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Research Article

Multi-Term Nonlinear Fractal-Orders Delayed Abel Integral Equation: Existence of Solutions, Stability, and Applications

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Abstract: In this paper we study a multi-term non-linear fractal orders delayed integral equation of the second kind. We examine the existence of integrable solution $x \in L_1[0, T]$. Under certain assumptions, the uniqueness of this solution will be demonstrated. Additionally, the continuous dependence on the problem's parameters will be investigated. The Hyers-Ulam stability of the problem itself will be established. Finally, as applications we study two initial value problems of multi-term fractal orders integro-differential equations.

Keywords: Abel integral equation, existence of solutions, delay integral equation, fractal operators, continuous dependence, Hyers-Ulam stability, nonlocal-integral problems

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1. Introduction

Abel integral equations play a fundamental role in solving various physical and engineering problems involving memory and hereditary properties. The second kind Abel integral equation typically includes both the unknown function and its integral, making it more complex and realistic for modelling real-world systems. When extended to multi-term, non-linear, and delayed forms, these equations can describe processes with multiple interacting effects, time delays, and non-linear behaviours [1–13].

Let $\beta \in (0, 1)$. The fractal operators, differential and integral of order β, are given, respectively [14–17]:

$$D_\beta x(t) = \frac{d}{dt^\beta} x(t), \qquad I_\beta x(t) = \int_0^t \beta\, s^{\beta-1}\, x(s)\, ds.$$

Consider the initial value problem

$$D_\beta\, x(t) + \lambda\, x(t) = g(t), \qquad x(0) = x_0,$$

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which can be written as

$$\frac{t^{1-\beta}}{\beta}\,\frac{dx}{dt} + \lambda\, x(t) = g(t), \qquad x(0) = x_0,$$

$$\frac{dx}{dt} + \lambda\,\beta\, t^{\beta-1}\, x(t) = \beta\, t^{\beta-1}\, g(t), \quad x(0) = x_0$$

and has the solution (the second kind Abel fractal integral equation)

$$x(t) + \lambda \int_0^t \beta\, s^{\beta-1}\, x(s)\, ds = G(t), \quad \text{where } G(t) = x_0 + \int_0^t \beta\, s^{\beta-1}\, g(s)\, ds.$$

The non-linear version with delay $(\vartheta(t) \le t)$ can be defined by

$$x(t) + f(\,t, \lambda \int_0^{\vartheta(t)} \beta\, s^{\beta-1}\, x(s)\, ds\,) = g(t, x(t)).$$

Our aim here is to study the multi-term non-linear fractal orders second kind Abel integral equation with delay

$$x(t) + f(t, \lambda_1 \int_0^{\vartheta_1(t)} \beta_1\, s^{\beta_1-1}\, x(s)\, ds, \ldots, \lambda_n \int_0^{\vartheta_n(t)} \beta_n\, s^{\beta_n-1}\, x(s)\, ds) = g(t, x(t)), \tag{1}$$

where $t \in J = [0, T]$, $\beta_i \in (0, 1)$ and the parameters $\lambda_i > 0$, $i = 1, \ldots, n$.

Here, under some given assumptions, the existence of an integrable solution to Equation (1) will be established within the space $L_1(J)$. Furthermore, the continuous dependence of the solution on the parameters λ_i, β_i as well as on the functions f, g, ϑ_i will be examined. Finally, the Hyers-Ulam stability of the problem itself will also be demonstrated.

As applications, we apply our obtained results to the two initial value problems of multi-term nonlinear fractal orders delayed integro-differential equations

$$D_\alpha\, u(t) + f(t,\ \lambda_1^* \int_0^{\vartheta_1(t)} D_{\alpha_1}\, u(s)\, ds, \ldots, \lambda_n^* \int_0^{\vartheta_n(t)} D_{\alpha_n}\, u(s)\, ds) = g(t,\ D_\alpha\, u(t)), \qquad t \in (0, T]$$

and

$$\frac{du}{dt} + f(t,\ \lambda_1 \int_0^{\vartheta_1(t)} \beta_1\, s^{\beta_1-1}\, \frac{du}{ds}\, ds, \ldots, \lambda_n \int_0^{\vartheta_n(t)} \beta_n\, s^{\beta_n-1}\, \frac{du}{ds}\, ds) = g(t,\ \frac{du}{dt}), \qquad t \in (0, T].$$

2. Existence of solution

We analyze equation (1) subject to the following assumptions:

(I) ϑ_i: $J \to J$ is a continuous function, where $\vartheta_i(t) \le t$, $i = 1, \ldots, n$.

(II)

• $f\colon J \times R^n \to R$ is measurable in $t \in J$, $\forall x \in R$ and continuous in $x \in R$, $\forall t \in J$, $g\colon J \times R \to R$ is measurable in $t \in J$, $\forall x \in R$ and continuous in $x \in R$, $\forall t \in J$.

• There exist $a_i\colon J \to R$, $i = 1, 2$ with $t^{\beta_0 - 1}|a_i(t)| \in L_1(J)$, $\int_0^T t^{\beta_0 - 1}|a_i(t)|\, dt \leq a$ and positive constants b_1, b_2 such that

$$|f(t, x_1, ...x_n)| \leq |a_1(t)| + b_1 \left(|x_1| + ... + |x_n| \right) \quad \text{and} \quad |g(t, x)| \leq |a_2(t)| + b_2\, |x|. \tag{2}$$

(III) $b\, (1 + n\, \lambda\, T^{\beta}) < 1$, where $b = \max\{b_1, b_2\}$, $\beta = \max\{\beta_i\}$ and $\lambda = \max\{\lambda_i\}$, $i = 1, ..., n$.

Theorem 1 Assuming that conditions (I)–(III) are met, equation (1) admits at least one solution. $x \in L_1(J)$.

Proof. Equation (1) can be expressed as

$$x(t) = -f(t, \lambda_1 \int_0^{\vartheta_1(t)} \beta_1\, s^{\beta_1 - 1}\, x(s)\, ds, ..., \lambda_n \int_0^{\vartheta_n(t)} \beta_n\, s^{\beta_n - 1}\, x(s)\, ds) + g(t, x(t))$$

which can be given by

$$\begin{aligned} t^{\beta_0 - 1}\, x(t) \quad &= \quad -\, t^{\beta_0 - 1}\, f(t, \lambda_1 \int_0^{\vartheta_1(t)} \beta_1\, s^{\beta_1 - 1}\, x(s)\, ds, ..., \lambda_n \int_0^{\vartheta_n(t)} \beta_n\, s^{\beta_n - 1}\, x(s)\, ds) \\ &\quad + t^{\beta_0 - 1}\, g(t, x(t)), \end{aligned}$$

where $\beta_0 = \min\{\beta_i, i = 1, ..., n\}$.

Let

$$\nu(t) = t^{\beta_0 - 1}\, x(t), \quad \text{then } x(t) = t^{1 - \beta_0}\, \nu(t) \tag{3}$$

and

$$\nu(t) = -t^{\beta_0 - 1} f(t, \lambda_1 \int_0^{\vartheta_1(t)} \beta_1 s^{\beta_1 - 1} s^{1 - \beta_0} \nu(s) ds, ..., \lambda_n \int_0^{\vartheta_n(t)} \beta_n s^{\beta_n - 1} s^{1 - \beta_0} \nu(s) ds) + t^{\beta_0 - 1} g(t, t^{1 - \beta_0} \nu(t)). \tag{4}$$

We introduce the operator F as follows

$$F\nu(t) = -t^{\beta_0 - 1}\, f(t, \lambda_1 \int_0^{\vartheta_1(t)} \beta_1 s^{\beta_1 - \beta_0} \nu(s) ds, ..., \lambda_n \int_0^{\vartheta_n(t)} \beta_n s^{\beta_n - \beta_0} \nu(s) ds) + t^{\beta_0 - 1} g(t, t^{1 - \beta_0} \nu(t))$$

and the non-empty, bounded, convex and closed ball

$$Q_r = \{\nu \in L_1(J)\colon ||\nu||_1 \leq r\}, \qquad \text{where } r = \frac{2\, a}{1 - b\, (1 + n\, \lambda\, T^{\beta})}.$$

Now, let $\nu \in Q_r$, then

$$
\begin{aligned}
|F\nu(t)| &= |-t^{\beta_0-1} f(t, \lambda_1 \int_0^{\vartheta_1(t)} \beta_1 s^{\beta_1-\beta_0} \nu(s)\, ds, ..., \lambda_n \int_0^{\vartheta_n(t)} \beta_n s^{\beta_n-\beta_0} \nu(s)\, ds) \\
&\quad + t^{\beta_0-1} g(t, t^{1-\beta_0} \nu(t))| \\
&\leq t^{\beta_0-1} | f(t, \lambda_1 \int_0^{\vartheta_1(t)} \beta_1 s^{\beta_1-\beta_0} \nu(s)\, ds, ..., \lambda_n \int_0^{\vartheta_n(t)} \beta_n s^{\beta_n-\beta_0} \nu(s)\, ds) | \\
&\quad + t^{\beta_0-1} | g(t, t^{1-\beta_0} \nu(t)) | \\
&\leq t^{\beta_0-1} |a_1(t)| + b_1 t^{\beta_0-1} \lambda_1 \int_0^{\vartheta_1(t)} \beta_1 s^{\beta_1-\beta_0} |\nu(s)|\, ds + ... \\
&\quad + b_1 t^{\beta_0-1} \lambda_n \int_0^{\vartheta_n(t)} \beta_n s^{\beta_n-\beta_0} |\nu(s)|\, ds + t^{\beta_0-1} |a_2(t)| + t^{\beta_0-1} b_2 t^{1-\beta_0} |\nu(t)| \\
&\leq t^{\beta_0-1} |a_1(t)| + b\, t^{\beta_0-1} \lambda \int_0^{t} \beta_1 s^{\beta_1-\beta_0} |\nu(s)|\, ds + ... \\
&\quad + b\, t^{\beta_0-1} \lambda \int_0^{t} \beta_n s^{\beta_n-\beta_0} |\nu(s)|\, ds + t^{\beta_0-1} |a_2(t)| + b\, |\nu(t)| \\
&\leq t^{\beta_0-1} |a_1(t)| + b\, t^{\beta_0-1} \lambda\, \beta_1 t^{\beta_1-\beta_0} \int_0^{T} |\nu(s)|\, ds + ... \\
&\quad + b\, t^{\beta_0-1} \lambda\, \beta_n t^{\beta_n-\beta_0} \int_0^{T} |\nu(s)|\, ds + t^{\beta_0-1} |a_2(t)| + b\, |\nu(t)| \\
&\leq t^{\beta_0-1} |a_1(t)| + n\, b\, \lambda\, \beta\, t^{\beta-1} ||\nu||_1 + t^{\beta_0-1} |a_2(t)| + b\, |\nu(t)|
\end{aligned}
$$

and

$$
\begin{aligned}
\int_0^T |F\nu(t)|\, dt &\leq \int_0^T t^{\beta_0-1} |a_1(t)|\, dt + n\, b\lambda\, ||\nu||_1 \int_0^T \beta\, t^{\beta-1} dt + \int_0^T t^{\beta_0-1} |a_2(t)|\, dt + b \int_0^T |\nu(t)|\, dt \\
&\leq a + n\, b\, \lambda\, T^{\beta}\, ||\nu||_1 + a + b\, ||\nu||_1 \\
&\leq 2\, a + b\, (1 + n\, \lambda\, T^{\beta})\, r = r.
\end{aligned}
$$

This confirms that the operator F maps Q_r into itself, and the family $\{F\nu\}$ remains uniformly bounded within Q_r.

Let $\Omega \in Q_r$ and $\nu \in \Omega$, then

$$\begin{aligned}
\int_0^T \mid (F\nu(s))_l - (F\nu(s)) \mid ds &= \int_0^T \mid \frac{1}{l} \int_t^{t+l} F\nu(\theta)\, d\theta - F\nu(s) \mid ds \\
&= \int_0^T \frac{1}{l} \mid \int_t^{t+l} (\, F\nu(\theta) - F\nu(s)\,)\, d\theta \mid ds \\
&\leq \int_0^T \frac{1}{l} \int_t^{t+l} \mid F\nu(\theta) - F\nu(s) \mid d\theta\, ds,
\end{aligned}$$

then

$$||(F\nu)_l - (F\nu)||_1 \leq \int_0^T \frac{1}{l} \int_t^{t+l} \mid F\nu(\theta) - F\nu(s) \mid d\theta\, ds.$$

Since $F\nu(t) \in Q_r \subset L_1(J),$ from Lebesgue point theorem [18], it follows that

$$\frac{1}{l} \int_t^{t+l} \mid F\nu(\theta) - F\nu(s) \mid d\theta \to 0, \quad \text{as } \ l \to 0,$$

then

$$||(F\nu)_l - (F\nu)||_1 \to 0, \quad \text{as } \ l \to 0.$$

Consequently, the family $\{F\nu\}$ is relatively compact [19] implying that F is compact operator [18, 20]. Take $\{\nu_m\} \subset Q_r$ such that $\nu_m \to \nu,$ thus

$$\begin{aligned}
|\nu_m(t)| &= |-\, t^{\beta_0 - 1}\, f(t, \lambda_1 \int_0^{\vartheta_1(t)} \beta_1\, s^{\beta_1 - \beta_0}\, \nu_m(s)\, ds, ..., \lambda_n \int_0^{\vartheta_n(t)} \beta_n\, s^{\beta_n - \beta_0}\, \nu_m(s)\, ds) \\
&\quad + t^{\beta_0 - 1}\, g(t, t^{1-\beta_0}\, \nu_m(t)) \mid \\
&\leq t^{\beta_0 - 1} \mid f(t, \lambda_1 \int_0^{\vartheta_1(t)} \beta_1\, s^{\beta_1 - \beta_0}\, \nu_m(s)\, ds, ..., \lambda_n \int_0^{\vartheta_n(t)} \beta_n\, s^{\beta_n - \beta_0}\, \nu_m(s)\, ds) \mid \\
&\quad + t^{\beta_0 - 1} \mid g(t, t^{1-\beta_0}\, \nu_m(t)) \mid \\
&\leq t^{\beta_0 - 1}\, |a_1(t)| + b\, t^{\beta_0 - 1}\, \lambda \int_0^t \beta_1\, s^{\beta_1 - \beta_0}\, |\nu_m(s)|\, ds\, + ...
\end{aligned}$$

$$+b\, t^{\beta_0-1}\, \lambda \int_0^t \beta_n\, s^{\beta_n-\beta_0}\, |\nu_m(s)|\, ds \,+\, t^{\beta_0-1}\, |a_2(t)| + \, b\, |\nu_m(t)|$$

$$\leq \quad t^{\beta_0-1}\, |a_1(t)| + n\, b\, \lambda\, \beta\, t^{\beta-1}\, ||\nu_m||_1 + t^{\beta_0-1}\, |a_2(t)| + b\, |\nu_m(t)|$$

and

$$(1-b)\, |\nu_m(t)| \,\leq\, t^{\beta_0-1}\, |a_1(t)| + n\, b\, \lambda\, \beta\, t^{\beta-1}\, r + t^{\beta_0-1}\, |a_2(t)|,$$

$$|\nu_m(t)| \,\leq\, \frac{1}{1-b}\, [\, t^{\beta_0-1}\, |a_1(t)| + n\, b\, \lambda\, \beta\, t^{\beta-1}\, r + t^{\beta_0-1}\, |a_2(t)|\,] \,=\, \gamma(t) \in L_1(J).$$

Now, we have

$$F\nu_m(t) = -\, t^{\beta_0-1}\, f(t,\, \lambda_1 \int_0^{\vartheta_1(t)} \beta_1\, s^{\beta_1-\beta_0}\, \nu_m(s)\, ds,\, ...,\, \lambda_n \int_0^{\vartheta_n(t)} \beta_n\, s^{\beta_n-\beta_0}\, \nu_m(s)\, ds)$$

$$+\, t^{\beta_0-1}\, g(t,\, t^{1-\beta_0}\, \nu_m(t)).$$

Subsequently, by invoking the Lebesgue Dominated Convergence Theorem, [18, 20], we derive

$$\begin{aligned}
\lim_{m\to\infty} F\nu_m(t) \quad &= \quad -\, t^{\beta_0-1}\, f(t,\, \lambda_1 \lim_{m\to\infty} \int_0^{\vartheta_1(t)} \beta_1\, s^{\beta_1-\beta_0}\, \nu_m(s)\, ds,\, ...,\, \lambda_n \lim_{m\to\infty} \int_0^{\vartheta_n(t)} \beta_n\, s^{\beta_n-\beta_0}\, \nu_m(s)\, ds) \\
&\quad\quad + t^{\beta_0-1}\, g(t,\, t^{1-\beta_0} \lim_{m\to\infty} \nu_m(t)) \\
&= \quad -\, t^{\beta_0-1}\, f(t,\, \lambda_1 \int_0^{\vartheta_1(t)} \beta_1\, s^{\beta_1-\beta_0} \lim_{m\to\infty} \nu_m(s)\, ds,\, ...,\, \lambda_n \int_0^{\vartheta_n(t)} \beta_n\, s^{\beta_n-\beta_0} \lim_{m\to\infty} \nu_m(s)\, ds) \\
&\quad\quad + t^{\beta_0-1}\, g(t,\, t^{1-\beta_0}\, \nu(t)) \\
&= \quad -\, t^{\beta_0-1}\, f(t,\, \lambda_1 \int_0^{\vartheta_1(t)} \beta_1\, s^{\beta_1-\beta_0}\, \nu(s)\, ds,\, ...,\, \lambda_n \int_0^{\vartheta_n(t)} \beta_n\, s^{\beta_n-\beta_0}\, \nu(s)\, ds) \\
&\quad\quad + t^{\beta_0-1}\, g(t,\, t^{1-\beta_0}\, \nu(t)) \\
&= \quad F\nu(t).
\end{aligned}$$

This signifies that $F\nu_m(t) \to F\nu(t)$. Then the operator F is continuous [21].

Now, Utilizing the Schauder Fixed Point Theorem [22], we confirm the existence of at least one integrable solution $v \in Q_r \subset L_1(J)$ of the integral Equation (4).

As a result, there exist at least one solution $x(t) = t^{1-\beta_0}\ v(t) \in L_1(J)$ of the fractal integral Equation (1).

2.1 *Uniqueness results*

Now, consider the following assumption **(II)***

- $f\colon J \times R^n \to R$ is measurable in $t \in J$ for every $x \in R$ and

$$|f(t, x_1, ..., x_n) - f(t, y_1, ..., y_n)| \le b_1\ (\ |x_1 - y_1| + ... + |x_n - y_n|\ ), \qquad b_1 > 0. \tag{5}$$

- $g\colon J \times R \to R$ is measurable in $t \in J$ for every $x \in R$ and

$$|g(t, x) - g(t, y)| \le b_2\ |x - y|, \qquad b_2 > 0. \tag{6}$$

Remark 1. From the Lipschitz conditions (5) and (6), we can get the growth conditions (2) as $f(t, 0) = a_1(t)$ and $g(t, 0) = a_2(t)$.

Theorem 2 Suppose the assumptions (I), (II)* and (III) be hold, the solution $v \in L_1(J)$ of equation (4) is unique. It follows that, the solution $x \in L_1(J)$ of (1) is unique.

Proof. Let the hypothesis of Theorem 1 be met, then the solution of (4) exist. Let v and $\bar{v}$ be two solutions of Equation (4), as a result

$$\begin{aligned}
|v(t) - \bar{v}(t)| \ &= \ |\ -\ t^{\beta_0 - 1}\ f(t,\ \lambda_1 \int_0^{\vartheta_1(t)} \beta_1\ s^{\beta_1 - \beta_0}\ v(s)\ ds,\ ...,\ \lambda_n \int_0^{\vartheta_n(t)} \beta_n\ s^{\beta_n - \beta_0}\ v(s)\ ds) \\
&\quad + t^{\beta_0 - 1}\ g(t,\ t^{1-\beta_0}\ v(t))\ -\ t^{\beta_0 - 1}\ g(t,\ t^{1-\beta_0}\ \bar{v}(t)) \\
&\quad + t^{\beta_0 - 1}\ f(t,\ \lambda_1 \int_0^{\vartheta_1(t)} \beta_1\ s^{\beta_1 - \beta_0}\ \bar{v}(s)\ ds,\ ...,\ \lambda_n \int_0^{\vartheta_n(t)} \beta_n\ s^{\beta_n - \beta_0}\ \bar{v}(s)\ ds)\ | \\
&\le \ t^{\beta_0 - 1}\ |\ f(t,\ \lambda_1 \int_0^{\vartheta_1(t)} \beta_1\ s^{\beta_1 - \beta_0}\ \bar{v}(s)\ ds,\ ...,\ \lambda_n \int_0^{\vartheta_n(t)} \beta_n\ s^{\beta_n - \beta_0}\ \bar{v}(s)\ ds) \\
&\quad - f(t,\ \lambda_1 \int_0^{\vartheta_1(t)} \beta_1\ s^{\beta_1 - \beta_0}\ v(s)\ ds,\ ...,\ \lambda_n \int_0^{\vartheta_n(t)} \beta_n\ s^{\beta_n - \beta_0}\ v(s)\ ds)\ | \\
&\quad + t^{\beta_0 - 1}\ |\ g(t,\ t^{1-\beta_0}\ v(t))\ -\ g(t,\ t^{1-\beta_0}\ \bar{v}(t))\ | \\
&\le \ t^{\beta_0 - 1}\ b_1\ \lambda_1 \int_0^{\vartheta_1(t)} \beta_1\ s^{\beta_1 - \beta_0}\ |\ \bar{v}(s) - v(s)\ |\ ds\ + ... \\
&\quad + t^{\beta_0 - 1}\ b_1\ \lambda_n \int_0^{\vartheta_n(t)} \beta_n\ s^{\beta_n - \beta_0}\ |\ \bar{v}(s) - v(s)\ |\ ds\ +\ t^{\beta_0 - 1}\ b_2\ t^{1-\beta_0}\ |\ v(t) - \bar{v}(t)\ |
\end{aligned}$$

$$
\begin{aligned}
&\leq \; t^{\beta_0-1}\, b\, \lambda \int_0^t \beta_1\, s^{\beta_1-\beta_0} \mid \bar{\nu}(s)-\nu(s) \mid ds \,+ \ldots. \\
&\quad +t^{\beta_0-1}\, b\, \lambda \int_0^t \beta_n\, s^{\beta_n-\beta_0} \mid \bar{\nu}(s)-\nu(s) \mid ds \,+\, b \mid \nu(t)-\bar{\nu}(t) \mid \\
&\leq \; t^{\beta_0-1}\, b\, \lambda\, \beta_1\, t^{\beta_1-\beta_0} \int_0^T \mid \bar{\nu}(s)-\nu(s) \mid ds \,+ \ldots. \\
&\quad +t^{\beta_0-1}\, b\, \lambda\, \beta_n\, t^{\beta_n-\beta_0} \int_0^T \mid \bar{\nu}(s)-\nu(s) \mid ds \,+\, b \mid \nu(t)-\bar{\nu}(t) \mid \\
&\leq \; b\, \lambda\, \beta_1\, t^{\beta_1-1}\, ||\bar{\nu}-\nu||_1 \,+ \ldots. +\, b\, \lambda\, \beta_n\, t^{\beta_n-1}\, ||\bar{\nu}-\nu||_1 \,+\, b \mid \nu(t)-\bar{\nu}(t) \mid \\
&\leq \; n\, b\, \lambda\, \beta\, t^{\beta-1}\, ||\bar{\nu}-\nu||_1 \,+\, b \mid \nu(t)-\bar{\nu}(t) \mid
\end{aligned}
$$

and

$$
\int_0^T |\nu(t)-\bar{\nu}(t)|\, dt \leq n\, b\, \lambda\, ||\nu-\bar{\nu}||_1 \int_0^T \beta\, t^{\beta-1} dt + b \int_0^T \mid \nu(t)-\bar{\nu}(t) \mid dt,
$$

$$
||\nu-\bar{\nu}||_1 \leq n\, b\, \lambda\, T^{\beta}\, ||\nu-\bar{\nu}||_1 + b\, ||\nu-\bar{\nu}||_1,
$$

then

$$
[1-b\,(1+n\,\lambda\, T^{\beta})]\; ||\nu-\bar{\nu}||_1 \leq 0,
$$

implies

$$
||\nu-\bar{\nu}||_1 = 0.
$$

This establishes $\nu=\bar{\nu}$, confirming the uniqueness of the solution of (4).
Consequently, for any two solutions $x, \bar{x} \in L_1(J)$ of the equation (3), we get

$$
\begin{aligned}
|x(t)-\bar{x}(t)| &= \mid t^{1-\beta_0}\, \nu(t) - t^{1-\beta_0}\, \bar{\nu}(t) \mid \\
&= t^{1-\beta_0} \mid \nu(t)-\bar{\nu}(t) \mid \\
&\leq T^{1-\beta_0} \mid \nu(t)-\bar{\nu}(t) \mid,
\end{aligned}
$$

then

$$\int_0^T |x(t)-\bar{x}(t)|\,dt \le T^{1-\beta_0} \int_0^T |\,v(t)-\bar{v}(t)\,|\,dt$$

$$||x-\bar{x}||_1 \ \le\ T^{1-\beta_0}\,||v-\bar{v}||_1 \ =\ 0, \qquad \Rightarrow \qquad x(t)\ =\ \bar{x}(t)$$

and the solution of (1) is unique.

2.2 *Continuous dependence*

Definition 1 The solution $x \in L_1(J)$ of equation (1) depends in a continuous manner on the parameters λ_i and on the functions f, g [23], if $\forall \varepsilon > 0,\ \exists\, \delta > 0$, where

$$\max\{\ |\lambda_i-\lambda_i^*|,\quad |f-f^*|,\quad |g-g^*|\ \} < \delta, \qquad i=1,\,...,\,n$$

implies

$$||x-x^*||_1 < \varepsilon,$$

where x^* is the unique solution of

$$x^*(t) = -f^*(t,\,\lambda_1^* \int_0^{\vartheta_1(t)} \beta_1\, s^{\beta_1-1}\, x^*(s)\,ds,\,...,\,\lambda_n^* \int_0^{\vartheta_n(t)} \beta_n\, s^{\beta_n-1} x^*(s)\,ds) + g^*(t,\,x^*(t)), \tag{7}$$

$$x^*(t) = t^{1-\beta_0}\, v^*(t), \tag{8}$$

where

$$v^*(t) = -\,t^{\beta_0-1}\, f^*(t,\,\lambda_1^* \int_0^{\vartheta_1(t)} \beta_1\, s^{\beta_1-\beta_0}\, v^*(s)\,ds,\,...,\,\lambda_n^* \int_0^{\vartheta_n(t)} \beta_n\, s^{\beta_n-\beta_0}\, v^*(s)\,ds)$$

$$+\,t^{\beta_0-1}\, g^*(t,\,t^{1-\beta_0}\, v^*(t)) \tag{9}$$

Theorem 3 Under the assumptions of Theorem 2, the unique solution of (1) depends continuously on the parameters $\lambda_i,\ i=1,\,...,\,n$ as well as the functions f, g.

Proof.

$$
\begin{aligned}
|\nu(t)-\nu^*(t)| &= \Big| -t^{\beta_0-1}\, f\Big(t,\, \lambda_1 \int_0^{\vartheta_1(t)} \beta_1\, s^{\beta_1-\beta_0}\, \nu(s)\, ds, \ldots, \lambda_n \int_0^{\vartheta_n(t)} \beta_n\, s^{\beta_n-\beta_0}\, \nu(s)\, ds\Big) \\
&\quad + t^{\beta_0-1}\, g(t,\, t^{1-\beta_0}\, \nu(t)) - t^{\beta_0-1}\, g^*(t,\, t^{1-\beta_0}\, \nu^*(t)) \\
&\quad + t^{\beta_0-1}\, f^*\Big(t,\, \lambda_1^* \int_0^{\vartheta_1(t)} \beta_1\, s^{\beta_1-\beta_0}\, \nu^*(s)\, ds, \ldots, \lambda_n^* \int_0^{\vartheta_n(t)} \beta_n\, s^{\beta_n-\beta_0}\, \nu^*(s)\, ds\Big) \Big| \\
&= \Big| -t^{\beta_0-1}\, f\Big(t,\, \lambda_1 \int_0^{\vartheta_1(t)} \beta_1\, s^{\beta_1-\beta_0}\, \nu(s)\, ds, \ldots, \lambda_n \int_0^{\vartheta_n(t)} \beta_n\, s^{\beta_n-\beta_0}\, \nu(s)\, ds\Big) \\
&\quad + t^{\beta_0-1}\, f\Big(t,\, \lambda_1^* \int_0^{\vartheta_1(t)} \beta_1\, s^{\beta_1-\beta_0}\, \nu(s)\, ds, \ldots, \lambda_n^* \int_0^{\vartheta_n(t)} \beta_n\, s^{\beta_n-\beta_0}\, \nu(s)\, ds\Big) \\
&\quad - t^{\beta_0-1}\, f\Big(t,\, \lambda_1^* \int_0^{\vartheta_1(t)} \beta_1\, s^{\beta_1-\beta_0}\, \nu(s)\, ds, \ldots, \lambda_n^* \int_0^{\vartheta_n(t)} \beta_n\, s^{\beta_n-\beta_0}\, \nu(s)\, ds\Big) \\
&\quad + t^{\beta_0-1}\, f\Big(t,\, \lambda_1^* \int_0^{\vartheta_1(t)} \beta_1\, s^{\beta_1-\beta_0}\, \nu^*(s)\, ds, \ldots, \lambda_n^* \int_0^{\vartheta_n(t)} \beta_n\, s^{\beta_n-\beta_0}\, \nu^*(s)\, ds\Big) \\
&\quad - t^{\beta_0-1}\, f\Big(t,\, \lambda_1^* \int_0^{\vartheta_1(t)} \beta_1\, s^{\beta_1-\beta_0}\, \nu^*(s)\, ds, \ldots, \lambda_n^* \int_0^{\vartheta_n(t)} \beta_n\, s^{\beta_n-\beta_0}\, \nu^*(s)\, ds\Big) \\
&\quad + t^{\beta_0-1}\, f^*\Big(t,\, \lambda_1^* \int_0^{\vartheta_1(t)} \beta_1\, s^{\beta_1-\beta_0}\, \nu^*(s)\, ds, \ldots, \lambda_n^* \int_0^{\vartheta_n(t)} \beta_n\, s^{\beta_n-\beta_0}\, \nu^*(s)\, ds\Big) \\
&\quad + t^{\beta_0-1}\, g(t,\, t^{1-\beta_0}\, \nu(t)) - t^{\beta_0-1}\, g(t,\, t^{1-\beta_0}\, \nu^*(t)) \\
&\quad + t^{\beta_0-1}\, g(t,\, t^{1-\beta_0}\, \nu^*(t)) - t^{\beta_0-1}\, g^*(t,\, t^{1-\beta_0}\, \nu^*(t)) \Big| \\
&\leq t^{\beta_0-1} \Big| f\Big(t,\, \lambda_1^* \int_0^{\vartheta_1(t)} \beta_1\, s^{\beta_1-\beta_0}\, \nu(s)\, ds, \ldots, \lambda_n^* \int_0^{\vartheta_n(t)} \beta_n\, s^{\beta_n-\beta_0}\, \nu(s)\, ds\Big) \\
&\quad - f\Big(t,\, \lambda_1 \int_0^{\vartheta_1(t)} \beta_1\, s^{\beta_1-\beta_0}\, \nu(s)\, ds, \ldots, \lambda_n \int_0^{\vartheta_n(t)} \beta_n\, s^{\beta_n-\beta_0}\, \nu(s)\, ds\Big) \Big| \\
&\quad + t^{\beta_0-1} \Big| f\Big(t,\, \lambda_1^* \int_0^{\vartheta_1(t)} \beta_1\, s^{\beta_1-\beta_0}\, \nu^*(s)\, ds, \ldots, \lambda_n^* \int_0^{\vartheta_n(t)} \beta_n\, s^{\beta_n-\beta_0}\, \nu^*(s)\, ds\Big) \\
&\quad - f\Big(t,\, \lambda_1^* \int_0^{\vartheta_1(t)} \beta_1\, s^{\beta_1-\beta_0}\, \nu(s)\, ds, \ldots, \lambda_n^* \int_0^{\vartheta_n(t)} \beta_n\, s^{\beta_n-\beta_0}\, \nu(s)\, ds\Big) \Big|
\end{aligned}
$$

$$
\begin{aligned}
&+t^{\beta_0-1} \mid f^*(t, \lambda_1^* \int_0^{\vartheta_1(t)} \beta_1\, s^{\beta_1-\beta_0}\, v^*(s)\, ds, ..., \lambda_n^* \int_0^{\vartheta_n(t)} \beta_n\, s^{\beta_n-\beta_0}\, v^*(s)\, ds)\\
&-f(t, \lambda_1^* \int_0^{\vartheta_1(t)} \beta_1\, s^{\beta_1-\beta_0}\, v^*(s)\, ds, ..., \lambda_n^* \int_0^{\vartheta_n(t)} \beta_n\, s^{\beta_n-\beta_0}\, v^*(s)\, ds) \mid\\
&+t^{\beta_0-1} \mid g(t, t^{1-\beta_0}\, v(t)) - g(t, t^{1-\beta_0}\, v^*(t)) \mid\\
&+t^{\beta_0-1} \mid g(t, t^{1-\beta_0}\, v^*(t)) - g^*(t, t^{1-\beta_0}\, v^*(t)) \mid\\
\leq\ & t^{\beta_0-1}\, b_1 \mid \lambda_1^* - \lambda_1 \mid \int_0^{\vartheta_1(t)} \beta_1\, s^{\beta_1-\beta_0}\, |v(s)|\, ds + ...\\
&+ t^{\beta_0-1}\, b_1 \mid \lambda_n^* - \lambda_n \mid \int_0^{\vartheta_n(t)} \beta_n\, s^{\beta_n-\beta_0}\, |v(s)|\, ds\\
&+t^{\beta_0-1}\, b_1\, \lambda_1^* \int_0^{\vartheta_1(t)} \beta_1\, s^{\beta_1-\beta_0} \mid v^*(s) - v(s) \mid ds + ...\\
&+ t^{\beta_0-1}\, b_1\, \lambda_n^* \int_0^{\vartheta_n(t)} \beta_n\, s^{\beta_n-\beta_0} \mid v^*(s) - v(s) \mid ds\\
&+t^{\beta_0-1}\, \delta + t^{\beta_0-1}\, b_2\, t^{1-\beta_0} \mid v(t) - v^*(t) \mid + t^{\beta_0-1}\, \delta\\
\leq\ & t^{\beta_0-1}\, b\, \delta \int_0^t \beta_1\, s^{\beta_1-\beta_0}\, |v(s)|\, ds + ... + t^{\beta_0-1}\, b\, \delta \int_0^t \beta_n\, s^{\beta_n-\beta_0}\, |v(s)|\, ds\\
&+t^{\beta_0-1}\, b\, \lambda_1^* \int_0^t \beta_1\, s^{\beta_1-\beta_0} \mid v^*(s) - v(s) \mid ds + ...\\
&+ t^{\beta_0-1}\, b\, \lambda_n^* \int_0^t \beta_n\, s^{\beta_n-\beta_0} \mid v^*(s) - v(s) \mid ds\\
&+2\, t^{\beta_0-1}\, \delta + b \mid v(t) - v^*(t) \mid\\
\leq\ & t^{\beta_0-1}\, b\, \delta\, \beta_1\, t^{\beta_1-\beta_0} \int_0^T |v(s)|\, ds + ... + t^{\beta_0-1}\, b\, \delta\, \beta_n\, t^{\beta_n-\beta_0} \int_0^T |v(s)|\, ds\\
&+t^{\beta_0-1}\, b\, \lambda_1^*\, \beta_1\, t^{\beta_1-\beta_0} \int_0^T \mid v^*(s) - v(s) \mid ds + ...\\
&+ t^{\beta_0-1}\, b\, \lambda_n^*\, \beta_n\, t^{\beta_n-\beta_0} \int_0^T \mid v^*(s) - v(s) \mid ds
\end{aligned}
$$

$$+2\, t^{\beta_0-1}\, \delta \;+\; b \mid \nu(t)-\nu^*(t) \mid$$

$$\leq \; b\, \delta\, \beta\, t^{\beta_1-1}\, ||\nu||_1 + ... + b\, \delta\, \beta\, t^{\beta_n-1}\, ||\nu||_1 \,+\, b\, \lambda\, \beta\, t^{\beta_1-1}\, ||\nu-\nu^*||_1 + ...$$

$$+\, b\, \lambda\, \beta\, t^{\beta_n-1}\, ||\nu-\nu^*||_1 + 2\, \delta\, t^{\beta_0-1} \;+\; b \mid \nu(t)-\nu^*(t) \mid$$

$$\leq \; n\, b\, \delta\, ||\nu||_1\, \beta\, t^{\beta-1} \,+\, n\, b\, \lambda\, ||\nu-\nu^*||_1\, \beta\, t^{\beta-1} \,+\, 2\, \delta\, t^{\beta-1} \,+\, b \mid \nu(t)-\nu^*(t) \mid,$$

then

$$\int_0^T |\nu(t)-\nu^*(t)|\, dt \leq n\, b\, \delta\, ||\nu||_1 \int_0^T \beta\, t^{\beta-1}\, dt \;+\; n\, b\, \lambda\, ||\nu-\nu^*||_1 \int_0^T \beta\, t^{\beta-1}\, dt$$

$$+2\, \delta \int_0^T t^{\beta-1}\, dt \;+\; b \int_0^T \mid \nu(t)-\nu^*(t) \mid dt,$$

$$||\nu-\nu^*||_1 \leq n\, b\, \delta\, T^{\beta}\, ||\nu||_1 \;+\; n\, b\, \lambda\, T^{\beta}\, ||\nu-\nu^*||_1 \,+\, 2\, \delta\, \frac{T^{\beta}}{\beta} \;+\; b\, ||\nu-\nu^*||_1$$

and

$$[1-b\,(1+n\, \lambda\, T^{\beta})]\, ||\nu-\nu^*||_1 \;\leq\; 2\, \delta\, \frac{T^{\beta}}{\beta} \,+\, n\, b\, \delta\, T^{\beta}\, ||\nu||_1,$$

thus

$$||\nu-\nu^*||_1 \;\leq\; \frac{2\, \delta\, T^{\beta} \,+\, n\, b\, \delta\, \beta\, T^{\beta}\, ||\nu||_1}{\beta\, [1-b\,(1+n\, \lambda\, T^{\beta})]} \;=\; \varepsilon_1.$$

Finally

$$\begin{aligned} |x(t)-x^*(t)| &= \mid t^{1-\beta_0}\, \nu(t) - t^{1-\beta_0}\, \nu^*(t) \mid \\ &= t^{1-\beta_0}\, |\nu(t)-\nu^*(t)| \\ &\leq T^{1-\beta_0}\, |\nu(t)-\nu^*(t)| \end{aligned}$$

and

$$
\begin{aligned}
\int_0^T |x(t) - x^*(t)|\, dt &\leq T^{1-\beta_0} \int_0^T |\nu(t) - \nu^*(t)|\, dt \\
&= T^{1-\beta_0}\, ||\nu - \nu^*||_1 \\
&\leq T^{1-\beta_0}\, \varepsilon_1 \;=\; \varepsilon,
\end{aligned}
$$

thus

$$
||x - x^*||_1 \;\leq\; \varepsilon.
$$

Definition 2 Let $\vartheta_i^*\colon J \to J$ be a continuous function and $\vartheta_i^*(t) \leq t$. The solution $x \in L_1(J)$ of equation (1) varies continuously with respect to the delay functions ϑ_i, if $\forall \varepsilon > 0$, $\exists \delta > 0$ such that

$$
|\vartheta_i - \vartheta_i^*| \;<\; \delta, \qquad i = 1, \ldots, n
$$

implies

$$
||x - x^*||_1 < \varepsilon,
$$

in which x^* uniquely satisfies the equation

$$
x^*(t) = -f\left(t,\, \lambda_1 \int_0^{\vartheta_1^*(t)} \beta_1\, s^{\beta_1 - 1}\, x^*(s)\, ds, \ldots, \lambda_n \int_0^{\vartheta_n^*(t)} \beta_n\, s^{\beta_n - 1}\, x^*(s)\, ds\right) + g(t,\, x^*(t)) \tag{10}
$$

which can be given by equation (8),

$$
\begin{aligned}
\nu^*(t) \;=& -t^{\beta_0 - 1}\, f\left(t,\, \lambda_1 \int_0^{\vartheta_1^*(t)} \beta_1\, s^{\beta_1 - \beta_0}\, \nu^*(s)\, ds, \ldots, \lambda_n \int_0^{\vartheta_n^*(t)} \beta_n\, s^{\beta_n - \beta_0}\, \nu^*(s)\, ds\right) \\
&+ t^{\beta_0 - 1}\, g(t,\, t^{1-\beta_0}\, \nu^*(t))
\end{aligned} \tag{11}
$$

Theorem 4 Suppose the hypotheses of Theorem 2 are fulfilled, the unique solution corresponding to equation (1) depends continuously with respect to the delay functions ϑ_i, $i = 1, \ldots, n$.

Proof. Let x and x^* be given by (3) and (8) and ν and ν^* by (4) and (11), then

$$
\begin{aligned}
|\nu(t)-\nu^*(t)| &= |-t^{\beta_0-1}\, f(t,\, \lambda_1 \int_0^{\vartheta_1(t)} \beta_1\, s^{\beta_1-\beta_0}\, \nu(s)\, ds,\, ...,\, \lambda_n \int_0^{\vartheta_n(t)} \beta_n\, s^{\beta_n-\beta_0}\, \nu(s)\, ds) \\
&\quad +t^{\beta_0-1}\, g(t,\, t^{1-\beta_0}\, \nu(t)) - t^{\beta_0-1}\, g(t,\, t^{1-\beta_0}\, \nu^*(t)) \\
&\quad +t^{\beta_0-1}\, f(t,\, \lambda_1 \int_0^{\vartheta_1^*(t)} \beta_1\, s^{\beta_1-\beta_0}\, \nu^*(s)\, ds,\, ...,\, \lambda_n \int_0^{\vartheta_n^*(t)} \beta_n\, s^{\beta_n-\beta_0}\, \nu^*(s)\, ds)\, | \\
&= |-t^{\beta_0-1}\, f(t,\, \lambda_1 \int_0^{\vartheta_1(t)} \beta_1\, s^{\beta_1-\beta_0}\, \nu(s)\, ds,\, ...,\, \lambda_n \int_0^{\vartheta_n(t)} \beta_n\, s^{\beta_n-\beta_0}\, \nu(s)\, ds) \\
&\quad +t^{\beta_0-1}\, f(t,\, \lambda_1 \int_0^{\vartheta_1^*(t)} \beta_1\, s^{\beta_1-\beta_0}\, \nu(s)\, ds,\, ...,\, \lambda_n \int_0^{\vartheta_n^*(t)} \beta_n\, s^{\beta_n-\beta_0}\, \nu(s)\, ds) \\
&\quad -t^{\beta_0-1}\, f(t,\, \lambda_1 \int_0^{\vartheta_1^*(t)} \beta_1\, s^{\beta_1-\beta_0}\, \nu(s)\, ds,\, ...,\, \lambda_n \int_0^{\vartheta_n^*(t)} \beta_n\, s^{\beta_n-\beta_0}\, \nu(s)\, ds) \\
&\quad +t^{\beta_0-1}\, f(t,\, \lambda_1 \int_0^{\vartheta_1^*(t)} \beta_1\, s^{\beta_1-\beta_0}\, \nu^*(s)\, ds,\, ...,\, \lambda_n \int_0^{\vartheta_n^*(t)} \beta_n\, s^{\beta_n-\beta_0}\, \nu^*(s)\, ds) \\
&\quad +t^{\beta_0-1}\, g(t,\, t^{1-\beta_0}\, \nu(t)) - t^{\beta_0-1}\, g(t,\, t^{1-\beta_0}\, \nu^*(t))\, | \\
&\le t^{\beta_0-1}\, |\, f(t,\, \lambda_1 \int_0^{\vartheta_1^*(t)} \beta_1\, s^{\beta_1-\beta_0}\, \nu(s)\, ds,\, ...,\, \lambda_n \int_0^{\vartheta_n^*(t)} \beta_n\, s^{\beta_n-\beta_0}\, \nu(s)\, ds) \\
&\quad -f(t,\, \lambda_1 \int_0^{\vartheta_1(t)} \beta_1\, s^{\beta_1-\beta_0}\, \nu(s)\, ds,\, ...,\, \lambda_n \int_0^{\vartheta_n(t)} \beta_n\, s^{\beta_n-\beta_0}\, \nu(s)\, ds)\, | \\
&\quad +t^{\beta_0-1}\, |\, f(t,\, \lambda_1 \int_0^{\vartheta_1^*(t)} \beta_1\, s^{\beta_1-\beta_0}\, \nu^*(s)\, ds,\, ...,\, \lambda_n \int_0^{\vartheta_n^*(t)} \beta_n\, s^{\beta_n-\beta_0}\, \nu^*(s)\, ds) \\
&\quad -f(t,\, \lambda_1 \int_0^{\vartheta_1^*(t)} \beta_1\, s^{\beta_1-\beta_0}\, \nu(s)\, ds,\, ...,\, \lambda_n \int_0^{\vartheta_n^*(t)} \beta_n\, s^{\beta_n-\beta_0}\, \nu(s)\, ds)\, | \\
&\quad +t^{\beta_0-1}\, |\, g(t,\, t^{1-\beta_0}\, \nu(t)) - g(t,\, t^{1-\beta_0}\, \nu^*(t))\, | \\
&\le t^{\beta_0-1}\, b_1 \sum_{i=1}^{n} \lambda_i \, |\int_{\vartheta_i(t)}^{\vartheta_i^*(t)} \beta_i\, s^{\beta_i-\beta_0}\, \nu(s)\, ds\, | \\
&\quad +t^{\beta_0-1}\, b_1 \sum_{i=1}^{n} \lambda_i \int_0^{\vartheta_i^*(t)} \beta_i\, s^{\beta_i-\beta_0}\, |\, \nu^*(s)-\nu(s)\, |\, ds + t^{\beta_0-1}\, b_2\, t^{1-\beta_0}\, |\, \nu(t)-\nu^*(t)\, |
\end{aligned}
$$

$$\leq \ t^{\beta_0-1}\, b\, \lambda \sum_{i=1}^{n} \Big| \int_{\vartheta_i(t)}^{\vartheta_i^*(t)} \beta_i\, s^{\beta_i-\beta_0}\, \nu(s)\, ds \Big|$$

$$+t^{\beta_0-1}\, b\, \lambda \sum_{i=1}^{n} \int_0^t \beta_i\, s^{\beta_i-\beta_0} \mid \nu(s)-\nu^*(s) \mid ds \ +\ b \mid \nu(t)-\nu^*(t) \mid$$

$$\leq \ t^{\beta_0-1}\, b\, \lambda \sum_{i=1}^{n} \beta_i\, t^{\beta_i-\beta_0} \Big| \int_{\vartheta_i(t)}^{\vartheta_i^*(t)} \nu(s)\, ds \Big|$$

$$+t^{\beta_0-1}\, b\, \lambda \sum_{i=1}^{n} \beta_i\, t^{\beta_i-\beta_0} \int_0^t \mid \nu(s)-\nu^*(s) \mid ds \ +\ b \mid \nu(t)-\nu^*(t) \mid$$

$$\leq \ b\, \lambda\, \beta \sum_{i=1}^{n} t^{\beta_i-1} \Big| \int_{\vartheta_i(t)}^{\vartheta_i^*(t)} \nu(s)\, ds \Big|$$

$$+\, b\, \lambda\, \beta \sum_{i=1}^{n} t^{\beta_i-1} \int_0^T \mid \nu(s)-\nu^*(s) \mid ds \ +\ b \mid \nu(t)-\nu^*(t) \mid$$

$$\leq \ b\, \lambda\, \beta\, t^{\beta-1} \sum_{i=1}^{n} \Big| \int_{\vartheta_i(t)}^{\vartheta_i^*(t)} \nu(s)\, ds \Big| \ +\ n\, b\, \lambda\, \beta\, t^{\beta-1}\, || \nu-\nu^* ||_1 \ +\ b \mid \nu(t)-\nu^*(t) \mid,$$

Since $\nu \in L_1(J)$ and $\mid \vartheta_i(t)-\vartheta_i^*(t) \mid < \delta$, then from the absolute continuity of Lebesgue integral [18], we can get $\mid \int_{\vartheta_i(t)}^{\vartheta_i^*(t)} \nu(s)\, ds \mid < \varepsilon^*,\ i=1, .., n$ and

$$|\nu(t)-\nu^*(t)| \ \leq\ n\, b\, \lambda\, \beta\, t^{\beta-1}\, \varepsilon^* \ +\ n\, b\, \lambda\, \beta\, t^{\beta-1}\, || \nu-\nu^* ||_1 \ +\ b \mid \nu(t)-\nu^*(t) \mid,$$

thus

$$||\nu-\nu^*||_1 \ \leq\ n\, b\, \lambda\, T^{\beta}\, \varepsilon^* \ +\ n\, b\, \lambda\, T^{\beta}\, ||\nu-\nu^*||_1 + b\, ||\nu-\nu^*||_1$$

and

$$[1-b\,(1+n\,\lambda\, T^{\beta})]\, ||\nu-\nu^*||_1 \ \leq\ n\, b\, \lambda\, T^{\beta}\, \varepsilon^*,$$

then

$$||\nu-\nu^*||_1 \ =\ \frac{n\, b\, \lambda\, T^{\beta}\, \varepsilon^*}{1-b\,(1+n\,\lambda\, T^{\beta})} \ =\ \varepsilon_1^*.$$

Now,

$$|x(t) - x^*(t)| \;=\; |\, t^{1-\beta_0}\, \nu(t) - t^{1-\beta_0}\, \nu^*(t) \,|$$

$$\leq\; T^{1-\beta_0}\, |\, \nu(t) - \nu^*(t) \,|,$$

then

$$||x - x^*||_1 \;\leq\; T^{1-\beta_0}\, ||\nu - \nu^*||_1 \;\leq\; T^{1-\beta_0}\, \varepsilon_1^* \;=\; \varepsilon.$$

Definition 3 The solution $x \in L_1(J)$ of equation (1) depends in a continuous manner on the parameters β_i [23], if $\forall \varepsilon > 0,\ \exists\, \delta > 0$, such that

$$|\beta_i - \beta_i^*| < \delta, \qquad i = 1, \ldots, n$$

implies

$$||x - x^*||_1 < \varepsilon,$$

where x^* is the unique solution of

$$x^*(t) = -f\Big(t,\, \lambda_1 \int_0^{\vartheta_1(t)} \beta_1^*\, s^{\beta_1^*-1}\, x^*(s)\, ds, \ldots, \lambda_n \int_0^{\vartheta_n(t)} \beta_n^*\, s^{\beta_n^*-1}\, x^*(s)\, ds\Big) + g(t, x^*(t)) \tag{12}$$

and

$$\nu^*(t) \;= -\; t^{\beta_0-1}\, f\Big(t,\, \lambda_1 \int_0^{\vartheta_1(t)} \beta_1^*\, s^{\beta_1^*-\beta_0}\, \nu^*(s)\, ds, \ldots, \lambda_n \int_0^{\vartheta_n(t)} \beta_n^*\, s^{\beta_n^*-\beta_0}\, \nu^*(s)\, ds\Big)$$

$$+\; t^{\beta_0-1}\, g(t, t^{1-\beta_0}\, \nu^*(t)) \tag{13}$$

Theorem 5 Under the assumptions of Theorem 2, the unique solution of (1) depends continuously on the parameters $\beta_i,\ i = 1, \ldots, n$.

Proof. Let x and x^* be as in (3) and (8) ν and ν^* be as in (4) and (13), then

$$|\nu(t) - \nu^*(t)| \;=\; \Big| - t^{\beta_0-1}\, f\Big(t,\, \lambda_1 \int_0^{\vartheta_1(t)} \beta_1\, s^{\beta_1-\beta_0}\, \nu(s)\, ds, \ldots, \lambda_n \int_0^{\vartheta_n(t)} \beta_n\, s^{\beta_n-\beta_0}\, \nu(s)\, ds\Big)$$

$$+ t^{\beta_0-1}\, g(t, t^{1-\beta_0}\, \nu(t)) - t^{\beta_0-1}\, g(t, t^{1-\beta_0}\, \nu^*(t))$$

$$
\begin{aligned}
& +t^{\beta_0-1}\ f(t,\ \lambda_1\ \int_0^{\vartheta_1(t)} \beta_1^*\ s^{\beta_1^*-\beta_0}\ \nu^*(s)\ ds,\ ...,\ \lambda_n\ \int_0^{\vartheta_n(t)} \beta_n^*\ s^{\beta_n^*-\beta_0}\ \nu^*(s)\ ds)\ | \\
= \quad & |-t^{\beta_0-1}\ f(t,\ \lambda_1\ \int_0^{\vartheta_1(t)} \beta_1\ s^{\beta_1-\beta_0}\ \nu(s)\ ds,\ ...,\ \lambda_n\ \int_0^{\vartheta_n(t)} \beta_n\ s^{\beta_n-\beta_0}\ \nu(s)\ ds) \\
& +t^{\beta_0-1}\ f(t,\ \lambda_1\ \int_0^{\vartheta_1(t)} \beta_1\ s^{\beta_1-\beta_0}\ \nu^*(s)\ ds,\ ...,\ \lambda_n\ \int_0^{\vartheta_n(t)} \beta_n\ s^{\beta_n-\beta_0}\ \nu^*(s)\ ds) \\
& -t^{\beta_0-1}\ f(t,\ \lambda_1\ \int_0^{\vartheta_1(t)} \beta_1\ s^{\beta_1-\beta_0}\ \nu^*(s)\ ds,\ ...,\ \lambda_n\ \int_0^{\vartheta_n(t)} \beta_n\ s^{\beta_n-\beta_0}\ \nu^*(s)\ ds) \\
& +t^{\beta_0-1}\ f(t,\ \lambda_1\ \int_0^{\vartheta_1(t)} \beta_1^*\ s^{\beta_1-\beta_0}\ \nu^*(s)\ ds,\ ...,\ \lambda_n\ \int_0^{\vartheta_n(t)} \beta_n^*\ s^{\beta_n-\beta_0}\ \nu^*(s)\ ds) \\
& -t^{\beta_0-1}\ f(t,\ \lambda_1\ \int_0^{\vartheta_1(t)} \beta_1^*\ s^{\beta_1-\beta_0}\ \nu^*(s)\ ds,\ ...,\ \lambda_n\ \int_0^{\vartheta_n(t)} \beta_n^*\ s^{\beta_n-\beta_0}\ \nu^*(s)\ ds) \\
& +t^{\beta_0-1}\ f(t,\ \lambda_1\ \int_0^{\vartheta_1(t)} \beta_1^*\ s^{\beta_1^*-\beta_0}\ \nu^*(s)\ ds,\ ...,\ \lambda_n\ \int_0^{\vartheta_n(t)} \beta_n^*\ s^{\beta_n^*-\beta_0}\ \nu^*(s)\ ds) \\
& +t^{\beta_0-1}\ g(t,\ t^{1-\beta_0}\ \nu(t))-t^{\beta_0-1}\ g(t,\ t^{1-\beta_0}\ \nu^*(t))\ | \\
\leq \quad & t^{\beta_0-1}\ |\ f(t,\ \lambda_1\ \int_0^{\vartheta_1(t)} \beta_1\ s^{\beta_1-\beta_0}\ \nu^*(s)\ ds,\ ...,\ \lambda_n\ \int_0^{\vartheta_n(t)} \beta_n\ s^{\beta_n-\beta_0}\ \nu^*(s)\ ds) \\
& -f(t,\ \lambda_1\ \int_0^{\vartheta_1(t)} \beta_1\ s^{\beta_1-\beta_0}\ \nu(s)\ ds,\ ...,\ \lambda_n\ \int_0^{\vartheta_n(t)} \beta_n\ s^{\beta_n-\beta_0}\ \nu(s)\ ds)\ | \\
& +t^{\beta_0-1}\ |\ f(t,\ \lambda_1\ \int_0^{\vartheta_1(t)} \beta_1^*\ s^{\beta_1-\beta_0}\ \nu^*(s)\ ds,\ ...,\ \lambda_n\ \int_0^{\vartheta_n(t)} \beta_n^*\ s^{\beta_n-\beta_0}\ \nu^*(s)\ ds) \\
& -f(t,\ \lambda_1\ \int_0^{\vartheta_1(t)} \beta_1\ s^{\beta_1-\beta_0}\ \nu^*(s)\ ds,\ ...,\ \lambda_n\ \int_0^{\vartheta_n(t)} \beta_n\ s^{\beta_n-\beta_0}\ \nu^*(s)\ ds)\ | \\
& +t^{\beta_0-1}\ |\ f(t,\ \lambda_1\ \int_0^{\vartheta_1(t)} \beta_1^*\ s^{\beta_1^*-\beta_0}\ \nu^*(s)\ ds,\ ...,\ \lambda_n\ \int_0^{\vartheta_n(t)} \beta_n^*\ s^{\beta_n^*-\beta_0}\ \nu^*(s)\ ds) \\
& -f(t,\ \lambda_1\ \int_0^{\vartheta_1(t)} \beta_1^*\ s^{\beta_1-\beta_0}\ \nu^*(s)\ ds,\ ...,\ \lambda_n\ \int_0^{\vartheta_n(t)} \beta_n^*\ s^{\beta_n-\beta_0}\ \nu^*(s)\ ds)\ | \\
& +t^{\beta_0-1}\ |\ g(t,\ t^{1-\beta_0}\ \nu(t))-g(t,\ t^{1-\beta_0}\ \nu^*(t))\ |
\end{aligned}
$$

$$
\begin{aligned}
&\leq \ t^{\beta_0-1}\, b_1 \sum_{i=1}^{n} \lambda_i \int_0^{\vartheta_i(t)} \beta_i\, s^{\beta_i-\beta_0}\, |\nu^*(s)-\nu(s)|\, ds \\
&\quad + t^{\beta_0-1}\, b_1 \sum_{i=1}^{n} \lambda_i \int_0^{\vartheta_i(t)} |\beta_i^*-\beta_i|\, s^{\beta_i-\beta_0}\, |\nu^*(s)|\, ds \\
&\quad + t^{\beta_0-1}\, b_1 \sum_{i=1}^{n} \lambda_i \int_0^{\vartheta_i(t)} \beta_i^*\, |s^{\beta_i^*-\beta_0}-s^{\beta_i-\beta_0}|\, |\nu^*(s)|\, ds \,+\, t^{\beta_0-1}\, b_2\, t^{1-\beta_0}\, |\nu(t)-\nu^*(t)| \\
&\leq \ t^{\beta_0-1}\, b\, \lambda\, \beta \sum_{i=1}^{n} \int_0^{t} s^{\beta-\beta_0}\, |\nu^*(s)-\nu(s)|\, ds \,+\, t^{\beta_0-1}\, b\, \lambda\, \delta \sum_{i=1}^{n} \int_0^{t} s^{\beta-\beta_0}\, |\nu^*(s)|\, ds \\
&\quad + t^{\beta_0-1}\, b\, \lambda \sum_{i=1}^{n} \int_0^{t} \beta_i^*\, |s^{\beta_i^*-\beta_0}-s^{\beta_i-\beta_0}|\, |\nu^*(s)|\, ds \,+\, b\, |\nu(t)-\nu^*(t)| \\
&\leq \ t^{\beta_0-1}\, n\, b\, \lambda\, \beta\, t^{\beta-\beta_0} \int_0^{T} |\nu^*(s)-\nu(s)|\, ds \,+\, t^{\beta_0-1}\, n\, b\, \lambda\, \delta\, t^{\beta-\beta_0} \int_0^{T} |\nu^*(s)|\, ds \\
&\quad + t^{\beta_0-1}\, b\, \lambda \sum_{i=1}^{n} \int_0^{T} |s^{\beta_i^*-\beta_0}-s^{\beta_i-\beta_0}|\, |\nu^*(s)|\, ds \,+\, b\, |\nu(t)-\nu^*(t)|.
\end{aligned}
$$

Now, [24] $\forall \varepsilon_1 > 0$, $\exists \delta > 0$ such that $|\beta_i^* - \beta_i| < \delta$ implies

$$
|s^{\beta_i^*-\beta_0} - s^{\beta_i-\beta_0}| = s^{\beta_i-\beta_0}\, |s^{\beta_i^*-\beta_i} - 1| < \varepsilon_1,
$$

thus

$$
\begin{aligned}
|\nu(t)-\nu^*(t)| \ &\leq \ n\, b\, \lambda\, \beta\, t^{\beta-1}\, ||\nu-\nu^*||_1 \,+\, n\, b\, \lambda\, \delta\, t^{\beta-1}\, ||\nu^*||_1 \\
&\quad + n\, b\, \lambda\, \varepsilon_1\, t^{\beta_0-1}\, ||\nu^*||_1 \,+\, b\, |\nu(t)-\nu^*(t)|,
\end{aligned}
$$

then

$$
\begin{aligned}
\int_0^T |\nu(t)-\nu^*(t)|\, dt &\leq n\, b\, \lambda\, ||\nu-\nu^*||_1 \int_0^T \beta\, t^{\beta-1}\, dt \,+\, n\, b\, \lambda\, \delta\, ||\nu^*||_1 \int_0^T t^{\beta-1}\, dt \\
&\quad + n\, b\, \lambda\, \varepsilon_1\, ||\nu^*||_1 \int_0^T t^{\beta-1}\, dt \,+\, b \int_0^T |\nu(t)-\nu^*(t)|\, dt
\end{aligned}
$$

$$\leq n\, b\, \lambda\, T^{\beta}\, ||v - v^*||_1 \, + \, n\, b\, \lambda\, \delta\, \frac{T^{\beta}}{\beta}\, ||v^*||_1$$

$$+ n\, b\, \lambda\, \varepsilon_1\, \frac{T^{\beta}}{\beta}\, ||v^*||_1 \, + \, b\, ||v - v^*||_1,$$

$$||v - v^*||_1 \, \leq \, b\, (1 + n\, \lambda\, T^{\beta})\, ||v - v^*||_1 \, + \, n\, b\, \lambda\, (\delta + \varepsilon_1)\, \frac{T^{\beta}}{\beta}\, ||v^*||_1$$

and

$$[1 - b\, (1 + n\, \lambda\, T^{\beta})]\, ||v - v^*||_1 \, \leq \, n\, b\, \lambda\, (\delta + \varepsilon_1)\, \frac{T^{\beta}}{\beta}\, ||v^*||_1,$$

then

$$||v - v^*||_1 \, \leq \, \frac{n\, b\, \lambda\, (\delta + \varepsilon_1)\, T^{\beta}\, ||v^*||_1}{\beta\, [1 - b\, (1 + n\, \lambda\, T^{\beta})]} \, = \, \varepsilon_2.$$

Now,

$$|x(t) - x^*(t)| \, = \, t^{1-\beta_0}\, |v(t) - v^*(t)| \, \leq \, T^{1-\beta_0}\, |v(t) - v^*(t)|$$

and

$$||x - x^*||_1 \, \leq \, T^{1-\beta_0}\, ||v - v^*||_1 \, \leq \, T^{1-\beta_0}\, \varepsilon_2 \, = \, \varepsilon.$$

2.3 *Hyers-Ulam stability*

Definition 4 [9, 15, 25, 26] Suppose equation (1) admits a solution $x \in L_1(J)$. The equation (1) is Hyers-Ulam stable provided that for each $\varepsilon > 0$, $\exists\, \delta > 0$ such that for any δ-approximate solution x_s of equation (1) satisfying

$$|\, x_s(t) + f(t,\, \lambda_1 \int_0^{\vartheta_1(t)} \beta_1\, \theta^{\beta_1 - 1}\, x_s(\theta)\, d\theta,\, ...,\, \lambda_n \int_0^{\vartheta_n(t)} \beta_n\, \theta^{\beta_n - 1}\, x_s(\theta)\, d\theta) - g(t,\, x_s(t))\, | < \delta,$$

implies

$$||x - x_s||_1 < \varepsilon.$$

Theorem 6 Assuming the conditions of Theorem 2 hold, it follows that the integral equation (1) possesses Hyers-Ulam stability.

Proof.

$$| x_s(t) + f(t, \lambda_1 \int_0^{\vartheta_1(t)} \beta_1\ \theta^{\beta_1 - 1}\ x_s(\theta)\ d\theta, ..., \lambda_n \int_0^{\vartheta_n(t)} \beta_n\ \theta^{\beta_n - 1}\ x_s(\theta)\ d\theta) - g(t, x_s(t))\ | < \delta,$$

then

$$-\delta < \ x_s(t) + f(t, \lambda_1 \int_0^{\vartheta_1(t)} \beta_1\ \theta^{\beta_1 - 1}\ x_s(\theta)\ d\theta, ..., \lambda_n \int_0^{\vartheta_n(t)} \beta_n\ \theta^{\beta_n - 1}\ x_s(\theta)\ d\theta) - g(t, x_s(t))\ < \ \delta$$

multiplying by $t^{\beta_0 - 1}$, we get

$$-\delta\ t^{\beta_0 - 1}\ <\ A\ <\ \delta\ t^{\beta_0 - 1},$$

where

$$A\ =\ t^{\beta_0 - 1}\ x_s(t)\ +\ t^{\beta_0 - 1}\ f(t, \lambda_1 \int_0^{\vartheta_1(t)} \beta_1 \theta^{\beta_1 - 1} x_s(\theta) d\theta, ..., \lambda_n \int_0^{\vartheta_n(t)} \beta_n \theta^{\beta_n - 1} x_s(\theta) d\theta)\ -\ t^{\beta_0 - 1}\ g(t, x_s(t))$$

and let $\nu_s(t) =\ t^{\beta_0 - 1}\ x_s(t)$, then

$$-\delta\ t^{\beta_0 - 1}\ <\ B\ <\ \delta\ t^{\beta_0 - 1},$$

where

$$B\ =\ \nu_s(t)\ +\ t^{\beta_0 - 1}\ f(t, \lambda_1 \int_0^{\vartheta_1(t)} \beta_1 \theta^{\beta_1 - \beta_0} \nu_s(\theta) d\theta, ..., \lambda_n \int_0^{\vartheta_n(t)} \beta_n \theta^{\beta_n - \beta_0} \nu_s(\theta) d\theta)\ -\ t^{\beta_0 - 1}\ g(t, t^{1 - \beta_0} \nu_s(t))$$

thus we obtain

$$|\nu_s(t) + t^{\beta_0 - 1} f(t, \lambda_1 \int_0^{\vartheta_1(t)} \beta_1 \theta^{\beta_1 - \beta_0} \nu_s(\theta) d\theta, ..., \lambda_n \int_0^{\vartheta_n(t)} \beta_n \theta^{\beta_n - \beta_0} \nu_s(\theta) d\theta) - t^{\beta_0 - 1} g(t, t^{1 - \beta_0} \nu_s(t))|\ <\ \delta t^{\beta_0 - 1}.$$

Now,

$$
\begin{aligned}
|\nu(t)-\nu_s(t)| &= |-t^{\beta_0-1}\, f(t,\lambda_1\int_0^{\vartheta_1(t)}\beta_1\,\theta^{\beta_1-\beta_0}\,\nu(\theta)\,d\theta,...,\lambda_n\int_0^{\vartheta_n(t)}\beta_n\,\theta^{\beta_n-\beta_0}\,\nu(\theta)\,d\theta) \\
&\quad +t^{\beta_0-1}\,g(t,t^{1-\beta_0}\,\nu(t))\,-\,\nu_s(t)\,| \\
&= |-t^{\beta_0-1}\, f(t,\lambda_1\int_0^{\vartheta_1(t)}\beta_1\,\theta^{\beta_1-\beta_0}\,\nu(\theta)\,d\theta,...,\lambda_n\int_0^{\vartheta_n(t)}\beta_n\,\theta^{\beta_n-\beta_0}\,\nu(\theta)\,d\theta) \\
&\quad +t^{\beta_0-1}\,g(t,t^{1-\beta_0}\,\nu(t))\,-\,t^{\beta_0-1}\,g(t,t^{1-\beta_0}\,\nu_s(t)) \\
&\quad +t^{\beta_0-1}\, f(t,\lambda_1\int_0^{\vartheta_1(t)}\beta_1\,\theta^{\beta_1-\beta_0}\,\nu_s(\theta)\,d\theta,...,\lambda_n\int_0^{\vartheta_n(t)}\beta_n\,\theta^{\beta_n-\beta_0}\,\nu_s(\theta)\,d\theta) \\
&\quad -(\,\nu_s(t)\,+\,t^{\beta_0-1}\, f(t,\lambda_1\int_0^{\vartheta_1(t)}\beta_1\,\theta^{\beta_1-\beta_0}\,\nu_s(\theta)\,d\theta,...,\lambda_n\int_0^{\vartheta_n(t)}\beta_n\,\theta^{\beta_n-\beta_0}\,\nu_s(\theta)\,d\theta) \\
&\quad -t^{\beta_0-1}\,g(t,t^{1-\beta_0}\,\nu_s(t))\,)\,| \\
&\leq t^{\beta_0-1}\,|\,f(t,\lambda_1\int_0^{\vartheta_1(t)}\beta_1\,\theta^{\beta_1-\beta_0}\,\nu_s(\theta)\,d\theta,...,\lambda_n\int_0^{\vartheta_n(t)}\beta_n\,\theta^{\beta_n-\beta_0}\,\nu_s(\theta)\,d\theta) \\
&\quad -f(t,\lambda_1\int_0^{\vartheta_1(t)}\beta_1\,\theta^{\beta_1-\beta_0}\,\nu(\theta)\,d\theta,...,\lambda_n\int_0^{\vartheta_n(t)}\beta_n\,\theta^{\beta_n-\beta_0}\,\nu(\theta)\,d\theta)\,| \\
&\quad +t^{\beta_0-1}\,|\,g(t,t^{1-\beta_0}\,\nu(t))\,-\,g(t,t^{1-\beta_0}\,\nu_s(t))\,| \\
&\quad +|\,\nu_s(t)\,+\,t^{\beta_0-1}\, f(t,\lambda_1\int_0^{\vartheta_1(t)}\beta_1\,\theta^{\beta_1-\beta_0}\,\nu_s(\theta)\,d\theta,...,\lambda_n\int_0^{\vartheta_n(t)}\beta_n\,\theta^{\beta_n-\beta_0}\,\nu_s(\theta)\,d\theta) \\
&\quad -t^{\beta_0-1}\,g(t,t^{1-\beta_0}\,\nu_s(t))\,)\,| \\
&\leq \delta\,t^{\beta_0-1}\,+\,t^{\beta_0-1}\,b_1\,\lambda_1\int_0^{\vartheta_1(t)}\beta_1\,\theta^{\beta_1-\beta_0}\,|\,\nu_s(\theta)-\nu(\theta)\,|\,d\theta\,+.... \\
&\quad +t^{\beta_0-1}\,b_1\,\lambda_n\int_0^{\vartheta_n(t)}\beta_n\,\theta^{\beta_n-\beta_0}\,|\,\nu_s(\theta)-\nu(\theta)\,|\,d\theta\,+\,t^{\beta_0-1}\,b_2\,t^{1-\beta_0}\,|\,\nu(t)-\nu_s(t)\,| \\
&\leq \delta\,t^{\beta_0-1}\,+\,t^{\beta_0-1}\,b\,\lambda\int_0^{t}\beta_1\,\theta^{\beta_1-\beta_0}\,|\,\nu_s(\theta)-\nu(\theta)\,|\,d\theta\,+.... \\
&\quad +t^{\beta_0-1}\,b\,\lambda\int_0^{t}\beta_n\,\theta^{\beta_n-\beta_0}\,|\,\nu_s(\theta)-\nu(\theta)\,|\,d\theta\,+\,b\,|\,\nu(t)-\nu_s(t)\,|
\end{aligned}
$$

$$
\begin{aligned}
&\leq \;\; \delta\, t^{\beta_0-1} + t^{\beta_0-1}\, b\, \lambda\, \beta_1\, t^{\beta_1-\beta_0} \int_0^T |\, \nu_s(\theta) - \nu(\theta)\,|\, d\theta + ...\\
&\quad + t^{\beta_0-1}\, b\, \lambda\, \beta_n\, t^{\beta_n-\beta_0} \int_0^T |\, \nu_s(\theta) - \nu(\theta)\,|\, d\theta + b\, |\, \nu(t) - \nu_s(t)\,|\\
&\leq \;\; \delta\, t^{\beta_0-1} + b\, \lambda\, \beta\, t^{\beta_1-1}\, ||\nu - \nu_s||_1 + ... + b\, \lambda\, \beta\, t^{\beta_n-1}\, ||\nu - \nu_s||_1 + b\, |\, \nu(t) - \nu_s(t)\,|\\
&\leq \;\; \delta\, t^{\beta-1} + n\, b\, \lambda\, \beta\, t^{\beta-1}\, ||\nu - \nu_s||_1 + b\, |\, \nu(t) - \nu_s(t)\,|,
\end{aligned}
$$

then

$$
\int_0^T |\nu(t) - \nu_s(t)|\, dt \;\leq\; \delta \int_0^T t^{\beta-1}\, dt + n\, b\, \lambda\, ||\nu - \nu_s||_1 \int_0^T \beta\, t^{\beta-1}\, dt + b \int_0^T |\, \nu(t) - \nu_s(t)\,|\, dt
$$

and

$$
||\nu - \nu_s||_1 \;\leq\; \delta\, \frac{T^\beta}{\beta} + n\, b\, \lambda\, T^\beta\, ||\nu - \nu_s||_1 + b\, ||\nu - \nu_s||_1,
$$

$$
[1 - b\, (1 + n\, \lambda\, T^\beta)]\, ||\nu - \nu_s||_1 \;\leq\; \delta\, \frac{T^\beta}{\beta},
$$

then

$$
||\nu - \nu_s||_1 \;\leq\; \frac{\delta\, T^\beta}{\beta - \beta\, b\, (1 + n\, \lambda\, T^\beta)} \;=\; \varepsilon_1. \tag{14}
$$

Finally

$$
\begin{aligned}
|x(t) - x_s(t)| \;\; &= \;\; |\, t^{1-\beta_0}\, \nu(t) - t^{1-\beta_0}\, \nu_s(t)\,|\\
&= \;\; t^{1-\beta_0}\, |\, \nu(t) - \nu_s(t)\,|\\
&\leq \;\; T^{1-\beta_0}\, |\, \nu(t) - \nu_s(t)\,|,
\end{aligned}
$$

then from (14), we get

$$
||x - x_s||_1 \;\leq\; T^{1-\beta_0}\, ||\nu - \nu_s||_1 \;\leq\; T\, \varepsilon_1 = \varepsilon.
$$

3. Examples

(I) Let $f_i\colon J\times R\to R,\ i=1, 2, ..., n$ and

$$f(t, u_1, u_2, ..., u_n) \;=\; \sum_{i=1}^{n} f_i(t, u_i),$$

the integral equation (1) may be reformulated as

$$x(t) \;+\; \sum_{i=1}^{n} f_i(t,\ \lambda_i \int_0^{\vartheta_i(t)} \beta_i\, s^{\beta_i-1}\, x(s)\, ds) \;=\; g(t, x(t)), \qquad t\in J.$$

(II) Let $f_i(t, u_i)=u_i,$ then

$$x(t) \;+\; \sum_{i=1}^{n} \lambda_i \int_0^{\vartheta_i(t)} \beta_i\, s^{\beta_i-1}\, x(s)\, ds \;=\; g(t, x(t)), \qquad t\in J.$$

(III) Here, we give a simple numerical example to illustrates our results
Take $n=2$ and $T=1$, then we have

$$x(t) \;+\; f(\,t, \lambda_1 \int_0^{\vartheta_1(t)} \beta_1\, s^{\beta_1-1}\, x(s)\, ds, \lambda_2 \int_0^{\vartheta_2(t)} \beta_2\, s^{\beta_2-1}\, x(s)\, ds\,) \;=\; g(t, x(t)), \qquad t\in[0,\, 1].$$

We choose the following parameters $\beta_1=0.2$, $\beta_2=0.4$, $\lambda_1=0.1$, $\lambda_2=0.3$,
the delay functions $\vartheta_1(t)=0.5\,t,\ \ \vartheta_2(t)=0.2\,t$, the function

$$f(t, u, v)=[2t]-0.4\,u+0.1\,v \quad \text{and} \quad g(t, x)=0.25\;+\;[t]+0.5\,x(t),$$

which all of them satisfies our assumed conditions as

$$b_1=0.4,\, b_2=0.5,\, a_1(t)=[2t] \;\text{ and }\; a_2(t)=0.25+[t]$$

Then we obtain $x\in L_1(J)$ as

$$x(t) \;+\; [2t] \;-\; 0.008\int_0^{\frac{1}{2}t} s^{-0.8}\, x(s)\, ds \;+\; 0.012\int_0^{\frac{1}{5}t} s^{-0.6}\, x(s)\, ds \;=\; 0.25 \;+\; [t]+0.5\, x(t)$$

from which we can obtain $x(0)=0.5$ and

$$x(t) \;=\; 0.5 \;+\; 0.016\int_0^{\frac{1}{2}t} s^{-0.8}\, x(s)\, ds \;-\; 0.024\int_0^{\frac{1}{5}t} s^{-0.6}\, x(s)\, ds, \qquad t\in[0,\, 0.5)$$

$$x(t) = -1.5 + 0.016 \int_0^{\frac{1}{2}t} s^{-0.8}\, x(s)\, ds - 0.024 \int_0^{\frac{1}{5}t} s^{-0.6}\, x(s)\, ds, \quad t \in [0.5, 1)$$

and

$$x(1) = -1.5 + 0.016 \int_0^{\frac{1}{2}} s^{-0.8}\, x(s)\, ds - 0.024 \int_0^{\frac{1}{5}} s^{-0.6}\, x(s)\, ds.$$

4. Applications

(a) For the initial value problem associated with the fractal-integro-differential equation.

$$\frac{du}{dt} + f\left(t,\ \lambda_1 \int_0^{\vartheta_1(t)} \beta_1\, s^{\beta_1 - 1} \frac{du}{ds}\, ds, \ldots, \lambda_n \int_0^{\vartheta_n(t)} \beta_n\, s^{\beta_n - 1} \frac{du}{ds}\, ds\right) = g\left(t,\ \frac{du}{dt}\right), \qquad t \in (0, T] \tag{15}$$

with

$$u(0) = u_0. \tag{16}$$

we have the following lemma.

Lemma 1 Let the assumptions of Theorem 1 (or Theorem 2) be satisfied, then the solution $u \in AC(J)$ of the problem (15)-(16) is given by

$$u(t) = u_0 + \int_0^t x(s)\, ds,$$

where $x \in L_1(J)$ is the solution of (1).

Proof. Assume that $\frac{du}{dt} = x(t),$ in (15)-(16), then we can get

$$u(t) = u(0) + \int_0^t x(s)\, ds$$

with the solution $u \in AC(J)$ of (15)-(16) represented by

$$u(t) = u_0 + \int_0^t x(s)\, ds \tag{17}$$

where $x \in L_1(J)$ denotes the solution to

$$x(t) + f\left(t,\ \lambda_1 \int_0^{\vartheta_1(t)} \beta_1\, s^{\beta_1 - 1}\, x(s)\, ds, \ldots, \lambda_n \int_0^{\vartheta_n(t)} \beta_n\, s^{\beta_n - 1}\, x(s)\, ds\right) = g(t,\ x(t)), \quad t \in J.$$

(b) For the initial value problem of the fractal orders integro-differential equation

$$D_\alpha\, u(t) \;+\; f(t,\; \lambda_1^* \int_0^{\vartheta_1(t)} D_{\alpha_1}\, u(s)\, ds,\, ...,\, \lambda_n^* \int_0^{\vartheta_n(t)} D_{\alpha_n}\, u(s)\, ds) \;=\; g(t,\; D_\alpha\, u(t)), \qquad t \in (0,\, T] \tag{18}$$

with

$$u(0) = x_0, \tag{19}$$

where $\alpha = \min\{\alpha_i,\; i = 1, .., n\}$ we have

$$D_\alpha\, u(t) = \frac{t^{1-\alpha}}{\alpha}\, \frac{du}{dt} = x(t),$$

then

$$D_{\alpha_i}\, u(t) \;=\; \frac{t^{1-\alpha_i}}{\alpha_i}\, \frac{du}{dt} \;=\; \alpha\, t^{\alpha-1}\, \frac{t^{1-\alpha_i}}{\alpha_i}\, (\, \frac{t^{1-\alpha}}{\alpha}\, \frac{du}{dt}\,) \;=\; \frac{\alpha}{\alpha_i}\, t^{\alpha-\alpha_i}\, x(t)$$

and let $\lambda_i = \frac{\alpha}{\alpha_i \beta_i} \lambda_i^*$, $\beta_i \in (0,\, 1)$ and $\beta_i = 1 + (\alpha - \alpha_i)$, then

$$\lambda_i^*\, D_{\alpha_i}\, u(t) \;=\; \lambda_i^*\, \frac{\alpha}{\alpha_i}\, t^{\alpha-\alpha_i}\, x(t) \;=\; \lambda_i^*\, \frac{\alpha}{\alpha_i}\, t^{\beta_i-1}\, x(t) \;=\; \lambda_i^*\, \frac{\alpha}{\alpha_i\, \beta_i}\, \beta_i\, t^{\beta_i-1}\, x(t) \;=\; \lambda_i\, \beta_i\, t^{\beta_i-1}\, x(t),$$

thus we get

$$\lambda_i^* \int_0^{\vartheta_i(t)} D_{\alpha_i}\, u(s)\, ds = \lambda_i \int_0^{\vartheta_i(t)} \beta_i\, s^{\beta_i-1}\, x(s)\, ds.$$

Finally, we obtain that

$$x(t) \;+\; f(t,\; \lambda_1 \int_0^{\vartheta_1(t)} \beta_1\, s^{\beta_1-1}\, x(s)\, ds,\, ...,\, \lambda_n \int_0^{\vartheta_n(t)} \beta_n\, s^{\beta_n-1}\, x(s)\, ds) \;=\; g(t,\; x(t)), \qquad t \in J$$

which is the multi-term non-linear second kind delayed Abel fractal integral equation (1) and

$$u(t) \;=\; u(0) \;+\; \int_0^t \alpha\, s^{\alpha-1}\, x(s)\, ds$$

which is the solution of problem (18)-(19) and can be written as

$$u(t) = x_0 + \int_0^t \alpha\, s^{\alpha-1}\, x(s)\, ds.$$

As a result, we arrive at the proof of the lemma below.

Lemma 2 Assuming the conditions of Theorem 1 (or Theorem 2) hold true, then the solution $u \in AC(J)$ of the problem (18)-(19) is given by

$$u(t) = x_0 + \int_0^t \alpha\, s^{\alpha-1}\, x(s)\, ds,$$

where $x \in L_1(J)$ is the solution of (1).

5. Conclusions

In recent years, the concept of fractional calculus (differential and integral of fractional orders) has become an important foundation in mathematics and its applications in most scientific fields, especially in modeling complex phenomena which cannot be accurately described using classical calculus. Fractional-order Abel integral equations are important examples of this [1].

Fractal Abel integral equations have been defined in [17]. Here, we try to present a theoretical study of the Abel fractal integral equation, which is an analogy to the case of fractional-orders equation.

In this paper, we presented a theoretical study on the analysis of the existence and stability of solution (and the problem itself) to the multi-term non-linear second kind delayed Abel integral equation of fractal orders (1) and properties associated with these solutions. Our analysis confirms that there is at least one solution $x \in L_1[0, T]$ of (1) by applying Schauder fixed point Theorem. In addition, we provided sufficient conditions that ensure the solution is unique and continuously depends on the parameters $\lambda_i,\ \beta_i,\ i = 1, 2, ..., n$ and on the functions f, g and the delay functions $\vartheta_1, ..., \vartheta_n$. Additionally, we explored the Hyers-Ulam stability of (1).

As applications, two initial value problems of multi-term fractal orders integro-differential equations have been studied. Some examples have been given also.

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Conflict of interest

The authors declare no competing financial interest.

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