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Research Article

A Priori and a Posteriori Error Estimates of Conforming Discontinuous Galerkin Method for Elliptic Problems

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Abstract: The Conforming Discontinuous Galerkin (CDG) method is an innovative numerical approach for solving partial differential equations. Based on the Weak Galerkin (WG) method, it simplifies the numerical scheme by eliminating the stabilizer, substituting the standard WG boundary function with the interior function's average. In this paper, we propose and analyze a CDG method for second order elliptic problems with variable coefficients. First, optimal a priori error estimates in both the energy norm and the L^2 norm are established. Then, a residual-type a posteriori error estimator is developed. Furthermore, we prove the efficiency of the a posteriori error estimator. Numerical experiments are conducted to validate the performance of the a priori and a posteriori error estimates.

Keywords: discontinuous Galerkin method, weak Galerkin method, weak gradient, a posteriori error estimate, elliptic problem

MSC: 65M60, 65M12

1. Introduction

In this paper, we develop a priori and a posteriori error estimates for a new kind of discontinuous Galerkin method applied to the following elliptic problems

$$\begin{cases} -\nabla\cdot(a\nabla u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$

Here, $\Omega \subset \mathbb{R}^2$ is a bounded polygonal domain. We further assume that the matrix $a(x) = (a_{ij}(x))_{1\le i,\, j\le 2}$ is a symmetric, bounded, and uniformly positive definite, i.e., there exist two constants $\alpha_i > 0$ $(i = 0,\, 1)$ such that,

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$$\alpha_0\|z\|^2 \le (az, z) \le \alpha_1\|z\|^2, \quad z \in \mathbb{R}^2. \tag{2}$$

Discontinuous Galerkin methods have attracted considerable attention over the past two decades and remain a topic of significant interest. While they offer advantages such as the ability to employ arbitrarily high-order elements and flexibility in parallel computing, they also come with certain drawbacks. Compared with conforming finite element methods, discontinuous Galerkin methods have more complicated variational forms. Another drawback is the need for tuning the penalty parameter, as it is sometimes not easy to determine how large this parameter should be to achieve stability. Adaptive finite element methods are widely employed in numerical PDEs to attain enhanced accuracy with reduced computational cost. Posteriori error estimators play a central role in adaptive finite element methods. Many researchers are interested in the a posteriori error estimates of the discontinuous Galerkin methods and have yielded significant results. For further details, we refer readers to [1–8] and the references therein.

Recently, a new family of discontinuous Galerkin schemes, known as Conforming Discontinuous Galerkin (CDG) methods, has been developed for the Poisson equation [9–11]. The CDG method is formulated within the weak Galerkin framework [12] and resembles conforming finite elements. A key simplification is that it requires no stabilizer or penalty parameters to guarantee coercivity. This is achieved through two core innovations: eliminating the stabilizer and replacing boundary functions with interior averages. These innovations reduce the boundary degrees of freedom, simplify the construction of the finite element space, and yield a readily parallelizable scheme. Consequently, the computational overhead is significantly reduced.

Now the CDG method has been successfully extended to a variety of problems. For instance, Ye and Zhang [13] proposed a CDG scheme for the Stokes equations. Wang et al. analyzed CDG methods for elliptic interface problems [14] and linear elasticity interface problems [15], respectively. Ye and Zhang [16] developed a superconvergent CDG method for non-self-adjoint and indefinite elliptic problems. Dang et al. [17] introduced a CDG method for the Brinkman equations, while Huo et al. [18] presented a locking-free CDG method for linear elasticity problems. Although the aforementioned works provide a priori error analysis for various mathematical models, none of them address the a posteriori error estimation of CDG methods.

Our aim of this paper is to propose a CDG method for second-order elliptic problems with variable coefficients. We establish both a priori error estimates and a residual-type a posteriori error estimate for the proposed CDG scheme. Furthermore, an adaptive CDG method is developed based on the aforementioned a posteriori error estimator. To the best of our knowledge, this constitutes the first work devoted to a posteriori error estimation of the CDG methods.

The remainder of this paper is organized as follows. In Section 2, we present the notation and propose the CDG methods for problem (1). Section 3 is devoted to the a priori error analysis of the proposed scheme in both the energy norm and the L^2 norm. A posteriori error analysis is provided in Section 4. Finally, numerical results are presented in Section 5 to validate the a priori and a posteriori error estimates.

In this work, $\|\cdot\|$ is used to denote the norm on $L^2(\Omega)$, while $\|\cdot\|_r$ refers to the norm on the Sobolev space $H^r(\Omega)$.

2. Preliminaries

In this section, we introduce the CDG finite element scheme associated with (1) and establish the unique solvability of the discrete problem.

By $\mathscr{T}_h$ we denote a shape-regular triangulation of the domain Ω with maximum mesh size h. For each element $K \in \mathscr{T}_h$, let h_K denotes its diameter. We write $\mathscr{E}_h$ for the set of all edges in $\mathscr{T}_h$, and define the set of interior edges as $\mathscr{E}_h^0 := \mathscr{E}_h \setminus \partial\Omega$. For any interior edge $e \in \mathscr{E}_h^0$, there exist two adjacent elements $K_1, K_2 \in \mathscr{T}_h$ such that $e = \partial K_1 \cap \partial K_2$; we denote the patch surrounding e by $\omega_e = \{K_1, K_2\}$. For each edge $e \in \mathscr{E}_h^0$, denote by h_e be the length of e. Owning to the shape regularity of $\mathscr{T}_h$, the following inequality holds:

$$\mu^{-1}h_K \le h_e \le \mu h_K, \quad \forall K \in \omega_e, \quad \forall e \in \mathscr{E}_h^0, \tag{3}$$

where the constant $\mu \ge 1$ doesn't depend on the local mesh size h_K.

For an arbitrary interior edge $e \in \mathscr{E}_h^0$, let K_1 and K_2 denote the two adjacent elements sharing the common edge e, and let $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ be the corresponding outward unit normal vectors on e from K_1 and K_2, respectively. The *average* and *jump* operators on e are defined as follows:

$$\{\!\{q\}\!\} := \frac{1}{2}(q|_{K_1} + q|_{K_2}), \quad [\![q]\!] := q|_{K_1}\boldsymbol{n}_1 + q|_{K_2}\boldsymbol{n}_2.$$

For edges on the boundary $\partial\Omega$, i.e., when $e = \partial K \cap \partial\Omega$, these operators are given by

$$\{\!\{q\}\!\} := q|_K, \quad [\![q]\!] := q|_K \boldsymbol{n}.$$

For the sake of clarity and brevity, we adopt the following notation:

$$(p, q)_h := \sum_{K \in \mathscr{T}_h} (p, q)_K = \sum_{K \in \mathscr{T}_h} \int_K pq\mathrm{d}x,$$

$$\langle p, q\rangle_h := \sum_{K \in \mathscr{T}_h} \langle p, q\rangle_{\partial K} = \sum_{K \in \mathscr{T}_h} \int_{\partial K} pq\mathrm{d}s.$$

Let $k(\ge 1)$ be a positive integer, and let $\mathbb{P}_k(K)$ denote the linear space of polynomials of degree at most k on K. The discontinuous finite element space is defined as

$$V_h := \{q \in L^2(\Omega) : q|_K \in \mathbb{P}_k(K), \forall K \in \mathscr{T}_h\},$$

and its subspace $V_h^0 \subset V_h$, consisting of functions vanishing on the boundary, is given by

$$V_h^0 := \{q \in V_h : q = 0 \text{ on } \partial\Omega\}.$$

In our analysis, the trace and inverse inequalities will play a crucial role. For any element $K \in \mathscr{T}_h$ with an edge e, the following estimates holds (cf. [19])

$$\|w\|_{L^2(e)} \le C(h_K^{-1/2}\|w\|_{L^2(K)} + h_K^{1/2}\|\nabla w\|_{L^2(K)}), \quad \forall w \in H^1(K), \tag{4}$$

$$\|q\|_{L^2(e)} \le Ch_K^{-1/2}\|q\|_{L^2(K)}, \quad \forall q \in \mathbb{P}_k(K). \tag{5}$$

Definition 1 (weak gradient) [9] For any $K \in \mathcal{T}_h$ and any function $\psi \in V_h$, the *weak gradient* $\nabla_w \psi|_K \in \mathbb{RT}_k(K)$ is defined as the unique solution of the following equation

$$(\nabla_w \psi, \boldsymbol{\chi})_K = -(\psi, \nabla \cdot \boldsymbol{\chi})_K + \langle \{\!\{\psi\}\!\}, \boldsymbol{\chi} \cdot \boldsymbol{n} \rangle_{\partial K}, \quad \boldsymbol{\chi} \in \mathbb{RT}_k(K), \tag{6}$$

where $\mathbb{RT}_k(K) := [\mathbb{P}_k(K)]^2 + \boldsymbol{x}\widehat{\mathbb{P}}_k(K)$ is the local Raviart-Thomas space defined on K and $\widehat{\mathbb{P}}_k(K)$ denotes the linear space of homogeneous polynomials of degree k defined on K.

Let $V = H_0^1(\Omega)$. For any $u, v \in V$, the variational formulation of problem (1) is to find $u \in V$ such that

$$(a\nabla u, \nabla v) = (f, v), \quad \forall v \in V.$$

The corresponding CDG finite elment scheme for problem (1) is then formulated as: find $u_h \in V_h^0$ such that

$$(a\nabla_w u_h, \nabla_w v_h)_h = (f, v_h), \quad \forall v_h \in V_h^0. \tag{7}$$

Remark 1 The symmetric Interior Penalty Discontinuous Galerkin (IPDG) method [20] for problem (1) is read as: find $u_h \in V_h$ such that

$$\sum_{K \in \mathcal{T}_h} (a\nabla u_h, \nabla v_h) - \sum_{e \in \mathcal{E}_h} (\langle \{\!\{a\nabla u_h\}\!\}, [\![v_h]\!] \rangle_e + \langle \{\!\{a\nabla v_h\}\!\}, [\![u_h]\!] \rangle_e)$$

$$+ \sum_{e \in \mathcal{E}_h} \sigma h_e^{-1} \langle [\![u_h]\!], [\![v_h]\!] \rangle_e = (f, v_h), \quad \forall v_h \in V_h, \tag{8}$$

where $\sigma > 0$ is a problem-dependent penalty parameter that must be chosen sufficiently large to ensure the stability and convergence of the method.

Evidently, compared to the IPDG scheme (8), the CDG scheme (7) features a simpler formulation and requires no parameter tuning. Additionally, the numerical experiments in Example 1 (Section 5) demonstrate that the CDG method yields higher accuracy than the IPDG method on the same mesh.

When considering vector-valued functions, we shall use the space

$$H(\text{div}; \Omega) := \{\boldsymbol{w} : \boldsymbol{w} \in [L^2(\Omega)]^2, \text{div}\boldsymbol{w} \in L^2(\Omega)\}.$$

In accordance with [21], we introduce a global interpolation operator for the space $H(\text{div}; \Omega)$. First, a local interpolation operator $\Pi_K : H(\text{div}; K) \to \mathbb{RT}_k(K)$ is defined such that

$$(\boldsymbol{\sigma} - \Pi_K \boldsymbol{\sigma}, \boldsymbol{\psi})_K = 0, \quad \forall \boldsymbol{\psi} \in [\mathbb{P}_{k-1}(K)]^2,$$

$$\langle (\boldsymbol{\sigma} - \Pi_K \boldsymbol{\sigma}) \cdot \boldsymbol{n}, \chi \rangle_e = 0, \quad \forall e \subset \partial K, \ \forall \chi \in \mathbb{P}_k(e).$$

The global interpolation operator Π_h: $H(\text{div}; \Omega) \to \mathbb{RT}_k(\mathscr{T}_h)$ is then given piecewise by

$$\Pi_h \boldsymbol{\sigma}|_K := \Pi_K(\boldsymbol{\sigma}|_K), \quad \forall K \in \mathscr{T}_h.$$

From Proposition 2.5.1 and Proposition 2.5.2 of [21], we know that the interpolation operator Π_K defined above has the following properties:

$$\|\boldsymbol{\sigma} - \Pi_K \boldsymbol{\sigma}\|_K \le C h_K^m |\boldsymbol{\sigma}|_{m,K}, \quad \text{for } 1 \le m \le k+1,$$

and

$$(\nabla \cdot (\boldsymbol{\sigma} - \Pi_K \boldsymbol{\sigma}), v_h)_K = 0, \quad \forall v_h \in \mathbb{P}_k(K). \tag{9}$$

Based on the property (9) of Π_h, we can conclude the following statements.

Lemma 1 [9] For any $\boldsymbol{\tau} \in H(\text{div}; \Omega)$, there holds

$$-(\nabla \cdot \boldsymbol{\tau}, v_h)_h = (\Pi_h \boldsymbol{\tau}, \nabla_w v_h)_h, \quad \forall v_h \in V_h^0. \tag{10}$$

Lemma 2 [9] Let $\widetilde{V}_h = V_h \cap H^1(\Omega)$. Then for any function $w \in \widetilde{V}_h$, there holds $\nabla_w w = \nabla w$.

For any $v \in H^m(\Omega)$ and $K \in \mathscr{T}_h$, let $I_K v$ be the local k-th order Lagrange interpolant. Then the global interpolant $I_h v$ is defined by $I_h v|_K = I_K v$ for all $K \in \mathscr{T}_h$. Then the interoplation operator I_h has the following approximation property [22]

$$\|v - I_h v\|_{s,K} \le C h_K^{m-s} |v|_{m,K}, \quad 0 \le s < m \le k+1. \tag{11}$$

Lemma 3 For any $v \in H^{1+s}(\Omega)$ with $s > 0$, there holds

$$\|\Pi_h(a\nabla v) - a\nabla_w(I_h v)\| \le C h^s |v|_{1+s}. \tag{12}$$

Proof. Since $I_h v \in \widetilde{V}_h$, from Lemma 2 we have

$$\nabla_w(I_h v) = \nabla(I_h v). \tag{13}$$

Then by invoking the triangle inequality and the approximation properties of Π_h and I_h, it can be deduced that

$$\begin{aligned}\|\Pi_h(a\nabla v)-a\nabla_w(I_hv)\| &= \|\Pi_h(a\nabla v)-a\nabla(I_hv)\|\\ &\le \|\Pi_h(a\nabla v)-a\nabla v\|+\|a\nabla v-a\nabla(I_hv)\|\\ &\le Ch^s|v|_{1+s}.\end{aligned}$$

The proof is ended. □

For any $q\in V_h$, we define two semi-norms $\|q\|_E$ and $\|q\|_{1,h}$ as follows:

$$\|q\|_E^2 := (a\nabla_w q,\, \nabla_w q)_h, \tag{14}$$

and

$$\|q\|_{1,h}^2 := \|\nabla q\|^2+\sum_{e\in\mathscr{E}_h} h_e^{-1}\|[\![q]\!]\|_e^2. \tag{15}$$

One may show that the seminorm $\|\cdot\|_{1,h}$ is, in fact, a norm on V_h^0. In what follows, we shall derive the two seminorms $\|\cdot\|_E$ and $\|\cdot\|_{1,h}$ are equivalent on the finite element space V_h^0 and then conclude that $\|\cdot\|_E$ is also a norm on V_h^0. To achieve this goal, we will construct a specific vector-valued polynomial in the next lemma.

Lemma 4 For each $K\in\mathscr{T}_h$ and every $w_h\in V_h^0$, there exists a vector-valued polynomial $\boldsymbol{q}\in\mathbb{RT}_k(K)$ such that

$$(\boldsymbol{q},\,\boldsymbol{r})_K=0,\quad \forall\boldsymbol{r}\in[\mathbb{P}_{k-1}(K)]^2, \tag{16}$$

$$\langle\boldsymbol{q}\cdot\boldsymbol{n},\, w_h-\{\!\{w_h\}\!\}\rangle_{\partial K}=\|w_h-\{\!\{w_h\}\!\}\|_{\partial K}^2, \tag{17}$$

and

$$\|\boldsymbol{q}\|_K\le Ch_K^{1/2}\|w_h-\{\!\{w_h\}\!\}\|_{\partial K}. \tag{18}$$

Proof. Consider a triangle element $K\in\mathscr{T}_h$ with edges denoted by e_1, e_2 and e_3. Following [23], we define vector $\boldsymbol{q}_i\in\mathbb{RT}_k(K)\,(i=1,\,2,\,3)$ by requiring that

$$\langle\boldsymbol{q}_i\cdot\boldsymbol{n},\, z\rangle_{e_i}=\langle w_h-\{\!\{w_h\}\!\},\, z\rangle_{e_i},\quad z\in\mathbb{P}_k(e_i), \tag{19a}$$

$$\langle\boldsymbol{q}_i\cdot\boldsymbol{n},\, z\rangle_{e_j}=0,\ j\neq i,\quad z\in\mathbb{P}_k(e_i), \tag{19b}$$

$$(\boldsymbol{q}_i,\,\boldsymbol{r})_K=0,\quad \forall\boldsymbol{r}\in[\mathbb{P}_{k-1}(K)]^2. \tag{19c}$$

If $k=0$, the condition (19c) is not necessary.

Obviously, (19a)-(19c) is a square linear system with

$$2\cdot k(k+1)/2+3(k+1)=(k+1)(k+3)$$

equations and unknowns. Hence for proving existence and uniqueness of $\boldsymbol{q}_i$ it is enough to consider the case $v_h=0$ and show that $\boldsymbol{q}_i=0$ is the unique solution.

In fact, condition (19a)-(19b) with $v_h=0$ imply $\boldsymbol{q}_i\cdot\boldsymbol{n}=0$ on ∂K. Hence, for any $\psi\in\mathbb{P}_k(K)$, it follows from (19c) that

$$(\psi,\,\operatorname{div}\boldsymbol{q}_i)_K=-(\nabla\psi,\,\boldsymbol{q}_i)_K+\langle\psi,\,\boldsymbol{q}_i\cdot\boldsymbol{n}\rangle_{\partial K}=0.$$

Due to the fact that $\operatorname{div}\boldsymbol{q}_i\in\mathbb{P}_k(K)$, we have $\operatorname{div}\boldsymbol{q}_i=0$. Thus, we may write $\boldsymbol{q}_i=\boldsymbol{curl}\,v$ with $v\in\mathbb{P}_{k+1}(K)$. Observe that $\frac{\partial v}{\partial \boldsymbol{t}}=\boldsymbol{q}_i\cdot\boldsymbol{n}=0$ on ∂K, where $\boldsymbol{t}$ stands for the tangential unit vector along ∂K. Thus, we may assume that $v=0$ on ∂K and write

$$v=\lambda_1\lambda_2\lambda_3\chi,\quad \chi\in\mathbb{P}_{k-2}(K)\quad (\chi=0\quad\text{for}\quad k=0,\,1),$$

where $\lambda_1,\,\lambda_2,\,\lambda_3$ denote the barycentric coordinates on K.

Using (19c) again, we obtain for any $\boldsymbol{\tau}\in[\mathbb{P}_{k-1}(K)]^2$,

$$0=(\boldsymbol{q}_i,\,\boldsymbol{\tau})_K=(\boldsymbol{curl}\,v,\,\boldsymbol{\tau})_K=(w,\,\nabla\times\boldsymbol{\tau})_K=(\lambda_1\lambda_2\lambda_3\chi,\,\nabla\times\boldsymbol{\tau})_K,$$

where $\nabla\times\boldsymbol{\tau}=\frac{\partial\tau_2}{\partial x_1}-\frac{\partial\tau_1}{\partial x_2}\in\mathbb{P}_{k-2}(K)$. By choosing $\boldsymbol{\tau}$ such that $\nabla\times\boldsymbol{\tau}=\chi$, we obtain

$$(\lambda_1\lambda_2\lambda_3,\,\chi^2)_K=0.$$

Therefore, we have $\chi=0$ so that $v=0$ and $\boldsymbol{q}_i=\boldsymbol{curl}\,v=0$.

Then, by (19a) and scaling arguments, we get

$$\|\boldsymbol{q}_i\|_K\le Ch_K^{1/2}\|w_h-\{\!\{w_h\}\!\}\|_{e_i}.$$

Finally, let $\boldsymbol{q}=\boldsymbol{q}_1+\boldsymbol{q}_2+\boldsymbol{q}_3$ ends the proof. □

Lemma 5 For any $\psi_h\in V_h^0$, there exist positive constants c_1 and c_2 such that

$$c_2\|\psi_h\|_{1,\,h}\le\|\psi_h\|_E\le c_1\|\psi_h\|_{1,\,h}. \tag{20}$$

Proof. Applying integration by parts and referring to (6), we obtain

$$(\nabla_w \psi_h, \boldsymbol{q})_K = (\nabla \psi_h, \boldsymbol{q})_K + \langle \{\!\{ \psi_h \}\!\} - \psi_h, \boldsymbol{q} \cdot \boldsymbol{n} \rangle_{\partial K}. \tag{21}$$

Letting $\boldsymbol{q} = \nabla_w \psi_h$ in (21), we get

$$\begin{aligned}
\|\nabla_w \psi_h\|_K^2 &= (\nabla \psi_h, \nabla_w \psi_h)_K + \langle \{\!\{ \psi_h \}\!\} - \psi_h, \nabla_w \psi_h \cdot \boldsymbol{n} \rangle_{\partial K} \\
&\leq \|\nabla \psi_h\|_K \|\nabla_w \psi_h\|_K + \|\{\!\{ \psi_h \}\!\} - \psi_h\|_{\partial K} \|\nabla_w \psi_h \cdot \boldsymbol{n}\|_{\partial K} \\
&\leq C(\|\nabla \psi_h\|_K + h_K^{-1/2} \|\{\!\{ \psi_h \}\!\} - \psi_h\|_{\partial K}) \|\nabla_w \psi_h\|_K.
\end{aligned}$$

Cancelling $\|\nabla_w \psi_h\|_K$ on both sides, we have

$$\|\nabla_w \psi_h\|_K \leq C(\|\nabla \psi_h\|_K + h_K^{-1/2} \|\{\!\{ \psi_h \}\!\} - \psi_h\|_{\partial K}),$$

squaring both sides and then summing over all $K \in \mathscr{T}_h$ leads to

$$\|\psi_h\|_E^2 \leq C(\|\nabla \psi_h\|^2 + \sum_{K \in \mathscr{T}_h} h_K^{-1} \|\{\!\{ \psi_h \}\!\} - \psi_h\|_{\partial K}^2).$$

Since $|\{\!\{ \psi_h \}\!\} - \psi_h|_e = \frac{1}{2} |[\![\psi_h]\!]|_e$, then

$$\|\psi_h\|_E^2 \leq C(\|\nabla \psi_h\|^2 + \sum_{e \in \mathscr{E}_h} h_e^{-1} \|[\![\psi_h]\!]\|_e^2),$$

thus

$$\|\psi_h\|_E \leq c_1 \|\psi_h\|_{1,h}.$$

Denote by $\boldsymbol{p}_h$ the vector-valued polynomial defined by Lemma 4. Taking $\boldsymbol{q} = \boldsymbol{p}_h$ in (21) implies

$$(\nabla_w \psi_h, \boldsymbol{p}_h)_K = (\nabla \psi_h, \boldsymbol{p}_h)_K + \langle \{\!\{ \psi_h \}\!\} - \psi_h, \boldsymbol{p}_h \cdot \boldsymbol{n} \rangle_{\partial K}.$$

By (16)–(18), one has

$$\|\{\!\{\psi_h\}\!\} - \psi_h\|_{\partial K}^2 = (\nabla_w \psi_h,\, \boldsymbol{p}_h)_K \le \|\nabla_w \psi_h\|_K \|\boldsymbol{p}_h\|_K$$

$$\le C h_K^{1/2} \|\nabla_w \psi_h\|_K \|\{\!\{\psi_h\}\!\} - \psi_h\|_{\partial K},$$

which leads to

$$\|\{\!\{\psi_h\}\!\} - \psi_h\|_{\partial K} \le C h_K^{1/2} \|\nabla_w \psi_h\|_K.$$

Thus,

$$\sum_{e\in\mathscr{E}_h} h_e^{-1} \|[\![\psi_h]\!]\|_e^2 \le C \sum_{K\in\mathscr{T}_h} h_K^{-1} \|\{\!\{\psi_h\}\!\} - \psi_h\|_{\partial K}^2 \le C \|\nabla_w \psi_h\|^2. \tag{22}$$

Taking $\boldsymbol{q} = \nabla \psi_h$ in (21) and using Cauchy-Schwarz inequality implies

$$\|\nabla \psi_h\|_K^2 = (\nabla_w \psi_h,\, \nabla \psi_h)_K + \langle \psi_h - \{\!\{\psi_h\}\!\},\, \nabla \psi_h \cdot \boldsymbol{n}\rangle_{\partial K}$$

$$\le \|\nabla_w \psi_h\|_K \|\nabla \psi_h\|_K + \|\psi_h - \{\!\{\psi_h\}\!\}\|_{\partial K} \|\nabla \psi_h \cdot \boldsymbol{n}\|_{\partial K}$$

$$\le C(\|\nabla_w \psi_h\|_K + h_K^{-1/2} \|\psi_h - \{\!\{\psi_h\}\!\}\|_{\partial K}) \|\nabla \psi_h\|_K.$$

Therefore, we have

$$\|\nabla \psi_h\|_K^2 \le C(\|\nabla_w \psi_h\|_K^2 + h_K^{-1} \|\psi_h - \{\!\{\psi_h\}\!\}\|_{\partial K}^2).$$

Taking the summation over all $K \in \mathscr{T}_h$ on both sides and using (22), we get

$$c_2 \|\psi_h\|_{1,\,h} \le \|\psi_h\|_E.$$

The concludes the proof. □

According to Lemma 5, the semi-norm $\|\cdot\|_E$ given by (14) is in fact a norm on V_h^0.

3. A priori error estimate

This section is devoted to deriving a priori error estimates for the CDG method (7) in both the energy-like norm and the L^2 norm, and to establishing its corresponding convergence rates.

Theorem 1 Let u and u_h denote the solutions of problem (1) and the CDG scheme (7), respectively. Assume the exact solution $u \in H_0^1(\Omega) \cap H^{k+1}(\Omega)$. Then there exists a constant $C > 0$, independent of h, such that

$$\|I_h u - u_h\|_E \le Ch^k |u|_{k+1}.$$

Proof. Testing the first equation of (1) by $v_h \in V_h^0$ gives

$$-(\nabla \cdot (a\nabla u),\, v_h) = (f,\, v_h).$$

Using (10) with $\boldsymbol{\tau} = a\nabla u$, we obtain

$$(\Pi_h(a\nabla u),\, \nabla_w v_h)_h = (f,\, v_h).$$

Adding $(a\nabla_w(I_h u),\, \nabla_w v_h)$ to the both sides of the above equation, we have

$$(a\nabla_w(I_h u),\, \nabla_w v_h) = (f,\, v_h) + (a\nabla_w(I_h u) - \Pi_h(a\nabla u),\, \nabla_w v_h)_h. \tag{23}$$

The difference of (23) and (7) gives

$$(a\nabla_w(I_h u - u_h),\, \nabla_w v_h) = (a\nabla_w(I_h u) - \Pi_h(a\nabla u),\, \nabla_w v_h)_h. \tag{24}$$

Choosing $v_h = I_h u - u_h$ in (24) and using (12) with $s = k$, we arrive at

$$\begin{aligned}
\|I_h u - u_h\|_E^2 &= (a\nabla_w(I_h u) - \Pi_h(a\nabla u),\, \nabla_w(I_h u - u_h))_h \\
&\le \|a\nabla_w(I_h u) - \Pi_h(a\nabla u)\| \|\nabla_w(I_h u - u_h)\| \\
&\le Ch^k |u|_{k+1} \|I_h u - u_h\|_E.
\end{aligned}$$

The proof is completed. □

We now proceed to establish an error estimate in the L^2 norm. Let

$$\widetilde{V}_h^0 = V_h \cap H_0^1(\Omega). \tag{25}$$

Consider the auxiliary problem defined by seeking $\widetilde{u}_h \in \widetilde{V}_h^0$ such that

$$(a\nabla \widetilde{u}_h,\, \nabla v_h) = (f,\, v_h), \quad \forall v_h \in \widetilde{V}_h^0. \tag{26}$$

Then we have the following important partial orthogonality result.

Lemma 6 Let u_h and $\widetilde{u}_h$ denote the solutions of the CDG scheme (7) and the auxiliary problem (26), respectively. Then

$$(a\nabla_w u_h - a\nabla\widetilde{u}_h, \nabla v_h)_h = 0, \quad \forall v_h \in \widetilde{V}_h^0. \tag{27}$$

Proof. Since $\widetilde{V}_h^0 \subset \widetilde{V}_h$, by Lemma 2, it is easy to see $\nabla_w v_h = \nabla v_h$ for $v_h \in \widetilde{V}_h^0$. It follows from (7) and $\nabla_w v_h = \nabla v_h$ that

$$(a\nabla_w u_h, \nabla v_h)_h = (a\nabla_w u_h, \nabla_w v_h)_h = (f, v_h), \quad \forall v_h \in \widetilde{V}_h^0.$$

Subtracting (26) from the equations above ends the proof. □

Theorem 2 (L^2 norm) Let u and u_h denote the solutions of problem (1) and the CDG scheme (7), respectively. Suppose that the exact solution $u \in H_0^1(\Omega) \cap H^{k+1}(\Omega)$. Then there exists a constant $C > 0$, independent of h, such that

$$\|u - u_h\| \leq Ch^{k+1}|u|_{k+1}.$$

Proof. Applying the triangle inequality gives

$$\|u - u_h\| \leq \|u - \widetilde{u}_h\| + \|\widetilde{u}_h - u_h\|,$$

where $\widetilde{u}_h$ is defined by (26). From classic conforming finite element approximation result (cf. [22]), we obtain

$$\|u - \widetilde{u}_h\| + h\|\nabla(u - \widetilde{u}_h)\| \leq Ch^{k+1}|u|_{k+1}. \tag{28}$$

To estimate $\|\widetilde{u}_h - u_h\|$, we consider the following problem:

$$\begin{cases} -\nabla\cdot(a\nabla w) = \widetilde{u}_h - u_h, & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases} \tag{29}$$

We now suppose that the problem (29) has the property that

$$|w|_2 \leq C\|\widetilde{u}_h - u_h\|. \tag{30}$$

Denote by w_h obtained by applying CDG scheme (7) to the problem (29), i.e., find $w_h \in V_h^0$ satisfying

$$(a\nabla_w w_h, \nabla_w v_h) = (\widetilde{u}_h - u_h, v_h), \quad \forall v_h \in V_h^0. \tag{31}$$

Setting $v_h = \widetilde{u}_h - u_h$ in (31) and invoking Lemma 2 yields

$$\|u_h - \widetilde{u}_h\|^2 = (a\nabla_w w_h, \nabla_w(\widetilde{u}_h - u_h))_h = (a(\nabla\widetilde{u}_h - \nabla_w u_h), \nabla_w w_h)_h. \tag{32}$$

Invoking Lemma 6 with $v_h = I_h w$ in (27) and using (13), we have

$$(a(\nabla\widetilde{u}_h - \nabla_w u_h), \nabla_w(I_h w))_h = (a(\nabla\widetilde{u}_h - \nabla_w u_h), \nabla(I_h w))_h = 0,$$

which combining with (32) yields

$$\begin{aligned}\|\widetilde{u}_h - u_h\|^2 &= (a(\nabla\widetilde{u}_h - \nabla_w u_h), \nabla_w(w_h - I_h w))_h \\ &\leq \|a^{1/2}(\nabla\widetilde{u}_h - \nabla_w u_h)\| \|w_h - I_h w\|_E.\end{aligned} \tag{33}$$

The definition of w_h, Theorem 1 and (30) implies

$$\|w_h - I_h w\|_E \leq Ch|w|_2 \leq Ch\|\widetilde{u}_h - u_h\|. \tag{34}$$

Invoking Lemma 2, we get $\nabla_w(I_h u) = \nabla(I_h u)$. By the triangle inequality, we have

$$\begin{aligned}\|a^{1/2}(\nabla\widetilde{u}_h - \nabla_w u_h)\| &\leq \|a^{1/2}\nabla(\widetilde{u}_h - I_h u)\| + \|a^{1/2}\nabla_w(I_h u - u_h)\| \\ &\leq \|a^{1/2}\nabla(\widetilde{u}_h - I_h u)\| + \|I_h u - u_h\|_E \\ &\leq \alpha_1(\|\nabla(\widetilde{u}_h - u)\| + \|\nabla(u - I_h u)\|) + \|I_h u - u_h\|_E.\end{aligned}$$

Then using (28), (11) and Theorem 1, we arrive at

$$\|a^{1/2}(\nabla\widetilde{u}_h - \nabla_w u_h)\| \leq Ch^k|u|_{k+1}. \tag{35}$$

Putting together (33), (34) and (35), we obtain

$$\|\widetilde{u}_h - u_h\| \leq Ch^{k+1}|u|_{k+1}.$$

Together with (28), this concludes the proof. □

On the basis of Theorem 1 and Theorem 2, we can conclude the following results.

Theorem 3 Let u and u_h denote the solutions of problem (1) and the CDG scheme (7), respectively. Suppose that the exact solution $u \in H_0^1(\Omega) \cap H^{k+1}(\Omega)$. Then there exists a constant $C > 0$, independent of h, such that

$$|||u - u_h||| \leq Ch^k |u|_{k+1},$$

where

$$|||u - u_h|||: = \left\{ \|a^{1/2}(\nabla u - \nabla_w u_h)\|^2 + s_h(u - u_h,\, u - u_h) \right\}^{1/2}, \tag{36}$$

and the bilinear form $s_h(\cdot,\, \cdot)$ is given by

$$s_h(w,\, v): = \sum_{e \in \mathscr{E}_h} h_e^{-1} \langle [\![w]\!],\, [\![v]\!] \rangle_e. \tag{37}$$

Proof. Combining the triangle inequality with (13), we deduce that

$$\|a^{1/2}(\nabla u - \nabla_w u_h)\| \leq \|a^{1/2} \nabla (u - I_h u)\| + \|a^{1/2} \nabla_w (I_h u - u_h)\|.$$

Then by (11) and Theorem 1, we have

$$\|a^{1/2}(\nabla u - \nabla_w u_h)\| \leq Ch^k |u|_{k+1}. \tag{38}$$

Given that $[\![u]\!]|_e = [\![I_h u]\!]|_e = 0$ for $e \in \mathscr{E}_h$, the inverse inequality (5) implies

$$\|[\![u - u_h]\!]\|_e = \|[\![I_h u - u_h]\!]\|_e \leq \sum_{e \subset \partial K} \|I_h u - u_h\|_{\partial K}$$

$$\leq Ch_K^{-1/2} \|I_h u - u_h\|_K.$$

Then,

$$s_h(u - u_h,\, u - u_h) \leq C \sum_{K \in \mathscr{T}_h} h_K^{-2} \|I_h u - u_h\|_K^2 \leq Ch^{-2} \|I_h u - u_h\|^2,$$

which together with the triangle inequality

$$\|I_h u - u_h\| \leq \|u - I_h u\| + \|u - u_h\|$$

and (11) and Theorem 2 yields

$$s_h(u-u_h,\, u-u_h) \le Ch^{2k}|u|_{k+1}^2. \tag{39}$$

Collecting (38) and (39) ends the proof. □

4. An a posteriori error estimate

The goal of this section is to introduce a reliable and efficient residual-type a posteriori error estimator for the CDG method formulated in (7).

For each interior edge $e \in \mathscr{E}_h^0$, let $\omega_e = K_1 \cup K_2$ be the patch of elements adjacent to e, where $e = \partial K_1 \cap \partial K_2$. For $K \in \mathscr{T}_h$, let ω_K be the set of K and its neighboring elements. Let a_K be the average of a over the element K, and define $|a_e^{\max}| = \max_{K\in\omega_e} |a_K|$, and $|a_e^{\min}| = \min_{K\in\omega_e} |a_K|$, where $|a_K|$ denotes its determinant.

We introduce a global error estimator on $\mathscr{T}_h$, which is defined as follows:

$$\eta(u_h,\, \mathscr{T}_h) := [\mathrm{res}_h^2(u_h,\, \mathscr{T}_h) + s_h(u_h,\, u_h)]^{1/2} \tag{40}$$

where

$$\mathrm{res}_h^2(u_h,\, \mathscr{T}_h) := \sum_{K\in\mathscr{T}_h} h_K^2 \|f + \mathrm{div}(a\nabla_w u_h)\|_K^2 + \sum_{e\in\mathscr{E}_h^0} h_e \|[\![a\nabla_w u_h]\!]\|_e^2. \tag{41}$$

In the following lemma, we state the Helmholtz decomposition for vector-valued L^2 function [24], which plays a key role in our a posteriori error analysis. For completeness, a brief proof is included below.

Lemma 7 For $\nabla u - \nabla_w u_h \in L^2(\Omega)$, there exist $\psi \in H_0^1(\Omega)$ and $v \in H^1(\Omega)$ satisfying $\int_\Omega v\mathrm{d}x = 0$, such that

$$\nabla u - \nabla_w u_h = \nabla\psi + a^{-1}\boldsymbol{curl}\, v \tag{42}$$

and

$$\|a^{1/2}(\nabla u - \nabla_w u_h)\|^2 = \|a^{1/2}\nabla\psi\|^2 + \|a^{-1/2}\boldsymbol{curl}\, v\|^2, \tag{43}$$

where $\boldsymbol{curl}\, v$ denotes the vector curl of the scalar function v, defined by $\boldsymbol{curl}\, v = (\frac{\partial v}{\partial x_2}, -\frac{\partial v}{\partial x_1})^T$.

Proof. Let $\psi \in H_0^1(\Omega)$ be the solution of the variational problem below

$$(a\nabla\psi,\, \nabla\chi) = (a(\nabla u - \nabla_w u_h),\, \nabla\chi)_h, \quad \forall\chi \in H_0^1(\Omega). \tag{44}$$

The existence and uniqueness of ψ can be obtained from Lax-Milgram theorem.

Denote $\boldsymbol{w} = a(\nabla u - \nabla_w u_h) - a\nabla\psi$. From (44), we have

$$(\boldsymbol{w}, \nabla\chi)_{\mathscr{T}_h} = 0, \quad \forall \chi \in H_0^1(\Omega), \tag{45}$$

and then integration by parts gives

$$-(\nabla\cdot\boldsymbol{w}, \chi) = 0, \quad \forall \chi \in H_0^1(\Omega).$$

Consequently, $\nabla\cdot\boldsymbol{w} = 0$ in Ω. One can then find a $v \in H^1(\Omega)$ such that

$$a(\nabla u - \nabla_w u_h) - a\nabla\psi = \boldsymbol{w} = \boldsymbol{curl}\, v.$$

Further, as $\boldsymbol{curl}\, C = \mathbf{0}$ for any $C \in \mathbb{R}$, we may choose a v with zero average.

Taking $\chi = \psi$ in (45), we get the orthogonality $(\nabla\psi, \boldsymbol{curl}\, v) = 0$, which implies (43). The proof is completed. □

Lemma 8 Let u and u_h be the solutions to problem (1) and the CDG scheme (7), respectively. And let ψ be given by the Helmholtz decomposition (42). Then there exists a constant $C > 0$ such that

$$(a(\nabla u - \nabla_w u_h), \nabla\psi)_h \le C\cdot \mathrm{res}_h(u_h, \mathscr{T}_h)\cdot\|a^{1/2}\nabla\psi\|. \tag{46}$$

Proof. Firstly, we introduce a robust interpolant $\psi_I \in \widetilde{V}_h^0$ as [25, 26], which satisfies the following approximation properties

$$|a_K|^{1/2}\|\psi - \psi_I\|_K \le C h_K \|a^{1/2}\nabla\psi\|_{\omega_K}, \tag{47}$$

$$|a_e^{\max}|^{1/2}\|\psi - \psi_I\|_e \le C h_e^{1/2}\|a^{1/2}\nabla\psi\|_{\omega_e}. \tag{48}$$

The left-hand side of (46) can then be rewritten as

$$\begin{aligned}(a(\nabla u - \nabla_w u_h), \nabla\psi)_h =& (a(\nabla u - \nabla_w u_h), \nabla(\psi - \psi_I))_h + (a(\nabla u - \nabla_w u_h), \nabla\psi_I)_h \\ =&: T_1 + T_2. \end{aligned} \tag{49}$$

For T_1, using $-\nabla\cdot(a\nabla u) = f$ and integration by parts, we get

$$\begin{aligned} T_1 &= \sum_{K\in\mathscr{T}_h} (-\nabla\cdot(a(\nabla u - \nabla_w u_h)), \psi - \psi_I)_K + \langle a(\nabla u - \nabla_w u_h)\cdot\boldsymbol{n}, \psi - \psi_I\rangle_{\partial K} \\ &= \sum_{K\in\mathscr{T}_h} (f + \nabla\cdot(a\nabla_w u_h), \psi - \psi_I)_K + \sum_{e\in\mathscr{E}_h^0} \langle [\![a\nabla_w u_h]\!], \psi - \psi_I\rangle_e. \end{aligned} \tag{50}$$

Combining Cauchy-Schwarz inequality with estimates (47), (48), and (50), we obtain

$$|T_1| \leq \sum_{K\in\mathscr{T}_h} \|f+\nabla\cdot(a\nabla_w u_h)\|_K \|\psi-\psi_I\|_K + \sum_{e\in\mathscr{E}_h^0} \|[\![a\nabla_w u_h]\!]\|_e \|\psi-\psi_I\|_e$$

$$= \sum_{K\in\mathscr{T}_h} |a_K|^{-1/2} h_K \|f+\nabla\cdot(a\nabla_w u_h)\|_K h_K^{-1} |a_K|^{1/2} \|\psi-\psi_I\|_K$$

$$+ \sum_{e\in\mathscr{E}_h^0} |a_e^{\max}|^{-1/2} h_e^{1/2} \|[\![a\nabla_w u_h]\!]\|_e h_e^{-1/2} |a_e^{\max}|^{1/2} \|\psi-\psi_I\|_e$$

$$\leq C \sum_{K\in\mathscr{T}_h} h_K \|f+\nabla\cdot(a\nabla_w u_h)\|_K \|a^{1/2}\nabla\psi\|_{\omega_K}$$

$$+ C \sum_{e\in\mathscr{E}_h^0} h_e^{1/2} \|[\![a\nabla_w u_h]\!]\|_e \|a^{1/2}\nabla\psi\|_{\omega_e}$$

$$\leq C\cdot \mathrm{res}_h(u_h,\, \mathscr{T}_h)\cdot \|a^{1/2}\nabla\psi\|. \tag{51}$$

Similar as (27), we can easily prove

$$(a(\nabla u-\nabla_w u),\, \nabla v_h)_h = 0, \quad \forall v_h \in \widetilde{V}_h^0.$$

Let $v_h = \psi_I$ in the above equation, we obtain

$$T_2 = (a(\nabla u-\nabla_w u),\, \nabla\psi_I)_h = 0,$$

which together with (49) and (51) ends the proof. □

Lemma 9 Let u and u_h be the solutions to problem (1) and the CDG scheme (7), respectively. And let v be given by the Helmholtz decomposition (42). Then there exists a constant $C > 0$ such that

$$(a(\nabla u-\nabla_w u_h),\, a^{-1}\boldsymbol{curl}\, v)_h \leq C s_h^{1/2}(u_h,\, u_h) \|a^{-1/2}\boldsymbol{curl}\, v\|. \tag{52}$$

Proof. Setting $\chi = u$ in (45) leads to $(\boldsymbol{curl}\, v,\, \nabla u) = (\boldsymbol{w},\, \nabla u) = 0$, which yields

$$(a(\nabla u-\nabla_w u_h),\, a^{-1}\boldsymbol{curl}\, v)_h = (\nabla u-\nabla_w u_h,\, \boldsymbol{curl}\, v)_h$$

$$= -(\nabla_w u_h,\, \boldsymbol{curl}\, v)_h. \tag{53}$$

Let $\mathbb{Q}_h$ denote the L^2-projection onto the space $[\mathbb{P}_{k-1}(K)]^2$. From (6) and integration by parts, it follows that

$$\begin{aligned}(\nabla_w u_h,\, \boldsymbol{curl}\, v)_K &= (\nabla_w u_h,\, \mathbb{Q}_h(\boldsymbol{curl}\, v))_K \\ &= -(u_h,\, \nabla\cdot\mathbb{Q}_h(\boldsymbol{curl}\, v))_K + \langle \{\!\{u_h\}\!\},\, \mathbb{Q}_h(\boldsymbol{curl}\, v)\cdot\boldsymbol{n}\rangle_{\partial K} \\ &= (\nabla u_h,\, \mathbb{Q}_h(\boldsymbol{curl}\, v))_K + \langle \{\!\{u_h\}\!\} - u_h,\, \mathbb{Q}_h(\boldsymbol{curl}\, v)\cdot\boldsymbol{n}\rangle_{\partial K}. \end{aligned} \tag{54}$$

Using the orthogonal property of $\mathbb{Q}_h$, integration by parts and noticing that $\nabla\cdot(\boldsymbol{curl}\, v)$ vanishes, we have

$$(\nabla u_h,\, \mathbb{Q}_h(\boldsymbol{curl}\, v))_K = (\nabla u_h,\, \boldsymbol{curl}\, v)_K = \langle u_h,\, \boldsymbol{curl}\, v\cdot\boldsymbol{n}\rangle_{\partial K}. \tag{55}$$

From (54), (55) and the equality

$$\sum_{K\in\mathscr{T}_h} \langle \{\!\{u_h\}\!\},\, \boldsymbol{curl}\, v\cdot\boldsymbol{n}\rangle_{\partial K} = 0,$$

we obtain

$$\begin{aligned}(\nabla_w u_h,\, \boldsymbol{curl}\, \varphi)_h = &\sum_{K\in\mathscr{T}_h} \langle \{\!\{u_h\}\!\} - u_h,\, \mathbb{Q}_h(\boldsymbol{curl}\, v)\cdot\boldsymbol{n}\rangle_{\partial K} \\ &- \sum_{K\in\mathscr{T}_h} \langle \{\!\{u_h\}\!\} - u_h,\, \boldsymbol{curl}\, v\cdot\boldsymbol{n}\rangle_{\partial K}. \end{aligned} \tag{56}$$

Combining the Cauchy-Schwarz inequality, the trace inequality, and the stability of the L^2-projection, we have

$$\begin{aligned}&|\langle \{\!\{u_h\}\!\} - u_h,\, \mathbb{Q}_h(\boldsymbol{curl}\, v)\cdot\boldsymbol{n}\rangle_{\partial K}| \\ \le&\|\{\!\{u_h\}\!\} - u_h\|_{\partial K}\|\mathbb{Q}_h(\boldsymbol{curl}\, v)\cdot\boldsymbol{n}\|_{\partial K} \\ \le& C h_e^{-1/2}\|\{\!\{u_h\}\!\} - u_h\|_{\partial K}\|\boldsymbol{curl}\, v\|_K \\ \le& C\Big(\sum_{e\subset\partial K} h_e^{-1}\|[\![u_h]\!]\|_e^2\Big)^{1/2}\|a^{-1/2}\boldsymbol{curl}\, v\|_K. \end{aligned} \tag{57}$$

Applying the trace inequality

$$\|\boldsymbol{q}\cdot\boldsymbol{n}\|_{H^{-1/2}(\partial K)} \le C(\|\boldsymbol{q}\|_K + h_K\|\nabla\cdot\boldsymbol{q}\|_K),$$

see [27], with $\boldsymbol{q} = \boldsymbol{curl}\, v$, together with $\nabla \cdot \boldsymbol{curl}\, v = 0$, we have

$$\|\boldsymbol{curl}\, v \cdot \boldsymbol{n}\|_{H^{-1/2}(\partial K)} \leq C\|\boldsymbol{curl}\, v\|_K. \tag{58}$$

Then, applying the inverse inequality and (58), we arrive at

$$\begin{aligned}
&|\langle \{\!\{u_h\}\!\} - u_h,\, \boldsymbol{curl}\, v \cdot \boldsymbol{n}\rangle_{\partial K}| \\
&\leq C\|\{\!\{u_h\}\!\} - u_h\|_{H^{1/2}(\partial K)}\|\boldsymbol{curl}\, v \cdot \mathbf{n}\|_{H^{-1/2}(\partial K)} \\
&\leq C(\sum_{e\subset\partial K} h_e^{-1}\|[\![u_h]\!]\|_e^2)^{1/2}\|a^{-1/2}\boldsymbol{curl}\, v\|_K.
\end{aligned} \tag{59}$$

Combining (56), (57) and (59) yields

$$(\nabla_w u_h,\, \boldsymbol{curl}\, v)_h \leq C s_h^{1/2}(u_h,\, u_h)\|a^{-1/2}\boldsymbol{curl}\, v\|,$$

which together with (53) completed the proof. □

Theorem 4 (Upper bound) Let u and u_h be the solutions to problem (1) and the CDG scheme (7), respectively. There exists a constant $C > 0$ such that

$$|\!|\!|u - u_h|\!|\!| \leq C\eta(u_h,\, \mathscr{T}_h). \tag{60}$$

Proof. From (42), we get

$$\begin{aligned}
\|a^{1/2}(\nabla u - \nabla_w u_h)\|^2 =&(a(\nabla u - \nabla_w u_h),\, \nabla\psi)_h \\
&+ (a(\nabla u - \nabla_w u_h),\, a^{-1}\boldsymbol{curl}\, v)_h.
\end{aligned} \tag{61}$$

Since $[\![u]\!]|_e = 0$, we have

$$s_h(u - u_h,\, u - u_h) = \sum_{e\in\mathscr{E}_h} h_e^{-1}\langle[\![u - u_h]\!],\, [\![u - u_h]\!]\rangle_e = \sum_{e\in\mathscr{E}_h} h_e^{-1}\langle[\![u_h]\!],\, [\![u_h]\!]\rangle_e = s_h(u_h,\, u_h),$$

which together with (61), (46), (52) and (43) yields the estimate (60). The proof is completed. □

Now we turn to the lower bound estimate. We apply the standard bubble function to derive an efficiency estimate.

Lemma 10 [25] For each $K \in \mathscr{T}_h$, let $\chi_K \in H_0^1(K)$ be its associated bubble function. Then the following holds:

$$C_1\|v_h\|_K^2 \le (\chi_K v_h, v_h)_K,$$

$$\|\chi_K v_h\|_K + h_K\|\nabla(\chi_K v_h)\|_K \le C_2\|v_h\|_K,$$

for any $v_h \in \mathbb{P}_k(K)$.

Lemma 11 [25] For each $K \in \mathscr{T}_h$ with an edge e, let $\chi_e \in H_0^1(\omega_e)$ be the associated bubble function. Then, for any $v_h \in \mathbb{P}_k(K)$, we have

$$C_1\|v_h\|_e^2 \le \langle \chi_e v_h, v_h\rangle_e,$$

$$h_e^{-1/2}\|\chi_e v_h\|_{\omega_e} + h_e^{1/2}\|\nabla(\chi_e v_h)\|_{\omega_e} \le C_2\|v_h\|_e.$$

Lemma 12 (local lower bound) There is a constant $C > 0$ such that

$$h_K\|f + \nabla\cdot(a\nabla_w u_h)\|_K \le C\|a^{1/2}(\nabla u - \nabla_w u_h)\|_K, \tag{62}$$

$$h_e^{1/2}\|[\![a\nabla_w u_h]\!]\|_e \le C\|a^{1/2}(\nabla u - \nabla_w u_h)\|_{\omega_K}. \tag{63}$$

Proof. By employing integration by parts, we have for any $v \in H_0^1(K)$:

$$\begin{aligned}(f + \nabla\cdot(a\nabla_w u_h), v)_K &= (a(\nabla u - \nabla_w u_h), \nabla v)_K \\ &\le \|a^{1/2}(\nabla u - \nabla_w u_h)\|_K\|\nabla v\|_K.\end{aligned}$$

Let χ_K be a bubble function defined on K. Taking $v = (f + \nabla\cdot(a\nabla_w u_h))\chi_K$ in the above equation, and then using Lemma 10 yields to the estimate (62).

For any $v_e \in H_0^1(\omega_e)$, using integration by parts, we get

$$\sum_{K\subset\omega_e}(a(\nabla u - \nabla_w u_h), \nabla v_e)_K = \langle[\![a\nabla_w u_h]\!], v_e\rangle_e + \sum_{K\subset\omega_e}(f + \nabla\cdot(a\nabla_w u_h), v_e)_K.$$

Let χ_e be an edge bubble function associated to e. Taking $v_e = [\![a\nabla_w u_h]\!]\chi_e$ in the above equation, and then invoking Lemma 11, we have

$$
\begin{aligned}
C_1\|[\![a\nabla_w u_h]\!]\|_e^2 \leq & \langle [\![a\nabla_w u_h]\!],\, v_e\rangle_e \\
\leq & \sum_{K\subset\omega_e} \|a^{1/2}(\nabla u-\nabla_w u_h)\|_K \|\nabla v_e\|_K \\
& + \sum_{K\subset\omega_e} \|f+\nabla\cdot(a\nabla_w u_h)\|_K \|v_e\|_K. \\
\leq & C_2 \sum_{K\subset\omega_e} h_e^{-1/2}\|a^{1/2}(\nabla u-\nabla_w u_h)\|_K \|[\![a\nabla_w u_h]\!]\|_e \\
& + C_2 \sum_{K\subset\omega_e} h_e^{1/2}\|f+\nabla\cdot(a\nabla_w u_h)\|_K \|[\![a\nabla_w u_h]\!]\|_e.
\end{aligned}
$$

Then, cancelling $\|[\![a\nabla_w u_h]\!]\|_e$ and then multiplying $h_e^{1/2}$, on both side of the above equation, yields to

$$
h_e^{1/2}\|[\![a\nabla_w u_h]\!]\|_e \leq C \sum_{K\subset\omega_e} [\|a^{1/2}(\nabla u-\nabla_w u_h)\|_K + h_e\|f+\nabla\cdot(a\nabla_w u_h)\|_K],
$$

which together with (62) leads to the estimate (63). The proof is completed. □

To show the efficiency of the estimator $\eta(u_h,\, \mathscr{T}_h)$ defined in (40), we introduce, for each element $K\in\mathscr{T}_h$, the local a priori error $|\!|\!|u-u_h|\!|\!|_K$ and the local error estimator $\eta(u_h,\, K)$ as follows:

$$
|\!|\!|u-u_h|\!|\!|_K^2 := \|a^{1/2}(\nabla u-\nabla_w u_h)\|_K^2 + s_K(u-u_h,\, u-u_h),
$$

with

$$
s_K(v,\, w) = \sum_{e\subset\partial K} h_e^{-1}\langle [\![v]\!],\, [\![w]\!]\rangle_e
$$

and

$$
\eta^2(u_h,\, K) := h_K^2\|f+\operatorname{div}(a\nabla_w u_h)\|_K^2 + \sum_{e\subset\partial K} h_e\|[\![a\nabla_w u_h]\!]\|_e^2 + s_K(u_h,\, u_h).
$$

Theorem 5 (Efficiency) Let u and u_h be the solutions to problem (1) and the CDG scheme (7), respectively. There exists a constant $C>0$ such that

$$
\eta(u_h,\, K) \leq C\left(\sum_{K\in\omega_K} |\!|\!|u-u_h|\!|\!|_K^2\right)^{1/2}.
$$

Proof. Since for the exact solution $u \in H_0^1(\Omega)$, we have $[\![u]\!]_e = 0$ on each $e \in \mathscr{E}_h$. This implies

$$\begin{aligned} s_K(u-u_h,\, u-u_h) &= \sum_{e\subset\partial K} h_e^{-1} \langle [\![u-u_h]\!],\, [\![u-u_h]\!] \rangle_e \\ &= \sum_{e\subset\partial K} h_e^{-1} \langle [\![u_h]\!],\, [\![u_h]\!] \rangle_e \\ &= s_K(u_h,\, u_h), \end{aligned}$$

which together with Lemma 12 completes the proof. □

Based on the proposed a posteriori error estimator, we develop an adaptive CDG method for problem (1) as described below.

Algorithm 1. Adaptive CDG FEM

Input

$\mathscr{T}_0$: initial triangulation of Ω; f : source term;

tol : tolerance level; $\theta \in (0,\, 1)$ marking parameter.

Output

$\mathscr{T}_I$: a triangulation of Ω;

u_I: conforming DG finite element solution on $\mathscr{T}_I$.

1. $i := 0$; $\eta := 1$;
2. **while** $\eta < tol$ **do**
3. *Solve* the CDG scheme (7) on the triangulation $\mathscr{T}_i$ to obtain the solution u_i;
4. *Estimate* the a posteriori error by $\eta = \eta(u_i,\, \mathscr{T}_i)$;
5. *Mark* a subset $\mathscr{M}_i$ of $\mathscr{T}_i$ with minimum number satisfying $\eta(u_i,\, \mathscr{M}_i) \geq \theta\eta(u_i,\, \mathscr{T}_i)$;
6. *Refine* the elements in $\mathscr{M}_i$, gather all new elements together with the remaining part $\mathscr{T}_i \setminus \mathscr{M}_i$ to build the updated triangulation $\mathscr{T}_{i+1}$;
7. Set $i := i+1$;
8. **end while**
9. Set $u_I := u_i$; $\mathscr{T}_I := \mathscr{T}_i$;

---

5. Numerical experiments

This section provides numerical investigations to demonstrate computational performance of the CDG scheme (7) and to verify the efficiency and reliability of the a posteriori error estimator η defined in (40).

Example 1 Let $\Omega = (0,\, 1)^2$ and $a(\boldsymbol{x}) = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$ in problem (1). We take suitable source function f so that the exact solution is

$$u(\boldsymbol{x}) = \sin(\pi x_1)\sin(\pi x_2).$$

In this example, we aim to verify the theoretical error estimates from Theorem 2 and 3 and to compare the performance of the CDG (7) and IPDG (8) schemes.

Table 1. Numerical errors and resulting convergence rates of CDG method for Example 1

k	Mesh N	$\|u-u_h\|_{1,h}$	Rate	$\|u-u_h\|$	Rate
1	4	6.599×10^{-1}	—	2.247×10^{-2}	—
	8	3.350×10^{-1}	0.978	7.279×10^{-3}	1.626
	16	1.680×10^{-1}	0.996	2.131×10^{-3}	1.772
	32	8.401×10^{-2}	1.000	5.744×10^{-4}	1.891
	64	4.200×10^{-2}	1.000	1.487×10^{-4}	1.950
	128	2.100×10^{-2}	1.000	3.782×10^{-5}	1.975
2	4	1.159×10^{-1}	—	2.459×10^{-3}	—
	8	2.970×10^{-2}	1.964	3.327×10^{-4}	2.886
	16	7.459×10^{-3}	1.993	4.307×10^{-5}	2.950
	32	1.867×10^{-3}	1.998	5.488×10^{-6}	2.972
	64	4.670×10^{-4}	1.999	6.931×10^{-7}	2.985
	128	1.168×10^{-4}	1.999	8.711×10^{-8}	2.992

Table 2. Numerical errors and resulting convergence rates of IPDG method for Example 1

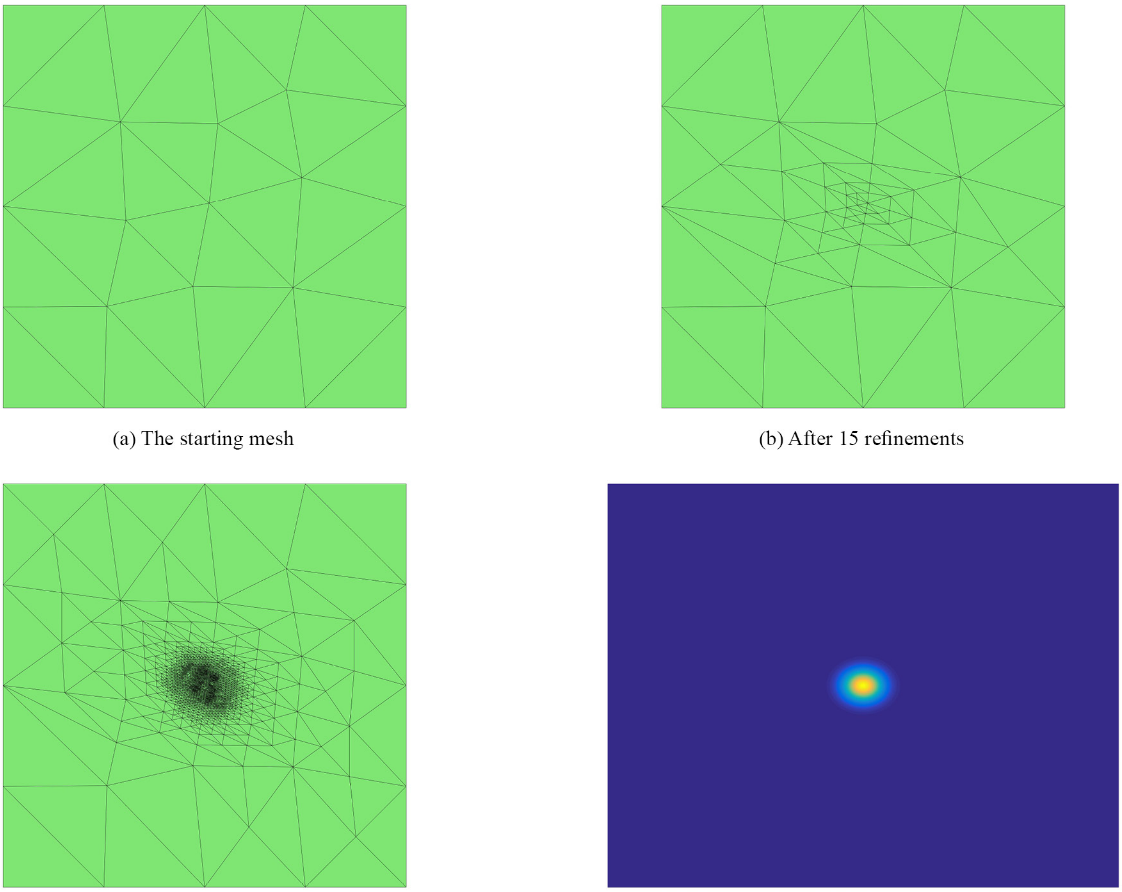
k	Mesh Nx	$\|u-u_h\|_{1,h}$	Rate	$\|u-u_h\|$	Rate
1	4	9.462×10^{-1}	—	4.811×10^{-2}	—
	8	4.677×10^{-1}	1.02	1.344×10^{-2}	1.84
	16	2.305×10^{-1}	1.02	3.505×10^{-3}	1.94
	32	1.142×10^{-2}	1.01	8.917×10^{-4}	1.97
	64	5.682×10^{-2}	1.01	2.247×10^{-4}	1.99
	128	2.834×10^{-2}	1.00	5.637×10^{-5}	1.99
2	4	1.803×10^{-1}	—	2.802×10^{-3}	—
	8	4.378×10^{-2}	2.04	3.604×10^{-4}	2.96
	16	1.072×10^{-2}	2.03	4.574×10^{-5}	2.98
	32	2.647×10^{-3}	2.02	5.764×10^{-6}	2.99
	64	6.572×10^{-4}	2.01	7.235×10^{-7}	2.99
	128	1.637×10^{-4}	2.01	9.062×10^{-8}	3.00

We conduct the numerical experiment for Example 1 using the following setup. The computational domain Ω is first partitioned into $N \times N$ congruent rectangles, each of which is then subdivided into two congruent triangles to form the final mesh. On this mesh, we employ both the CDG scheme (7) and the IPDG scheme (8) with $\mathbb{P}_1$ and $\mathbb{P}_2$ finite elements. We take the penalty parameter $\sigma = 10$ in the IPDG scheme (8). Tables 1 and 2 present the numerical convergence histories of the CDG and IPDG methods for Example 1, respectively. The results in Table 1 demonstrate that the convergence rates of the errors in both the L^2 norm and the energy norm (36) closely align with theoretical predictions of Theorem 2 and 3. Meanwhile, Table 2 indicates that while the IPDG method achieves the same convergence orders, its errors are slightly larger than those of the CDG method on identical meshes.

Example 2 Let $\Omega = (0, 1)^2$ and $a(\boldsymbol{x}) = I$. The source term f is chosen such that the exact solution of problem (1) is given by

$$u(\boldsymbol{x}) = 10x_1(1-x_1)x_2(1-x_2)(1-2x_2)e^{-200((2x_1-1)^2+(2x_2-1)^2)}.$$

Figure 1 shows a series of adaptive meshes with initial mesh, 15-th level and 45-th level and the numerical solution computed by $\mathbb{P}_2$ finite element on mesh level 45 and $\theta = 0.25$. It can be observed that the singularity in the solution is accurately captured by the a posteriori error estimator during adaptive mesh refinement.



(a) The starting mesh

(b) After 15 refinements

(c) After 45 refinements

(d) Numerical solution on 45-th level mesh

Figure 1. Adaptive refinement mesh and the corresponding numerical solution for Example 2, $\mathbb{P}_2$ finite element

Figure 2 demonstrates the error curves of $\|u-u_h\|$ and the a posterior error estimator (40) using $\mathbb{P}_2$ element on both uniform and adaptive meshes. It can be seen that the errors obtained on adaptive meshes are significantly smaller than those on uniform meshes. Furthermore, the adaptive CDG algorithm achieves the optimal convergence rate.

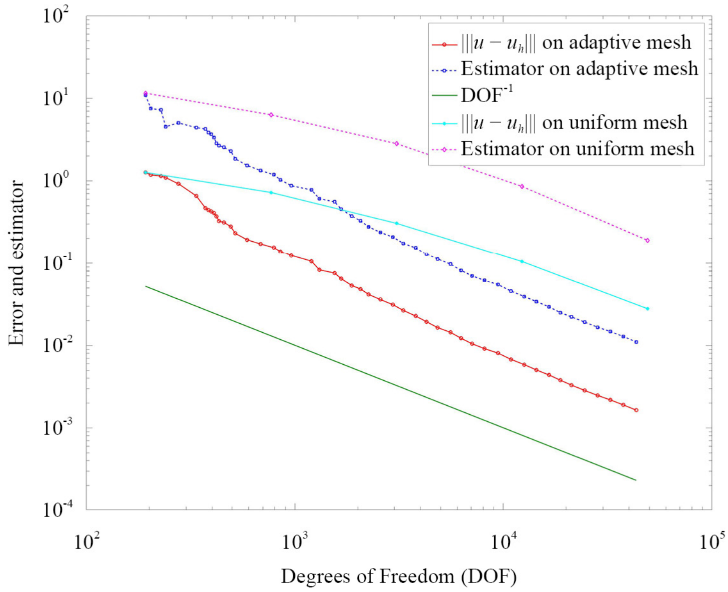


Figure 2. Example 2: Error curves of $\|u-u_h\|$ and the a posteriori estimator (40) on both uniform meshes and adaptive meshes

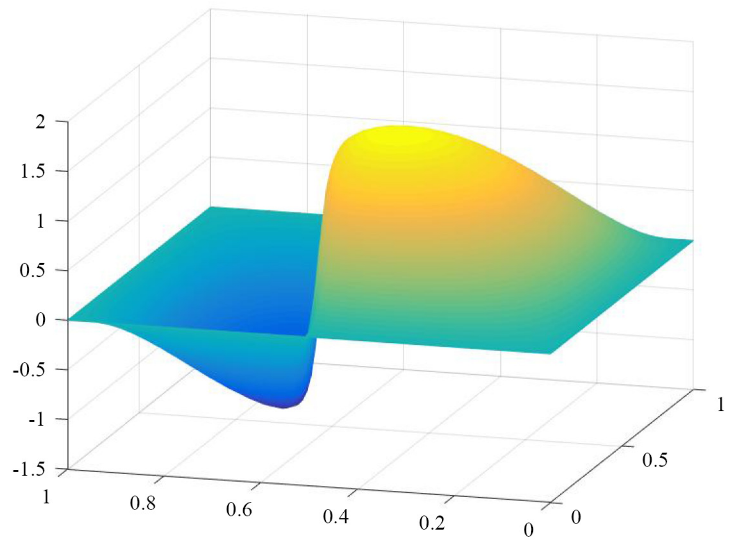


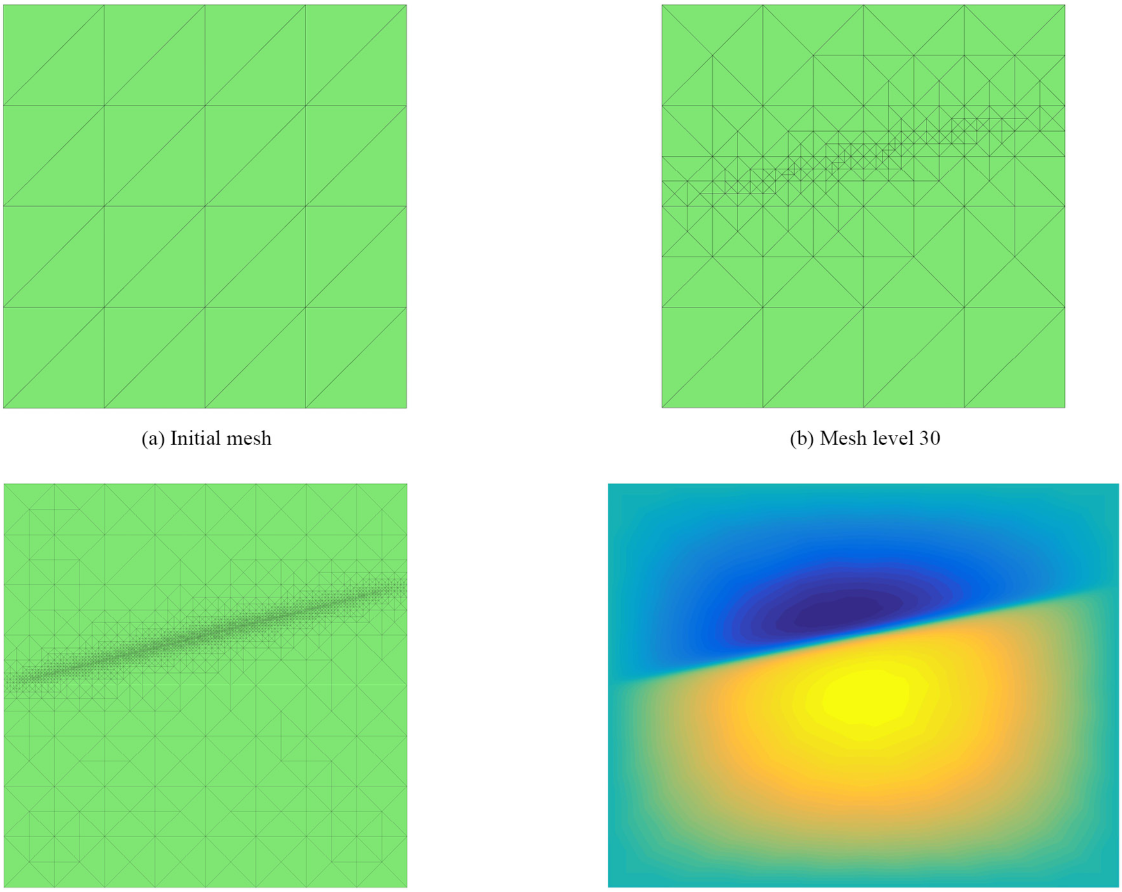
Figure 3. Exact solution of Example 3

Example 3 Let $\Omega=(0,1)^2$ and $a(\boldsymbol{x})=I$. The source term f is chosen such that the exact solution of problem (1) is given by

$$u(\boldsymbol{x}) = 16x_1(1-x_1)x_2(1-x_2)\arctan(25x_1 - 100x_2 + 50).$$

As illustrated in Figure 3, the exact solution $u(\boldsymbol{x})$ exhibits a sharp interior layer.

Figure 4 presents the meshes of different levels obtained by adaptive refinement with $\mathbb{P}_2$ finite element and $\theta = 0.25$. Once again, the a posteriori error estimator (40) effectively captures the interior layer present in the exact solution.



(a) Initial mesh

(b) Mesh level 30

(c) Mesh level 50

(d) Numerical solution on Mesh level 50

Figure 4. Adaptive refinement mesh and the corresponding numerical solution for Example 3, $\mathbb{P}_2$ finite element

6. Conclusion

In this work, we have introduced a conforming discontinuous Galerkin method for solving second-order elliptic partial differential equations. First, we established a priori error analysis under standard regularity assumptions, proving that the method converges optimally in both the energy norm and the L^2 norm. Second, we developed a reliable and efficient residual-based a posteriori error estimator. We rigorously proved that the estimator serves as both an upper bound (reliability) and a lower bound (efficiency) for the true error, up to generic constants, ensuring it is a robust guide for adaptive mesh refinement. The numerical experiments successfully confirmed the optimal convergence rates predicted by the a priori analysis and demonstrated the practical effectiveness and reliability of the a posteriori error estimator in driving adaptive mesh refinement strategies. Although our analysis focused on 2D cases for simplicity, the findings can be directly generalized to three dimensions. Future work will focus on extending this framework to more complex problems, such as nonlinear elliptic equations and systems, and on designing efficient adaptive algorithms powered by the proven error estimator.

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Conflict of interest

The authors declare no competing financial interest.

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