,\quad x\geq0$$

or

$$K_2(x+1)>K_2(x+2)>\cdots>K_2(x+i),\quad i\in N;\ x\geq0.$$

But

$$K_2(x)=\frac{964337}{495684000x^8}-\frac{38950081}{696436020000x^9}+O(x^{-10}),\quad x\to\infty$$

and hence $\lim_{x\to\infty}K_2(x)=0$. Therefore $K_2(x)>0$ for $x\geq1$, or

$$\psi(x) > \ln x - \frac{5x(7x(8430x-17)+32682)}{30x(x(14x(1405x-237)+11447)-1580)+6241}, \quad x \ge 1. \tag{12}$$

Using inequality (12), we have

$$K_1'(x) > \left(\frac{461}{5040x^2} - \frac{1}{2}\right)\left[\ln\left(\frac{x^2+\frac{151}{504}}{x^2+\frac{67}{504}}\right) - \frac{K_5(x)}{K_6(x)}\right] =: \left(\frac{461}{5040x^2} - \frac{1}{2}\right)K_7(x),$$

where

$$\begin{aligned} K_5(x) =& 504x(124912368000x^7 - 21070627200x^6 + 76856774400x^5 \\ & - 10735986240x^4 - 1776091310x^3 + 596725710x^2 - 1914376032x \\ & + 315700985) \end{aligned}$$

and

$$\begin{aligned} K_6(x) = & \left(2520x^2-461\right)\left(590100x^4 - 99540x^3 + 343410x^2 - 47400x \right. \\ & \left. +6241\right)\left(504x^2+67\right)\left(504x^2+151\right). \end{aligned}$$

Now

$$K_7'(x+1) = \frac{K_8(x)}{K_9(x)} > 0, \quad x \ge 0$$

where

$$\begin{aligned} K_8(x) =& 504(3.0922\times10^{24}x^{11} + 3.3256\times10^{25}x^{10} + 1.64367\times10^{26}x^9 \\ & + 4.92624\times10^{26}x^8 + 9.94264\times10^{26}x^7 + 1.41806\times10^{27}x^6 \\ & + 1.45737\times10^{27}x^5 + 1.07849\times10^{27}x^4 + 5.62792\times10^{26}x^3 \\ & + 1.9709\times10^{26}x^2 + 4.16576\times10^{25}x + 4.02278\times10^{24}) \end{aligned}$$

and

$$K_9(x) = \left(504x^2+1008x+571\right)^2\left(504x^2+1008x+655\right)^2\left(2520x^2+5040x\right.$$

$$\left.+2059\right)^2\left(590100x^4+2260860x^3+3585390x^2+2701200x+792811\right)^2.$$

Also,

$$K_7(x) = -\frac{438544523}{321203232000x^8} - \frac{38950081}{348218010000x^9} + O(x^{-10}), \quad x\to\infty$$

and hence $\lim_{x\to\infty} K_7(x) = 0$. Therefore $K_7(x) < 0$ for $x \geq 1$, then $K_1'(x) > 0$ for $x \geq 1$. Also,

$$K_1(x) = \frac{198128243}{253487646720x^9} - \frac{34333}{91445760x^7} + O(x^{-11})$$

and hence $\lim_{x\to\infty} K_1(x) = 0$. Therefore $K_1(x) > 0$ for $x \geq 1$ and

$$\ln\left(\frac{\left(\frac{571}{655}\right)^{2981/5040} e}{\sqrt{2\pi}}\right) = K_1(1) < K_1(x) < K_1(\infty) = 0,$$

with sharp bounds which completes the proof. □

Remark. Beyond the specific formulas obtained here, such Stirling-type approximations have several applications in analytic number theory: they refine bounds for factorials and binomial coefficients, improve asymptotic estimates of partition functions and divisor sums, and sharpen evaluations in formulas involving the Riemann zeta function. Moreover, the general scheme (9) provides a framework for constructing entire hierarchies of refined approximations, suggesting new directions for inequalities and computational methods.

4. A monotonicity result

Related to the approximation formula (1), Mansour et al. [6, Theorem 1] proved that the function

$$P_1(x) = \frac{\Gamma(x+1)}{\sqrt{2\pi x}\left(\frac{x}{e}\right)^x\left(\frac{x^2+\frac{7}{60}}{x^2-\frac{1}{20}}\right)^{\frac{x}{2}}}$$

is strictly decreasing for $x \geq 1$. We give here an alternate solution for this fact. In this sense, let us define the function

$$f(x) = \ln P_1(x) = \ln\Gamma(x+1) - \frac{1}{2}\ln 2\pi x - x\ln x + x - \frac{x}{2}\ln\frac{x^2+\frac{7}{60}}{x^2-\frac{1}{20}}.$$

We have:

$$f''(x) = \psi'(x) - \frac{1}{x^2} - \frac{P(x)}{2x^2 (60x^2+7)^2 (20x^2-1)^2},$$

where

$$P(x) = 98x + 1120x^2 + 6160x^3 + 10400x^4 - 52\,800x^5$$

$$- 192\,000x^6 + 864\,000x^7 - 1440\,000x^8 + 2880\,000x^9 - 49.$$

Here, ψ is the logarithmic derivative of the gamma function

$$\psi(x) = (\ln\Gamma(x))' = \frac{\Gamma'(x)}{\Gamma(x)}.$$

The function ψ is also called the digamma function, while its derivative ψ' is known as the trigamma function and admits the following asymptotic expansion as $x \to \infty$:

$$\psi'(x) = \frac{1}{x} + \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^5} + \frac{1}{42x^7} - \frac{1}{30x^9} + \frac{5}{66x^{11}} + \dots .$$

The complete series can be expressed in terms of Bernoulli numbers. Successive lower and upper approximations of ψ' can be obtained by truncating its series. What we need here is the inequality:

$$\psi'(x) > \frac{1}{x} + \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^5} + \frac{1}{42x^7} - \frac{1}{30x^9}. \tag{13}$$

For proofs and further properties, please see [10].

Now we deduce that $f''(x) > g(x)$, where

$$g(x) = \frac{1}{x} + \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^5} + \frac{1}{42x^7} - \frac{1}{30x^9}$$

$$- \frac{1}{x^2} - \frac{P(x)}{2x^2 (60x^2+7)^2 (20x^2-1)^2}.$$

We have

$$g(x) = \frac{t(x)}{210x^9 (60x^2+7)^2 (20x^2-1)^2},$$

where

$$
\begin{aligned}
t(x+2) =& 1890000x^{14} + 52920000x^{13} + 687960000x^{12} + 5503680000x^{11} \\
& + 30274850000x^{10} + 121173160000x^{9} + 364063593600x^{8} \\
& + 834549657600x^{7} + 1467442056755x^{6} + 1970383055460x^{5} \\
& + 1988993872157x^{4} + 1463247554856x^{3} + 740964277053x^{2} \\
& + 230795020724x + 33272650829 > 0, \quad x \geq 0.
\end{aligned}
$$

Evidently, $g(x) > 0$, then $f''(x) > 0$, for $x \geq 2$. Thus f' is strictly increasing, for $x \geq 2$. As $f'(2) = 1.3456... > 0$, we deduce that $f'(x) > 0$, for $x \geq 2$. In conclusion, f is strictly increasing on $[2, \infty)$.

The function f is also strictly increasing on $[1, \infty)$ (as Theorem 1 in [6] asserts) and this fact can be proven using the above method by considering more terms in (13) from the truncation of the trigamma function ψ''.

Note that, moreover, the function P_1 is logarithmically convex, as $f = \ln P_1$ is convex.

Remark 1 Some calculations in this paper were performed by using the Maple software for symbolic computation.

5. Conclusion

The main objective of this work is to develop highly accurate and efficient approximations for the Gamma function. These approximations have applications in mathematics, statistics, physics, and engineering, wherever the Gamma function arises in computations or modeling. The main conclusions of this study are presented in Theorem 1 and formulas (7) and (8), where we provide several highly accurate approximation formulas for the Gamma function. Theorem 2 presents some new sharp bounds of the Gamma function. Some of our new approximations achieve a rate of convergence of order n^{-7}, which is significantly better than the classical Stirling's formula of order n^{-1} and the recent Mahmoud et al. formula (1) of order n^{-5}. The methods used in this work can be adapted to other special functions, such as the polygamma and incomplete Gamma functions. Finally, we propose an open question concerning the general form of several approximations of the Gamma function.

Conflict of interest

The authors declare no competing financial interest.

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