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# On $(a, d)$-Total Neighborhood-Antimagic Labelings

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**Abstract:** Suppose $G = (V, E)$ is a graph of $p$ vertices and $q$ edges. Let $f: V \cup E \to \{1, 2, \ldots, p+q\}$ be a bijection such that $WT(u) = \sum[f(ux) + f(x)]$ (over every neighbor $x$ of $u$) is the total weight of vertex $u$ induced by $f$. We say $G$ is $(a, d)$-total neighborhood-antimagic if all the total weights form an arithmetic progression with first term $a$ and common difference $d$. In this paper, we obtain many necessary and sufficient conditions for 1- and 2-regular graphs, and the one point union of such graphs to admit $(a, d)$-total neighborhood-antimagic labeling.

***Keywords***: $(a, d)$-total neighborhood-antimagic, regular graphs, one point union

**MSC:** 05C78

## 1. Introduction

For a graph $G = (V, E)$ with vertex set $V(G)$ and edge set $E(G)$ that has order $p$ and size $q$, a bijective labeling $g: E(G) \to \{1, 2, \ldots, q\}$ is an *antimagic labeling* if all the induced vertex labels, each given by the sum of the incident edge labels, are distinct. The most famous unsolved problem is the conjecture that all connected graphs, except the graph $K_2$, are antimagic (see [1]). We also say $g$ is a magic labeling if all the induced vertex labels are equal. In [2], the authors introduced the concept of $(a, d)$-antimagic labeling in which the induced vertex labels form an arithmetic progression with first term $a$ and common difference $d$. Motivated by this, the authors in [3] introduced the concept of *vertex-antimagic total labeling* (respectively, the $(a, d)$-*vertex-antimagic total labeling*). A bijective total labeling $f: V(G) \cup E(G) \to \{1, 2, \ldots, p+q\}$ is vertex-antimagic total (respectively, $(a, d)$-vertex-antimagic total) if the weight of all the vertices are distinct (respectively, form an arithmetic progression with first term $a$ and common difference $d$), where the weight of a vertex $u$ is $f(u) + \sum f(ux)$ over all the vertices $x$ adjacent to $u$. Further, we say a bijective total labeling $f$ is a *total neighborhood-antimagic labeling* if for every two distinct vertices $u$, $v$, $WT(u) \neq WT(v)$ where the total weight $WT(u) = \sum(f(ux) + f(x))$, over every neighbor $x$ of $u$. Moreover, we say a total neighborhood-antimagic labeling is also an $(a, d)$-*total neighborhood-antimagic* if all the total weights form an arithmetic progression with first term $a$ and common difference $d \geq 1$ (see [4]). Note that if we allow $d = 0$, the labeling is also known as a *total neighborhood-magic labeling* (with magic constant $a$) since all the vertices have total weights $a$ (see [5, 6]). Interested readers may refer to [7–14] for more known results. The disjoint union of graphs $G$ and $H$ is denoted $G+H$. For $n \geq 2$, the disjoint union of $n$


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copies of $G$ is denoted $nG$. Let $[a, b] = \{a, a+1, \dots, b\}$ for integers $a$, $b$ and $a < b$. For a vertex $v$ of $G$, let $\deg(v)$ be the degree of $v$ in $G$, $\Delta$ and $\delta$ be the maximum and minimum degree of vertices in $G$.

In [4], the authors obtained necessary or sufficient conditions for a cycle to admit $(a, d)$-total neighborhood-antimagic labeling. Motivated by the fact that a cycle is a 2-regular graph, we study the $(a, d)$-total neighborhood-antimagic labeling of 1-regular graphs in Section 2. In Section 3, we obtain necessary or sufficient conditions for 2-regular graphs to admit an $(a, d)$-total neighborhood-antimagic labeling. Consequently, we completely determine the $(a, d)$-total neighborhood-antimagic labeling of $nC_3$ and $nC_4$ for some even $d$. As a by-product, we obtain necessary and sufficient condition for $nC_3$ to admit a total neighborhood-magic labeling. In Section 4, we completely determine the $(a, d)$-total neighborhood-antimagic labeling of the one point union of the three families of graphs in Sections 2 and 3. Suitable problems for further research are given in Section 5.

The following theorem in [4] is needed.

**Theorem 1** Suppose $G$ is a graph of order $p \geq 2$ and size $q \geq 1$. If $G$ admits an $(a, d)$-total neighborhood-antimagic labeling $f$, then

(a) the sum of all the total weights is $\sum_{v\in V(G)} deg(v)f(v) + 2S_e = pa + \frac{p(p-1)}{2}d$, where $S_e$ is the sum of all the edge labels,

(b) $a \geq \delta(2\delta+1)$,

(c) $d \leq \frac{\Delta[2(p+q)-2\Delta+1]-\delta(2\delta+1)}{p-1}$.

## 2. 1-regular graphs

In this section, we study 1-regular graphs $nP_2$, $n \geq 1$ with vertex set $\{u_i, v_i \mid 1 \leq i \leq n\}$ and edge set $\{e_i = u_iv_i \mid 1 \leq i \leq n\}$.

**Lemma 1** If $nP_2$ is $(a, d)$-total neighborhood-antimagic, then $1 \leq d \leq 2$.

**Proof.** By Theorem 1(c), we have $d \leq \frac{1(6n-1)-1(2+1)}{2n-1} = \frac{6n-4}{2n-1} = 3 - \frac{1}{2n-1} < 3$. Thus, $d = 1$ or 2. □

**Theorem 2** If $n \geq 1$ is odd, $nP_2$ is $\left(\frac{3(n+1)}{2}, 1\right)$-total neighborhood-antimagic.

**Proof.** Define a labeling $f$: $V(nP_2) \cup E(nP_2) \to [1, 3n]$ as follows:

(i) $f(e_i) = i$, $1 \leq i \leq n$,

(ii) $f(u_{2i-1}) = \frac{3n+3}{2} - i$, $1 \leq i \leq \frac{n+1}{2}$,

(iii) $f(v_{2i-1}) = 2n+2-i$, $1 \leq i \leq \frac{n+1}{2}$,

(iv) $f(u_{2i}) = \frac{5n+3}{2} - i$, $1 \leq i \leq \frac{n-1}{2}$,

(v) $f(v_{2i}) = 3n+1-i$, $1 \leq i \leq \frac{n-1}{2}$.

Thus, $\{f(e_i) \mid 1 \leq i \leq n\} = [1, n]$, $\{f(u_{2i-1}), f(v_{2i-1}) \mid 1 \leq i \leq \frac{n+1}{2}\} = [n+1, 2n+1]$, $\{f(u_{2i}), f(v_{2i}) \mid 1 \leq i \leq \frac{n-1}{2}\} = [2n+2, 3n]$.

Thus, $f$ is a bijective total labeling of $nP_2$. Moreover, for $1 \leq i \leq \frac{n+1}{2}$, $WT(u_{2i-1}) = f(e_{2i-1}) + f(v_{2i-1}) = 2n+1+i$. So $\{WT(u_{2i-1}) \mid 1 \leq i \leq \frac{n+1}{2}\} = [2n+2, \frac{5n+3}{2}]$. Similarly, we have

$$\{WT(u_{2i}) \mid 1 \leq i \leq \frac{n-1}{2}\} = [3n+2, \frac{7n+1}{2}];$$

$$\{WT(v_{2i-1}) \mid 1 \leq i \leq \frac{n+1}{2}\} = [\frac{3n+3}{2}, 2n+1];$$

$$\{WT(v_{2i}) \mid 1 \leq i \leq \frac{n-1}{2}\} = [\frac{5n+5}{2}, 3n+1].$$

Thus, $f$ is a $(\frac{3n+3}{2}, 1)$-total neighborhood-antimagic labeling. This completes the proof. □

**Lemma 2** $2P_2$ is not $(a, 1)$-total neighborhood-antimagic.

**Proof.** Suppose $2P_2$ admits an $(a, 1)$-total neighborhood-antimagic $f$. So the sum of all the total weights is $4a+6 = 2(1+2+\cdots+6) - \sum_{i=1}^{2}[f(u_i)+f(v_i)]$. Thus, $\sum_{i=1}^{2}[f(u_i)+f(v_i)] = 36-4a$. Since $10 \le \sum_{i=1}^{2}[f(u_i)+f(v_i)] \le 18$, we have $10 \le 36-4a \le 18$ so that $5 \le a \le 6$. Note that the labels set is $[1, 6]$.

**Case 1.** Suppose $a = 5$. So the total weights are 5 to 8. Without loss of generality, assume $WT(u_1) = 5$ so that $f(e_1) \in \{1, 2, 3, 4\}$. If $f(e_1) = 1$, then $f(v_1) = 4$ and $f(u_1) \in \{5, 6\}$. If $f(u_1) = 5$, then $WT(v_1) = 6$, $\{f(u_2), f(v_2), f(e_2)\} = \{2, 3, 6\}$ and the total weight 7 does not exist, a contradiction. If $f(u_1) = 6$, then $WT(v_1) = 7$, $\{f(u_2), f(v_2), f(e_2)\} = \{2, 3, 5\}$ and the total weight 6 does not exist, a contradiction. If $f(e_1) = 2$, 3 or 4, then $f(v_1) = 3$, 2 or 1. Consequently, we can conclude that total weight 8, 6 or 7 does not exist.

**Case 2.** Suppose $a = 6$. So that total weights are 6 to 9. Without loss of generality, assume $WT(u_1) = 6$ so that $f(e_1) \in \{1, 2, 4, 5\}$. For each possible values of $f(e_1)$, we can get a contradiction similar to Case 1. The details are thus omitted. □

**Theorem 3** For even $n \ge 4$, $nP_2$ is $(\frac{3n}{2}+2, 1)$-total neighborhood-antimagic.

**Proof.** For $n = 4$, Figure 1 gives an $(8, 1)$-total neighborhood-antimagic labeling for $4P_2$:

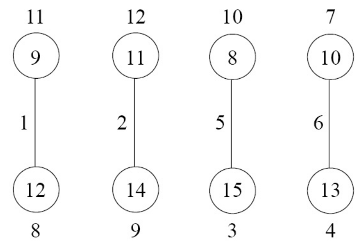


**Figure 1.** The total weight of each vertex is indicated on the vertex

Suppose $n \ge 6$, define a labeling $f$: $V(nP_2) \cup E(nP_2) \to [1, 3n]$ as follows:

(i) $f(e_i) = i$ for $i \in [1, n]\setminus\{\frac{n}{2}+1, \frac{n}{2}+2\}$, and $f(e_{\frac{n}{2}+1}) = n+1$, $f(e_{\frac{n}{2}+2}) = n+2$. The labels used are in $\{[1, \frac{n}{2}], n+1, n+2, [\frac{n}{2}+3, n]\}$.

(ii) $f(u_i) = \frac{3n}{2}+1+i$ for $i \in [1, \frac{n}{2}]$, $f(u_{\frac{n}{2}+1}) = \frac{n}{2}+1$, $f(u_{\frac{n}{2}+2}) = \frac{n}{2}+2$, and $f(u_i) = \frac{n}{2}+i$ for $\frac{n}{2}+3 \le i \le n$. The labels used are in $\{\frac{n}{2}+1, \frac{n}{2}+2, [\frac{3n}{2}+2, 2n+1], [n+3, \frac{3n}{2}]\}$.

(iii) $f(v_i) = \frac{5n}{2}+i$ for $i \in [1, \frac{n}{2}]$, $f(v_{\frac{n}{2}+1}) = \frac{5n}{2}$, $f(v_{\frac{n}{2}+2}) = \frac{3n}{2}+1$, and $f(v_i) = \frac{3n}{2}-1+i$ for $\frac{n}{2}+3 \le i \le n$. The labels used are in $\{\frac{3n}{2}+1, [\frac{5n}{2}, 3n], [2n+2, \frac{5n}{2}-1]\}$.

Thus, $f$ is a bijective total labeling of $nP_2$. Moreover,

(1) $WT(u_i) = \frac{5n}{2}+2i$ for $i \le \frac{n}{2}$, $WT(u_{\frac{n}{2}+1}) = \frac{7n}{2}+1$, $WT(u_{\frac{n}{2}+2}) = \frac{5n}{2}+3$, and $WT(u_i) = \frac{3n}{2}-1+2i$ for $\frac{n}{2}+3 \le i \le n$;

(2) $WT(v_i) = \frac{3n}{2}+1+2i$ for $1 \le i \le \frac{n}{2}$, $WT(v_{\frac{n}{2}+1}) = \frac{3n}{2}+2$, $WT(v_{\frac{n}{2}+2}) = \frac{3n}{2}+4$, $WT(v_i) = \frac{n}{2}+2i$ for $\frac{n}{2}+3 \le i \le n$.

Thus, the total weights set is $[\frac{3n}{2}+2, \frac{7n}{2}+1]$. Therefore, $f$ is a $(\frac{3n}{2}+2, 1)$-total neighborhood-antimagic labeling. This completes the proof. □

**Lemma 3** For $n \ge 1$, $nP_2$ is $(n+2, 2)$-total neighborhood-antimagic.

**Proof.** Define a bijective total labeling $f$: $V(nP_2) \cup E(nP_2) \to [1, 3n]$ such that $f(u_i) = i$, $f(e_i) = n+i$ and $f(v_i) = 2n+i$ for $1 \le i \le n$. It is easy to verify that $WT(v_i) = n+2i$ and $WT(u_i) = 3n+2i$ for $1 \le i \le n$. So $f$ is an $(n+2, 2)$-total neighborhood-antimagic labeling. This completes the proof. □

**Corollary 1** Suppose $n \ge 1$, $nP_2$ is (i) $(a, 1)$-total neighborhood-antimagic if and only if $n \ne 2$, and (ii) $(a, 2)$-total neighborhood-antimagic, for all $n$ and some suitable $a$.

## 3. 2-regular graphs $nC_3$ and $nC_4$

For $m \geq 3$, let $V(nC_m) = \{v_{i,\, j} \mid 1 \leq i \leq n,\, 1 \leq j \leq m\}$ and $E(nC_m) = \{e_{i,\, j} = v_{i,\, j}v_{i,\, j+1} \mid 1 \leq i \leq n,\, 1 \leq j \leq m,\, v_{i,\, m+1} = v_{i,\, 1}\}$.

**Lemma 4** If $nC_m$ is $(a,\, d)$-total neighborhood-antimagic, then

$$d \leq \begin{cases} 4 & \text{if } mn = 3, \\ 5 & \text{if } mn = 4, \\ 6 & \text{if } 5 \leq mn \leq 8 \\ 7 & \text{if } mn \geq 9. \end{cases}$$

**Proof.** Since $\Delta = \delta = 2$ and $p = q = mn$, by Theorem 1, $d \leq \frac{2(4mn-3)-10}{mn-1} = \frac{8mn-16}{mn-1} = 8 - \frac{8}{mn-1}$. It is easy to verify that the lemma holds. □

**Lemma 5** For $n \geq 1$, $m \geq 3$, if $nC_m$ is $(a,\, d)$-total neighborhood-antimagic, then $a = 4mn + 2 - \frac{1}{2}(mn-1)d$. Moreover, (i) $nC_m$ is not $(a,\, d)$-total neighborhood-antimagic if $mn$ is even and $d$ is odd; (ii) $nC_3$ is not $(a,\, d)$-total neighborhood-antimagic for even $n$ and $d = 2k$, $k$ is odd.

**Proof.** Suppose $nC_m$ is $(a,\, d)$-total neighborhood-antimagic, then sum of all the total weights is $\sum\limits_{k=1}^{mn}[a + (k-1)d] = 2\sum\limits_{k=1}^{2mn} k$. Thus $\frac{mn}{2}[2a + (mn-1)d] = 2mn(2mn+1)$. Hence,

$$2a + (mn-1)d = 4(2mn+1). \tag{1}$$

From (1), we have $(mn-1)d \equiv 0 \pmod 2$. Thus, if $nC_m$ is $(a,\, d)$-total neighborhood-antimagic, then either $mn$ is odd or $d$ is even. Hence we have (i).

Suppose $m = 3$ and $d \equiv 2 \pmod 4$. If $nC_3$ is $(a,\, d)$-total neighborhood-antimagic, then from (1) we have

$$\begin{aligned} & 2a + (3n-1)d \equiv 0 \pmod 4 \\ \Longleftrightarrow\ & 2a - 2(n+1) \equiv 0 \pmod 4 \\ \Longleftrightarrow\ & a \equiv n+1 \pmod 2. \end{aligned}$$

If $n$ is even, then $a$ is odd. This implies that all the total weights are odd. Without loss of generality, we have $\sum\limits_{j=1}^{3} WT(v_{1,\, j}) = 2\sum\limits_{j=1}^{3}[f(v_{1,\, j}) + f(e_{1,\, j})]$ is odd, a contradiction. □

By the same argument as in proving Lemma 5 (i), we have the following corollary.

**Corollary 2** Let $G$ be a 2-regular graph with even number of edges, then $G$ is not $(a,\, d)$-total neighborhood-antimagic for odd $d$.

### 3.1 $nC_3$

**Theorem 4** $3C_3$ is not $(a, 7)$-total neighborhood-antimagic.

**Proof.** Suppose $3C_3$ is $(a, 7)$-total neighborhood-antimagic. From the proof of Lemma 5, we have $2a+56 = 4(19)$ so that $a = 10$. Without loss of generality, let $WT(v_{1,3}) = f(v_{1,1}) + f(v_{1,2}) + f(e_{1,2}) + f(e_{1,3}) = a = 10$. Since the labels are in $[1, 18]$, we must have $\{f(v_{1,1}), f(v_{1,2}), f(e_{1,2}), f(e_{1,3})\} = \{1, 2, 3, 4\}$. Moreover, $WT(v_{1,1}) = 10 + 7i$, $WT(v_{1,2}) = 10+7j$ for $i \neq j$. Thus, $WT(v_{1,1}) - WT(v_{1,2}) = f(v_{1,2}) + f(e_{1,3}) - f(v_{1,2}) - f(e_{1,2}) \equiv 0 \pmod 7$. Therefore, $WT(v_{1,1}) - WT(v_{1,2}) \leq 4+3-2-1 = 4 \not\equiv 0 \pmod 7$, a contradiction. □

For convenience, let $A([i, j]; d)$ be the arithmetic progression with common difference $d$, the first term $i$ and the last term $j$.

**Theorem 5** For $n \geq 1$, $nC_3$ is $(a, 4)$-total neighborhood-antimagic if and only if $a = 6n+4$.

**Proof.** By Lemma 5, the necessity holds. To prove the sufficiency, we give a $(6n+4, 2)$-total neighborhood-antimagic of $nC_3$.

Suppose $n$ is even. Define a labeling $f: V(nC_3) \cup E(nC_3) \to [1, 6n]$ as follows.

(1) For $1 \leq i \leq \frac{n}{2}$, $f(v_{i,1}) = 2i-1$, $f(v_{i,2}) = 2n+2i-1$, $f(v_{i,3}) = 4n+2i-1$. The labels used are odd numbers in $[1, n] \cup [2n+1, 3n] \cup [4n+1, 5n]$.

(2) For $\frac{n}{2}+1 \leq i \leq n$, $f(v_{i,1}) = 2i-n$, $f(v_{i,2}) = n+2i$, $f(v_{i,3}) = 3n+2i$. The labels used are even numbers in $[1, n] \cup [2n+1, 3n] \cup [4n+1, 5n]$.

(3) For $1 \leq i \leq \frac{n}{2}$, $f(e_{i,1}) = 5n+2i-1$, $f(e_{i,2}) = n+2i-1$, $f(e_{i,3}) = 3n+2i-1$. The labels used are odd numbers in $[n+1, 2n] \cup [3n+1, 4n] \cup [5n+1, 6n]$.

(4) For $\frac{n}{2}+1 \leq i \leq n$, $f(e_{i,1}) = 4n+2i$, $f(e_{i,2}) = 2i$, $f(e_{i,3}) = 2n+2i$. The labels used are even numbers in $[n+1, 2n] \cup [3n+1, 4n] \cup [5n+1, 6n]$.

Clearly, $f$ is a bijective total labeling. Moreover, for $1 \leq i \leq n$, $WT(v_{i,1}) = f(v_{i,2}) + f(v_{i,3}) + f(e_{i,1}) + f(e_{i,3})$, $WT(v_{i,2}) = f(v_{i,1}) + f(v_{i,3}) + f(e_{i,1}) + f(e_{i,2})$ and $WT(v_{i,3}) = f(v_{i,1}) + f(v_{i,2}) + f(e_{i,2}) + f(e_{i,3})$.

Suppose $1 \leq i \leq \frac{n}{2}$.

(a) $WT(v_{i,1}) = (2n+2i-1) + (4n+2i-1) + (5n+2i-1) + (3n+2i-1) = 14n+8i-4$.

So $\{WT(v_{i,1}) \mid 1 \leq i \leq \frac{n}{2}\} = A([14n+4, 18n-4]; 8)$.

(b) $WT(v_{i,2}) = (2i-1) + (4n+2i-1) + (5n+2i-1) + (n+2i-1) = 10n+8i-4$.

So $\{WT(v_{i,2}) \mid 1 \leq i \leq \frac{n}{2}\} = A([10n+4, 14n-4]; 8)$.

(c) $WT(v_{i,3}) = (2i-1) + (2n+2i-1) + (n+2i-1) + (3n+2i-1) = 6n+8i-4$.

So $\{WT(v_{i,3}) \mid 1 \leq i \leq \frac{n}{2}\} = A([6n+4, 10n-4]; 8)$.

Suppose $\frac{n}{2}+1 \leq i \leq n$. Similarly, we have

(d) $\{WT(v_{i,1}) \mid \frac{n}{2}+1 \leq i \leq n\} = A([14n+8, 18n]; 8)$;

(e) $\{WT(v_{i,2}) \mid \frac{n}{2}+1 \leq i \leq n\} = A([10n+8, 14n]; 8)$;

(f) $\{WT(v_{i,3}) \mid \frac{n}{2}+1 \leq i \leq n\} = A([6n+8, 10n]; 8)$.

Thus, $f$ is a $(6n+4, 4)$-total neighborhood-antimagic labeling.

Suppose $n$ is odd. Define a labeling $f: V(nC_3) \cup E(nC_3) \to [1, 6n]$ as follows.

(1) For $1 \leq i \leq n$, $f(v_{i,1}) = 2i-1$, $f(v_{i,2}) = 2n+2i-1$, $f(v_{i,3}) = 4n+2i-1$. The labels used are odd numbers in $[1, 6n]$.

(2) For $1 \leq i \leq \frac{n+1}{2}$, $f(e_{i,1}) = 6n+4-4i$, $f(e_{i,2}) = 2n+4-4i$, $f(e_{i,3}) = 4n+4-4i$. The labels used are in $A([2, 2n]; 4) \cup A([2n+2, 4n]; 4) \cup A([4n+2, 6n]; 4)$.

(3) For $\frac{n+3}{2} \leq i \leq n$, $f(e_{i,1}) = 8n+4-4i$, $f(e_{i,2}) = 4n+4-4i$, $f(e_{i,3}) = 6n+4-4i$. The labels used are in $A([4, 2n-2]; 4) \cup A(2n+4, 4n-2]; 4) \cup A(4n+4, 6n-2]; 4)$.

Clearly, $f$ is a bijective total labeling. By a similar computation as above, we have

(a) $\{WT(v_{i,1}) \mid 1 \leq i \leq \frac{n+1}{2}\} = A([14n+4, 16n+2]; 4)$;

(b) $\{WT(v_{i,2}) \mid 1 \leq i \leq \frac{n+1}{2}\} = A([10n+4, 12n+2]; 4)$;

(c) $\{WT(v_{i,3}) \mid 1 \leq i \leq \frac{n+1}{2}\} = A([6n+4, 8n+2]; 4)$, and

(d) $\{WT(v_{i,1}) \mid \frac{n+3}{2} \leq i \leq n\} = A([16n+6, 18n]; 4)$;

(e) $\{WT(v_{i,2}) \mid \frac{n+3}{2} \le i \le n\} = A([12n+6, 14n]; 4)$;

(f) $\{WT(v_{i,3}) \mid \frac{n+3}{2} \le i \le n\} = A([8n+6, 10n]; 4)$.

Thus, $f$ is a $(6n+4, 4)$-total neighborhood-antimagic labeling. This completes the proof. □

**Theorem 6** For $n \ge 1$, $nC_3$ is $(a, 2)$-total neighborhood-antimagic if and only if $n$ is odd and $a = 9n+3$.

**Proof.** By Lemma 5, the necessity holds. To prove the sufficiency, we give a $(9n+3, 2)$-total neighborhood-antimagic of $nC_3$. Now consider odd $n \ge 1$. Define a labeling $f: V(nC_3) \cup E(nC_3) \to [1, 6n]$ as follows:

(1) For $1 \le i \le n$, $f(v_{i,1}) = i$, $f(v_{i,2}) = n+i$, $f(v_{i,3}) = 2n+i$. The labels used are in $[1, 3n]$.

(2) For $1 \le i \le \frac{n+1}{2}$, $f(e_{i,1}) = \frac{11n-1}{2} + i$, $f(e_{i,2}) = \frac{7n-1}{2} + i$, $f(e_{i,3}) = \frac{9n-1}{2} + i$. The labels used are in $[\frac{7n+1}{2}, 4n\} \cup [\frac{9n+1}{2}, 5n] \cup [\frac{11n+1}{2}, 6n]$.

(3) For $\frac{n+3}{2} \le i \le n$, $f(e_{i,1}) = \frac{9n-1}{2} + i$, $f(e_{i,2}) = \frac{5n-1}{2} + i$, $f(e_{i,3}) = \frac{7n-1}{2} + i$. The labels used are in $[3n+1, \frac{7n-1}{2}] \cup [4n+1, \frac{9n-1}{2}] \cup [5n+1, \frac{11n-1}{2}]$.

Clearly, $f$ is a bijective total labeling.

Suppose $1 \le i \le \frac{n+1}{2}$.

(a) $WT(v_{i,1}) = (n+i) + (2n+i) + (\frac{11n-1}{2} + i) + (\frac{9n-1}{2} + i) = 13n - 1 + 4i$.

So $\{WT(v_{i,1}) \mid 1 \le i \le \frac{n+1}{2}\} = A([13n+3, 15n+1]; 4)$.

(b) $WT(v_{i,2}) = \{i + (2n+i) + (\frac{11n-1}{2} + i) + (\frac{7n-1}{2} + i) = 11n - 1 + 4i$.

So $\{WT(v_{i,2}) \mid 1 \le i \le \frac{n+1}{2}\} = A([11n+3, 13n+1]; 4)$.

(c) $WT(v_{i,3}) = \{i + (n+i) + (\frac{7n-1}{2} + i) + (\frac{9n-1}{2} + i) = 9n - 1 + 4i$.

So $\{WT(v_{i,3}) \mid 1 \le i \le \frac{n+1}{2}\} = A([9n+3, 11n+1]; 4)$.

Suppose $\frac{n+3}{2} \le i \le n$. Similarly we have

(d) $\{WT(v_{i,1}) \mid \frac{n+3}{2} \le i \le n\} = A([13n+5, 15n-1]; 4)$,

(e) $\{WT(v_{i,2}) \mid \frac{n+3}{2} \le i \le n\} = A([11n+5, 13n-1]; 4)$,

(f) $\{WT(v_{i,3}) \mid \frac{n+3}{2} \le i \le n\} = A([9n+5, 11n-1]; 4)$.

Thus, $f$ is a $(9n+3, 2)$-total neighborhood-antimagic labeling. This completes the proof. □

**Theorem 7** $nC_3$ is $(a, 6)$-total neighborhood-antimagic if and only if $n \ge 3$ is odd and $a = 3n+5$.

**Proof.** By Lemmas 4 and 5, the necessity holds. To prove the sufficiency, we give a $(3n+5, 6)$-total neighborhood-antimagic of $nC_3$ for odd $n \ge 3$.

**Case (1).** $n = 6t+1 \ge 7$. We shall show that $nC_3$ admits an $(18t+8, 6)$-total neighborhood-antimagic labeling. Suppose $n = 7$. A required labeling is given in Figure 2 below.

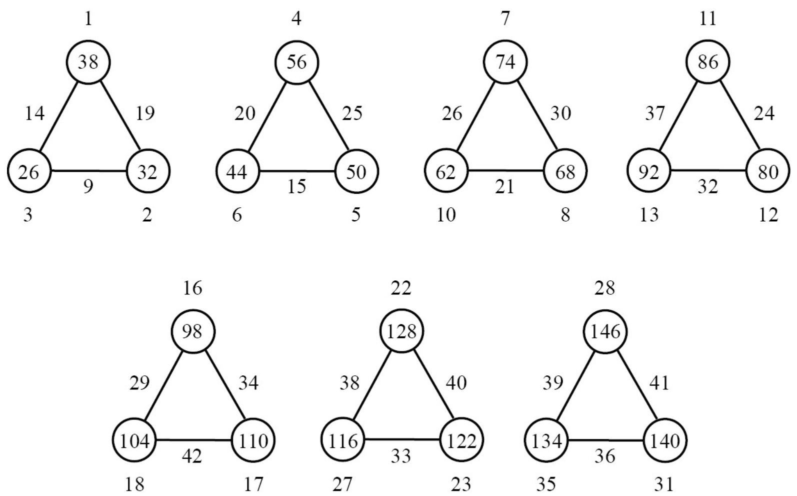


**Figure 2.** The total weights set is $A([26, 146]; 6)$

Suppose $n \geq 13$ (i.e., $t \geq 2$). We shall define a total labeling $f\colon V(nC_3) \cup E(nC_3) \to [1, 36t+6]$.
(a) Firstly, we label the vertices and edges of the first $3t+1$ $C_3$'s.
For vertices $v_{k,\,j}$, where $1 \leq k \leq 3t+1$ and $1 \leq j \leq 3$,

| $j$ | 1 | 2 | 3 |
|---|---|---|---|
| $f(v_{3i-2,\,j})$ | $9i-8$ | $9i-7$ | $9i-6$ |
| $f(v_{3i-1,\,j})$ | $9i-5$ | $9i-4$ | $9i-3$ |
| $f(v_{3i,\,j})$ | $9i-2$ | $9i-1$ | $9i$ |

here $1 \leq i \leq t-1$,

| $j$ | 1 | 2 | 3 |
|---|---|---|---|
| $f(v_{3t-2,\,j})$ | $9t-8$ | $9t-7$ | $9t-6$ |
| $f(v_{3t-1,\,j})$ | $9t-5$ | $9t-4$ | $9t-3$ |
| $f(v_{3t,\,j})$ | $9t-2$ | $9t+1$ | $9t+4$ |
| $f(v_{3t+1,\,j})$ | $9t-1$ | $9t+2$ | $9t+3$ |

The labels used are in

$$[1, 9t+4] \setminus \{9t\}. \tag{2}$$

For edges $e_{k,\,j}$, where $1 \leq k \leq 3t-3$ and $1 \leq j \leq 3$,

| $j$ | 2 | 3 | 1 |
|---|---|---|---|
| $f(e_{3i-2,\,j})$ | $9t+18i-18$ | $9t+18i-13$ | $9t+18i-8$ |
| $f(e_{3i-1,\,j})$ | $9t+18i-12$ | $9t+18i-7$ | $9t+18i-2$ |
| $f(e_{3i,\,j})$ | $9t+18i-6$ | $9t+18i-1$ | $9t+18i+4$ |

here $1 \leq i \leq t-1$. The labels used are in

| $j$ | 2 | 3 | 1 |
|---|---|---|---|
| $f(e_{3i-2,\,j})$ | $A([9t, 27t-36]; 18)$ | $A([9t+5, 27t-31]; 18)$ | $A([9t+10, 27t-26]; 18)$ |
| $f(e_{3i-1,\,j})$ | $A([9t+6, 27t-30]; 18)$ | $A([9t+11, 27t-25]; 18)$ | $A([9t+16, 27t-20]; 18)$ |
| $f(e_{3i,\,j})$ | $A([9t+12, 27t-24]; 18)$ | $A([9t+17, 27t-19]; 18)$ | $A([9t+22, 27t-14]; 18)$ |

For edges $e_{k,\,j}$, where $3t-2 \leq k \leq 3t+1$ and $1 \leq j \leq 3$,

| $j$ | 2 | 3 | 1 |
|---|---|---|---|
| $f(e_{3t-2,\,j})$ | $27t-18$ | $27t-13$ | $27t-8$ |
| $f(e_{3t-1,\,j})$ | $27t-12$ | $27t-7$ | $27t-2$ |
| $f(e_{3t,\,j})$ | $27t-6$ | $27t-3$ | $27t$ |
| $f(e_{3t+1,\,j})$ | $27t+2$ | $27t+5$ | $27t+10$ |

Combining with the above edge labels, the labels used are in

$$\{27t+2,\, 27t-3\} \cup A([9t,\, 27t];\, 18) \cup A([9t+5,\, 27t+5];\, 18) \cup A([9t+6,\, 27t-12];\, 18)$$

$$\cup A([9t+10,\, 27t+10];\, 18) \cup A([9t+11,\, 27t-7];\, 18) \cup A([9t+12,\, 27t-6];\, 18) \tag{3}$$

$$\cup A([9t+16,\, 27t-2];\, 18) \cup A([9t+17,\, 27t-19];\, 18) \cup A([9t+22,\, 27t-14];\, 18).$$

(b) Secondly, we label the vertices and edges of the next $3t$ $C_3$'s. For vertices $v_{k,\, j}$, where $3t+2 \leq k \leq 6t+1$ and $1 \leq j \leq 3$,

| $j$ | 1 | 2 | 3 |
|---|---|---|---|
| $f(v_{3i-1,\, j})$ | $18i-9t-11$ | $18i-9t-10$ | $18i-9t-9$ |
| $f(v_{3i,\, j})$ | $18i-9t-5$ | $18i-9t-4$ | $18i-9t-3$ |
| $f(v_{3i+1,\, j})$ | $18i-9t+1$ | $18i-9t+2$ | $18i-9t+3$ |

here $t+1 \leq i \leq 2t-1$,

| $j$ | 1 | 2 | 3 |
|---|---|---|---|
| $f(v_{6t-1,\, j})$ | $27t-11$ | $27t-10$ | $27t-9$ |
| $f(v_{6t,\, j})$ | $27t-5$ | $27t-4$ | $27t-1$ |
| $f(v_{6t+1,\, j})$ | $27t+1$ | $27t+4$ | $27t+7$ |

The labels used are in

$$A([9t+7,\, 27t+7];\, 18) \cup A([9t+8,\, 27t-10];\, 18) \cup A([9t+9,\, 27t-9];\, 18)$$

$$\cup A([9t+13,\, 27t-5];\, 18) \cup A([9t+14,\, 27t-4];\, 18) \cup A([9t+15,\, 27t-21];\, 18) \tag{4}$$

$$\cup A([9t+19,\, 27t+1];\, 18) \cup A([9t+20,\, 27t-16];\, 18) \cup A([9t+21,\, 27t-15];\, 18)$$

$$\cup \{27t-1,\, 27t+4\}.$$

Combining (3) and (4) we have the label used in

$$[9t,\, 27t+2] \cup \{27t+4,\, 27t+5,\, 27t+7,\, 27t+10\}. \tag{5}$$

For edges $e_{k,\, j}$, where $3t+2 \leq k \leq 6t+1$ and $1 \leq j \leq 3$,

| $j$ | 2 | 3 | 1 |
|---|---|---|---|
| $f(e_{3i-1,\,j})$ | $18t+9i-6$ | $18t+9i-1$ | $18t+9i+4$ |
| $f(e_{3i,\,j})$ | $18t+9i-3$ | $18t+9i+2$ | $18t+9i+7$ |
| $f(e_{3i+1,\,j})$ | $18t+9i$ | $18t+9i+5$ | $18t+9i+10$ |

here $t+1 \le i \le 2t-1$,

| $j$ | 2 | 3 | 1 |
|---|---|---|---|
| $f(e_{6t-1,\,j})$ | $36t-6$ | $36t-1$ | $36t+4$ |
| $f(e_{6t,\,j})$ | $36t-3$ | $36t+2$ | $36t+5$ |
| $f(e_{6t+1,\,j})$ | $36t$ | $36t+3$ | $36t+6$ |

The labels used are in

$$A([27t+3,\,36t+3];\,9) \cup A([27t+6,\,36t+6];\,9) \cup A([27t+8,\,36t-1];\,9)$$

$$\cup A([27t+9,\,36t];\,9) \cup A([27t+11,\,36t+2];\,9) \cup A([27t+13,\,36t+4];\,9)$$

$$\cup A([27t+14,\,36t+5];\,9) \cup A([27t+16,\,36t-2];\,9) \cup A([27t+19,\,36t+1];\,9)$$

$$= [27t+11,\,36t+6] \cup \{27t+3,\,27t+6,\,27t+8,\,27t+9\}. \tag{6}$$

Combining (2), (5) and (6), we see that all labels in $[1,\,36t+6]$ are used.

We shall now determine the total weights of the vertices.

1) Consider $1 \le i \le t-1$. The vertices of the $(3i-2)$-nd $C_3$ have total weights $18t+54i-46$, $18t+54i-40$ and $18t+54i-34$. The vertices of the $(3i-1)$-st $C_3$ have total weights $18t+54i-28$, $18t+54i-22$ and $18t+54i-16$. The vertices of the $(3i)$-th $C_3$ have total weights $18t+54i-10$, $18t+54i-4$ and $18t+54i+2$. So the total weights set for these vertices is $A([18t+8,\,72t-52];\,6)$.

2) For vertices in the $(3t-2)$-nd to the $(3t+1)$-st $C_3$, one may check that their total weights set is $A([72t-46,\,72t+20];\,6)$.

3) Consider $t+1 \le i \le 2t-1$. The vertices of the $(3i-1)$-st $C_3$ have total weights $18t+54i-28$, $18t-54i-22$ and $18t+54i-16$. The vertices of the $(3i)$-th $C_3$ have total weights $18t+54i-10$, $18t+54i-4$ and $18t+54i+2$. The vertices of the $(3i+1)$-st $C_3$ have total weights $18t+54i+8$, $18t+54i+14$ and $18t+54i+20$. So the total weights set for these vertices is $A([72t+26,\,126t-34];\,6)$.

4) For vertices in the $(6t-1)$-st to $(6t+1)$-st $C_3$, one may check that their total weights set is $A([126t-28,\,126t+20];\,6)$.

Thus, $f$ is an $(18t+8,\,6)$-total neighborhood-antimagic labeling.

**Case (2).** $n = 6t+3 \ge 3$. We shall show that $nC_3$ admits an $(18t+14,\,6)$-total neighborhood-antimagic labeling. Suppose $n = 3$. A required labeling is given in Figure 3 below.

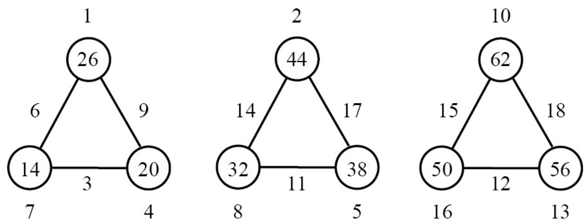


**Figure 3.** The total weights set is $A([14,\ 62];\ 6)$

Suppose $n = 9$. A required labeling is given in Figure 4 below.

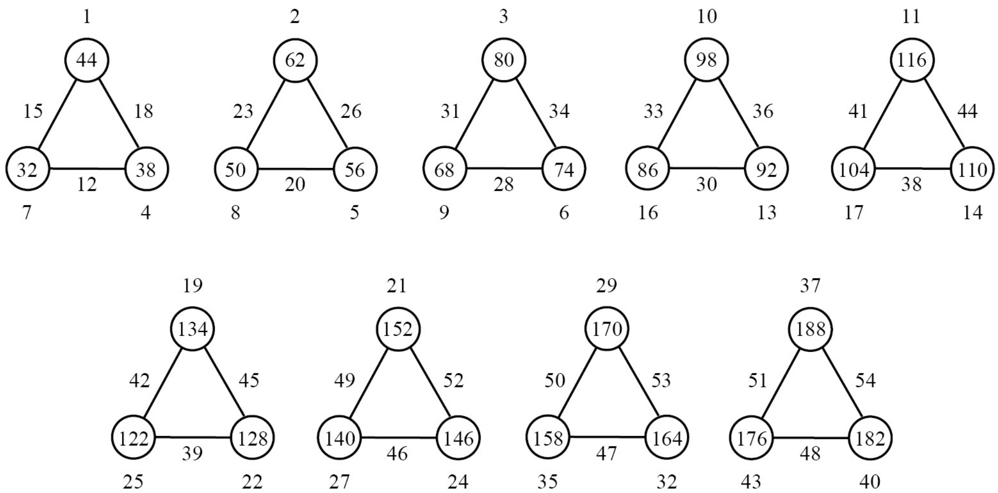


**Figure 4.** The total weights set is $A([32,\ 188];\ 6)$

Suppose $n \geq 15$ (or $t \geq 2$). We shall define a total labeling $f\colon V(nC_3) \cup E(nC_3) \to [1,\ 36t+18]$.
(a) Firstly, we label the vertices and edges of the first $3t+3$ $C_3$'s.
For vertices $v_{k,\ j}$, where $1 \leq k \leq 3t+3$ and $1 \leq j \leq 3$,

| $j$ | 1 | 2 | 3 |
|---|---|---|---|
| $f(v_{3i-2,\ j})$ | $9i-8$ | $9i-5$ | $9i-2$ |
| $f(v_{3i-1,\ j})$ | $9i-7$ | $9i-4$ | $9i-1$ |
| $f(v_{3i,\ j})$ | $9i-6$ | $9i-3$ | $9i$ |

here $1 \leq i \leq t$,

| $j$ | 1 | 2 | 3 |
|---|---|---|---|
| $f(v_{3t+1,\ j})$ | $9t+1$ | $9t+4$ | $9t+7$ |
| $f(v_{3t+2,\ j})$ | $9t+2$ | $9t+5$ | $9t+8$ |
| $f(v_{3t+3,\ j})$ | $9t+10$ | $9t+13$ | $9t+16$ |

The set of all vertex labels of the first $3t+3$ $C_3$'s is

$$[1,\, 9t+16] \setminus \{9t+3,\, 9t+6,\, 9t+9,\, 9t+11,\, 9t+12,\, 9t+14,\, 9t+15\}. \tag{7}$$

For edges $e_{k,\, j}$, where $1 \le k \le 3t+3$ and $1 \le j \le 3$,

| $j$ | 2 | 3 | 1 |
|---|---|---|---|
| $f(e_{3i-2,\, j})$ | $9t+18i-15$ | $9t+18i-12$ | $9t+18i-9$ |
| $f(e_{3i-1,\, j})$ | $9t+18i-7$ | $9t+18i-4$ | $9t+18i-1$ |
| $f(e_{3i,\, j})$ | $9t+18i+1$ | $9t+18i+4$ | $9t+18i+7$ |

here $1 \le i \le t$,

| $j$ | 2 | 3 | 1 |
|---|---|---|---|
| $f(e_{3t+1,\, j})$ | $27t+3$ | $27t+6$ | $27t+9$ |
| $f(e_{3t+2,\, j})$ | $27t+11$ | $27t+14$ | $27t+17$ |
| $f(e_{3t+3,\, j})$ | $27t+12$ | $27t+15$ | $27t+18$ |

The edge labels used in the first $3t$ $C_3$'s are in

| $j$ | 2 | 3 | 1 |
|---|---|---|---|
| $f(e_{3i-2,\, j})$ | $A([9t+3,\, 27t-15];\, 18)$ | $A([9t+6,\, 27t-12];\, 18)$ | $A([9t+9,\, 27t-9];\, 18)$ |
| $f(e_{3i-1,\, j})$ | $A([9t+11,\, 27t-7];\, 18)$ | $A([9t+14,\, 27t-4];\, 18)$ | $A([9t+17,\, 27t-1];\, 18)$ |
| $f(e_{3i,\, j})$ | $A([9t+19,\, 27t+1];\, 18)$ | $A([9t+22,\, 27t+4];\, 18)$ | $A([9t+25,\, 27t+7];\, 18)$ |

here $1 \le i \le t$.

Combining with the next 3 $C_3$'s, the set of all edge labels of the first $3t+3$ $C_3$'s is

$$\begin{aligned} & A([9t+3,\, 27t+3];\, 18) \cup A([9t+6,\, 27t+6];\, 18) \cup A([9t+9,\, 27t+9];\, 18) \\ & \cup A([9t+11,\, 27t+11];\, 18) \cup A([9t+14,\, 27t+14];\, 18) \cup A([9t+17,\, 27t+17];\, 18) \\ & \cup A([9t+19,\, 27t+1];\, 18) \cup A([9t+22,\, 27t+4];\, 18) \cup A([9t+25,\, 27t+7];\, 18) \\ & \cup \{27t+12,\, 27t+15,\, 27t+18\} \end{aligned} \tag{8}$$

(b) Secondly, we label the vertices and edges of the last $3t$ $C_3$'s.
For vertices $v_{k,\, j}$ and edges $e_{k,\, j}$ , where $3t+4 \le k \le 6t+3$ and $1 \le j \le 3$,

| $j$ | 1 | 2 | 3 |
|---|---|---|---|
| $f(v_{3i-2,\,j})$ | $18i-9t-24$ | $18i-9t-21$ | $18i-9t-18$ |
| $f(v_{3i-1,\,j})$ | $18i-9t-16$ | $18i-9t-13$ | $18i-9t-10$ |
| $f(v_{3i,\,j})$ | $18i-9t-8$ | $18i-9t-5$ | $18i-9t-2$ |

here $t+2 \le i \le 2t+1$,

| $j$ | 2 | 3 | 1 |
|---|---|---|---|
| $f(e_{3i-2,\,j})$ | $18t+9i+1$ | $18t+9i+4$ | $18t+9i+7$ |
| $f(e_{3i-1,\,j})$ | $18t+9i+2$ | $18t+9i+5$ | $18t+9i+8$ |
| $f(e_{3i,\,j})$ | $18t+9i+3$ | $18t+9i+6$ | $18t+9i+9$ |

here $t+2 \le i \le 2t+1$.

From the very top array above, we have that the labels used are in

| $j$ | 1 | 2 | 3 |
|---|---|---|---|
| $f(v_{3i-2,\,j})$ | $A([9t+12,\, 27t-6];\, 18)$ | $A([9t+15,\, 27t-3];\, 18)$ | $A([9t+18,\, 27t];\, 18)$ |
| $f(v_{3i-1,\,j})$ | $A([9t+20,\, 27t+2];\, 18)$ | $A([9t+23,\, 27t+5];\, 18)$ | $A([9t+26,\, 27t+8];\, 18)$ |
| $f(v_{3i,\,j})$ | $A([9t+28,\, 27t+10];\, 18)$ | $A([9t+31,\, 27t+13];\, 18)$ | $A([9t+34,\, 27t+16];\, 18)$ |

here $t+2 \le i \le 2t+1$.

Combining the above array with (7) and (8), we see that the labels used are in $[1,\, 27t+18]$.

From the second top array, we have that the labels used are in

| $j$ | 2 | 3 | 1 |
|---|---|---|---|
| $f(e_{3i-2,\,j})$ | $A([27t+19,\, 36t+10];\, 9)$ | $A([27t+22,\, 36t+13];\, 9)$ | $A([27t+25,\, 36t+16];\, 9)$ |
| $f(e_{3i-1,\,j})$ | $A([27t+20,\, 36t+11];\, 9)$ | $A([27t+23,\, 36t+14];\, 9)$ | $A([27t+26,\, 36t+17];\, 9)$ |
| $f(e_{3i,\,j})$ | $A([27t+21,\, 36t+12];\, 9)$ | $A([27t+24,\, 36t+15];\, 9)$ | $A([27t+27,\, 36t+18];\, 9)$ |

here $t+2 \le i \le 2t+1$.

Thus the set of edge labels of the last $3t$ $C_3$'s is $[27t+19,\, 36t+18]$.

Thus, $f$ is bijective. We shall now determine the total weights of the vertices.

1) For $1 \le i \le t$, the vertices of the $(3i-2)$-nd $C_3$ have total weights $18t+54i-40$, $18t+54i-34$ and $18t+54i-28$. The vertices of the $(3i-1)$-st $C_3$ have total weights $18t+54i-22$, $18t+54i-16$ and $18t+54i-10$. The vertices of the $(3i)$-th $C_3$ have total weights $18t+54i-4$, $18t+54i+2$ and $18t+54i+8$. So the total weights set is $A([18t+14,\, 72t+8];\, 6)$.

2) For the $(3t+1)$-st to $(3t+3)$-rd $C_3$, their total weights set is $A([72t+14,\, 72t+62];\, 6)$.

3) For $t+2 \le i \le 2t+1$, the vertices of the $(3i-2)$-nd $C_3$ have total weights $18t+54i-40$, $18t+54i-34$ and $18t+54i-28$. The vertices of the $(3i-1)$-st $C_3$ have total weights $18t+54i-22$, $18t+54i-16$ and $18t+54i-10$. The vertices of the $(3i)$-th $C_3$ have total weights $18t+54i-4$, $18t+54i+2$ and $18t+54i+8$. So, the total weights set is $A([72t+68,\, 126t+8];\, 6)$.

Thus, $f$ is an $(18t+14,\, 6)$-total neighborhood-antimagic labeling.

**Case (3).** $n = 6t+5 \ge 5$. We shall show that $nC_3$ admits an $(18t+20,\, 6)$-total neighborhood-antimagic labeling. A required labeling for $n = 5$ and $n = 11$ are given in Figure 5 and Figure 6 below.

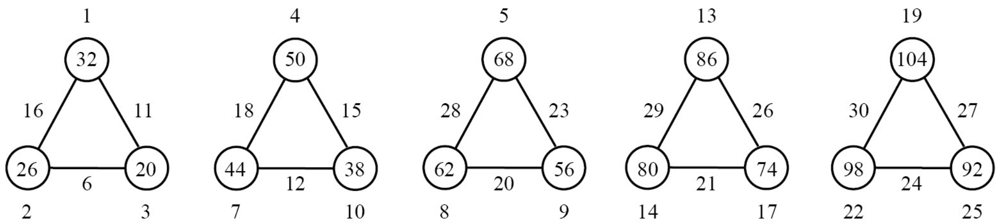


**Figure 5.** The total weights set is $A([20,\ 104];\ 6)$

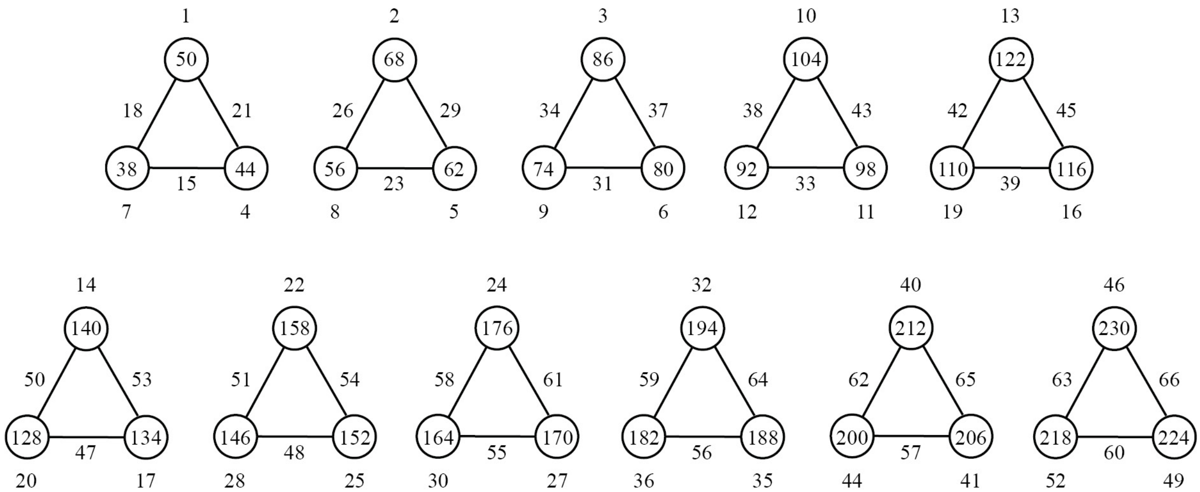


**Figure 6.** The total weights set is $A([38,\ 230];\ 6)$

Consider $n \geq 17$ (or $t \geq 2$). We shall define a total labeling $f$: $V(nC_3) \cup E(nC_3) \to [1,\ 36t+30]$.
(a) Firstly, we label the vertices and edges of the first $3t+4$ $C_3$'s.
For vertices $v_{k,\ j}$, where $1 \leq k \leq 3t+4$ and $1 \leq j \leq 3$,

| $j$ | 1 | 2 | 3 |
|---|---|---|---|
| $f(v_{3i-2,\ j})$ | $9i-8$ | $9i-5$ | $9i-2$ |
| $f(v_{3i-1,\ j})$ | $9i-7$ | $9i-4$ | $9i-1$ |
| $f(v_{3i,\ j})$ | $9i-6$ | $9i-3$ | $9i$ |

here $1 \leq i \leq t$,

| $j$ | 1 | 2 | 3 |
|---|---|---|---|
| $f(v_{3t+1,\ j})$ | $9t+1$ | $9t+2$ | $9t+3$ |
| $f(v_{3t+2,\ j})$ | $9t+4$ | $9t+7$ | $9t+10$ |
| $f(v_{3t+3,\ j})$ | $9t+5$ | $9t+8$ | $9t+11$ |
| $f(v_{3t+4,\ j})$ | $9t+13$ | $9t+16$ | $9t+19$ |

The set of all vertex labels of the first $3t+4$ $C_3$'s is

$$[1,\, 9t+19] \setminus \{9t+6,\, 9t+9,\, 9t+12,\, 9t+14,\, 9t+15,\, 9t+17,\, 9t+18\}. \tag{9}$$

For edges $e_{k,\, j}$, where $1 \le k \le 3t+3$ and $1 \le j \le 3$,

| $j$ | 2 | 3 | 1 |
|---|---|---|---|
| $f(e_{3i-2,\, j})$ | $9t+18i-12$ | $9t+18i-9$ | $9t+18i-6$ |
| $f(e_{3i-1,\, j})$ | $9t+18i-4$ | $9t+18i-1$ | $9t+18i+2$ |
| $f(e_{3i,\, j})$ | $9t+18i+4$ | $9t+18i+7$ | $9t+18i+10$ |

here $1 \le i \le t$,

| $j$ | 2 | 3 | 1 |
|---|---|---|---|
| $f(e_{3t+1,\, j})$ | $27t+6$ | $27t+11$ | $27t+16$ |
| $f(e_{3t+2,\, j})$ | $27t+12$ | $27t+15$ | $27t+18$ |
| $f(e_{3t+3,\, j})$ | $27t+20$ | $27t+23$ | $27t+26$ |
| $f(e_{3t+4,\, j})$ | $27t+21$ | $27t+24$ | $27t+27$ |

The edge labels used in the first $3t$ $C_3$'s are in

| $j$ | 2 | 3 | 1 |
|---|---|---|---|
| $f(e_{3i-2,\, j})$ | $A([9t+6,\, 27t-12];\, 18)$ | $A([9t+9,\, 27t-9];\, 18)$ | $A([9t+12,\, 27t-6];\, 18)$ |
| $f(e_{3i-1,\, j})$ | $A([9t+14,\, 27t-4];\, 18)$ | $A([9t+17,\, 27t-1];\, 18)$ | $A([9t+20,\, 27t+2];\, 18)$ |
| $f(e_{3i,\, j})$ | $A([9t+22,\, 27t+4];\, 18)$ | $A([9t+25,\, 27t+7];\, 18)$ | $A([9t+28,\, 27t+10];\, 18)$ |

here $1 \le i \le t$.

Combining with the next 4 $C_3$'s, the set of all edge labels of the first $3t+4$ $C_3$'s is

$$\begin{aligned} &A([9t+6,\, 27t+24];\, 18) \cup A([9t+9,\, 27t-9];\, 18) \cup A([9t+12,\, 27t+12];\, 18) \\ &\cup A([9t+14,\, 27t+-4];\, 18) \cup A([9t+17,\, 27t-1];\, 18) \cup A([9t+20,\, 27t+20];\, 18) \\ &\cup A([9t+22,\, 27t+4];\, 18) \cup A([9t+25,\, 27t+7];\, 18) \cup A([9t+28,\, 27t+10];\, 18) \\ &\cup \{27t+11,\, 27t+15,\, 27t+16,\, 27t+18,\, 27t+21,\, 27t+23,\, 27t+26,\, 27t+27\}. \end{aligned} \tag{10}$$

(b) Secondly, we label the vertices and edges of the last $3t+1$ $C_3$'s.
For vertices $v_{k,\, j}$, where $3t+5 \le k \le 6t+5$ and $1 \le j \le 3$,

| $j$ | 1 | 2 | 3 |
|---|---|---|---|
| $f(v_{3i-1,\, j})$ | $18i-9t-2$ | $18i-9t-18$ | $18i-9t-15$ |
| $f(v_{3i,\, j})$ | $18i-9t-13$ | $18i-9t-10$ | $18i-9t-7$ |
| $f(v_{3i+1,\, j})$ | $18i-9t-5$ | $18i-9t-2$ | $18i-9t+1$ |

here $t+2 \le i \le 2t$,

| $j$ | 1 | 2 | 3 |
|---|---|---|---|
| $f(v_{6t+2,\, j})$ | $27t-3$ | $27t$ | $27t+3$ |
| $f(v_{6t+3,\, j})$ | $27t+5$ | $27t+8$ | $27t+9$ |
| $f(v_{6t+4,\, j})$ | $27t+13$ | $27t+14$ | $27t+17$ |
| $f(v_{6t+5,\, j})$ | $27t+19$ | $27t+22$ | $27t+25$ |

The set of all vertex labels of the first $3t+1$ $C_3$'s is

$$A([9t+15,\, 27t-3];\, 18)\cup A([9t+18,\, 27t];\, 18)\cup A([9t+21,\, 27t+3];\, 18)$$

$$\cup A([9t+23,\, 27t+5];\, 18)\cup A([9t+26,\, 27t+8];\, 18)\cup A([9t+29,\, 27t-7];\, 18) \tag{11}$$

$$\cup A([9t+31,\, 27t+13];\, 18)\cup A([9t+34,\, 27t-2];\, 18)\cup A([9t+37,\, 27t+19];\, 18)$$

$$\cup \{27t+9,\, 27t+14,\, 27t+17,\, 27t+22,\, 27t+25\}.$$

Combining (9), (10) and (11), we see that the labels used in $[1,\, 27t+27]$.
For edges $e_{k,\, j}$, where $3t+5\le k\le 6t+5$ and $1\le j\le 3$,

| $j$ | 2 | 3 | 1 |
|---|---|---|---|
| $f(e_{3i-1,\, j})$ | $18t+9i+10$ | $18t+9i+13$ | $18t+9i+16$ |
| $f(e_{3i,\, j})$ | $18t+9i+11$ | $18t+9i+14$ | $18t+9i+17$ |
| $f(e_{3i+1,\, j})$ | $18t+9i+12$ | $18t+9i+15$ | $18t+9i+18$ |

here $t+2\le i\le 2t$,

| $j$ | 2 | 3 | 1 |
|---|---|---|---|
| $f(e_{6t+2,\, j})$ | $36t+19$ | $36t+22$ | $36t+25$ |
| $f(e_{6t+3,\, j})$ | $36t+20$ | $36t+23$ | $36t+28$ |
| $f(e_{6t+4,\, j})$ | $36t+21$ | $36t+26$ | $36t+29$ |
| $f(e_{6t+5,\, j})$ | $36t+24$ | $36t+27$ | $36t+30$ |

The set of all edge labels of the last $3t+1$ $C_3$'s is $[27t+28,\, 36t+30]$. Thus $f$ is bijective.

We shall now determine the total weights of the vertices.

1) For $1\le i\le t$, the vertices of the $(3i-2)$-nd $C_3$ have total weights $18t+54i-34$, $18t+54i-28$ and $18t+54i-22$. The vertices of the $(3i-1)$-st $C_3$ have total weights $18t+54i-16$, $18t+54i-10$ and $18t+54i-4$. The vertices of the $(3i)$-th $C_3$ have total weights $18t+54i+2$, $18t+54i+8$ and $18t+54i+14$. So the total weights set is $A([18t+20,\, 72t+14];\, 6)$.

2) For the $(3t+1)$-st to $(3t+4)$-th $C_3$, their total weights set is $A([72t+20,\, 72t+86];\, 6)$.

3) For $t+2\le i\le 2t$, the vertices of the $(3i-1)$-st $C_3$ have total weights $18t+54i-16$, $18t+54i-10$ and $18t+54i-4$. The vertices of the $(3i)$-th $C_3$ have total weights $18t+54i+2$, $18t+54i+8$ and $18t+54i+14$. The vertices of the $(3i+1)$-st $C_3$ have total weights $18t+54i+20$, $18t+54i+26$ and $18t+54i+32$. So the total weights set is $A([72t+92,\, 126t+32];\, 6)$.

4) For the $(6t+2)$-nd to $(6t+5)$-th $C_3$, their total weights set is $A([126t+38,\, 126t+104];\, 6)$.

Thus, $f$ is an $(18t+20,\, 6)$-total neighborhood-antimagic labeling. This completes the proof. □

As a by-product, we also have the following theorem on total neighborhood-magic.

**Theorem 8** For $n \geq 1$, $nC_3$ is total neighborhood-magic if and only if the magic constant is $12n+2$.

**Proof.** By Lemma 5 (i), the necessity holds. To prove the sufficiency, we give a total neighborhood-magic labeling of $nC_3$ with magic constant $12n+2$. For $1 \leq i \leq n$, define a labeling $f\colon V(nC_3) \cup E(nC_3) \to [1,\, 6n]$ as follows:

(1) $f(v_{i,\,1}) = 3i-2$, $f(v_{i,\,2}) = 3i-1$, $f(v_{i,\,3}) = 3i$,

(2) $f(e_{i,\,1}) = 6n-3i+1$, $f(e_{i,\,2}) = 6n-3i+3$, $f(e_{i,\,3}) = 6n-3i+2$.

Clearly, $f$ is a bijective total labeling. For $1 \leq i \leq n$,

$$WT(v_{i,\,1}) = (3i-1)+(3i)+(6n-3i+1)+(6n-3i+2) = 12n+2,$$

$$WT(v_{i,\,2}) = (3i-2)+(3i)+(6n-3i+1)+(6n-3i+3) = 12n+2,$$

$$WT(v_{i,\,3}) = (3i-2)+(3i-1)+(6n-3i+3)+(6n-3i+2) = 12n+2.$$

This completes the proof. □

### 3.2 $nC_4$

**Lemma 6** For $n \geq 1$, if $nC_4$ is $(a,\, d)$-total neighborhood-antimagic, then $d$ is even.

**Proof.** Suppose $nC_4$ is $(a,\, d)$-total neighborhood-antimagic. The sum of all the total weights is $\sum_{i=1}^{4n}[a+(n-1)d] = 2\sum_{i=1}^{8n} i$. So $2n[2a+(4n-1)d] = 8n(8n+1)$ or $2a+(4n-1)d = 4(8n+1)$. Thus, $2a-d \equiv 0 \pmod 4$. So $d$ is even. □

**Lemma 7** Suppose $nC_4$ is $(a,\, d)$-total neighborhood-antimagic, then

$$d \leq \begin{cases} 4 & \text{if } n = 1, \\ 6 & \text{if } n \geq 2. \end{cases}$$

**Proof.** From the proof above, we have $a \equiv \frac{d}{2} \pmod 2$. Combining with Lemma 4, we have the conclusion. □

**Theorem 9** For $n \geq 1$, $nC_4$ is $(a,\, 2)$-total neighborhood-antimagic if and only if $a = 12n+3$.

**Proof.** By Lemma 5, we have $a = 12n+3$. This proves the necessity.

We shall now give a $(12n+3,\, 2)$-total neighborhood-antimagic labeling of $nC_4$. Consider $f\colon V(nC_4) \cup E(nC_4) \to [1,\, 8n]$ as follows.

| $j$ | 1 | 2 | 3 | 4 | |
|---|---|---|---|---|---|
| $f(v_{i,\,j})$ | $6n+3-6i$ | $6n+2-6i$ | $6n-1+2i$ | $6n+2i$ | where $1 \leq i \leq n-1$ |
| $f(v_{n,\,j})$ | $8n$ | $8n-1$ | $2$ | $3$ | |
| $f(e_{i,\,j})$ | $6i-2$ | $6i-1$ | $6i$ | $6i-5$ | where $1 \leq i \leq n$ |

We have that the labels used are in

| $j$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $f(v_{i,j})$ | $A([9, 6n-3]; 6)$ | $A([8, 6n-4]; 6)$ | $A([6n+1, 8n-3]; 2)$ | $A([6n+2, 8n-2]; 2)$ |
| $f(v_{n,j})$ | $8n$ | $8n-1$ | $2$ | $3$ |
| $f(e_{i,j})$ | $A([4, 6n-2]; 6)$ | $A([5, 6n-1]; 6)$ | $A([6, 6n]; 6)$ | $A([1, 6n-5]; 6)$ |

Thus, all labels in $[1, 8n]$ are used and hence $f$ is a bijective total labeling.

We now determine the total weights of the vertices.

(a) For $1 \le i \le n-1$,

$$WT(v_{i,1}) = (6n+2-6i)+(6n+2i)+(6i-2)+(6i-5) = 12n+8i-5,$$

$$WT(v_{i,2}) = (6n+3-6i)+(6n+2i-1)+(6i-2)+(6i-1) = 12n+8i-1,$$

$$WT(v_{i,3}) = (6n+2-6i)+(6n+2i)+(6i-1)+(6i) = 12n+8i+1,$$

$$WT(v_{i,4}) = (6n+2i-1)+(6n+3-6i)+(6i)+(6i-5) = 12n+8i-3.$$

So the total weights set is

$$A([12n+3, 20n-13]; 8) \cup A([12n+7, 20n-9]; 8) \cup A([12n+9, 20n-7]; 8) \cup A([12n+5, 20n-11]; 8).$$

(b) $WT(v_{n,1}) = (8n-1)+3+(6n-2)+(6n-5) = 20n-5$, $WT(v_{n,2}) = 8n+2+(6n-2)+(6n-1) = 20n-1$, $WT(v_{n,3}) = (8n-1)+3+(6n-1)+6n = 20n+1$, $WT(v_{n,4}) = 8n+2+6n+(6n-5) = 20n-3$.

Hence, the total weights set is $\{20n-5, 20n-3, 20n-1, 20n+1\}$.

Clearly, the total weights set is $A([12n+3, 20n+1]; 2)$. Thus, $f$ is a $(12n+3, 2)$-total neighborhood-antimagic labeling. This completes the proof. □

## 4. Some one point union graphs

Let $H_i$ be a graph and $v_i \in V(H_i)$ be fixed, $1 \le i \le n$. A *one point union* of $H_i$, $1 \le i \le n$, is the graph obtained from the disjoint union of $H_i$ by merging all $v_i$ into a single vertex which is called the *merged vertex* or *core vertex*. We denote the one point union of $H_i$, $1 \le i \le n$, by $\biguplus_{i=1}^{n} H_i$ for $n \ge 2$.

Specially, if $H_i = P_2$, then $\biguplus_{i=1}^{n} H_i$, $n \ge 2$, is a star, denoted $St(n)$. If $H_i = C_3$, then $\biguplus_{i=1}^{n} H_i$, $n \ge 2$, is called a *friendship graph*, denoted $F_n$. For convenience, we let $St(1) = P_2$ and $F_1 = C_3$.

In this section, we investigate the one point union of the three families of graphs in Sections 2 and 3.

**Theorem 10** For $n \ge 1$, $St(n)$ is $(a, d)$-total neighborhood-antimagic if and only if

$$(n, a, d) \in \{(1, 3, 1), (1, 3, 2), (1, 4, 1), (2, 4, 4), (2, 6, 2), (2, 7, 2), (2, 8, 1)\}.$$

**Proof.** Let the vertex set and the edge set of $St(n)$ be $\{v_i \mid 1 \le i \le n\} \cup \{u\}$ and $\{e_i = uv_i \mid 1 \le i \le n\}$, respectively.

It is easy to verify case $n = 1$. Now, we consider $n \ge 2$. Suppose $St(n)$ admits an $(a, d)$-total neighborhood-antimagic labeling $f$: $V(St(n)) \cup E(St(n)) \to [1, 2n+1]$. Now,

$$WT(u) = \sum_{i=1}^{n}[f(v_i) + f(e_i)] = \sum_{i=1}^{2n+1} i - f(u) = (n+1)(2n+1) - f(u),$$

$$WT(v_i) = f(u) + f(e_i) \text{ for } 1 \le i \le n.$$

Thus, $WT(u) - WT(v_i) = (n+1)(2n+1) - 2f(u) - f(e_i) \ge (n+1)(2n+1) - 2(2n+1) - 2n = (n-1)(2n+1) - 2n > 0$. So, $WT(u) = a + nd$ is the largest total weight of all the vertices.

Without loss of generality, let $WT(v_i) = a + (i-1)d$ for $1 \le i \le n$. Thus, $d = f(e_{i+1}) - f(e_i)$ for $1 \le i < n$. Since all the values of $f(e_1), \ldots, f(e_n)$ are in $[1, 2n+1]$ and form an arithmetic progression with common difference $d$, we have $d \le 2$. Recall that $WT(u) = (n+1)(2n+1) - f(u)$. Since $WT(u) = a + nd$ and $a = WT(v_1) = f(u) + f(e_1)$, we now have $(n+1)(2n+1) - f(u) = f(u) + f(e_1) + nd$. Thus,

$$0 = (n+1)(2n+1) - 2f(u) - f(e_1) - nd$$

$$= (n-2)(2n+1) + 2[(2n+1) - f(u)] + [(2n+1) - f(e_1)] - nd$$

$$> (n-2)(2n+1) - nd > 2n(n-2) - nd.$$

So, $d > 2n - 4$. Therefore, $2n - 3 \le d \le 2$. Hence $n \le 2$. Consequently, $n = 2$ and $d = 1, 2$.

Suppose $St(2)$ admits an $(a, d)$-total neighborhood-antimagic $f$. Now the labels set is $[1, 5]$. Thus, the sum of all the total weights is $a + (a+d) + (a+2d) = 2f(u) + f(v_1) + f(v_2) + 2f(e_1) + 2f(e_2)$. This gives $3a + 3d = 2(1+2+3+4+5) - f(v_1) - f(v_2)$. Thus, $f(v_1) + f(v_2) \equiv 0 \pmod 3$. Therefore, we must have $f(v_1) + f(v_2) = 1+2$ or $2+4$ or $1+5$. By symmetry, we may assume that $f(v_1) < f(v_2)$.

(1) $f(v_1) = 1$, $f(v_2) = 2$.

If $f(u) = 5$, $f(e_1) = 3$ (or 4), $f(e_2) = 4$, (or 3), then the total weights of the vertices are 8, 9 and 10. So, $St(2)$ is (8, 1)-total neighborhood-antimagic.

If $f(u) = 4$, $f(e_1) = 3$, $f(e_2) = 5$, then the total weights of the vertices are 7, 9 and 11. So, $St(2)$ is (7, 2)-total neighborhood-antimagic.

If $f(u) = 3$, then it is easy to check that $f$ is not a required labeling.

(2) $f(v_1) = 2$, $f(v_2) = 4$.

If $f(u) = 5$, $f(e_1) = 1$ (or 3), $f(e_2) = 3$ (or 1), then the total weights of the vertices are 6, 8 and 10. So, $St(2)$ is (6, 2)-total neighborhood-antimagic.

If $f(u) = 3$, $f(e_1) = 1$ (or 5), $f(e_2) = 5$ (or 1), then the total weights of the vertices are 4, 8 and 12. So, $St(2)$ is (4, 4)-total neighborhood-antimagic.

If $f(u) = 1$, then it is easy to check that $f$ is not a required labeling.

(3) $f(v_1) = 1$, $f(v_2) = 5$. By enumeration, it is easy to check that $f$ is not a required labeling.

This completes the proof. □

**Theorem 11** For $n \ge 2$, $F_n$ is $(a, d)$-total neighborhood-antimagic if and only if $n = 2$ and $(a, d) \in \{(24, 3), (22, 4), (20, 5), (18, 6)\}$.

**Proof.** Let the vertex set and the edge set of $F_n$ be $\{c, u_i, v_i \mid 1 \leq i \leq n\}$ and $\{cu_i, cv_i, u_iv_i \mid 1 \leq i \leq n\}$, respectively. Suppose $F_n$ admits an $(a, d)$-total neighborhood-antimagic labeling $f$: $V(F_n) \cup E(F_n) \to [1, 5n+1]$. Without loss of generality, we may assume that $WT(u_1)$ is the largest total weight among all total weights of non-core vertices. Then $WT(c) = \sum_{i=1}^{n} [f(u_i) + f(v_i) + f(cu_i) + f(cv_i)]$ and $WT(u_1) = f(c) + f(v_1) + f(cu_1) + f(u_1v_1)$. Now,

$$WT(c) - WT(u_1) = \sum_{i=1}^{n} [f(u_i) + f(v_i) + f(cu_i) + f(cv_i)] - [f(c) + f(v_1) + f(cu_1) + f(u_1v_1)]$$

$$\geq [1 + 2 + \cdots + (4n-2)] - [5n + (5n+1)] = 8n(n-2) \geq 0.$$

Since $WT(c) \neq WT(u_1)$, $WT(c) > WT(u_1)$. Thus $WT(c) = a + 2nd$ is the largest total weight. From Theorem 1 (b), $a \geq 10$. Thus $WT(c) = a + 2nd \geq 10 + 2nd$. Moreover,

$$WT(c) = \sum_{i=1}^{n} [f(u_i) + f(v_i) + f(cu_i) + f(cv_i)]$$

$$= \sum_{i=1}^{5n+1} i - [f(c) + \sum_{i=1}^{n} f(u_iv_i)] \leq \frac{1}{2}(5n+1)(5n+2) - [1 + 2 + \cdots + (n+1)]$$

$$= 12n^2 + 6n.$$

Thus, $12n^2 + 6n \geq 10 + 2nd$. Hence $d \leq 6n + 3 - \frac{5}{n}$. Since $d = WT(c) - WT(u_1) \geq 8n(n-2)$, $6n + 3 - \frac{5}{n} \geq 8n(n-2)$ so that $8n^2 - 22n - 3 \leq -\frac{5}{n} < 0$. Thus $n \leq 2$. Under the hypothesis, $n = 2$.

Observe that the sum of total weights is $4f(c) + 2\sum_{i=1}^{2} [f(u_i) + f(v_i) + f(cu_i) + f(cv_i) + f(u_iv_i)] = \frac{5}{2}(2a + 4d) = 5(a + 2d)$. Thus, $2f(c) + 2(1 + 2 + \cdots + 11) = 5a + 10d$. So that

$$5a = 2f(c) + 132 - 10d. \tag{12}$$

Since $1 \leq f(c) \leq 11$ and $f(c) \equiv 4 \pmod 5$, $f(c) \in \{4, 9\}$. Also, from Theorem 1 (c), $d \leq 8$.

(1) Suppose $f(c) = 4$. We have $WT(c)$ is the sum of eight integers in $\{1, 2, 3, 5, 6, 7, 8, 9, 10, 11\}$ so that $WT(c) \geq 41$. Therefore, $(a, d, WT(c)) \in \{(14, 7, 42), (12, 8, 44)\}$. Thus, (i) $WT(c) = 1+2+3+5+6+7+8+10 = 42$ or (ii) $WT(c) = 1+2+3+5+6+8+9+10 = 44$ or (iii) $WT(c) = 1+2+3+5+6+7+9+11 = 44$.

In (i), $(a, d) = (14, 7)$ and $\{f(u_1v_1), f(u_2v_2)\} = \{9, 11\}$. Now $35 = a + 3d = WT(u_1) \leq 11 + 4 + f(v_1) + f(cu_1)$. This implies that $f(v_1) + f(cu_1) \geq 20$ which is not possible. In (ii) and (iii), by a similar argument, we have the same conclusion.

(2) Suppose $f(c) = 9$. Since $a$ is a total weight of a non-core vertex, $a$ is the sum of 9 and three integers in $[1, 11] \setminus \{9\}$. We have $a \geq 9 + 1 + 2 + 3 = 15$. From (12), $a + 2d = 30$. Thus $d \leq 7$. Since $a + 4d = WT(c) \geq 36$, $d \geq 3$. Thus $3 \leq d \leq 7$. Therefore, $(a, d, WT(c)) \in \{(24, 3, 36), (22, 4, 38), (20, 5, 40), (18, 6, 42), (16, 7, 44)\}$.

For $(a, d) = (24, 3)$, $(22, 4)$, $(20, 5)$, $(18, 6)$, a required $(a, d)$-total neighborhood-antimagic labeling is given in Figure 7.

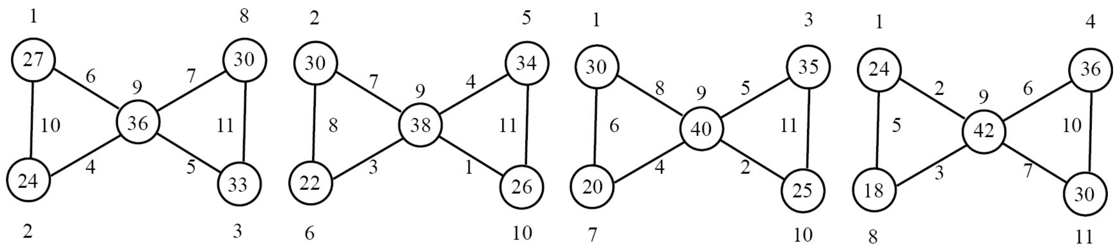


**Figure 7.** From left to right: $(a, d) = (24, 3)$, $(22, 4)$, $(20, 5)$, $(18, 6)$

Consider $a = 16$, $d = 7$. Now, $WT(c) = 44$. There are four cases: (i) $WT(c) = 1+2+3+4+5+8+10+11 = 44$; (ii) $WT(c) = 1+2+3+4+6+7+10+11 = 44$; (iii) $WT(c) = 1+2+4+5+6+7+8+11 = 44$; (iv) $WT(c) = 1+3+4+5+6+7+8+10 = 44$.

Since $37 = WT(u_1) = 9 + f(v_1) + f(cu_1) + f(u_1v_1) \le 9 + 11 + 10 + f(u_1v_1) = 30 + f(u_1v_1)$, $f(u_1v_1) \ge 7$.

In (i), $f(u_1v_1) = 7$ and $f(u_2v_2) = 6$. Thus, $a$ is at least the sum of 6, 9 and two integers in $\{1, 2, 3, 4, 5, 8, 10, 11\}$ which is greater than 16. It is a contradiction. In (ii) and (iii), by a similar argument, we have the same conclusion.

In (iv), we have $f(u_1v_1) = 11$ and $f(u_2v_2) = 2$. Hence $\{f(v_1), f(cu_1)\} = \{7, 10\}$. Note that, $a = 16$ is the sum of 9 and three integers in $[1, 11] \setminus \{9\}$. Thus $a = 9+1+2+4$ which is the smallest total weight of a non-core vertex. Without loss of generality, we may assume $WT(v_2) = a$. Hence $\{f(u_1), f(cv_2)\} = \{1, 4\}$.

Now, the unused integers are 3, 5, 6, 8. $WT(v_1) = f(c) + f(u_1v_1) + f(u_1) + f(cv_1) \ge 9 + 11 + 3 + 5 = 28$. Thus $WT(v_1) = 30$ and $f(u_1) + f(cv_1) = 10$ which is impossible.

This completes the proof. □

Let $H_i = C_4$. For $n \ge 2$, let $\{c, u_i, v_i, w_i \mid 1 \le i \le n\}$ and $\{cu_i, cw_i, u_iv_i, v_iw_i \mid 1 \le i \le n\}$ be the vertex set and edge set of $B_n = \biguplus_{i=1}^{n} H_i$, respectively.

**Theorem 12** For $n \ge 3$, $B_n$ is not $(a, d)$-total neighborhood-antimagic.

**Proof.** Suppose $B_n$ admits an $(a, d)$-total neighborhood-antimagic labeling $f$: $V(B_n) \cup E(B_n) \to [1, 7n+1]$. Then $WT(c) = \sum_{i=1}^{n} [f(u_i) + f(w_i) + f(cu_i) + f(cw_i)]$, $WT(u_i) = f(c) + f(v_i) + f(cu_i) + f(u_iv_i)$, $WT(v_i) = f(u_i) + f(w_i) + f(u_iv_i) + f(v_iw_i)$ and $WT(w_i) = f(c) + f(v_i) + f(cw_i) + f(v_iw_i)$. From Theorem 1 (b) and (c), we have $a \ge 10$ and $d \le \frac{20}{3}n + 2 - \frac{10}{3n}$.

Without loss of generality, we may assume that the largest total weight among the non-core vertex is $WT(u_1)$ or $WT(v_1)$. Now,

$$WT(c) - WT(u_1) \ge [1 + 2 + \cdots + (4n-1)] - [(7n+1) + 7n + (7n-1)] = n(8n-23), \tag{13}$$

$$WT(c) - WT(v_1) \ge [1 + 2 + \cdots + (4n-2)] - [(7n+1) + 7n] = 4n(2n-5) > n(8n-23). \tag{14}$$

Therefore, $WT(c)$ is the largest total weight for $n \ge 3$, and hence $WT(c) = a + 3nd$.

From (13) and (14), we get that $d \ge n(8n-23)$. Now $n(8n-23) \le d \le \frac{20n}{3} + 2 - \frac{10}{3n} < \frac{20n}{3} + 2$. It is equivalent to $24n^2 - 89n - 6 < 0$ or $(24n+7)(n-4) < -22$. Hence $n \le 3$. Thus $n = 3$.

Observe that the sum of the total weights is

$$5(2a+9d) = 6f(c) + 2\sum_{i=1}^{3}[f(u_i)+f(v_i)+f(w_i)+f(cu_i)+f(u_iv_i)+f(v_iw_i)+f(cw_i)]$$

$$= 4f(c) + 2\sum_{x\in V(B_3)\cup E(B_3)} f(x) = 4f(c) + 2\sum_{j=1}^{22} j = 4f(c) + 506.$$

Thus,

$$4f(c) + 506 = 10a + 45d. \tag{15}$$

So $d$ is even and $f(c) \equiv 1 \pmod 5$.

Suppose $WT(v_1)$ is the second large weight. From (14), we have $d \geq 12$. But $a+96 \leq a+8d = WT(v_1) \leq 22+21+20+19 = 82$ which is impossible. Thus $WT(u_1)$ is the second large weight.

Now $a+8d = WT(u_1) \leq 22+21+20+f(c) = f(c)+63$, we have $\frac{4f(c)+506-45d}{10} + 8d \leq f(c)+63$. This implies that

$$35d - 124 \leq 6f(c) \leq 126. \tag{16}$$

Hence $d < 8$.

Substituting (15) in $WT(c)$, we have $WT(c) = \frac{4f(c)+506+45d}{10}$. Then $78 = \sum_{i=1}^{12} i \leq WT(c) = \frac{4f(c)+506+45d}{10}$. This implies that $d > 4$. Consequently, $d = 6$ only.

From (16) we have $f(c) > 14$. Thus $f(c) = 16$, 21.

1) $f(c) = 16$. From (15), we have $a = 30$. Let $M$ be the maximum label that gives to $WT(c)$. Since $84 = WT(c) \geq 1+2+\cdots+11+M$, $M \leq 18$.

So, $78 = WT(u_1) = 16+f(v_1)+f(u_1v_1)+f(cu_1)$. Hence $f(v_1)+f(u_1v_1)+f(cu_1) = 62$. Thus, $\{f(v_1), f(u_1v_1), f(cu_1)\} = \{22, 21, 19\}$. Thus $f(cu_1) \geq 19$, which contradicts $M \leq 18$.

2) $f(c) = 21$. From (15), we have $a = 32$. Let $M$ be the maximum label contribute to $WT(c)$. Since $86 = WT(c) \geq 1+2+\cdots+11+M$, $M \leq 20$.

So, $80 = WT(u_1) = 21+f(v_1)+f(u_1v_1)+f(cu_1)$. Hence $f(v_1)+f(u_1v_1)+f(cu_1) = 59$. Thus, $\{f(v_1), f(u_1v_1), f(cu_1)\} = \{22, 20, 17\}$ or $\{22, 19, 18\}$. Since $M \leq 20$, $20 \geq f(cu_1) \geq 17$.

2-1) Suppose $f(cu_1) = 20$. Then $WT(c) - f(cu_1) = 66$. That means $\{f(u_j), f(w_j), f(cu_j), f(cw_j) \mid 1 \leq j \leq 3\} \setminus \{f(cu_1)\} = [1, 11]$. Now

$$WT(v_j) = f(u_j)+f(w_j)+f(w_jv_j)+f(u_jv_j) \leq 11+10+22+19 = 62,\ j = 1, 2, 3,$$

$$WT(w_j) = f(c)+f(v_j)+f(w_jv_j)+f(cw_j) \leq 21+22+19+11 = 73,\ j = 1, 2, 3,$$

$$WT(u_i) = f(c)+f(v_i)+f(u_iv_i)+f(cu_i) \leq 21+22+19+11 = 73,\ i = 2, 3.$$

No total weight is 74, the third largest total weight, a contradiction.

2-2) Suppose $f(cu_1) = 19$, i.e., $\{f(v_1),\, f(u_1v_1)\} = \{22,\, 18\}$. So, $WT(c) - f(cu_1) = 67$. That means $\{f(u_j),\, f(w_j),\, f(cu_j),\, f(cw_j) \mid 1 \le j \le 3\} \setminus \{f(cu_1)\} = [1,\, 12] \setminus \{11\}$. Now

$$WT(v_j) = f(u_j) + f(w_j) + f(w_jv_j) + f(u_jv_j) \le 12 + 10 + 20 + 22 = 64,\; j = 1,\, 2,\, 3,$$

$$WT(w_1) = f(c) + f(v_1) + f(w_1v_1) + f(cw_1) \le 21 + 22 + 17 + 12 = 72,$$

$$WT(w_i) = f(c) + f(v_i) + f(w_iv_i) + f(cw_i) \le 21 + 20 + 17 + 12 = 70,\; i = 2,\, 3,$$

$$WT(u_i) = f(c) + f(v_i) + f(u_iv_i) + f(cu_i) \le 21 + 20 + 17 + 12 = 70,\; i = 2,\, 3.$$

Similarly, it is impossible.

2-3) Suppose $f(cu_1) = 17$, i.e., $\{f(v_1),\, f(u_1v_1)\} = \{22,\, 20\}$. So, $WT(c) - f(cu_1) = 69$. We may see that $\{f(u_j),\, f(w_j),\, f(cu_j),\, f(cw_j) \mid 1 \le j \le 3\} \setminus \{f(cu_1)\}$ is (i) $[1,\, 10] \cup \{14\}$, (ii) $[1,\, 9] \cup \{11,\, 13\}$ or (iii) $[1,\, 8] \cup \{10,\, 11,\, 12\}$. Thus $f(u_i) + f(w_i) \le 24$. Now

$$WT(v_j) = [f(u_j) + f(w_j)] + f(w_jv_j) + f(u_jv_j) \le 24 + 19 + 22 = 65,\; j = 1,\, 2,\, 3,$$

$$WT(w_i) = f(c) + f(v_i) + f(w_iv_i) + f(cw_i) \le 21 + 19 + 18 + 14 = 72,\; i = 2,\, 3,$$

$$WT(u_i) = f(c) + f(v_i) + f(u_iv_i) + f(cu_i) \le 21 + 19 + 18 + 14 = 72,\; i = 2,\, 3.$$

Thus, $WT(w_1)$ is the third largest total weight and $WT(v_i)$ is not the fourth largest total weight for each $i$. By symmetry, we may assume the fourth largest total weight is $WT(u_2)$. Now $53 = 74 - 21 = WT(w_1) - f(c) = f(v_1) + f(w_1v_1) + f(cw_1) \le 22 + 19 + f(cw_1) = 41 + f(cw_1)$. So $f(cw_1) \ge 12$. We consider the following three cases.

(i) $f(cw_1) = 14$. Thus $WT(w_1) = 21 + 20 + 19 + 14$ so that $WT(u_2) = f(c) + f(v_2) + f(u_2v_2) + f(cu_2) \le 21 + 18 + 16 + 10 = 65$.

(ii) $f(cw_1) = 13$. Thus $WT(w_1) = 21 + 22 + 18 + 13$ so that $WT(u_2) = f(c) + f(v_2) + f(u_2v_2) + f(cu_2) \le 21 + 19 + 16 + 11 = 67$.

(iii) $f(cw_1) = 12$. Thus $WT(w_1) = 21 + 22 + 19 + 12$ so that $WT(u_2) = f(c) + f(v_2) + f(u_2v_2) + f(cu_2) \le 21 + 18 + 16 + 11 = 66$.

We get a contraction for each case.

2-4) Suppose $f(cu_1) = 18$, i.e., $\{f(v_1),\, f(u_1v_1)\} = \{22,\, 19\}$. So $WT(c) - f(cu_1) = 68$. We may see that $\{f(u_j),\, f(w_j),\, f(cu_j),\, f(cw_j) \mid 1 \le j \le 3\} \setminus \{f(cu_1)\}$ is (i) $[1,\, 10] \cup \{13\}$, (ii) $[1,\, 9] \cup \{11,\, 12\}$. Thus $f(u_i) + f(w_i) \le 23$. Now

$$WT(v_j) = [f(u_j) + f(w_j)] + f(w_jv_j) + f(u_jv_j) \le 23 + 20 + 22 = 65,\; j = 1,\, 2,\, 3,$$

$$WT(w_i) = f(c) + f(v_i) + f(w_iv_i) + f(cw_i) \le 21 + 20 + 18 + 14 = 73,\; i = 2,\, 3,$$

$$WT(u_i) = f(c) + f(v_i) + f(u_iv_i) + f(cu_i) \le 21 + 20 + 18 + 14 = 73,\; i = 2,\, 3.$$

Thus, $WT(w_1)$ is the third largest total weight and $WT(v_i)$ is not the fourth largest total weight for each $i$. By symmetry, we may assume the fourth largest total weight is $WT(u_2)$. Now $53 = 74 - 21 = WT(w_1) - f(c) = f(v_1) + f(w_1v_1) + f(cw_1) \leq 22 + 20 + f(cw_1) = 42 + f(cw_1)$. So $f(cw_1) \geq 11$.

(i) In this case, $f(cw_1)$ must be 13. So, $f(v_1) + f(w_1v_1) = 40$. There is no solution since $f(cu_1) = 18$ and $f(c) = 21$.
(ii) In this case, $f(cw_1)$ is 11 or 12.

If $f(cw_1) = 11$, then $f(v_1) + f(w_1v_1) = 42$. Since $f(w_1v_1) \leq 20$, $f(v_1) = 22$ and $f(w_1v_1) = 20$. Then $WT(u_2) = f(c) + f(v_2) + f(u_2v_2) + f(cu_2) \leq 21 + 17 + 16 + 12 = 66$.

If $f(cw_1) = 12$, then $f(v_1) + f(w_1v_1) = 41$. Since $f(w_1v_1) \leq 20$ and $f(c) \in \{22, 19\}$, there is no solution.

This completes the proof. □

**Theorem 13** $B_2$ is $(a, d)$-total neighborhood-antimagic if and only if

$$(a, d) \in \{(33, 1), (30, 2), (27, 3), (24, 4), (21, 5), (18, 6), (15, 7),$$

$$(35, 1), (32, 2), (29, 3), (26, 4), (23, 5), (20, 6), (17, 7).\}$$

**Proof.** From the labelings in Figures 8 and 9, we get the sufficiency.

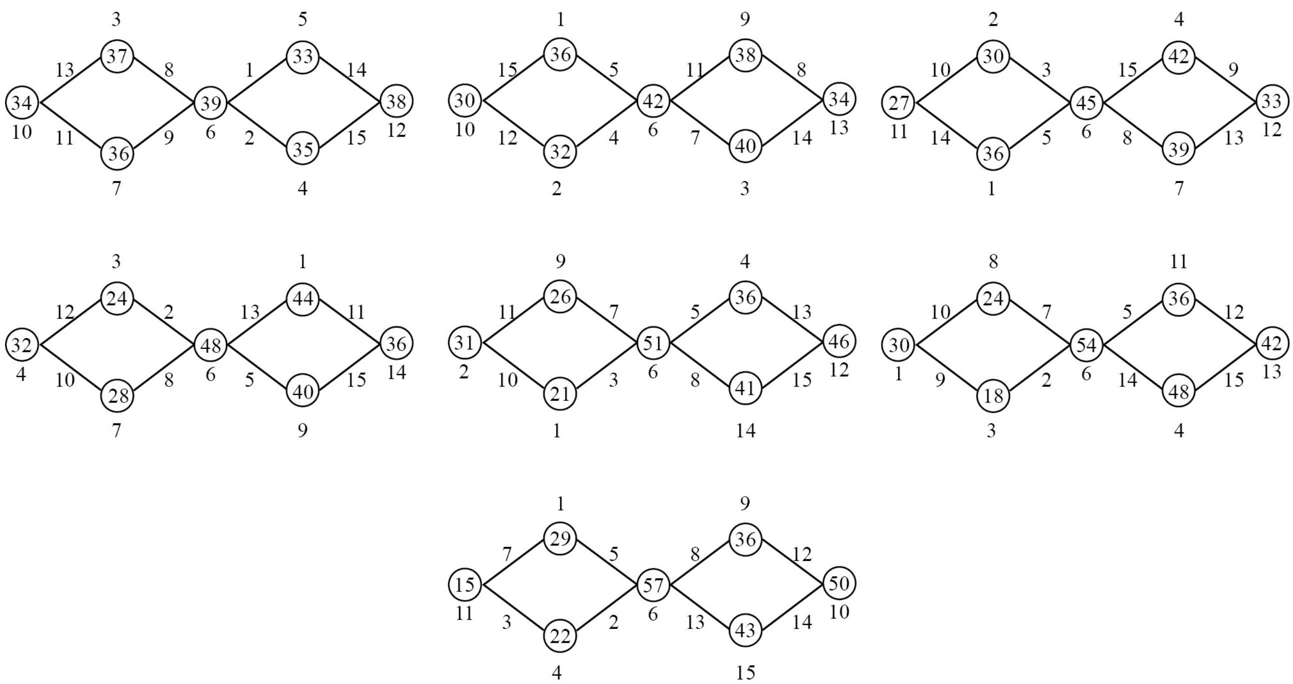


**Figure 8.** From left to right, and top to bottom: $(a, d) = (33, 1), (30, 2), (27, 3), (24, 4), (21, 5), (18, 6), (15, 7)$

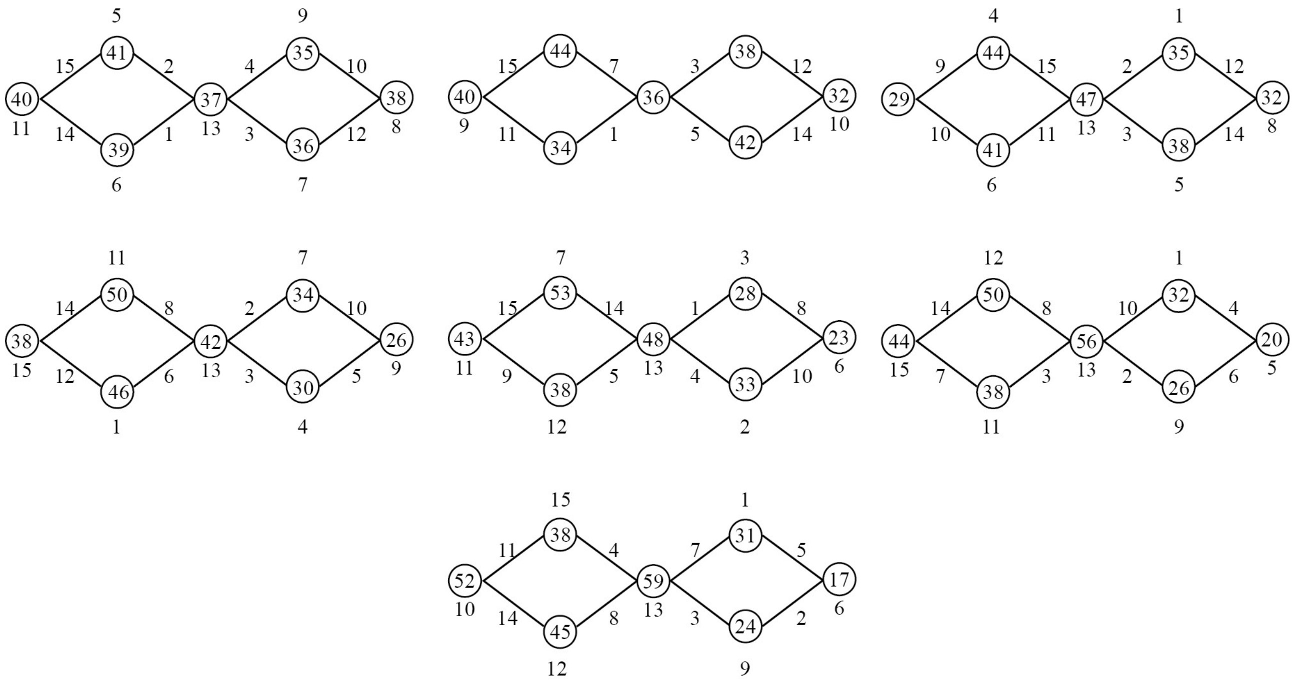


**Figure 9.** From left to right, and top to bottom: $(a, d) = (35, 1), (32, 2), (29, 3), (26, 4), (23, 5), (20, 6), (17, 7)$

We now prove the necessity. Similar to the proof of Theorem 12, calculating the sum of the total weights gives $2f(c)+2[1+2+\cdots+15] = 7(a+3d)$. Thus, $7a = 2f(c)+240-21d$ so that $f(c) \in \{6, 13\}$. Moreover, $a \geq 10$.

When $f(c) = 6$, we get $a = 36-3d \geq 10$ so that $d \leq 8$. Hence $(a, d) = (33, 1), (30, 2), (27, 3), (24, 4), (21, 5), (18, 6), (15, 7), (12, 8)$. Thus, it suffices to show that the case $(a, d) = (12, 8)$ does not exists.

When $(a, d) = (12, 8)$, the total weights are 12, 20, 28, 36, 44, 52, 60. Observe that for $i = 1, 2$, $WT(u_i), WT(w_i) \leq 6+13+14+15 = 48$ and $WT(v_i) \leq 12+13+14+15 = 54$. Therefore, $WT(c) = 60$. Without loss of generality, $WT(v_1) = 52 = 15+14+12+11$ or $15+14+13+10$. Now, the corresponding remaining vertex and edge labels set must be $[1, 5] \cup [7, 10] \cup \{13\}$ or $[1, 5] \cup [7, 9] \cup \{11, 12\}$. Moreover, $WT(u_1) \geq 10+6+2+1 = 19$. Similarly, $WT(w_1) \geq 19$. Thus the smallest weight, $a = 12$, is $WT(u_2)$, $WT(w_2)$ or $WT(v_2)$. By symmetry, we always assume that $WT(u_i) > WT(w_i)$ for $i = 1, 2$. Thus, $12 \in \{WT(w_2), WT(v_2)\}$. We have the following two cases.

If the smallest total weight is the sum of 1, 2, 3, 6, then only $WT(w_2) = a$. We get

$$WT(v_2) \leq \begin{cases} 3+13+10+9 = 35, \\ \\ 3+12+11+9 = 35, \end{cases} \quad \text{and} \quad WT(u_2) \leq \begin{cases} 6+13+10+3 = 32, \\ \\ 6+12+11+3 = 32. \end{cases}$$

Thus $\{WT(v_2), WT(u_2)\} = \{20, 28\}$.

If the smallest total weight is the sum of 1, 2, 4, 5, then only $WT(v_2) = a$. We get

$$WT(u_2),\ WT(w_2) \leq \begin{cases} 6+13+10+5 = 34, \\ \\ 6+12+11+5 = 34. \end{cases}$$

Thus $\{WT(u_2), WT(w_2)\} = \{20, 28\}$.

From the above cases, we obtain that $WT(u_1) = 44$ and $WT(w_1) = 36$.

(a) Suppose $WT(w_2) = 1+2+3+6$, i.e., $\{f(cw_2), f(v_2w_2), f(v_2)\} = \{1, 2, 3\}$.

a.1) Suppose $WT(v_1) = 15+14+12+11$. Since there is only the term $f(u_1v_1)$ of $WT(u_1)$ and $WT(v_1)$ in common, $WT(u_1) = 15+6+13+10$. Hence $f(u_1v_1) = 15$. Now, $\{f(v_1), f(cu_1)\} = \{13, 10\}$, $\{f(v_1w_1), f(u_1), f(w_1)\} = \{14, 12, 11\}$ and $f(cw_1) \in \{9, 8, 7, 5, 4\}$. So, $WT(w_1) = f(c) + f(v_1) + f(v_1w_1) + f(cw_1)$, where $(f(v_1), f(v_1w_1), f(cw_1)) = (10, 11, 9)$, $(10, 12, 8)$ or $(13, 12, 5)$. Thus, $\{f(cu_2), f(u_2v_2), f(u_2), f(w_2)\} = \{8, 7, 5, 4\}$, $\{9, 7, 5, 4\}$ or $\{9, 8, 7, 4\}$, respectively.

Therefore, $WT(v_2) \leq 9+8+7+3 = 27$ and $WT(u_2) \leq 6+9+8+3 = 26$, a contraction.

a.2) Suppose $WT(v_1) = 15+14+13+10$. Similarly, we have $WT(u_1) = 15+6+12+11$. Hence $f(u_1v_1) = 15$. Now, $\{f(v_1), f(cu_1)\} = \{12, 11\}$, $\{f(v_1w_1), f(u_1), f(w_1)\} = \{14, 13, 10\}$ and $f(cw_1) \in \{9, 8, 7, 5, 4\}$. Similarly, we have $(f(v_1), f(v_1w_1), f(cw_1)) = (12, 10, 8)$, $(12, 13, 5)$, $(12, 14, 4)$, $(11, 10, 9)$ or $(11, 14, 5)$. Similar to case (a.1), we have $WT(v_2) \leq 9+8+7+3 = 27$ and $WT(u_2) \leq 6+9+8+3 = 26$, a contraction.

(b) Suppose $WT(v_2) = 1+2+4+5$, i.e., $\{f(u_2), f(w_2), f(u_2v_2), f(v_2w_2)\} = \{1, 2, 4, 5\}$.

b.1) Suppose $WT(v_1) = 15+14+12+11$. By the same argument in case (a), we have $WT(u_1) = 15+6+13+10$ and $f(u_1v_1) = 15$. Now, $\{f(v_1), f(cu_1)\} = \{13, 10\}$, $\{f(v_1w_1), f(u_1), f(w_1)\} = \{14, 12, 11\}$ and $f(cw_1) \in \{9, 8, 7, 3\}$. So, $(f(v_1), f(v_1w_1), f(cw_1)) = (10, 11, 9)$, $(10, 12, 8)$ or $(13, 14, 3)$. Therefore, $\{f(cu_2), f(cw_2), f(v_2)\} = \{8, 7, 3\}$, $\{9, 7, 3\}$ or $\{9, 8, 7\}$, respectively.

When $\{f(cu_2), f(cw_2), f(v_2)\} = \{8, 7, 3\}$ or $\{9, 7, 3\}$, we have $WT(u_2), WT(w_2) \leq 6+8+7+5 = 27$, a contraction.

When $\{f(cu_2), f(cw_2), f(v_2)\} = \{9, 8, 7\}$, we have $WT(u_2), WT(w_2) \leq 6+9+8+5 = 28$. Thus, $28 = WT(u_2)$. Hence $\{f(cu_2), f(v_2)\} = \{9, 8\}$ and $f(u_2v_2) = 5$ so that $f(cw_2) = 7$. However, $WT(w_2) \geq 6+7+8+1 = 22$, a contradiction.

b.2) Suppose $WT(v_1) = 15+14+13+10$. Similarly, we have $WT(u_1) = 6+15+12+11$. Hence $f(u_1v_1) = 15$. Now, $\{f(v_1), f(cu_1)\} = \{12, 11\}$, $\{f(v_1w_1), f(u_1), f(w_1)\} = \{14, 13, 10\}$ and $f(cw_1) \in \{9, 8, 7, 3\}$. So, $(f(v_1), f(v_1w_1), f(cw_1)) = (12, 10, 8)$ or $(11, 10, 9)$. Similar to the above case, we get $WT(u_2), WT(w_2) \leq 6+9+7+5 = 27$, a contradiction.

When $f(c) = 13$, we get $a = 38 - 3d \geq 10$. So that $d \leq 9$. Therefore, $(a, d) \in \{(35, 1), (32, 2), (29, 3), (26, 4), (23, 5), (20, 6), (17, 7), (14, 8), (11, 9)\}$. It suffices to show that the cases $(a, d) = (14, 8)$ and $(a, d) = (11, 9)$ do not exist.

Consider $(a, d) = (11, 9)$. We must have $a = 1+2+3+5$. Since $f(c) = 13$, $WT(u_i), WT(w_i) \geq 19$ for $i = 1, 2$. Without loss of generality, let $WT(v_1) = f(u_1) + f(w_1) + f(u_1v_1) + f(v_1w_1) = 11$. Thus, $f(u_1v_1) \geq 1$, $f(v_1) + f(cu_1) \geq 4+6$ so that $WT(u_1) \geq 1+4+6+13 = 24 > 20$. Similarly, $WT(u_2), WT(w_1), WT(w_2), WT(c) > 20$. Therefore, total weight 20 does not exist, a contradiction.

Consider $(a, d) = (14, 8)$. The total weights are 14, 22, 30, 38, 46, 54, 62. Now $WT(v_i) \leq 15+14+12+11 = 52$ and $WT(u_i), WT(w_i) \leq 12+13+14+15 = 54$, for $i = 1, 2$. Thus $WT(c) = 62$. Without loss of generality, let $WT(u_1) = 54$. Thus, each remaining total weight is at most $15+11+10+9 = 45$. So that total weight 46 does not exist, a contradiction.

□

## 5. Conclusion and open problems

In this paper, we have obtained necessary and / or sufficient condition for 1-regular graphs, 2-regular graphs $nC_3$, $nC_4$, and one point union of $n \geq 2$ copies of $P_2$, $C_3$, $C_4$ respectively, to admit an $(a, d)$-total neighborhood-antimagic labeling. Particularly, (1) from Lemma 4, we know that if $nC_3$ is $(a, d)$-total neighborhood-antimagic, then $d \leq 7$; (2) from Lemma 5, we know that if $n$ is even and $nC_3$ is $(a, d)$-total neighborhood-antimagic, then $d \equiv 0 \pmod 4$. We have completely studied the cases when $d = 2, 4, 6$ from Theorem 3.5 to Theorem 3.7. The remaining cases is $d = 1, 3, 5, 7$. The following question and problems arise naturally.

**Question 1** Do there exist odd $n$, $d$ such that $nC_3$ is $(a, d)$-total neighborhood-antimagic?

**Problem 1** For $d = 1, 2$, determine all the possible $a$ such that $nP_2$ admits an $(a, d)$-total neighborhood-antimagic labeling.

**Problem 2** Study the $(a, d)$-total neighborhood-antimagic labelings of regular graphs.

We end this paper with the following conjecture.

**Conjecture 1** The one point union of cycles with order at least 5 is not $(a, d)$-total neighborhood-antimagic for all $a, d \geq 1$.

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## Confilict of interest

The authors declare no competing financial interest.

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