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Research Article

Stability and Uniqueness of Fractional Order Newton-Raphson's Method for Nonlinear Equations

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Abstract: The study proposes a fractional-order derivative-based Newton-Raphson algorithm for solving nonlinear fractional-order systems. The algorithm focuses on polynomial, exponential, and trigonometric functions and demonstrates its $2\alpha^{\text{th}}$ convergence with a Gamma operation damping bound. The method is effective in solving complex nonlinear differential equations of fractional orders, with rigorous mathematical results consistent across both partial subordinates. The technique is particularly useful for complex roots and precise fractional calculus, making it an effective substitute for the traditional Newton-Raphson method. This method is useful for solving complex nonlinear differential equations in engineering and physics due to its more precise computation than most usual methods. The method's effectiveness is influenced by the specific issue, precision level, and computing power available.

***Keywords*:** nonlinear equations, fractional order operators, Newton-Raphson, convergence

MSC: 65H04, 46T99, 26A33

1. Introduction

The origin of differential equations can be traced back to the seventeenth century when Gottfried Leibniz and Isaac Newton first introduced calculus. In the eighteenth century, Leonhard Euler contributed significantly to the development of differential equation theory by designing methods for solving both linear and nonlinear Ordinary Differential Equations (ODEs). These methods were also used to represent the motion of objects in mechanics. The nineteenth century is

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known as the era of Fourier series that came into being through the efforts of mathematicians like Carl Gustav Jacobi, Jean-Baptiste Fourier, and Joseph Fourier, who extended the theory. Computers and numerical methods for analyzing complex systems have brought about a huge change in differential equation evaluation in the 20th century [1]. Differential equations progress alongside the development of computers. They have become a core of theoretical physics and applied mathematics [2]. These equations pose tough mathematical problems that have numerous solutions when explored. It is indeed difficult to find the exact solution for linear and nonlinear differential equations. The pursuit of exact and accurate results in applied mathematics has led to the innovation of new analytical methods. Fluid dynamics, quantum physics, and gravitational waves are a few research areas that have benefited from employing differential equations. They are applied in coming up with practical computer programs that deal with situations where an analytical solution is absent, and also help in simulating complex systems. They are essential for comprehending physical phenomena, and they find application in various fields.

A subfield of mathematics known as non-integer calculus makes use of fractional-order operators to comprehend and model physical phenomena. Additionally, it applies to non-integer order differentiation problems, recovering zero-order derivatives, and meeting certain requirements for fractional derivatives [3]. There are two primary kinds of fractional derivatives: Liouville-Caputo and Riemann-Liouville. Fractional calculus theory is developed by creating new definitions of fractional operators and integrals. A very interesting fact here is that those problems with fractional derivatives have already gathered more enthusiastic scholars, since the traditional derivative equations are not sufficient to describe reality. The field of fractional calculus is a potential source of inspiration for scholars due to the multitude of applications mentioned and its frequent usage in nonlinear complex systems. Because Fractional Differential Equations (FDEs) find their use in multiple scientific and engineering areas, they have seen an upsurge of development in the last few decades. As FDEs help in lowering the errors coming from the omission of insignificant parameters, they are quite helpful in the simulation of reality. A number of math models in areas such as biology, physics, nature, and other scientific fields make use of FDEs (for instance [4–12]). This research paper deals with time-FDEs in an attempt to represent the memory and hereditary characteristics of various materials and systems. Such equations are very suitable when one is trying to depict a case where the speed of change is affected by the total record of the event.

In consideration of the remarkable progress of analytic and numerical algorithms for obtaining solutions of FDEs, a robust and reliable numerical method is here proposed by the authors of this paper. Fractional-order nonlinear equations have been solved numerically using various methods, e.g., the Adomian Decomposition Method (ADM), Variational Iteration Method (VIM), fractional finite difference methods, and spectral approaches. To enhance the accuracy and effectiveness of their results, authors usually combine the two analytical and numerical techniques. For instance, it is possible to mix spectrum methods with optimization algorithms or ADM and VIM with Laplace transforms. The nature of the fractional-order nonlinear equation, thus its order, type of nonlinearity, and the accuracy needed, determines the choice of the method. Kumar et al. [13] have converted the fractional-order problems into algebraic equations by the collocation method and the Genocchi wavelets for the numerical simulations of the disease model. The accuracy and applicability of the Genocchi wavelets method were examined after the model was solved with different parameters using the Adams-Bashforth-Moulton (ABM) numerical scheme. A novel fractional chaotic system was presented and examined by Baleanu et al. [14], taking into account cubic and quadratic nonlinearities. They examined the chaotic behavior of an effective nonstandard finite difference scheme in both the time domain and the phase plane. The parameters' estimation for a fractional-order nonlinear finite impulse response system having colored noise was examined by Wang et al. [15]. By using key term separation to express the system's output as a linear combination of unknown parameters, they sought to minimize redundant parameter estimation. In order to increase estimation accuracy, they presented a multi-innovation gradient-based iterative algorithm using multi-innovation theory and created a gradient-based iterative algorithm employing negative gradient search. By contrasting it with well-known non-local Caputo fractional derivatives and Atangana-Baleanu (ABC) fractional derivatives, Chauhan et al. [16] presented the AD Formable Transform Method for resolving the fractional order Sharma-Tasso-Olver problem. Using Newton's interpolation polynomial and a fractal-fractional order system, Saber [17] resolved the Burke-Shaw-type nonlinear chaotic system. He solved the fractional Burke-Shaw model using a numerical power series approach and a new methodology based on an efficient polynomial. Researchers [18] updated the Predictor-Corrector scheme to numerically solve a delay Differential Equations (DEs) in the

Caputo derivative sense and provided numerical examples to show the method's availability. To solve the Caputo-Fabrizio (CF) fractional order partial differential equation in series form, Ahmad et al. [19] presented the Yang transform homotopy perturbation approach. They showed that the approach converges to the exact solution and provided comprehensive results. The Swift-Hohenberg equations were solved using the Elzaki transform decomposition method in [20], with an emphasis on temperature and thermal convection in fluid dynamics. This paradigm can explain formation processes on liquid surfaces enclosed by a horizontally well-conducting border. By fusing the integral transform with the homotopy perturbation method, Fareed et al. [21] presented a novel analytical-numerical approach to solving nonlinear fractional differential equations. They illustrated the method's versatility by using it to solve the Kawahara issue. A fractional shifted Legendre neural network and an extreme machine learning algorithm were used by Fathima et al. [22] to create a novel approach to solving fractional differential equations. For linear differential equations, they used the pseudo-inverse of the activation function output matrix to find the unknown coefficient matrix; for nonlinear equations, they employed a nonlinear least squares perturbation technique. In the study [23], fourth-order time-fractional Cahn-Hilliard models were solved and analyzed using the "Tantawy Technique," a quick and precise method, in conjunction with the variational iteration transform method and the homotopy perturbation transform method within the Yang transform framework. The article [24] discusses the Aboodh residual power series method and Aboodh transform iteration method, which are effective techniques for solving fractional-order linear and nonlinear partial differential equations. Ebrahimzadeh et al. [25] proposed a mathematical model to analyze water-related contamination transmission dynamics, relying on a trapezoidal method for effective numerical scheme execution and undertaking extensive error analysis and convergence analyses. Baleanu et al. [26] developed a fractional-order mathematical model for tumor-immune monitoring, examining the relationships between tumor cell populations and the immune system using fractional differential equations. They devised an efficient numerical approach that takes into account both singular and nonsingular derivative operators, as well as an optimal control approach for studying the influence of treatment on the model.

The goal of this article is to solve nonlinear differential equations using the fractional-order Newton-Raphson technique. The Fractional Newton-Raphson (FNR) method is a variant of the Newton-Raphson approach that provides flexibility in convergence by finding the roots of equations, including complicated ones, using fractional derivatives rather than integer-order derivatives. One effective method for locating the roots of both linear and nonlinear equations is the Newton-Raphson method. Complex roots can be found from real initial conditions using the fractional-order technique, which substitutes fractional-order derivatives for integer-order derivatives. The proposed method is a robust mathematical framework for solving fractional-order derivatives nonlinear differential equations, offering high accuracy, fast convergence, and computational efficiency. With only a few iterations, the suggested approach effectively solves nonlinear differential equations and yields a Taylor series solution in a rapidly convergent series. Discretizing and linearizing procedures are not necessary for the nonlinear issue, and a few repetitions can result in a solution that is readily estimable with these methods. We will use numerical examples with known exact answers to evaluate the accuracy and efficacy of the procedure.

2. Fundamental ideas of fractional-order operators

Fractional calculus has become widely used in many sectors of science and engineering, with many derivatives relevant for a variety of situations. This paper examines the effect of fractional derivatives on nonlinear equation roots and their dependency on initial estimates. It also includes an extended mean value theorem and various examples of Taylor's formula. The following is a discussion of the popular notion of fractional-order derivatives, primarily the Caputo and Riemann-Liouville [27, 28].

Definition 1 The Caputo derivative of $f(x)$, with $\beta > 0$, a, β, $x \in$ real numbers, is

$$
{}_{a}^{C}D_{x}^{\beta} f(x) = \begin{cases} \dfrac{1}{\Gamma(m-\beta)} \displaystyle\int_{a}^{x} \dfrac{\frac{df^{(m)}(t)}{dt^{(m)}}}{(x-t)^{\alpha+1-m}} dt, & m-1 < \alpha \leq m \in \mathbb{N}, \\ \\ \dfrac{df^{(m)}(t)}{dt^{(m)}}, & \alpha = m \in \mathbb{N}. \end{cases} \tag{1}
$$

Because it forms zero of the constant value and meets the ordinary or classic derivative features, the Caputo derivative is essential.

Definition 2 The Riemann-Liouville (RL) derivative of $f(x)$ of order β, with $0 < \beta \leq 1$, is

$$
D_{a+}^{k\beta} = \begin{cases} \dfrac{1}{\Gamma(1+k\beta)} \dfrac{d}{dx} \displaystyle\int_{a}^{x} \dfrac{f(x)}{(x-t)^{\alpha}} dt, & 0 < \beta \leq 1, \\ \\ \dfrac{df(t)}{dt}, & \alpha = 1. \end{cases} \tag{2}
$$

Additionally, it is now necessary to use fractional derivatives of Riemann-Liouville to extend the nonlinear $f(x)$ in a Taylor series to permit the converging outcome.

Definition 3 Caputo-Fabrizio fractional derivative having a non-singular kernel, for a function $f(t)$ of C^1, is

$$
{}_{a}^{CF}D_{t}^{\beta} f(t) = \frac{1}{1-\beta} \int_{a}^{t} f'(\tau) \exp\left(-\beta \frac{t-\tau}{1-\alpha}\right) d\tau, \quad \text{where } a > 0,\ \beta \in (0,\ 1]. \tag{3}
$$

Definition 4 Newton's method, sometimes referred to as the Newton-Raphson method, is a mathematical technique that starts with a single variable breaking point, a derivative f', and a central notion for $f(x)$ and continuously improves approximations to the roots of a valid cutoff using root-discovery computations. If the hidden speculation is near and the limit satisfies satisfactory doubts, then

$$
x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.
$$

The root is not at all a better prediction than x_0. The additional evolved guess is the particular foundation for the direct prediction at the actual point. It repeats the process to be

$$
x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.
$$

Until an accurate enough worth is reached.

3. Analysis of convergence in the Newton-Raphson method for fractional order

Theorem 1 [27] Let us suppose that $D_a^{j\beta} g(x) \in C([a, b])$ for $j = 1, 2, \ldots, n+1$, where, $\beta \in (0, 1]$, then we have

$$g(x) = \sum_{i=0}^{n} D_a^{i\beta} f(a) \frac{(x-a)^{i\beta}}{\Gamma(\beta+1)} D_a^{(n+1)\beta} g(\xi) \frac{(x-a)^{(n+1)\beta}}{\Gamma((n+1)\beta+1)}, \tag{4}$$

with $a \le \xi \le x$, for all $x \in (a, b]$ where $D_a^{n\beta} = D_a . D_a^{i\beta} . D_a^{i\beta} \ldots D_a^{i\beta}$. ($n$ time).

The proof of this generalized Taylor's formula is given in Theorem 3 of [27].

Considering the Taylor progression of a limit $g(x)$ around $a = \bar{x}$ to the extent of fragmentary auxiliaries of Caputo-type.

$$g(x) = \frac{D_{\bar{x}}^{\beta} g(\bar{x})}{\Gamma(\beta+1)} (x-\bar{x})^{\beta} + \frac{D_{\bar{x}}^{2\beta} g(\bar{x})}{\Gamma(2\beta+1)} (x-\bar{x})^{2\beta} + \frac{D_{\bar{x}}^{3\beta} g(\bar{x})}{\Gamma(3\beta+1)} (x-\bar{x})^{3\beta} + O((x-\bar{x})^{4\beta}). \tag{5}$$

By extracting, this Taylor development can be stated as a common factor, as,

$$g(x) = \left[(x-\bar{x})^{\beta} + C_2 (x-\bar{x})^{2\beta} + C_3 (x-\bar{x})^{3\beta}\right] \frac{D_{\bar{x}}^{\beta} g(\bar{x})}{\Gamma(\beta+1)} + O((x-\bar{x})^{4\beta}), \tag{6}$$

where $C_j = \frac{\Gamma(\beta+1)}{\Gamma(j\beta+1)} \frac{D_{\bar{x}}^{j\beta} g(\bar{x})}{D_{\bar{x}}^{\beta} g(\bar{x})}$, for $j \ge 2$.

From (5) and (6), we have

$$D_a^{\beta} g(x) = \frac{D_{\bar{x}}^{\beta} g(\bar{x})}{\Gamma(\beta+1)} \left[\Gamma(\beta+1) + \frac{\Gamma(2\beta+1)}{\Gamma(\beta+1)} C_2 (x-\bar{x})^{\beta} + \frac{\Gamma(3\beta+1)}{\Gamma(2\beta+1)} C_3 (x-\bar{x})^{2\beta}\right] + O\big((x-\bar{x})^{3\beta}\big). \tag{7}$$

As a result, the following conclusion describes the design and convergence analysis of a Newton-type method based on Caputo fractional derivatives, abbreviated Caputo Fractional Newton (CFN).

Theorem 2 Let g: $D \subseteq \mathbb{R} \to \mathbb{R}$ have $k\beta$-order fractional derivatives, with k a positive integer and $0 < \beta < 1$ in the domain D with a zero $\bar{x}$ of $g(x)$. Also, let ${}_C D_{\bar{x}}^{\beta} g(\bar{x})$ be persistent and not invalid in $\bar{x}$. On the off chance that a first guess x_0 is suitably close to $\bar{x}$, the local convergence order of the fractional Newton technique for Caputo-type.

$$x_{k+1} = x_k - \Gamma(\beta+1) \frac{g(x_k)}{{}_C D_a^{\beta} g(x_k)}, \quad k = 0, 1 \tag{8}$$

Eq. (8) is at least 2β with the error equation:

$$e_{k+1}^{\beta} = \frac{\Gamma(2\beta+1) - (\Gamma(\beta+1))^2}{(\Gamma(\beta+1))^3} C_2 e_k^{2\beta} + O(e_k^{3\beta}). \tag{9}$$

Proof. Allow us to consider a succession of repeats $\{x_k\} \geq 0$, obtained by utilizing (3), that seeks to estimate the $\bar{x}$ zero having a non-linear capacity $g(x)$. Taylor formulation and its Caputo derivative at x_k, around $\bar{x}$, can be communicated as

$$g(x_k) = \left[(x_k-\bar{x})^{\beta} + C_2(x_k-\bar{x})^{2\beta} + C_3(x_k-\bar{x})^{3\beta}\right] \frac{{}_C D^{\beta}_{\bar{x}} g(\bar{x})}{\Gamma(\beta+1)} + O((x_k-\bar{x})^{4\beta}), \tag{10}$$

and

$${}_C D^{\beta}_{a} g(x_k) = \left[\Gamma(\beta+1) + \frac{\Gamma(2\beta+1)}{\Gamma(\beta+1)} C_2(x-\bar{x})^{\beta} + \frac{\Gamma(3\beta+1)}{\Gamma(2\beta+1)} C_3(x-\bar{x})^{2\beta}\right] \frac{{}_C D^{\beta}_{\bar{x}} g(\bar{x})}{\Gamma(\beta+1)} + O((x-\bar{x})^{3\beta}), \tag{11}$$

where $C_j = \frac{\Gamma(\beta+1)}{\Gamma(j\beta+1)} \frac{{}_C D^{j\beta}_{\bar{x}} g(\bar{x})}{{}_C D^{j\beta}_{\bar{x}} g(\bar{x})}$, for $j \geq 2$. Accordingly, a Newton-like remainder can be determined, and expressed in terms of error at the $K-th$ iteration $e_k = x_k - \bar{x}$ as

$$\begin{aligned} \frac{g(x_k)}{{}_C D^{\beta}_{a} g(x_k)} &= \frac{e_k^{\beta} + C_2 e_k^{2\beta} + C_3 e_k^{3\beta} + O e_k^{3\beta}}{\Gamma(\beta+1) + \frac{\Gamma(2\beta+1)}{\Gamma(\beta+1)} C_2 e_k^{\beta} + \frac{\Gamma(3\beta+1)}{\Gamma(2\beta+1)} C_3 e_k^{2\beta} + O(e_k^{3\beta})} \\ &= \frac{1}{\Gamma(\beta+1)} e_k^{\beta} + \frac{(\Gamma(\beta+1))^2 - \Gamma(2\beta+1)}{(\Gamma(\beta+1))^3} C_2 e_k^{2\beta} + O(e_k^{3\beta}). \end{aligned} \tag{12}$$

In this way, to accomplish the 2βth order of convergence, a Caputo-fractional Newton's technique ought to incorporate $\Gamma(\beta+1)$ as a damping boundary, being the subsequent error equation

$$e_{k+1}^{\beta} = x_{k+1} - \bar{x} = e_k^{\beta} - \Gamma(\beta+1) \frac{g(x_k)}{{}_C D^{\beta}_{a} g(x_k)} = -\frac{(\Gamma(\beta+1))^2 - \Gamma(2\beta+1)}{(\Gamma(\beta+1))^3} C_2 e_k^{2\beta} + O(e_k^{3\beta}).$$

We now examine if Newton's method's second order of convergence can be attained with RL fractional derivatives or if further adjustments are required. We explore the relevance of the RL derivative of the first type of function $g(x)$ with request β to arrive at this conclusion:

$$D^{\beta}_{a+} g(x) = \begin{cases} \frac{1}{\Gamma(1+\beta)} \frac{d}{dx} \int_a^x \frac{g(t)}{(x-t)^{\beta}} dt, & 0 < \beta \leq 1, \\ \\ \frac{dg(t)}{dt}, & \beta = 1. \end{cases} \tag{13}$$

Besides, to achieve convergence, expand the nonlinear function in Taylor series using Riemann-Liouville fractional derivatives.

Theorem 3 The mapping g: $R \to R$ has fragmentary subordinates of request $k\beta$, for any sure number k and any β, $0 < \beta \leq 1$, then, at that point, the accompanying equation holds,

$$g(x+h)=\sum_{k=0}^{\infty}\frac{(h)^{k\beta}}{\Gamma(1+k\beta)}D_{a+}^{k\beta}g(x),\ 0<\beta\leq 1, \tag{14}$$

where $D_{a+}^{k\beta}g(x)$ represents the RL derivative of order $k\beta$. So, by utilizing this Taylor expansion in the same way as it has been expressed for Caputo fractional derivatives, it is easy to verify that the iterative expression

$$x_{k+1}=x_k-\Gamma(\beta+1)\frac{g(x_k)}{D_{a+}^{\beta}g(x_k)},\ k=0,\ 1,\ \ldots \tag{15}$$

The following result summarizes the process of achieving 2βtht order of convergence from an initial estimation. It is represented by Riemann-Liouville Fractional Newton (R-LFN).

Theorem 4 Let f: $D\subseteq R\rightarrow R$ be a continuous function such that, for any positive integer k and β, $0<\beta<1$, in the interval D containing the zero $\bar{x}$ of $g(x)$, has fractional derivatives of order $k\beta$. Additionally, let's assume that is nonsingular and continuous in $\bar{x}$. The fractional Newton method's local convergence order can be determined if an initial approximation x_0 is close enough to $\bar{x}$

$$x_{k+1}=x_k-\Gamma(\beta+1)\frac{g(x_k)}{D_{a+}^{\beta}g(x_k)},\ k=0,\ 1 \tag{16}$$

of RL-type of 2β, being $0<\beta\leq 1$, with the error expression:

$$e_{k+1}^{\beta}=\frac{\Gamma(2\beta+1)-(\Gamma(\beta+1))^2}{(\Gamma(\beta+1))^3}C_2e_k^{2\beta}+O(e_k^{3\beta}). \tag{17}$$

Proof. For a continuous function f: $R\rightarrow R$ with fractional derivatives of order $k\beta$, the following equality is true for every positive integer k and any β, $0<\beta<1$,

$$f(x+m)=\sum_{k=0}^{\infty}\frac{m^{k\beta}}{\Gamma(1+k\beta)}D_{a+}^{k\beta}f(x). \tag{18}$$

Therefore, it is simple to demonstrate that the iterative expression is correct by applying this Taylor expansion like that which has been stated for Caputo fractional derivatives. The above result (17) summarizes that, given an initial estimation, it achieves th order of convergence. R-LFN is the symbol for it. The initial estimation dependence has a substantial impact on the numerical estimation of roots and will be discussed in more detail in the next section.

Next, we present a convergence analysis for several scenarios using the proposed fractional derivative on the Newton-Raphson method. In practical applications such as computational fluid dynamics and structural analysis, convergence is essential for precise engineering judgments since it is a reliable and stable method that requires minor input changes to achieve modest output changes. A high-order, quickly convergent iterative technique for resolving fractional differential equations is the Newton-Raphson (NR) method. It works especially well for determining complex roots from real initial conditions, which is something that the traditional method frequently lacks. The fractional order utilized determines the method's order of convergence, and changing the fractional order and initial circumstances is necessary for a critical interpretation of the results Nevertheless, its use is restricted to particular issues or necessitates adjustments to guarantee stability. Its performance varies according to the problem and comparison method, and it is frequently compared to other iterative approaches for solving reduced systems of non-linear algebraic equations. Predictor-corrector techniques, product-integration rules, and specialized software like FdeSolver are further broad strategies. The Newton-Raphson approach offers flexibility in convergence by finding roots of equations using fractional derivatives instead of integer-order derivatives. This method is a robust mathematical framework for solving fractional-order derivatives nonlinear differential equations, offering high accuracy, fast convergence, and computational efficiency. With only a few iterations, the proposed approach effectively solves nonlinear differential equations and yields a Taylor series solution in a rapidly convergent series. Discretizing and linearizing procedures are not necessary for nonlinear issues, and a few repetitions can result in a solution that is readily estimable with these methods.

4. Example 1 (Polynomial function)

The first thing we do is solve polynomial functions.

4.1 *Analysis of Newton-type method convergence [Theorem 1]*

Let the polynomial:

$$g(x) = x^4 7x^2 + 6$$

Replace x by a:

$$g(a) = a^4 7a^2 + 6$$

$$g(\xi) = \xi^4 7\xi^2 + 6$$

Because $a \leq \xi \leq x$ for all equal intervals, we apply Taylor development of the function $g(x)$

$$g(x) = \sum_{i=0}^{n} {}_C D_a^{i\beta}(a^4 - 7a^2 + 6)\frac{(x-a)^{i\beta}}{\Gamma(i\beta+1)} + {}_C D_a^{(n+1)\beta}(\xi^4 - 7\xi^2 + 6)\frac{(x-a)^{(n+1)\beta}}{\Gamma((n+1)\beta+1)}.$$

Because it has 3 orders

$$x^4 - 7x^2 + 6 = {}_C D_{\bar{x}}^{2\beta} x^4 \frac{(x-\bar{x})^{\beta}}{\Gamma(\beta+1)} - 7{}_C D_{\bar{x}}^{2\beta} x^2 \frac{(x-\bar{x})^{\beta}}{\Gamma(\beta+1)} + 6{}_C D_{\bar{x}}^{2\beta} \frac{(x-\bar{x})^{\beta}}{\Gamma(\beta+1)} + {}_C D_{\bar{x}}^{2\beta} x^4 \frac{(x-\bar{x})^{2\beta}}{\Gamma(2\beta+1)}$$

$$- 7{}_C D_{\bar{x}}^{2\beta} x^2 \frac{(x-\bar{x})^{2\beta}}{\Gamma(2\beta+1)} + 6{}_C D_{\bar{x}}^{2\beta} \frac{(x-\bar{x})^{2\beta}}{\Gamma(2\beta+1)} + {}_C D_{\bar{x}}^{2\beta} x^4 \frac{(x-\bar{x})^{3\beta}}{\Gamma(3\beta+1)} + {}_C D_{\bar{x}}^{2\beta} x^4 \frac{(x-\bar{x})^{3\beta}}{\Gamma(3\beta+1)}$$

$$- 7{}_C D_{\bar{x}}^{2\beta} x^2 \frac{(x-\bar{x})^{3\beta}}{\Gamma(3\beta+1)} + 6{}_C D_{\bar{x}}^{2\beta} \frac{(x-\bar{x})^{3\beta}}{\Gamma(3\beta+1)}$$

$\frac{{}_C D_{\bar{x}}^{\beta}(\bar{x}^4-7\bar{x}^2+6)}{\Gamma(\beta+1)}$ is a typical variable; the Taylor advancement can be communicated as:

$$x^4-7x^2+6=\frac{{}_C D_{\bar{x}}^{\beta}(\bar{x}^4-7\bar{x}^2+6)}{\Gamma(\beta+1)}\left[(x-\bar{x})^{\beta}+C_2(x-\bar{x})^{2\beta}+C_3(x-\bar{x})^{3\beta}\right]+O((x-\bar{x})^{4\beta}),$$

where $C_j=\frac{\Gamma(\beta+1)}{\Gamma(j\beta+1)}\frac{{}_C D_{\bar{x}}^{j\beta}(\bar{x}^4-7\bar{x}^2+6)}{{}_C D_{\bar{x}}^{\beta}(\bar{x}^4-7\bar{x}^2+6)}$ for $j\geq 2$.

Using a fractional derivative around $\bar{x}$ gives us.

$${}_C D_a^{\beta}(x^4-7x^2+6)=\frac{{}_C D_{\bar{x}}^{\beta}(\bar{x}^4-7\bar{x}^2+6)}{\Gamma(\beta+1)}\left[\Gamma(\beta+1)+\frac{\Gamma(2\beta+1)}{\Gamma(\beta+1)}C_2(x-\bar{x})^{\beta}+\frac{\Gamma(3\beta+1)}{\Gamma(2\beta+1)}C_3(x-\bar{x})^{2\beta}\right]+O((x-\bar{x})^{3\beta}).$$

4.1.1 *Existence and uniqueness of solutions*

1. For a fractional-order derivative ${}_C D_a^{\beta}(x^4-7x^2+6)$ to exist, we need
2. x^4-7x^2+6 to be n-times differentiable
3. The integral of ${}_C D_a^{\beta}(x^4-7x^2+6)$ must converge.

Since x^4-7x^2+6 is a polynomial, it is differentiable infinitely. Therefore, $D^n(x^4-7x^2+6)$ exists. The subsequent integrand converges near $\varpi=x$:

$$\frac{D^n(t^4-7t^2+6)}{(x-\varpi)^{\beta-n+1}}.$$

As a result, for each $\beta>0$, ${}_C D_a^{\beta}(x^4-7x^2+6)$ exists. Also, $D^n(x^4-7x^2+6)$ is distinct, since the integer-order derivative is distinct for x^4-7x^2+6 and the integral is clearly defined, and convergence occurs to exactly one value of β.

Now, we let

$${}_0^C D_x^{\beta} g(x)=x^4-7x^2+6,$$

$${}_0^C D_x^{\beta} g(0)=g_0.$$

For $0<\beta\leq 1$,

$$g(x)=g_0+\frac{1}{\Gamma(\beta)}\int_0^x(x-\varpi)^{\beta-1}(\varpi^4-7\varpi^2+6)d\varpi.$$

$\varpi^4-7\varpi^2+6$ is continuous and differentiable infinitely. Also, for $\beta>0$, the kernel $(x-\varpi)^{\beta-1}$ is integrable. Therefore, a solution exists.

Let an operator F such that

$$F[g(x)]=g_0+\frac{1}{\Gamma(\beta)}\int_0^x(x-\varpi)^{\beta-1}(\varpi^4-7\varpi^2+6)d\varpi.$$

To show that F is a contraction on an appropriate function space, we do the following:

• On a finite interval, $\varpi^4 - 7\varpi^2 + 6$ is Lipschitz continuous, so for g_1, g_2

$$|F[g_1(x)] - F[g_2(x)]| \leq \Xi \frac{1}{\Gamma(\beta)} \int_0^x (x-\varpi)^{\beta-1} |g_1(\varpi) - g_2(\varpi)| d\varpi,$$

where Ξ is the Lipschitz constant.

• Utilizing Grönwall's inequality, we have

$$|[F[g_1(x)] - F[g_2(x)]]| \leq M \|g_1 - g_2\|.$$

For sufficiently small x, $M < 1$.

This confirms that F is a contraction, ensuring a fixed point.

4.2 *Convergence order of Fractional Newton (F. N)-Caputo-type method [Theorem 2]*

Let

$$g(\bar{x}) = \bar{x}^4 - 7\bar{x}^2 + 6$$

$$g(x_k) = x_k^4 - 7x_k^2 + 6$$

Now, putting in the above, we get

$$g(x_k) = \frac{{}_C D_{\bar{x}}^{\beta} g(\bar{x})}{\Gamma(\beta+1)} \left[(x_k - \bar{x})^{\beta} + C_2 (x_k - \bar{x})^{2\beta} + C_3 (x_k - \bar{x})^{3\beta} \right] + O((x_k - \bar{x})^{4\beta})$$

$$x_k^4 - 7x_k^2 + 6 = \frac{{}_C D_{\bar{x}}^{\beta} (\bar{x}^4 - 7\bar{x}^2 + 6)}{\Gamma(\beta+1)} \left[(x_k - \bar{x})^{\beta} + C_2 (x_k - \bar{x})^{2\beta} + C_3 (x_k - \bar{x})^{3\beta} \right] + O((x_k - \bar{x})^{4\beta}),$$

and also

$${}_C D_a^{\beta} g(x_k) = \frac{{}_C D_{\bar{x}}^{\beta} g(\bar{x})}{\Gamma(\beta+1)} \left[\Gamma(\beta+1) + \frac{\Gamma(2\beta+1)}{\Gamma(\beta+1)} C_2 (x_k - \bar{x})^{\beta} + \frac{\Gamma(3\beta+1)}{\Gamma(2\beta+1)} C_3 (x_k - \bar{x})^{2\beta} \right] + O((x_k - \bar{x})^{3\beta})$$

$${}_C D_a^{\beta} g(x_k^4 - 7x_k^2 + 6) = \frac{{}_C D_{\bar{x}}^{\beta} (\bar{x}^4 - 7\bar{x}^2 + 6)}{\Gamma(\beta+1)} \left[\Gamma(\beta+1) + \frac{\Gamma(2\beta+1)}{\Gamma(\beta+1)} C_2 (x_k - \bar{x})^{\beta} + \frac{\Gamma(3\beta+1)}{\Gamma(2\beta+1)} C_3 (x_k - \bar{x})^{2\beta} \right]$$

$$+ O((x_k - \bar{x})^{3\beta}).$$

Hence, a Newton-like remainder can be determined, and communicated as far as the mistake at the k-the repeat. $e_k = x_k - \bar{x}$ as

$$\left| x_k^4 - 7x_k^2 + 6 = \frac{{}_C D_{\bar{x}}^{\beta}(\bar{x}^4 - 7\bar{x}^2 + 6)}{\Gamma(\beta+1)} e_k^{\beta} + C_2 e_k^{2\beta} + C_3 e_k^{3\beta} \right| + O(e_k^{4\beta})$$

$$x_k^4 - 7x_k^2 + 6 = \frac{{}_C D_{\bar{x}}^{\beta}(\bar{x}^4)}{\Gamma(\beta+1)} \left[e_k^{\beta} + C_2 e_k^{2\beta} + C_3 e_k^{3\beta} \right] - 7\bar{x}^2 \frac{{}_C D_{\bar{x}}^{\beta}}{\Gamma(\beta+1)} \left[e_k^{\beta} + C_2 e_k^{2\beta} + C_3 e_k^{3\beta} \right]$$

$$+ 6 \frac{{}_C D_{\bar{x}}^{\beta}}{\Gamma(\beta+1)} \left[e_k^{\beta} + C_2 e_k^{2\beta} + C_3 e_k^{3\beta} \right] + O(e_k^{4\beta}).$$

$$x_k^4 - 7x_k^2 + 6 = \frac{{}_C D_{\bar{x}}^{\beta}(\bar{x}^4)}{\Gamma(\beta+1)} e_k^{\beta} + \frac{{}_C D_{\bar{x}}^{\beta}(\bar{x}^4)}{\Gamma(\beta+1)} C_2 e_k^{2\beta} + \frac{{}_C D_{\bar{x}}^{\beta}(\bar{x}^4)}{\Gamma(\beta+1)} C_3 e_k^{3\beta} - 7\bar{x}^2 \frac{{}_C D_{\bar{x}}^{\beta}}{\Gamma(\beta+1)} e_k^{\beta} - 7\bar{x}^2 \frac{{}_C D_{\bar{x}}^{\beta}}{\Gamma(\beta+1)} C_2 e_k^{2\beta}$$

$$- 7\bar{x}^2 \frac{{}_C D_{\bar{x}}^{\beta}}{\Gamma(\beta+1)} C_3 e_k^{3\beta} + 6 \frac{{}_C D_{\bar{x}}^{\beta}}{\Gamma(\beta+1)} e_k^{\beta} + 6 \frac{{}_C D_{\bar{x}}^{\beta}}{\Gamma(\beta+1)} C_2 e_k^{2\beta} + 6 \frac{{}_C D_{\bar{x}}^{\beta}}{\Gamma(\beta+1)} C_3 e_k^{3\beta} + O(e_k^{4\beta}).$$

Also from Eq. (11):

$$g(x_k^4 - 7x_k^2 + 6) = \frac{{}_C D_{\bar{x}}^{\beta}(\bar{x}^4 - 7\bar{x}^2 + 6)}{\Gamma(\beta+1)} \left[\Gamma(\beta+1) + \frac{\Gamma(2\beta+1)}{\Gamma(\beta+1)} C_2 (x_k - \bar{x})^{\beta} + \frac{\Gamma(3\beta+1)}{\Gamma(2\beta+1)} C_3 (x_k - \bar{x})^{2\beta} \right] + O((x_k - \bar{x})^{3\beta})$$

$${}_C D_a^{\beta} g(x_k^4 - 7x_k^2 + 6) = \frac{{}_C D_{\bar{x}}^{\beta}(\bar{x}^4 - 7\bar{x}^2 + 6)}{\Gamma(\beta+1)} \left[\Gamma(\beta+1) + \frac{\Gamma(2\beta+1)}{\Gamma(\beta+1)} C_2 e_k^{\beta} + \frac{\Gamma(3\beta+1)}{\Gamma(2\beta+1)} C_3 e_k^{2\beta} \right] + O e_k^{3\beta}$$

$${}_C D_a^{\beta} g(x_k^4 - 7x_k^2 + 6) = {}_C D_{\bar{x}}^{\beta}(\bar{x}^4) + {}_C D_{\bar{x}}^{\beta}(\bar{x}^4) \frac{\Gamma(2\beta+1)}{[\Gamma(\beta+1)]^2} C_2 e_k^{\beta} + {}_C D_{\bar{x}}^{\beta}(\bar{x}^4) \frac{\Gamma(3\beta+1)}{\Gamma(\beta+1)\Gamma(2\beta+1)} C_3 e_k^{2\beta} - 7 {}_C D_{\bar{x}}^{\beta}(\bar{x}^2) -$$

$$7 {}_C D_{\bar{x}}^{\beta}(\bar{x}^2) \frac{\Gamma(2\beta+1)}{[\Gamma(\beta+1)]^2} C_2 e_k^{\beta} - 7 {}_C D_{\bar{x}}^{\beta}(\bar{x}^2) \frac{\Gamma(3\beta+1)}{\Gamma(\beta+1)\Gamma(2\beta+1)} C_3 e_k^{2\beta} + 6 {}_C D_{\bar{x}}^{\beta} +$$

$$6 {}_C D_{\bar{x}}^{\beta} \frac{\Gamma(2\beta+1)}{[\Gamma(\beta+1)]^2} C_2 e_k^{\beta} + 6 {}_C D_{\bar{x}}^{\beta} \frac{\Gamma(3\beta+1)}{\Gamma(\beta+1)\Gamma(2\beta+1)} C_3 e_k^{2\beta} + O(e_k^{3\beta}).g$$

Now we divide the Eq. (12) by Eq. (14)

$$\frac{g(x_k)}{{}_C D_{\bar{x}}^{\beta} g(x_k)} = \frac{x_k^4 - 7x_k^2 + 6}{{}_C D_{\bar{x}}^{\beta} x_k^4 - 7x_k^2 + 6}$$

$$\frac{g(x_k)}{{}_CD^{\beta}_{\bar{x}}g(x_k)}=\frac{\frac{{}_CD^{\beta}_{\bar{x}}(\bar{x}^4)}{\Gamma(\beta+1)}e_k^{\beta}}{{}_CD^{\beta}_{\bar{x}}(\bar{x}^4)}+\frac{\frac{{}_CD^{\beta}_{\bar{x}}(\bar{x}^4)}{\Gamma(\beta+1)}C_2e_k^{2\beta}}{{}_CD^{\beta}_{\bar{x}}(\bar{x}^4)\frac{\Gamma(2\beta+1)}{[\Gamma(\beta+1)]^2}C_2e_k^{\beta}}+\frac{\frac{{}_CD^{\beta}_{\bar{x}}(\bar{x}^4)}{\Gamma(\beta+1)}C_3e_k^{3\beta}}{{}_CD^{\beta}_{\bar{x}}(\bar{x}^4)\frac{\Gamma(3\beta+1)}{\Gamma(\beta+1)\Gamma(2\beta+1)}C_3e_k^{2\beta}}+\frac{-7\bar{x}^2\frac{{}_CD^{\beta}_{\bar{x}}}{\Gamma(\beta+1)}e_k^{\beta}}{-7\,{}_CD^{\beta}_{\bar{x}}(\bar{x}^2)}$$

$$+\frac{-7\bar{x}^2\frac{{}_CD^{\beta}_{\bar{x}}}{\Gamma(\beta+1)}C_2e_k^{2\beta}}{-7\,{}_CD^{\beta}_{\bar{x}}(\bar{x}^2)\frac{\Gamma(2\beta+1)}{[\Gamma(\beta+1)]^2}C_2e_k^{\beta}}+\frac{-7\bar{x}^2\frac{{}_CD^{\beta}_{\bar{x}}}{\Gamma(\beta+1)}C_3e_k^{3\beta}}{-7\,{}_CD^{\beta}_{\bar{x}}(\bar{x}^2)\frac{\Gamma(3\beta+1)}{\Gamma(\beta+1)\Gamma(2\beta+1)}C_3e_k^{2\beta}}\frac{6\frac{{}_CD^{\beta}_{\bar{x}}}{\Gamma(\beta+1)}e_k^{\beta}}{{}_CD^{\beta}_{\bar{x}}}$$

$$+\frac{6\frac{{}_CD^{\beta}_{\bar{x}}}{\Gamma(\beta+1)}C_2e_k^{2\beta}}{6\,{}_CD^{\beta}_{\bar{x}}\frac{\Gamma(2\beta+1)}{[\Gamma(\beta+1)]^2}C_2e_k^{\beta}}+\frac{6\frac{{}_CD^{\beta}_{\bar{x}}}{\Gamma(\beta+1)}C_3e_k^{3\beta}}{6\,{}_CD^{\beta}_{\bar{x}}\frac{\Gamma(3\beta+1)}{\Gamma(\beta+1)\Gamma(2\beta+1)}C_3e_k^{2\beta}}$$

$$\frac{g(x_k)}{{}_CD^{\beta}_{\bar{x}}g(x_k)}=\frac{e_k^{\beta}}{\Gamma(\beta+1)}+\frac{C_2e_k^{2\beta}}{\frac{\Gamma(2\beta+1)}{\Gamma(\beta+1)}C_2e_k^{\beta}}+\frac{C_3e_k^{3\beta}}{\frac{\Gamma(3\beta+1)}{\Gamma(2\beta+1)}C_3e_k^{2\beta}}+\frac{e_k^{\beta}}{\Gamma(\beta+1)}+\frac{C_2e_k^{2\beta}}{\frac{\Gamma(2\beta+1)}{\Gamma(\beta+1)}C_2e_k^{\beta}}+\frac{C_3e_k^{3\beta}}{\frac{\Gamma(3\beta+1)}{\Gamma(2\beta+1)}C_3e_k^{2\beta}}$$

$$+\frac{e_k^{\beta}}{\Gamma(\beta+1)}+\frac{C_2e_k^{2\beta}}{\frac{\Gamma(2\beta+1)}{\Gamma(\beta+1)}C_2e_k^{\beta}}+\frac{C_3e_k^{3\beta}}{\frac{\Gamma(3\beta+1)}{\Gamma(2\beta+1)}C_3e_k^{2\beta}}+\frac{O(e_k^{4\beta})}{O(e_k^{3\beta})},$$

where we use $C_j=\frac{\Gamma(\beta+1)}{\Gamma(j\beta+1)}\frac{{}_CD^{j\beta}_{\bar{x}}g(\bar{x})}{{}_CD^{\beta}_{\bar{x}}g(\bar{x})}$, for $j\geq 2$, then

$$=\frac{e_k^{\beta}}{\Gamma(\beta+1)}+\frac{e_k^{2\beta}}{\frac{\Gamma(2\beta+1)}{\Gamma(\beta+1)}e_k^{\beta}}+\frac{e_k^{3\beta}}{\frac{\Gamma(3\beta+1)}{\Gamma(2\beta+1)}e_k^{2\beta}}+\frac{O(e_k^{4\beta})}{O(e_k^{3\beta})}=\frac{e_k^{\beta}+e_k^{2\beta}+e_k^{3\beta}+O(e_k^{4\beta})}{\Gamma(\beta+1)+\frac{\Gamma(2\beta+1)}{\Gamma(\beta+1)}e_k^{\beta}+\frac{\Gamma(3\beta+1)}{\Gamma(2\beta+1)}e_k^{2\beta}+O(e_k^{3\beta})}.$$

4.3 *Solution with RL [Theorem 3]*

Allow us to expect that the constant capacity $f\colon R\to R$ has partial subsidiaries of request $k\beta$, for any sure number k and any β, $0<\beta\leq 1$, then, at that point, the accompanying equity holds,

$$g(x+h)=\sum_{k=0}^{\infty}\frac{(h)^{k\beta}}{\Gamma(1+k\beta)}D^{k\beta}_{a+}g(x),\ 0<\beta\leq 1,$$

where $D^{k\beta}_{a+}g(x)$ is the RL derivative of order $k\beta$ of $g(x)$. Let

$$g(x)=x^4-7x^2+6,$$

$$g(x+h)=\left((x+h)^4-7(x+h)^2+6\right),$$

$$g\left\{\left((x+h)^4-7(x+h)^2+6\right)\right\}=\sum_{k=0}^{\infty}\frac{(h)^{k\beta}}{\Gamma(1+k\beta)}D_{a+}^{k\beta}\left(x^4-7x^2+6\right), 0<\beta\leq 1,$$

where $D_{a+}^{k\beta}\left(x^4-7x^2+6\right)$ is the Riemann-Liouville derivative of order $k\beta$ of $g(x)$.

The iterative articulation can be demonstrated using the Taylor extension, as expressed for Caputo fragmentary subordinates.

$$x_{k+1}=x_k-\Gamma(\beta+1)\frac{\left(x_k^4-7x_k^2+6\right)}{D_{a+}^{k\beta}\left(x_k^4-7x_k^2+6\right)}, k=0, 1, \ldots$$

It is called RL fractional Newton.

4.4 *The RL-type F. N-method's local convergence [Theorem 4]*

Let the persistent capacity g: $D\subseteq R\to R$ have fragmentary subsidiaries of request $k\alpha$, for any sure number k and some β, $0<\beta<1$, in D having the zero $\bar{x}$ of $g(x)$. Allow us likewise to assume that $D_{a+}^{k\beta}g(x)$ is persistent and nonsingular in $\bar{x}$. If an underlying estimate. x_0 is adequately near $\bar{x}$, the nearby combination request of the partial Newton strategy

$$x_{k+1}=x_k-\Gamma(\beta+1)\frac{g\left(x_k\right)}{D_{a+}^{k\beta}g\left(x_k\right)}, k=0, 1, \ldots$$

Let

$$g(x)=x^4-7x^2+6$$

$$g\left(x_k\right)=x_k^4-7x_k^2+6$$

$$x_{k+1}=x_k-\Gamma(\beta+1)\frac{\left(x_k^4-7x_k^2+6\right)}{D_{a+}^{k\beta}\left(x_k^4-7x_k^2+6\right)}, k=0, 1$$

Of RL is 2β, being $0<\beta\leq 1$, being again the blunder condition

$$e_{k+1}^{\beta}=\frac{\Gamma(2\beta+1)-(\Gamma(\beta+1))^2}{(\Gamma(\beta+1))^3}C_2e_k^{2\beta}+O(e_k^{3\beta}).$$

When the union request is set up, another significant perspective for the mathematical assessment of roots is the reliance on the underlying assessment. This angle is examined in the accompanying area.

Now we construct a table of the above polynomial,

$$g(x) = x^4 - 7x^2 + 6$$

Since $g(1)$ and $g(2)$ are of opposite signs, there is a root between 1 and 2. Let us take $x_0 = 1.5$, shown in Tables 1 and 2 respectively.

Table 1. Table with CFN Method: Fragmentary Newton results for $g(x)$ with starting $x_0 = 1.5$

β	$\bar{x}$	$\lvert g(x_{k+1})\rvert$	$\lvert x_{k+1} - x_k\rvert$	Iterations
0.7	1	0	1.5	500
0.72	1	0	0	500
0.74	1	0	0	500
0.76	1	$2.15105711021124 \times 10^{-16}$	2.15105711021124	500
0.78	1	$2.15105711021124 \times 10^{-16}$	0	500
0.8	1	$2.15105711021124 \times 10^{-16}$	0	500
0.82	1	$2.15105711021124 \times 10^{-16}$	0	500
0.84	1	$2.15105711021124 \times 10^{-16}$	0	500
0.86	1	$2.15105711021124 \times 10^{-16}$	0	500
0.88	1	$2.15105711021124 \times 10^{-16}$	0	500
0.9	1	$2.15105711021124 \times 10^{-16}$	0	500
0.92	1	$2.15105711021124 \times 10^{-16}$	0	500
0.94	1	$2.15105711021124 \times 10^{-16}$	0	500
0.96	1	$2.15105711021124 \times 10^{-16}$	0	500
0.98	1	$2.15105711021124 \times 10^{-16}$	0	500
1	1	$2.15105711021124 \times 10^{-16}$	0	500

Table 2. Table with R-LFN Method: Fragmentary Newton results for $g(x)$ with starting $x_0 = 1.5$

β	$\bar{x}$	$\lvert g(x_{k+1})\rvert$	$\lvert x_{k+1} - x_k\rvert$	Iterations
0.7	1	0	1.5	500
0.72	1	0	0	500
0.74	1	0	0	500
0.76	1	$2.15105711021124 \times 10^{-16}$	2.15105711021124	500
0.78	1	$2.15105711021124 \times 10^{-16}$	0	500
0.8	1	$2.15105711021124 \times 10^{-16}$	0	500
0.82	1	$2.15105711021124 \times 10^{-16}$	0	500
0.84	1	$2.15105711021124 \times 10^{-16}$	0	500
0.86	1	$2.15105711021124 \times 10^{-16}$	0	500

Table 2. (cont.)

β	$\bar{x}$	$\lvert g(x_{k+1})\rvert$	$\lvert x_{k+1}-x_k\rvert$	Iterations
0.88	1	$2.15105711021124\times10^{-16}$	0	500
0.9	1	$2.15105711021124\times10^{-16}$	0	500
0.92	1	$2.15105711021124\times10^{-16}$	0	500
0.94	1	$2.15105711021124\times10^{-16}$	0	500
0.96	1	$2.15105711021124\times10^{-16}$	0	500
0.98	1	$2.15105711021124\times10^{-16}$	0	500
1	1	$2.15105711021124\times10^{-16}$	0	500

5. Example 2 (Trigonometric function)

$$g(x)=\sin x-\cos x$$

5.1 *Solution with RL [Theorem 3]*

Let

$$g(x)=\sin x-\cos x$$

$$g(x+h)=\sin(x+h)-\cos(x+h)$$

$$g\left(\sin(x+h)-\cos(x+h)\right)=\sum_{k=0}^{\infty}\frac{(h)^{k\beta}}{\Gamma(1+k\beta)}D_{a+}^{k\beta}\left(\sin x-\cos x\right),\ 0<\beta\leq 1,$$

where $D_{a+}^{k\beta}\left(\sin x-\cos x\right)$ is the RL derivative of order $k\beta$ of $g(x)$.

In light of this, it is easy to show that the iterative articulation of this Taylor extension, as it has been articulated for Caputo's fragmentary subordinates,

$$x_{k+1}=x_k-\Gamma(\beta+1)\frac{(\sin x_k-\cos x_k)}{D_{a+}^{k\beta}\left(\sin x_k-\cos x_k\right)},\ k=0,\ 1,\ \ldots$$

This is represented as R-LFN.

5.2 *The RL-type F. N-method's local convergence [Theorem 4]*

Let

$$g(x) = \sin x - \cos x$$

$$g\left(x_k\right) = \sin x_k - \cos x_k$$

$$x_{k+1} = x_k - \Gamma(\beta+1) \frac{(\sin x_k - \cos x_k)}{D_{a+}^{k\beta}\left(\sin x_k - \cos x_k\right)},\ k = 0,\ 1,\ ...$$

of RL is 2β, being $0 < \beta \leq 1$, with

$$e_{k+1}^{\beta} = \frac{\Gamma(2\beta+1) - (\Gamma(\beta+1))^2}{(\Gamma(\beta+1))^3} C_2 e_k^{2\beta} + O(e_k^{3\beta}).$$

When the request for a combination is set up, another significant perspective for the mathematical assessment of roots is the reliance on the underlying assessment. This perspective is broken down in the accompanying area and results shown in Tables 3 and 4 respectively.

Table 3. Table with CFN Method: Fragmentary Newton results for $g(x)$ with starting estimate = -2

β	x	$\lvert g(x_{k+1})\rvert$	$\lvert x_{k+1} - x_k\rvert$	Iterations
0.7	$-4.2123922302322 + 2.25072861154451i$	$1.14384387239291 \times 10^{-09}$	2	500
0.72	$-4.21239223060365 + 2.25072861111207i$	$2.17174665775416 \times 10^{-09}$	0.000126853	500
0.74	$-4.21239223153796 + 2.25072861153265i$	$4.53480801253855 \times 10^{-09}$	0.000291624	500
0.76	$-4.21239223198018 + 2.2507286135061i$	$1.04453825614085 \times 10^{-08}$	0.000209236	500
0.78	$-4.2123923065015 + 2.25072861784736i$	$2.69928588181368 \times 10^{-08}$	0.000555105	500
0.8	$-4.21239219848104 - 2.25072861749299i$	$1.40622339776359 \times 10^{-07}$	0.0003768	500
0.82	$-4.21239222230516 - 2.25072857899927i$	$1.45233413438701 \times 10^{-07}$	0.000042047	500
0.84	$-53.6273218012895 + 24.0264628012996i$	13599727845.8567	13599727846	500
0.86	$-136.268573801364 - 6.15800603111927i$	117.37439596805	13599727728	500
0.88	$-118.030837529062 - 6.99625452129768i$	433.879147796998	316.5047518	500
0.9	$-130.705549527417 + 8.641125520455404i$	2709.43688661118	2275.557739	500
0.92	$-4.21194895861812 + 2.25179395864061i$	0.00498810312887983	2709.431899	500
0.94	$-4.23061746598058 - 2.24934623234154i$	0.0790555299744162	0.074067427	500
0.96	$-124.711130426757 - 6.19545750838747i$	166.293739840848	166.2146843	500
0.98	$-241.632068822952 - 17.3597314847355i$	17306402.3561964	17306236.06	500
1	$-45192024.9160925 + 0i$	45192025.7386553	27885623.38	500

Table 4. Table with RLFN method: Partial Newton results for $g(x)$ and beginning assessment $x_0 = -2$

β	$\bar{x}$	$\lvert g(x_{k+1})\rvert$	$\lvert x_{k+1} - x_k\rvert$	Iterations
0.7	$-4.2123922302322 + 2.25072861154451i$	$1.14384387239291 \times 10^{-09}$	2	500
0.72	$-4.21239223060365 + 2.25072861111207i$	$2.17174665775416 \times 10^{-09}$	0.000126853	500
0.74	$-4.21239223153796 + 2.25072861153265i$	$4.53480801253855 \times 10^{-09}$	0.000291624	500
0.76	$-4.21239223198018 + 2.2507286135061i$	$1.04453825614085 \times 10^{-08}$	0.000209236	500
0.78	$-4.2123923065015 + 2.25072861784736i$	$2.69928588181368 \times 10^{-08}$	0.000555105	500
0.8	$-4.21239219848104 - 2.25072861749299i$	$1.40622339776359 \times 10^{-07}$	0.0003768	500
0.82	$-4.21239222230516 - 2.25072857899927i$	$1.45233413438701 \times 10^{-07}$	0.000042047	500
0.84	$-53.6273218012895 + 24.0264628012996i$	13599727845.8567	13599727846	500
0.86	$-136.268573801364 - 6.15800603111927i$	117.37439596805	13599727728	500
0.88	$-118.030837529062 - 6.99625452129768i$	433.879147796998	316.5047518	500
0.9	$-130.705549527417 + 8.641125520455404i$	2709.43688661118	2275.557739	500
0.92	$-4.21194895861812 + 2.25179395864061i$	0.00498810312887983	2709.431899	500
0.94	$-4.23061746598058 - 2.24934623234154i$	0.0790555299744162	0.074067427	500
0.96	$-124.711130426757 - 6.19545750838747i$	166.293739840848	166.2146843	500
0.98	$-241.632068822952 - 17.3597314847355i$	17306402.3561964	17306236.06	500
1	$-45192024.9160925 + 0i$	45192025.7386553	27885623.38	500

6. Example 3 (Exponential function)

$$g(x) = e^x - \sin x$$

6.1 *Solution with RL [Theorem 3]*

Let

$$g(x) = e^x - \sin x$$

$$g(x+h) = e^{(x+h)} - \sin(x+h)$$

$$g\left\{\left(e^{(x+h)} - \sin(x+h)\right)\right\} = \sum_{k=0}^{\infty} \frac{(h)^{k\beta}}{\Gamma(1+k\beta)} D_{a+}^{k\beta}\left(e^x - \sin x\right), \, 0 < \beta \leq 1,$$

where $D_{a+}^{k\beta}(e^x - \sin x)$ is the RL derivative of order $k\beta$ of $g(x)$.

Therefore, it is easy to show that the iterative articulation of this Taylor extension, as it has been articulated for Caputo's fragmentary subordinates,

$$x_{k+1} = x_k - \Gamma(\beta+1)\frac{(e^{x_k} - \sin x_k)}{D_{a+}^{k\beta}(e^{x_k} - \sin x_k)},\ k = 0,\ 1,\ ...$$

It is called RL fractional Newton.

6.2 *The RL-type F. N-method's local convergence [Theorem 4]*

Let

$$g(x) = e^x - \sin x$$

$$g(x_k) = e^{x_k} - \sin x_k$$

$$x_{k+1} = x_k - \Gamma(\beta+1)\frac{(e^{x_k} - \sin x_k)}{D_{a+}^{k\beta}(e^{x_k} - \sin x_k)},\ k = 0,\ 1,\ ...$$

of RL is 2β, being $0 < \beta \leq 1$, with

$$e_{k+1}^{\beta} = \frac{\Gamma(2\beta+1) - (\Gamma(\beta+1))^2}{(\Gamma(\beta+1))^3} C_2 e_k^{2\beta} + O(e_k^{2\beta}).$$

When the request for a combination is set up, another significant perspective for the mathematical assessment of roots is the reliance on the underlying assessment. This perspective is investigated in the accompanying segment.

6.3 *Solution with F. N method [by applying Theorem 1]*

Convergence analysis for the Newton method and CFN.
Let

$$g(x) = e^x - \sin x$$

Since $g(-1.5)$ and $g(-4)$ are of opposite signs, there is a root between -1.5 and -4. Let $x \in (-1.5, -4)$. We take $x_0 = -1.5$, results shown in Tables 5 and 6 respectively.

Table 5. Table with CFN Method: Fragmentary Newton results for $g(x)$ with starting $x_0 = -1.5$

β	$\bar{x}$	$\lvert g(x_{k+1})\rvert$	$\lvert x_{k+1} - x_k\rvert$	Iterations
0.7	$-3.18306301193336 + 0i$	$2.15105711021124 \times 10^{-16}$	1.5	500
0.72	$-3.18306301193336 + 0i$	$2.15105711021124 \times 10^{-16}$	0	500
0.74	$-3.18306301193336 + 0i$	$2.15105711021124 \times 10^{-16}$	0	500

Table 5. (cont.)

β	$\bar{x}$	$\lvert g(x_{k+1})\rvert$	$\lvert x_{k+1}-x_k\rvert$	Iterations
0.76	$-3.18306301193336+0i$	$2.15105711021124\times 10^{-16}$	0	500
0.78	$-3.18306301193336+0i$	$2.15105711021124\times 10^{-16}$	0	500
0.8	$-3.18306301193336+8.58595029646951\times 10^{-312}i$	$2.15105711021124\times 10^{-16}$	0	500
0.82	$-3.18306301193336+9.94044979283937\times 10^{-285}i$	$2.15105711021124\times 10^{-16}$	0	500
0.84	$-3.18306301193336-2.1631132910515\times 10^{-262}i$	$2.15105711021124\times 10^{-16}$	0	500
0.86	$-3.18306301193336+1.09039463020986\times 10^{-242}i$	$2.15105711021124\times 10^{-16}$	0	500
0.88	$-3.18306301193336+3.54451323329003\times 10^{-226}i$	$2.15105711021124\times 10^{-16}$	0	500
0.9	$-3.18306301193336+8.20838171041579\times 10^{-212}i$	$2.15105711021124\times 10^{-16}$	0	500
0.92	$-3.18306301193336-3.15071044622119\times 10^{-199}i$	$2.15105711021124\times 10^{-16}$	0	500
0.94	$-3.18306301193336-3.7295342492801\times 10^{-188}i$	$2.15105711021124\times 10^{-16}$	0	500
0.96	$-3.18306301193336-1.899435570133589\times 10^{-178}i$	$2.15105711021124\times 10^{-16}$	0	500
0.98	$-3.18306301193336-5.43584239395708\times 10^{-170}i$	$2.15105711021124\times 10^{-16}$	0	500
1	$-3.18306301193336+0i$	$2.15105711021124\times 10^{-16}$	0	500

Table 6. Table with R-LFN Method: Fragmentary Newton results for $g(x)$ with $x_0=-1.5$

β	$\bar{x}$	$\lvert g(x_{k+1})\rvert$	$\lvert x_{k+1}-x_k\rvert$	Iterations
0.7	$-3.18306301193336+0i$	$2.15105711021124\times 10^{-16}$	1.5	500
0.72	$-3.18306301193336+0i$	$2.15105711021124\times 10^{-16}$	0	500
0.74	$-3.18306301193336+0i$	$2.15105711021124\times 10^{-16}$	0	500
0.76	$-3.18306301193336+0i$	$2.15105711021124\times 10^{-16}$	0	500
0.78	$-3.18306301193336+0i$	$2.15105711021124\times 10^{-16}$	0	500
0.8	$-3.18306301193336+8.58595029646951\times 10^{-312}i$	$2.15105711021124\times 10^{-16}$	0	500
0.82	$-3.18306301193336+9.94044979283937\times 10^{-285}i$	$2.15105711021124\times 10^{-16}$	0	500
0.84	$-3.18306301193336-2.1631132910515\times 10^{-262}i$	$2.15105711021124\times 10^{-16}$	0	500
0.86	$-3.18306301193336+1.09039463020986\times 10^{-242}i$	$2.15105711021124\times 10^{-16}$	0	500
0.88	$-3.18306301193336+3.54451323329003\times 10^{-226}i$	$2.15105711021124\times 10^{-16}$	0	500
0.9	$-3.18306301193336+8.20838171041579\times 10^{-212}i$	$2.15105711021124\times 10^{-16}$	0	500
0.92	$-3.18306301193336-3.15071044622119\times 10^{-199}i$	$2.15105711021124\times 10^{-16}$	0	500
0.94	$-3.18306301193336-3.7295342492801\times 10^{-188}i$	$2.15105711021124\times 10^{-16}$	0	500
0.96	$-3.18306301193336-1.899435570133589\times 10^{-178}i$	$2.15105711021124\times 10^{-16}$	0	500
0.98	$-3.18306301193336-5.43584239395708\times 10^{-170}i$	$2.15105711021124\times 10^{-16}$	0	500
1	$-3.18306301193336+0i$	$2.15105711021124\times 10^{-16}$	0	500

We present the application of the Fractional Order Newton-Raphson method for solving the equation $(x) = x^4 - 7x^2 + 6$. The fractional derivative is computed using the Caputo definition, and the convergence behavior is analyzed over a range of fractional orders β and initial guesses x_0. We obtain the following figure.

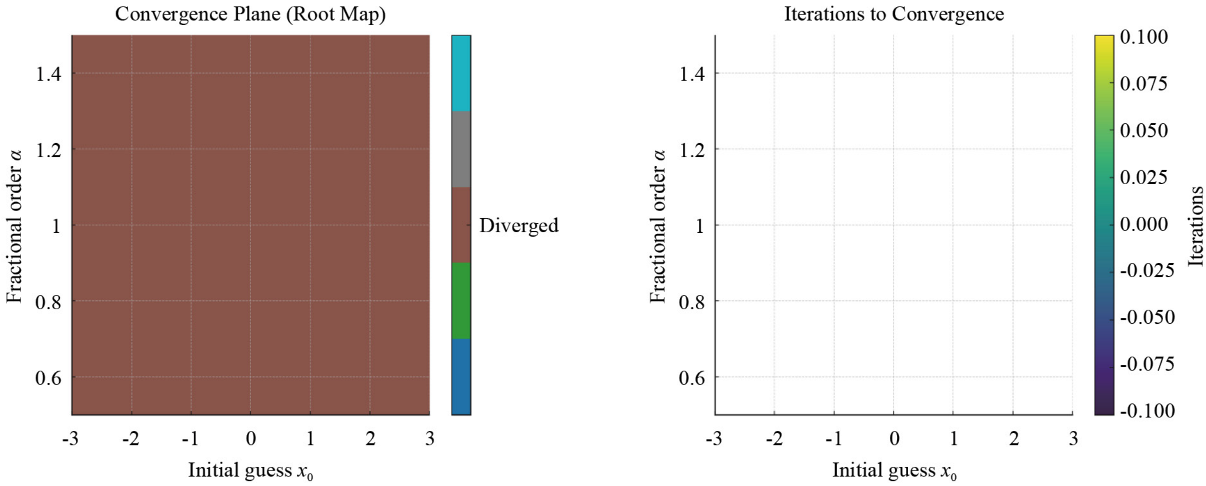

Figure 1. Convergence Plane of the example

This Figure 1 shows two plots:

1. Convergence plane (root map): Regions are color-coded according to the root the method converged to, with black indicating divergence.

2. Iterations to convergence: Color scale indicates how many iterations were required, and NaN indicates where divergence occurred.

The first figure shows a convergence plane with beginning estimates on the abscissa axis and fractional derivative index values on the ordinate axis. Each mesh point is utilized to launch the iterative procedure, which assigns colors based on the convergent root. If no root is discovered within 500 iterations, the point is colored black. The second figure indicates that convergence is lost for low parameter values, and the root convergence is determined by the initial estimation and derivative order. However, convergence can be obtained without modifying the original estimation value.

The Fractional Order Newton-Raphson method is a generalization of the classical Newton-Raphson method using fractional calculus. In this approach, fractional (non-integer) order derivatives are employed, providing enhanced convergence properties and better stability in solving nonlinear equations. This method is particularly useful when the classical Newton-Raphson method fails to converge or is unstable.

Basic Attractions:

- Enhanced convergence with fractional derivative order $\alpha \in (0, 2)$.
- Extra degree of freedom using α as a tuning parameter.
- Better stability near saddle points or multiple roots.
- Handles complex roots effectively.
- Wider convergence domain.
- Suitable for various applications (engineering, physics, biology, economics).
- Incorporates memory effects inherent in fractional calculus.
- Supports multiple derivative definitions (Caputo, Riemann-Liouville, Grünwald-Letnikov).

We consider solving the nonlinear equation:

$$f(x) = x^3 - 2x + 2 = 0$$

Using the method with fractional order $\alpha = 0.9$, we aim to find the root.

The following Python code implements the Fractional Order Newton-Raphson method using the Grünwald-Letnikov approximation for fractional derivatives.

```
import numpy as np
import matplotlib.pyplot as plt
from math import gamma
# Define the function
def f(x):
    return x**3 - 2*x + 2
# Grünwald-Letnikov approximation of the fractional derivative
def frac_deriv(f, x, alpha, h=1e-3, N=100):
    coeffs = [(-1)**k * gamma(alpha + 1) / (gamma(k + 1) * gamma(alpha - k + 1)) for k in range(N)]
    return sum(coeffs[k] * f(x - k * h) for k in range(N)) / h**alpha
# Fractional Newton-Raphson method
def fractional_newton(f, x0, alpha=0.9, tol=1e-6, max_iter=100):
    x = x0
    for i in range(max_iter):
        df = frac_deriv(f, x, alpha)
        if abs(df) < 1e-12:
            print("Derivative too small. Exiting.")
            break
        x_new = x - f(x) / df
        if abs(x_new - x) < tol:
            print(f"Converged to x_new in i+1 iterations.")
            return x_new
        x = x_new
    print(f"Did not converge. Last x = x")
    return x
# Initial guess and fractional order
x0 = -1.5
alpha = 0.9
# Run FONR
root = fractional_newton(f, x0, alpha).
```

- $\alpha = 1.0$: Recovers classical Newton-Raphson method.
- $\alpha < 1.0$: Slower steps, better global behavior, avoids local traps.
- $\alpha > 1.0$: More aggressive, may converge faster but less stable.

Tip: Start with $\alpha = 0.9$ or $\alpha = 1.1$ and adjust based on convergence behavior.

7. Conclusion

This study examines polynomial, exponential, and trigonometric functions using the Newton-Raphson method with RL-Caputo fractional derivative. It illustrates the $2\alpha^{\text{th}}$ convergence of the two techniques using Gamma action as a damping boundary. A Gamma function is used as a dampening boundary in mathematical experiments that concentrate on interacting planes. Through numerical examples, the suggested techniques' efficacy in resolving fractional-order nonlinear differential equations is illustrated. According to the study, CFN produces superior outcomes for larger root bowls with greater α-value autonomy, and both partial subordinates produce distinct results even when they are theoretically equal. This study provides a guide for researchers to investigate and apply this method in various applications, resulting in accurate and approximate results. Many high-dimensional fractional-order differential and partial differential equations that are frequently found in applied science, engineering, and other scientific fields can be studied using the current methods. With applications in fluid mechanics, nonlinear dynamics, and wave propagation, the CFN is a potent mathematical algorithm for solving nonlinear FDEs. It is a possible substitute for fractional differential equation modeling because of its excellent accuracy, quick convergence, and computational efficiency. This method may be extended in future studies to chaotic fractional-order models, nonlinear Schrödinger equations, and higher-dimensional fractional systems. Additionally, incorporating machine learning techniques into the CFN may improve its efficacy and flexibility in handling a greater variety of nonlinear problems. The Newton-Raphson method is difficult to apply to systems of equations because of its processing expense and inevitability requirement. Higher-dimensional applications confront issues such as initial guess sensitivity, potential divergence if the Jacobian is unique or near-singular, and the difficulty of calculating the function and its Jacobian for large systems. Future efforts include creating resilient methods such as backtracking line search and adaptive step-size algorithms to ensure convergence in complex circumstances while also dealing with weak starting guesses or ill-conditioned Jacobians.

Confilict of interest

The authors declare no competing financial interest.

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