Contemporary Mathematics
https://ojs.wiserpub.com/index.php/CM/

UNIVERSAL WISER PUBLISHER

Research Article

Majorization-Based Conticrete Inequalities Involving the Caputo-Fabrizio Fractional Operators with Applications to Modified Bessel Functions and Special Means

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Received: 24 August 2025; **Revised:** 17 October 2025; **Accepted:** 24 October 2025

Abstract: The Hermite-Hadamard inequality is universally recognized as a highly influential inequality in mathematics. Nowadays, researchers are actively engaged in exploring its various improvements, generalizations and refinements. This article focuses on determining the results of Hermite-Hadamard-Mercer type in conticrete settings within a fractional framework. The approach combines the ideas of majorization, convexity, and Caputo-Fabrizio fractional operators. New weighted versions are also presented by employing certain monotonic tuples together with weighted majorized Jensen-Mercer inequalities. In addition, an integral identity is established for a differentiable function. This identity is further applied to obtain estimates for the discrepancy in terms related to the major result. The obtained bounds rely on the convex nature of $|f'|$, $|f'|^q$, $(1 < q)$, along with power mean, Hölder, and Young's inequalities. The paper further demonstrates applications of the main findings to modified Bessel functions and special means. Several existing results are recovered as special cases, while new inequalities are also established.

Keywords: Hermite-Hadamard inequality, Jensen's inequality, Mercer's inequality, Majorization, Caputo-Fabrizio fractional integral operators

MSC: 26D15, 26A51, 26A33, 26A42

1. Introduction

The theory of convexity has stood as one of the fundamental pillars of mathematical analysis for more than a century, shaping the study of extremal problems and influencing diverse areas of pure and applied mathematics [1, 2]. Its elegant structure and wide-ranging adaptability have made it a powerful analytical framework, equally valuable in theoretical explorations and in addressing problems across engineering [3], economics [4], and the physical sciences [5]. Over the years, mathematicians have explored numerous generalizations of convexity, each enriching the theory and expanding its applicability. For example, classical convex functions and their practical implications can be found in detail in [6], while Ostrowski type results have been extended to s-convex functions [7] and p-convex settings [8] . Further refinements include Geometric-Arithmetic means convexity [9] and broader generalized convexity frameworks, for which modified

DOI: https://doi.org/10.37256/cm.7120268367

integral inequalities have been established [10]. Convexity theory has also been adapted to hyperbolic geometry, leading to Hermite-Hadamard (H-H) inequalities for hyperbolic p-convex functions [11], and has been linked to higher-order expansions such as Steffensen's inequality and generalized Taylor's formula [12]. Developments in modern analysis have connected convexity with quantum calculus, producing quantum H-H inequalities via Green's functions [13], while coordinate-wise convexity has been employed to derive Hadamard-type inequalities for products of s-convex functions [14].

The strength of convexity lies not only in its own intrinsic properties but also in the way it seamlessly leads to inequality theory. Convex functions possess geometric and analytic features such as those reflected in Jensen's inequality that make them a natural tool for deriving sharp bounds, estimating functional values, and establishing stability results. This deep and inherent relationship ensures that many classical inequalities are, in essence, manifestations of convexity principles, while newer inequalities often arise from its modern generalizations. In this sense, convexity acts as a conceptual bridge, connecting pure mathematical structures with a vast array of applied problems.

From this foundation, inequalities emerge as a direct extension of convexity theory, growing into one of the versatile and broadly applicable branches of analysis. Integral inequalities, in particular, have far-reaching applications: they are instrumental in optimization theory for designing reliable algorithms, in probability and statistics for determining expectations and bounds, in information theory and economics for modeling uncertainty, and in mathematical finance for quantifying risk. Their relevance extends into engineering, data science, and physics, making them indispensable across disciplines.

Among the wide range of inequalities influenced by convexity, several occupy a central place in both theoretical development and practical application. Notably, Jensen's inequality [15, 16], Fejer inequality [17], Slater's inequality [18, 19], the H-H inequality [20], Ostrowski inequality [21], and majorization inequality [22], have each inspired substantial research activity, and lead to further refinements and extensions.

Fractional calculus is a well-established branch of mathematical analysis that extends the classical notions of differentiation and integration to arbitrary, non-integer orders [23, 24]. The concept was first introduced in 1695 by Leibniz and L'Hopital. It sparked curiosity among mathematicians for more than three centuries. Since then, it has evolved through the contributions of many prominent scholars. It experienced significant theoretical advancements in the nineteenth century and attracted renewed interest in recent decades. Its remarkable versatility has led to applications across numerous scientific and engineering domains, including geophysics [25],biology [26], mathematical analysis [27], engineering [28], control theory [29] and medicine [30]. Fractional calculus, unlike integer-order calculus, naturally considers memory effects and long-range dependencies. This property makes it a powerful tool for modeling processes influenced by their past states. Furthermore, it plays a pivotal role in signal processing, particularly in the study of self-similar and scale-invariant signals [31, 32].

One of the earliest and most influential developments in this area was the introduction of the Riemann-Liouville fractional integrals [33], which formed the basis for subsequent extensions and generalizations. In the present day, researchers frequently employ a variety of fractional operators, such as the Caputo [34], Hadamard [35], k-Caputo [36], Caputo-Fabrizio [37, 38], Katugampola [39], and Atangana-Baleanu [40] formulations. These operators offer distinct properties that aid in resolving diverse mathematical problems with greater precision and versatility.

Within this framework, the relationship between fractional calculus and convexity theory forms an active and compelling research direction. By bridging ideas from classical calculus and modern fractional analysis, this connection enables the formulation of models that incorporate non-local effects, thereby enriching both optimization theory and mathematical modeling [41–44]. Fractional operators have the capacity to modify the geometric properties of functions. As a result, they can produce generalized notions of convexity and concavity that extend beyond classical definitions [45, 46]. The past few years have witnessed significant progress in extending the classical Hermite-Hadamard inequality to the domian of fractional calculus, particularly through the use of fractional operators. Researchers have introduced a variety of generalized forms, applying them to both fractional integrals and derivatives. These contributions not only deepen the understanding of convex functions in fractional contexts but also offer directions for theoretical exploration and practical implementation.

Researchers commonly employ generalized convexity, generalized integrals, or a combination of both to establish inequalities in either the continuous or discrete setting, which are recognized as the two primary categories of inequalities. At this point, there is a growing need for concepts capable of producing inequalities that bridge the gap between these two settings. The theory of majorization satisfies this requirement. To fulfill this requirement, Faisal et al. [47–49] recently formulated generalized inequalities by means of the notions of convexity and majorization that unifies continuous and discrete versions.

Majorization represents a partial order structure involving two tuples. It measures how one tuple differs from another, or how closely the entries of one tuple approximate those of the other. This concept is useful for transforming complex optimization problems into simpler ones that can be solved more readily [50, 51]. Modern applications of majorization theory can be found in areas such as signal processing and communication [52, 53].

The primary aim of this work is to establish conticrete Hermite-Hadamard-Mercer inequalities by combining the concepts of convexity, Caputo-Fabrizio fractional integral operators, and majorization theory. For specific parameter choices, the proposed inequalities reduce to known results in the literature. In addition, the paper presents weighted extensions of the main results, formulates integral identities, and derives further consequences of these identities using Hölder, power mean, and Young's inequalities. Applications of the principal findings are also provided, including those related to Bessel functions and various special means.

The paper is structured as follows. Section 2 reviews the fundamental concepts and preliminary definitions. Section 3 establishes the two main results by employing fractional integrals, convexity, and majorization theory. In Section 4, weighted extensions of the principal findings are derived for certain monotonic tuples with the aid of majorized weighted Jensen-Mercer results presented in Lemma 1 and Lemma 2. Section 5 introduces an integral identity for a differentiable function, from which additional results are obtained by applying Hölder, power mean, and Young's inequalities. Applications of the main results to modified Bessel functions and special means are discussed in Section 6. The paper is finalized in Section 7, where conclusions are given and directions for future research are proposed.

2. Preliminaries

This section is devoted to recalling the fundamental concepts, definitions, and preliminary results required for establishing our main findings. We begin with a discussion of the classical Jensen-Mercer inequality and the Hermite-Hadamard inequality. The notion of majorization is then introduced, followed by the majorized form of the Jensen-Mercer inequality. Weighted versions of the majorized Jensen-Mercer inequality for different tuples are also presented and formulated as Lemma 1 and Lemma 2. Finally, we provide the definition of the Caputo-Fabrizio fractional integral operator, which plays a central role in the subsequent analysis.

Jensen-Mercer Inequality ([54, 55])**:**

Let f represents a convex function throughout the interval $[\vartheta_1,\, \vartheta_2] \subseteq \mathbb{R}$. Also, let $y_{\varsigma^\cdot} \in [\vartheta_1,\, \vartheta_2]$, and non-negative weights $\sigma_{\varsigma^\cdot}$ for $\varsigma^\cdot = 1, 2, \ldots, \varphi^\cdot$ such that $\sum\limits_{\varsigma^\cdot=1}^{\varphi^\cdot} \sigma_{\varsigma^\cdot} = 1$. Then

$$f\left(\vartheta_1+\vartheta_2-\sum_{\varsigma^\cdot=1}^{\varphi^\cdot}\sigma_{\varsigma^\cdot}y_{\varsigma^\cdot}\right)\le f(\vartheta_1)+f(\vartheta_2)-\sum_{\varsigma^\cdot=1}^{\varphi^\cdot}\sigma_{\varsigma^\cdot}f(y_{\varsigma^\cdot}).$$

Hermite-Hadamard Inequality ([56])**:**

Let f: $[\vartheta_1,\, \vartheta_2] \to \mathbb{R}$ represents a convex function. Then

$$f\left(\frac{\vartheta_1+\vartheta_2}{2}\right)\le\frac{1}{\vartheta_2-\vartheta_1}\int_{\vartheta_1}^{\vartheta_2}f(u)du\le\frac{f(\vartheta_1)+f(\vartheta_2)}{2}.$$

Majorization ([57])**:**

Consider two real tuples $\boldsymbol{\delta} = (\delta_1, \delta_2, \ldots, \delta_{\varphi^\cdot})$ and $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \ldots, \gamma_{\varphi^\cdot})$, both arranged in decreasing order. Then $\boldsymbol{\delta}$ is said to majorize $\boldsymbol{\gamma}$, written as $\boldsymbol{\gamma} \prec \boldsymbol{\delta}$, if the following requirements are satisfied:

$$\sum_{\varsigma^\cdot=1}^{s} \delta_{[\varsigma^\cdot]} \geq \sum_{\varsigma^\cdot=1}^{s} \gamma_{[j]} \quad (s = 1, 2, \ldots, \varphi^\cdot - 1), \text{ and } \sum_{\varsigma^\cdot=1}^{\varphi^\cdot} \delta_{\varsigma^\cdot} = \sum_{\varsigma^\cdot=1}^{\varphi^\cdot} \gamma_{\varsigma^\cdot}.$$

Within the setting of majorization, the Jensen-Mercer inequality admits the following extended formulation.

Theorem 1 ([58]) Let $f\colon I \to \mathbb{R}$ denotes a function throughout the interval I, where f is assumed to be convex. Consider an $n \times \varphi^\cdot$ matrix Y with entries $y_{i\varsigma^\cdot} \in I$ for all $i = 1, 2, \ldots, n$ and $\varsigma^\cdot = 1, 2, \ldots, \varphi^\cdot$. Also, let

- a tuple $\boldsymbol{\omega} = (\omega_1, \omega_2, \ldots, \omega_{\varphi^\cdot})$ where $\varsigma^\cdot = 1, 2, \ldots, \varphi^\cdot$.
- a tuple of non-negative weights $(\sigma_1, \sigma_2, \ldots, \sigma_n)$ such that $\sum_{i=1}^{n} \sigma_i = 1$.

Suppose further that $\boldsymbol{\omega}$ majorizes every row of the matrix Y. Then

$$f\left(\sum_{\varsigma^\cdot=1}^{\varphi^\cdot} \omega_{\varsigma^\cdot} - \sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1} \sum_{i=1}^{n} \sigma_i y_{i\varsigma^\cdot}\right) \leq \sum_{\varsigma^\cdot=1}^{\varphi^\cdot} f(\omega_{\varsigma^\cdot}) - \sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1} \sum_{i=1}^{n} \sigma_i f(y_{i\varsigma^\cdot}).$$

The following lemmas are employed as fundamental tools in establishing our new results [47].

Lemma 1 Let $f\colon I \to \mathbb{R}$ denotes a function throughout the interval I, where f is assumed to be convex. Consider an $n \times \varphi^\cdot$ matrix with entries $y_{i\varsigma^\cdot} \in I$ for all $\varsigma^\cdot = 1, 2, \ldots, \varphi^\cdot$, $i = 1, 2, \ldots, n$. Also, let

- a tuple $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_\zeta)$, where $\varsigma^\cdot = 1, 2, \ldots, \varphi^\cdot$.
- a tuple of weights $\mathbf{p} = (p_1, \ldots, p_{\varphi^\cdot})$ with $p_{\varsigma^\cdot} \geq 0$, $p_{\varphi^\cdot} \neq 0$ for all $\varsigma^\cdot = 1, \ldots, \varphi^\cdot$. Define $\eta = \frac{1}{p_{\varphi^\cdot}}$.
- a tuple of non-negative scalars $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ with $\sum_{i=1}^{n} \sigma_i = 1$.

Suppose that the tuple $(y_{i1}, y_{i2}, \ldots, y_{i\varphi^\cdot})$ is decreasing for each $i = \{1, 2, \ldots, n\}$ and satisfies

$$\sum_{\varsigma^\cdot=1}^{s} p_{\varsigma^\cdot} y_{i\varsigma^\cdot} \leq \sum_{\varsigma^\cdot=1}^{s} p_{\varsigma^\cdot} \omega_{\varsigma^\cdot}, \ (1 \leq s \leq \varphi^\cdot - 1) \text{ and } \sum_{\varsigma^\cdot=1}^{\varphi^\cdot} p_{\varsigma^\cdot} \omega_{\varsigma^\cdot} = \sum_{\varsigma^\cdot=1}^{\varphi^\cdot} p_{\varsigma^\cdot} y_{i\varsigma^\cdot}.$$

Then,

$$f\left(\sum_{\varsigma^\cdot=1}^{\varphi^\cdot} \eta p_{\varsigma^\cdot} \omega_{\varsigma^\cdot} - \sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1} \sum_{i=1}^{n} \eta \sigma_i p_{\varsigma^\cdot} y_{i\varsigma^\cdot}\right) \leq \sum_{\varsigma^\cdot=1}^{\varphi^\cdot} \eta p_{\varsigma^\cdot} f(\omega_{\varsigma^\cdot}) - \sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1} \sum_{i=1}^{n} \eta \sigma_i p_{\varsigma^\cdot} f(y_{i\varsigma^\cdot}).$$

Lemma 2 Let $f\colon I \to \mathbb{R}$ denotes a function throughout the interval I, where f is assumed to be convex. Consider an $n \times \varphi^\cdot$ matrix with entries $y_{i\varsigma^\cdot} \in I$ for all $\varsigma^\cdot = 1, 2, \ldots, \varphi^\cdot$, $i = 1, 2, \ldots, n$. Also, let

- a tuple $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_\zeta)$, where $\varsigma^\cdot = 1, 2, \ldots, \varphi^\cdot$.
- a tuple of weights $\mathbf{p} = (p_1, \ldots, p_{\varphi^\cdot})$ with $p_{\varsigma^\cdot} \geq 0$, $p_{\varphi^\cdot} \neq 0$ for all $\varsigma^\cdot = 1, \ldots, \varphi^\cdot$. Define $\eta = \frac{1}{p_{\varphi^\cdot}}$.
- a tuple of non-negative scalars $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ with $\sum_{i=1}^{n} \sigma_i = 1$.

Suppose that both $y_{i\varsigma^\cdot}$ and $(\omega_{\varsigma^\cdot} - y_{i\varsigma^\cdot})$ exhibit identical monotonic behavior for each $i = 1, 2, \ldots, n$, and $\sum_{\varsigma^\cdot=1}^{\varphi^\cdot} p_{\varsigma^\cdot} \omega_{\varsigma^\cdot} = \sum_{\varsigma^\cdot=1}^{\varphi^\cdot} p_{\varsigma^\cdot} y_{i\varsigma^\cdot}$ holds. Then,

$$f\left(\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}}\eta p_{\varsigma^{\cdot}}\omega_{\varsigma^{\cdot}}-\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1}\sum_{i=1}^{n}\eta\sigma_i p_{\varsigma^{\cdot}}y_{i\varsigma^{\cdot}}\right)\leq\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}}\eta p_{\varsigma^{\cdot}}f(\omega_{\varsigma^{\cdot}})-\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1}\sum_{i=1}^{n}\eta\sigma_i p_{\varsigma^{\cdot}}f(y_{i\varsigma^{\cdot}}).$$

Now, we recall the definition of fractional operator which will be utilized to present the results of this paper.

Definition 1 ([37, 38]) If $f\in H(\zeta_1,\,\zeta_2)$ where $\zeta_1<\zeta_2$ and $0\leq\alpha\leq1$, the corresponding left and right Caputo-Fabrizio fractional integrals take the form:

$$ {}^{CF}_{\zeta_1}I^{\alpha}f(r)=\frac{1-\alpha}{B(\alpha)}f(r)+\frac{\alpha}{B(\alpha)}\int_{\zeta_1}^{r}f(s)ds$$

and

$$ {}^{CF}I^{\alpha}_{\zeta_2}f(r)=\frac{1-\alpha}{B(\alpha)}f(r)+\frac{\alpha}{B(\alpha)}\int_{r}^{\zeta_2}f(s)ds,$$

where $0<B(\alpha)$ denotes a normalization function such that $B(0)=B(1)=1$.

3. Main results

We formulate our new findings in the setting of Caputo-Fabrizio fractional operators, as presented below.

Theorem 2 Let $f\colon I\to\mathbb{R}$ denote a function throughout the interval I, where f is assumed to be convex. Let the tuples $\boldsymbol{\omega}$, ρ, $\varrho\in I^{\varphi^{\cdot}}$ be defined by $\boldsymbol{\omega}=(\omega_1,\,\omega_2,\,\ldots,\,\omega_{\varphi^{\cdot}})$, $\rho=(\rho_1,\,\rho_2,\,\ldots,\,\rho_{\varphi^{\cdot}})$ and $\varrho=(\varrho_1,\,\varrho_2,\,\ldots,\,\varrho_{\varphi^{\cdot}})$ with $\rho_{\varphi^{\cdot}}>\varrho_{\varphi^{\cdot}}$ and $\omega_{\varsigma^{\cdot}},\,\rho_{\varsigma^{\cdot}},\,\varrho_{\varsigma^{\cdot}}\in I$ for all $\varsigma^{\cdot}=1,\,2,\,\ldots,\,\varphi^{\cdot}$. Suppose further that $\rho\prec\boldsymbol{\omega}$ and $\varrho\prec\boldsymbol{\omega}$ and let $0\leq\alpha\leq1$, while $B(\alpha)$ denotes normalization mapping. Then

$$\begin{aligned}
f\left(\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}}\omega_{\varsigma^{\cdot}}-\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1}\left(\frac{\rho_{\varsigma^{\cdot}}+\varrho_{\varsigma^{\cdot}}}{2}\right)\right)\leq&\frac{B(\alpha)}{\alpha\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1}(\varrho_{\varsigma^{\cdot}}-\rho_{\varsigma^{\cdot}})}\left[{}^{CF}I^{\alpha}_{\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}}\omega_{\varsigma^{\cdot}}-\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1}\left(\frac{\rho_{\varsigma^{\cdot}}+\varrho_{\varsigma^{\cdot}}}{2}\right)}f\left(\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}}\omega_{\varsigma^{\cdot}}-\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1}\varrho_{\varsigma^{\cdot}}\right)\right.\\
&+{}^{CF}_{\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}}\omega_{\varsigma^{\cdot}}-\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1}\left(\frac{\rho_{\varsigma^{\cdot}}+\varrho_{\varsigma^{\cdot}}}{2}\right)}I^{\alpha}f\left(\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}}\omega_{\varsigma^{\cdot}}-\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1}\rho_{\varsigma^{\cdot}}\right)\\
&\left.-\frac{1-\alpha}{B(\alpha)}\left\{f\left(\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}}\omega_{\varsigma^{\cdot}}-\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1}\rho_{\varsigma^{\cdot}}\right)+f\left(\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}}\omega_{\varsigma^{\cdot}}-\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1}\varrho_{\varsigma^{\cdot}}\right)\right\}\right]\\
\leq&\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}}f(\omega_{\varsigma^{\cdot}})-\frac{1}{2}\left[\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1}f(\rho_{\varsigma^{\cdot}})+\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1}f(\varrho_{\varsigma^{\cdot}})\right].
\end{aligned}\tag{1}$$

Proof. Let $0\leq t\leq1$. The proof of the desired result is carried out in the following manner:

$$
\begin{aligned}
& f\left(\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}} \omega_{\varsigma^{\cdot}}-\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1}\left(\frac{\rho_{\varsigma^{\cdot}}+\varrho_{\varsigma^{\cdot}}}{2}\right)\right) \\
& =f\left\{\frac{1}{2}\left(\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}} \omega_{\varsigma^{\cdot}}-\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1} \rho_{\varsigma^{\cdot}}+\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}} \omega_{\varsigma^{\cdot}}-\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1} \varrho_{\varsigma^{\cdot}}\right)\right\} \\
& =f\left\{\frac{1}{2}\left(\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}} \omega_{\varsigma^{\cdot}}-\left(\frac{t}{2} \sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1} \rho_{\varsigma^{\cdot}}+\frac{2-t}{2} \sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1} \varrho_{\varsigma^{\cdot}}\right)\right.\right. \\
& \left.\left.\quad+\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}} \omega_{\varsigma^{\cdot}}-\left(\frac{t}{2} \sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1} \varrho_{\varsigma^{\cdot}}+\frac{2-t}{2} \sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1} \rho_{\varsigma^{\cdot}}\right)\right)\right\} .
\end{aligned} \tag{2}
$$

The convexity of f ensures that (2) results in the below inequality.

$$
\begin{aligned}
& f\left(\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}} \omega_{\varsigma^{\cdot}}-\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1}\left(\frac{\rho_{\varsigma^{\cdot}}+\varrho_{\varsigma^{\cdot}}}{2}\right)\right) \\
& \leq \frac{1}{2}\left\{f\left(\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}} \omega_{\varsigma^{\cdot}}-\left(\frac{t}{2} \sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1} \rho_{\varsigma^{\cdot}}+\frac{2-t}{2} \sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1} \varrho_{\varsigma^{\cdot}}\right)\right)\right. \\
& \left.\quad+f\left(\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}} \omega_{\varsigma^{\cdot}}-\left(\frac{t}{2} \sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1} \varrho_{\varsigma^{\cdot}}+\frac{2-t}{2} \sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1} \rho_{\varsigma^{\cdot}}\right)\right)\right\} .
\end{aligned}
$$

By integrating over $t \in [0, 1]$, we obtain

$$
\begin{aligned}
& f\left(\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}} \omega_{\varsigma^{\cdot}}-\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1}\left(\frac{\rho_{\varsigma^{\cdot}}+\varrho_{\varsigma^{\cdot}}}{2}\right)\right) \\
& \leq \frac{1}{\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1}\left(\varrho_{\varsigma^{\cdot}}-\rho_{\varsigma^{\cdot}}\right)}\left[\int_{\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}} \omega_{\varsigma^{\cdot}}-\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1} \varrho_{\varsigma^{\cdot}}}^{\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}} \omega_{\varsigma^{\cdot}}-\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1}\left(\frac{\rho_{\varsigma^{\cdot}}+\varrho_{\varsigma^{\cdot}}}{2}\right)} f(u) d u+\int_{\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}} \omega_{\varsigma^{\cdot}}-\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1}\left(\frac{\rho_{\varsigma^{\cdot}}+\varrho_{\varsigma^{\cdot}}}{2}\right)}^{\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}} \omega_{\varsigma^{\cdot}}-\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1} \rho_{\varsigma^{\cdot}}} f(u) d u\right] .
\end{aligned} \tag{3}
$$

The fact that $\rho \prec \boldsymbol{\omega}$ and $\varrho \prec \boldsymbol{\omega}$, leads to $\sum_{\varsigma\cdot=1}^{\varphi\cdot} \omega_{\varsigma\cdot} - \sum_{\varsigma\cdot=1}^{\varphi\cdot-1} \varrho_{\varsigma\cdot} = \varrho_{\varphi\cdot}$, $\sum_{\varsigma\cdot=1}^{\varphi\cdot} \omega_{\varsigma\cdot} - \sum_{\varsigma\cdot=1}^{\varphi\cdot-1} \rho_{\varsigma\cdot} = \rho_{\varphi\cdot}$. Also, taking $\rho_{\varphi\cdot} > \varrho_{\varphi\cdot}$ and doing simple calculation, we can prove that $\sum_{\varsigma\cdot=1}^{\varphi\cdot} \omega_{\varsigma\cdot} - \sum_{\varsigma\cdot=1}^{\varphi\cdot-1} \left(\frac{\rho_{\varsigma\cdot}+\varrho_{\varsigma\cdot}}{2}\right) > \sum_{\varsigma\cdot=1}^{\varphi\cdot} \omega_{\varsigma\cdot} - \sum_{\varsigma\cdot=1}^{\varphi\cdot-1} \varrho_{\varsigma\cdot}$ and $\sum_{\varsigma\cdot=1}^{\varphi\cdot} \omega_{\varsigma\cdot} - \sum_{\varsigma\cdot=1}^{\varphi\cdot-1} \left(\frac{\rho_{\varsigma\cdot}+\varrho_{\varsigma\cdot}}{2}\right) < \sum_{\varsigma\cdot=1}^{\varphi\cdot} \omega_{\varsigma\cdot} - \sum_{\varsigma\cdot=1}^{\varphi\cdot-1} \rho_{\varsigma\cdot}$. This enables us to apply the definition of Caputo-Fabrizio fractional integral operator.

Now, by multiplying both sides of (3) by $\frac{\alpha\sum_{\varsigma\cdot=1}^{\varphi\cdot-1}(\varrho_{\varsigma\cdot}-\rho_{\varsigma\cdot})}{B(\alpha)}$ and then adding $\frac{1-\alpha}{B(\alpha)} f\left(\sum_{\varsigma\cdot=1}^{\varphi\cdot} \omega_{\varsigma\cdot} - \sum_{\varsigma\cdot=1}^{\varphi\cdot-1} \rho_{\varsigma\cdot}\right) + \frac{1-\alpha}{B(\alpha)} f\left(\sum_{\varsigma\cdot=1}^{\varphi\cdot} \omega_{\varsigma\cdot} - \sum_{\varsigma\cdot=1}^{\varphi\cdot-1} \varrho_{\varsigma\cdot}\right)$, we have

$$
\begin{aligned}
&\frac{\alpha\sum_{\varsigma\cdot=1}^{\varphi\cdot-1}(\varrho_{\varsigma\cdot}-\rho_{\varsigma\cdot})}{B(\alpha)} f\left(\sum_{\varsigma\cdot=1}^{\varphi\cdot} \omega_{\varsigma\cdot} - \sum_{\varsigma\cdot=1}^{\varphi\cdot-1} \left(\frac{\rho_{\varsigma\cdot}+\varrho_{\varsigma\cdot}}{2}\right)\right)\\
&+\frac{1-\alpha}{B(\alpha)} f\left(\sum_{\varsigma\cdot=1}^{\varphi\cdot} \omega_{\varsigma\cdot} - \sum_{\varsigma\cdot=1}^{\varphi\cdot-1} \rho_{\varsigma\cdot}\right) + \frac{1-\alpha}{B(\alpha)} f\left(\sum_{\varsigma\cdot=1}^{\varphi\cdot} \omega_{\varsigma\cdot} - \sum_{\varsigma\cdot=1}^{\varphi\cdot-1} \varrho_{\varsigma\cdot}\right)\\
&\leq \frac{1-\alpha}{B(\alpha)} f\left(\sum_{\varsigma\cdot=1}^{\varphi\cdot} \omega_{\varsigma\cdot} - \sum_{\varsigma\cdot=1}^{\varphi\cdot-1} \varrho_{\varsigma\cdot}\right) + \frac{\alpha}{B(\alpha)} \int_{\sum_{\varsigma\cdot=1}^{\varphi\cdot} \omega_{\varsigma\cdot} - \sum_{\varsigma\cdot=1}^{\varphi\cdot-1} \varrho_{\varsigma\cdot}}^{\sum_{\varsigma\cdot=1}^{\varphi\cdot} \omega_{\varsigma\cdot} - \sum_{\varsigma\cdot=1}^{\varphi\cdot-1}\left(\frac{\rho_{\varsigma\cdot}+\varrho_{\varsigma\cdot}}{2}\right)} f(u)du\\
&+\frac{1-\alpha}{B(\alpha)} f\left(\sum_{\varsigma\cdot=1}^{\varphi\cdot} \omega_{\varsigma\cdot} - \sum_{\varsigma\cdot=1}^{\varphi\cdot-1} \rho_{\varsigma\cdot}\right) + \frac{\alpha}{B(\alpha)} \int_{\sum_{\varsigma\cdot=1}^{\varphi\cdot} \omega_{\varsigma\cdot} - \sum_{\varsigma\cdot=1}^{\varphi\cdot-1}\left(\frac{\rho_{\varsigma\cdot}+\varrho_{\varsigma\cdot}}{2}\right)}^{\sum_{\varsigma\cdot=1}^{\varphi\cdot} \omega_{\varsigma\cdot} - \sum_{\varsigma\cdot=1}^{\varphi\cdot-1} \rho_{\varsigma\cdot}} f(u)du.\\
&= {}^{CF}I^{\alpha}_{\sum_{\varsigma\cdot=1}^{\varphi\cdot} \omega_{\varsigma\cdot} - \sum_{\varsigma\cdot=1}^{\varphi\cdot-1}\left(\frac{\rho_{\varsigma\cdot}+\varrho_{\varsigma\cdot}}{2}\right)} f\left(\sum_{\varsigma\cdot=1}^{\varphi\cdot} \omega_{\varsigma\cdot} - \sum_{\varsigma\cdot=1}^{\varphi\cdot-1} \varrho_{\varsigma\cdot}\right) + {}^{CF}_{\sum_{\varsigma\cdot=1}^{\varphi\cdot} \omega_{\varsigma\cdot} - \sum_{\varsigma\cdot=1}^{\varphi\cdot-1}\left(\frac{\rho_{\varsigma\cdot}+\varrho_{\varsigma\cdot}}{2}\right)} I^{\alpha} f\left(\sum_{\varsigma\cdot=1}^{\varphi\cdot} \omega_{\varsigma\cdot} - \sum_{\varsigma\cdot=1}^{\varphi\cdot-1} \rho_{\varsigma\cdot}\right)
\end{aligned}
$$

Re-arrangement of the above inequality yields

$$
\begin{aligned}
&f\left(\sum_{\varsigma\cdot=1}^{\varphi\cdot} \omega_{\varsigma\cdot} - \sum_{\varsigma\cdot=1}^{\varphi\cdot-1} \left(\frac{\rho_{\varsigma\cdot}+\varrho_{\varsigma\cdot}}{2}\right)\right)\\
&\leq \frac{B(\alpha)}{\alpha\sum_{\varsigma\cdot=1}^{\varphi\cdot-1}(\varrho_{\varsigma\cdot}-\rho_{\varsigma\cdot})} \left[{}^{CF}I^{\alpha}_{\sum_{\varsigma\cdot=1}^{\varphi\cdot} \omega_{\varsigma\cdot} - \sum_{\varsigma\cdot=1}^{\varphi\cdot-1}\left(\frac{\rho_{\varsigma\cdot}+\varrho_{\varsigma\cdot}}{2}\right)} f\left(\sum_{\varsigma\cdot=1}^{\varphi\cdot} \omega_{\varsigma\cdot} - \sum_{\varsigma\cdot=1}^{\varphi\cdot-1} \varrho_{\varsigma\cdot}\right)\right.
\end{aligned}
$$

$$+{}^{CF}_{\sum_{\varsigma\cdot=1}^{\varphi\cdot}\omega_{\varsigma\cdot}-\sum_{\varsigma\cdot=1}^{\varphi\cdot-1}\left(\frac{\rho_{\varsigma\cdot}+\varrho_{\varsigma\cdot}}{2}\right)}I^{\alpha}f\left(\sum_{\varsigma\cdot=1}^{\varphi\cdot}\omega_{\varsigma\cdot}-\sum_{\varsigma\cdot=1}^{\varphi\cdot-1}\rho_{\varsigma\cdot}\right)$$

$$-\frac{1-\alpha}{B(\alpha)}\left\{f\left(\sum_{\varsigma\cdot=1}^{\varphi\cdot}\omega_{\varsigma\cdot}-\sum_{\varsigma\cdot=1}^{\varphi\cdot-1}\rho_{\varsigma\cdot}\right)+f\left(\sum_{\varsigma\cdot=1}^{\varphi\cdot}\omega_{\varsigma\cdot}-\sum_{\varsigma\cdot=1}^{\varphi\cdot-1}\varrho_{\varsigma\cdot}\right)\right\}\Bigg]. \tag{4}$$

This ends the first part of (1). By employing Theorem 1 with the substitutions $\sigma_1=\frac{t}{2}$, $\sigma_2=\frac{2-t}{2}$, and $n=2$, we establish the second part of inequality (1) as outlined below:

$$\begin{aligned}&f\left(\sum_{\varsigma\cdot=1}^{\varphi\cdot}\omega_{\varsigma\cdot}-\left(\frac{t}{2}\sum_{\varsigma\cdot=1}^{\varphi\cdot-1}\rho_{\varsigma\cdot}+\frac{2-t}{2}\sum_{\varsigma\cdot=1}^{\varphi\cdot-1}\varrho_{\varsigma\cdot}\right)\right)\\&\leq\sum_{\varsigma\cdot=1}^{\varphi\cdot}f(\omega_{\varsigma\cdot})-\left(\frac{t}{2}\sum_{\varsigma\cdot=1}^{\varphi\cdot-1}f(\rho_{\varsigma\cdot})+\frac{2-t}{2}\sum_{\varsigma\cdot=1}^{\varphi\cdot-1}f(\varrho_{\varsigma\cdot})\right)\end{aligned} \tag{5}$$

and

$$\begin{aligned}&f\left(\sum_{\varsigma\cdot=1}^{\varphi\cdot}\omega_{\varsigma\cdot}-\left(\frac{t}{2}\sum_{\varsigma\cdot=1}^{\varphi\cdot-1}\varrho_{\varsigma\cdot}+\frac{2-t}{2}\sum_{\varsigma\cdot=1}^{\varphi\cdot-1}\rho_{\varsigma\cdot}\right)\right)\\&\leq\sum_{\varsigma\cdot=1}^{\varphi\cdot}f(\omega_{\varsigma\cdot})-\left(\frac{t}{2}\sum_{\varsigma\cdot=1}^{\varphi\cdot-1}f(\varrho_{\varsigma\cdot})+\frac{2-t}{2}\sum_{\varsigma\cdot=1}^{\varphi\cdot-1}f(\rho_{\varsigma\cdot})\right).\end{aligned} \tag{6}$$

Adding (5) and (6), we get

$$\begin{aligned}&f\left(\sum_{\varsigma\cdot=1}^{\varphi\cdot}\omega_{\varsigma\cdot}-\left(\frac{t}{2}\sum_{\varsigma\cdot=1}^{\varphi\cdot-1}\rho_{\varsigma\cdot}+\frac{2-t}{2}\sum_{\varsigma\cdot=1}^{\varphi\cdot-1}\varrho_{\varsigma\cdot}\right)\right)\\&+f\left(\sum_{\varsigma\cdot=1}^{\varphi\cdot}\omega_{\varsigma\cdot}-\left(\frac{t}{2}\sum_{\varsigma\cdot=1}^{\varphi\cdot-1}\varrho_{\varsigma\cdot}+\frac{2-t}{2}\sum_{\varsigma\cdot=1}^{\varphi\cdot-1}\rho_{\varsigma\cdot}\right)\right)\\&\leq 2\sum_{\varsigma\cdot=1}^{\varphi\cdot}f(\omega_{\varsigma\cdot})-\left(\sum_{\varsigma\cdot=1}^{\varphi\cdot-1}f(\rho_{\varsigma\cdot})+\sum_{\varsigma\cdot=1}^{\varphi\cdot-1}f(\varrho_{\varsigma\cdot})\right).\end{aligned}$$

Integrating over $t\in[0,\,1]$, and adopting the above mentioned procedure, we get

$$
{}^{CF}I^{\alpha}_{\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}}\omega_{\varsigma^{\cdot}}-\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1}\left(\frac{\rho_{\varsigma^{\cdot}}+\varrho_{\varsigma^{\cdot}}}{2}\right)} f\left(\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}}\omega_{\varsigma^{\cdot}}-\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1}\varrho_{\varsigma^{\cdot}}\right)
$$

$$
+ {}^{CF}_{\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}}\omega_{\varsigma^{\cdot}}-\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1}\left(\frac{\rho_{\varsigma^{\cdot}}+\varrho_{\varsigma^{\cdot}}}{2}\right)}I^{\alpha} f\left(\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}}\omega_{\varsigma^{\cdot}}-\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1}\rho_{\varsigma^{\cdot}}\right)
$$

$$
\leq \frac{\alpha\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1}\left(\varrho_{\varsigma^{\cdot}}-\rho_{\varsigma^{\cdot}}\right)}{B(\alpha)}\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}} f(\omega_{\varsigma^{\cdot}})-\frac{\alpha\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1}\left(\varrho_{\varsigma^{\cdot}}-\rho_{\varsigma^{\cdot}}\right)}{B(\alpha)}\left(\frac{\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1} f(\rho_{\varsigma^{\cdot}})+\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1} f(\varrho_{\varsigma^{\cdot}})}{2}\right) \tag{7}
$$

$$
+\frac{1-\alpha}{B(\alpha)}\left[f\left(\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}}\omega_{\varsigma^{\cdot}}-\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1}\rho_{\varsigma^{\cdot}}\right)+f\left(\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}}\omega_{\varsigma^{\cdot}}-\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1}\varrho_{\varsigma^{\cdot}}\right)\right].
$$

Now, by re-arranging (7) and then joining with (4), we get complete result of (1). □

Remark 1

(i) If we substitute $\varphi^{\cdot}=2$ in (1), then we get the following inequality of [59].

$$
f\left(\omega_1+\omega_2-\left(\frac{\rho_1+\varrho_1}{2}\right)\right)
$$

$$
\leq \frac{B(\alpha)}{(\varrho_1-\rho_1)}\left[{}^{CF}I^{\alpha}_{\omega_1+\omega_2-\frac{\rho_1+\varrho_1}{2}} f(\omega_1+\omega_2-\varrho_1)+{}^{CF}_{\omega_1+\omega_2-\frac{\rho_1+\varrho_1}{2}}I^{\alpha} f(\omega_1+\omega_2-\rho_1)\right.
$$

$$
\left.-\frac{1-\alpha}{B(\alpha)}\left[f(\omega_1+\omega_2-\rho_1)+f(\omega_1+\omega_2-\varrho_1)\right]\right]
$$

$$
\leq f(\omega_1)+f(\omega_2)-\frac{f(\rho_1)+f(\varrho_1)}{2}.
$$

(ii) If we substitute $\varphi^{\cdot}=2$, $\rho_1=\omega_1$, $\varrho_1=\omega_2$ in (1), then we get the following inequality of [60].

$$
f\left(\frac{\omega_1+\omega_2}{2}\right)
$$

$$
\leq \frac{B(\alpha)}{\alpha(\omega_2-\omega_1)}\left[{}^{CF}I^{\alpha}_{\left(\frac{\omega_1+\omega_2}{2}\right)} f(\omega_1)+{}^{CF}_{\left(\frac{\omega_1+\omega_2}{2}\right)}I^{\alpha} f(\omega_2)-\frac{1-\alpha}{B(\alpha)}\left[f(\omega_1)+f(\omega_2)\right]\right]
$$

$$\leq \frac{f(\omega_1)+f(\omega_2)}{2}.$$

We present another version of our result by employing a procedure similar to that used in the previous theorem.

Theorem 3 Let $f\colon I \to \mathbb{R}$ denote a function throughout the interval I, where f is assumed to be convex. Let the tuples $\boldsymbol{\omega}$, ρ, $\varrho \in I^{\varphi^\cdot}$ be defined by $\boldsymbol{\omega} = (\omega_1, \omega_2, \ldots, \omega_{\varphi^\cdot})$, $\rho = (\rho_1, \rho_2, \ldots, \rho_{\varphi^\cdot})$ and $\varrho = (\varrho_1, \varrho_2, \ldots, \varrho_{\varphi^\cdot})$ with $\rho_{\varphi^\cdot} > \varrho_{\varphi^\cdot}$ and $\omega_{\varsigma^\cdot}, \rho_{\varsigma^\cdot}, \varrho_{\varsigma^\cdot} \in I$ for all $\varsigma^\cdot = 1, 2, \ldots, \varphi^\cdot$. Suppose further that $\rho \prec \boldsymbol{\omega}$ and $\varrho \prec \boldsymbol{\omega}$ and let $0 \leq \alpha \leq 1$, while $B(\alpha)$ denotes a normalization mapping. Then

$$\begin{aligned}
& f\left(\sum_{\varsigma^\cdot=1}^{\varphi^\cdot} \omega_{\varsigma^\cdot} - \sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1} \left(\frac{\rho_{\varsigma^\cdot}+\varrho_{\varsigma^\cdot}}{2}\right)\right) \\
\leq & \frac{B(\alpha)}{2\alpha \sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1} \left(\varrho_{\varsigma^\cdot} - \rho_{\varsigma^\cdot}\right)} \left[{}^{CF}_{\left(\sum_{\varsigma^\cdot=1}^{\varphi^\cdot} \omega_{\varsigma^\cdot} - \sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1} \varrho_{\varsigma^\cdot}\right)} I^{\alpha} f\left(\sum_{\varsigma^\cdot=1}^{\varphi^\cdot} \omega_{\varsigma^\cdot} - \sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1} \rho_{\varsigma^\cdot}\right) \right. \\
& + {}^{CF}I^{\alpha}_{\left(\sum_{\varsigma^\cdot=1}^{\varphi^\cdot} \omega_{\varsigma^\cdot} - \sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1} \rho_{\varsigma^\cdot}\right)} f\left(\sum_{\varsigma^\cdot=1}^{\varphi^\cdot} \omega_{\varsigma^\cdot} - \sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1} \varrho_{\varsigma^\cdot}\right) \\
& \left. - \frac{1-\alpha}{B(\alpha)} \left\{ f\left(\sum_{\varsigma^\cdot=1}^{\varphi^\cdot} \omega_{\varsigma^\cdot} - \sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1} \rho_{\varsigma^\cdot}\right) + f\left(\sum_{\varsigma^\cdot=1}^{\varphi^\cdot} \omega_{\varsigma^\cdot} - \sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1} \varrho_{\varsigma^\cdot}\right) \right\} \right] \\
\leq & \frac{1}{2}\left[f\left(\sum_{\varsigma^\cdot=1}^{\varphi^\cdot} \omega_{\varsigma^\cdot} - \sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1} \rho_{\varsigma^\cdot}\right) + f\left(\sum_{\varsigma^\cdot=1}^{\varphi^\cdot} \omega_{\varsigma^\cdot} - \sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1} \varrho_{\varsigma^\cdot}\right) \right] \\
\leq & \sum_{\varsigma^\cdot=1}^{\varphi^\cdot} f(\omega_{\varsigma^\cdot}) - \frac{1}{2}\left(\sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1} f(\rho_{\varsigma^\cdot}) + \sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1} f(\varrho_{\varsigma^\cdot}) \right).
\end{aligned} \tag{8}$$

Proof. It can be written as:

$$\begin{aligned}
& f\left(\sum_{\varsigma^\cdot=1}^{\varphi^\cdot} \omega_{\varsigma^\cdot} - \sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1} \left(\frac{\rho_{\varsigma^\cdot}+\varrho_{\varsigma^\cdot}}{2}\right)\right) \\
= & f\left\{ \frac{1}{2} \left\{ \sum_{\varsigma^\cdot=1}^{\varphi^\cdot} \omega_{\varsigma^\cdot} - \sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1} \rho_{\varsigma^\cdot} + \sum_{\varsigma^\cdot=1}^{\varphi^\cdot} \omega_{\varsigma^\cdot} - \sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1} \varrho_{\varsigma^\cdot} \right\} \right\}
\end{aligned}$$

$$=f\left\{\frac{1}{2}\left\{t\left(\sum_{\varsigma\cdot=1}^{\varphi\cdot}\omega_{\varsigma\cdot}-\sum_{\varsigma\cdot=1}^{\varphi\cdot-1}\rho_{\varsigma\cdot}\right)+(1-t)\left(\sum_{\varsigma\cdot=1}^{\varphi\cdot}\omega_{\varsigma\cdot}-\sum_{\varsigma\cdot=1}^{\varphi\cdot-1}\varrho_{\varsigma\cdot}\right)\right.\right.$$

$$\left.\left.+t\left(\sum_{\varsigma\cdot=1}^{\varphi\cdot}\omega_{\varsigma\cdot}-\sum_{\varsigma\cdot=1}^{\varphi\cdot-1}\varrho_{\varsigma\cdot}\right)+(1-t)\left(\sum_{\varsigma\cdot=1}^{\varphi\cdot}\omega_{\varsigma\cdot}-\sum_{\varsigma\cdot=1}^{\varphi\cdot-1}\rho_{\varsigma\cdot}\right)\right\}\right\}. \tag{9}$$

The convexity of f ensures that (9) results in the below inequality.

$$f\left(\sum_{\varsigma\cdot=1}^{\varphi\cdot}\omega_{\varsigma\cdot}-\sum_{\varsigma\cdot=1}^{\varphi\cdot-1}\left(\frac{\rho_{\varsigma\cdot}+\varrho_{\varsigma\cdot}}{2}\right)\right)$$

$$\leq\frac{1}{2}\left\{f\left\{t\left(\sum_{\varsigma\cdot=1}^{\varphi\cdot}\omega_{\varsigma\cdot}-\sum_{\varsigma\cdot=1}^{\varphi\cdot-1}\rho_{\varsigma\cdot}\right)+(1-t)\left(\sum_{\varsigma\cdot=1}^{\varphi\cdot}\omega_{\varsigma\cdot}-\sum_{\varsigma\cdot=1}^{\varphi\cdot-1}\varrho_{\varsigma\cdot}\right)\right\}\right. \tag{10}$$

$$\left.+f\left\{t\left(\sum_{\varsigma\cdot=1}^{\varphi\cdot}\omega_{\varsigma\cdot}-\sum_{\varsigma\cdot=1}^{\varphi\cdot-1}\varrho_{\varsigma\cdot}\right)+(1-t)\left(\sum_{\varsigma\cdot=1}^{\varphi\cdot}\omega_{\varsigma\cdot}-\sum_{\varsigma\cdot=1}^{\varphi\cdot-1}\rho_{\varsigma\cdot}\right)\right\}\right\}.$$

On multiplying (10) with $\frac{\alpha\sum_{\varsigma\cdot=1}^{\varphi\cdot-1}(\varrho_{\varsigma\cdot}-\rho_{\varsigma\cdot})}{B(\alpha)}$ and subsequently integrating with respect to t on [0, 1], we obtain

$$\frac{\alpha\sum_{\varsigma\cdot=1}^{\varphi\cdot-1}\left(\varrho_{\varsigma\cdot}-\rho_{\varsigma\cdot}\right)}{B(\alpha)}f\left(\sum_{\varsigma\cdot=1}^{\varphi\cdot}\omega_{\varsigma\cdot}-\sum_{\varsigma\cdot=1}^{\varphi\cdot-1}\left(\frac{\rho_{\varsigma\cdot}+\varrho_{\varsigma\cdot}}{2}\right)\right)$$

$$\leq\frac{\alpha\sum_{\varsigma\cdot=1}^{\varphi\cdot-1}\left(\varrho_{\varsigma\cdot}-\rho_{\varsigma\cdot}\right)}{2B(\alpha)}\left[\int_0^1 f\left(t\left(\sum_{\varsigma\cdot=1}^{\varphi\cdot}\omega_{\varsigma\cdot}-\sum_{\varsigma\cdot=1}^{\varphi\cdot-1}\rho_{\varsigma\cdot}\right)+(1-t)\left(\sum_{\varsigma\cdot=1}^{\varphi\cdot}\omega_{\varsigma\cdot}-\sum_{\varsigma\cdot=1}^{\varphi\cdot-1}\varrho_{\varsigma\cdot}\right)\right)dt\right.$$

$$\left.+\int_0^1 f\left(t\left(\sum_{\varsigma\cdot=1}^{\varphi\cdot}\omega_{\varsigma\cdot}-\sum_{\varsigma\cdot=1}^{\varphi\cdot-1}\varrho_{\varsigma\cdot}\right)+(1-t)\left(\sum_{\varsigma\cdot=1}^{\varphi\cdot}\omega_{\varsigma\cdot}-\sum_{\varsigma\cdot=1}^{\varphi\cdot-1}\rho_{\varsigma\cdot}\right)\right)dt\right]$$

By making a suitable substitution, we have

$$\frac{\alpha\sum\limits_{\varsigma\cdot=1}^{\varphi\cdot-1}\left(\varrho_{\varsigma\cdot}-\rho_{\varsigma\cdot}\right)}{B(\alpha)}f\left(\sum_{\varsigma\cdot=1}^{\varphi\cdot}\omega_{\varsigma\cdot}-\sum_{\varsigma\cdot=1}^{\varphi\cdot-1}\left(\frac{\rho_{\varsigma\cdot}+\varrho_{\varsigma\cdot}}{2}\right)\right)$$

$$\leq\frac{\alpha\sum\limits_{\varsigma\cdot=1}^{\varphi\cdot-1}\left(\varrho_{\varsigma\cdot}-\rho_{\varsigma\cdot}\right)}{2B(\alpha)}\left[\int_{\sum\limits_{\varsigma\cdot=1}^{\varphi\cdot}\omega_{\varsigma\cdot}-\sum\limits_{\varsigma\cdot=1}^{\varphi\cdot-1}\varrho_{\varsigma\cdot}}^{\sum\limits_{\varsigma\cdot=1}^{\varphi\cdot}\omega_{\varsigma\cdot}-\sum\limits_{\varsigma\cdot=1}^{\varphi\cdot-1}\rho_{\varsigma\cdot}}f(u)\frac{du}{\sum\limits_{\varsigma\cdot=1}^{\varphi\cdot-1}\left(\varrho_{\varsigma\cdot}-\rho_{\varsigma\cdot}\right)}+\int_{\sum\limits_{\varsigma\cdot=1}^{\varphi\cdot}\omega_{\varsigma\cdot}-\sum\limits_{\varsigma\cdot=1}^{\varphi\cdot-1}\varrho_{\varsigma\cdot}}^{\sum\limits_{\varsigma\cdot=1}^{\varphi\cdot}\omega_{\varsigma\cdot}-\sum\limits_{\varsigma\cdot=1}^{\varphi\cdot-1}\rho_{\varsigma\cdot}}f(u)\frac{du}{\sum\limits_{\varsigma\cdot=1}^{\varphi\cdot-1}\left(\varrho_{\varsigma\cdot}-\rho_{\varsigma\cdot}\right)}\right] \tag{11}$$

$$\frac{2\alpha\sum\limits_{\varsigma\cdot=1}^{\varphi\cdot-1}\left(\varrho_{\varsigma\cdot}-\rho_{\varsigma\cdot}\right)}{B(\alpha)}f\left(\sum_{\varsigma\cdot=1}^{\varphi\cdot}\omega_{\varsigma\cdot}-\sum_{\varsigma\cdot=1}^{\varphi\cdot-1}\left(\frac{\rho_{\varsigma\cdot}+\varrho_{\varsigma\cdot}}{2}\right)\right)$$

$$\leq\frac{\alpha}{B(\alpha)}\int_{\sum\limits_{\varsigma\cdot=1}^{\varphi\cdot}\omega_{\varsigma\cdot}-\sum\limits_{\varsigma\cdot=1}^{\varphi\cdot-1}\varrho_{\varsigma\cdot}}^{\sum\limits_{\varsigma\cdot=1}^{\varphi\cdot}\omega_{\varsigma\cdot}-\sum\limits_{\varsigma\cdot=1}^{\varphi\cdot-1}\rho_{\varsigma\cdot}}f(u)du+\frac{\alpha}{B(\alpha)}\int_{\sum\limits_{\varsigma\cdot=1}^{\varphi\cdot}\omega_{\varsigma\cdot}-\sum\limits_{\varsigma\cdot=1}^{\varphi\cdot-1}\varrho_{\varsigma\cdot}}^{\sum\limits_{\varsigma\cdot=1}^{\varphi\cdot}\omega_{\varsigma\cdot}-\sum\limits_{\varsigma\cdot=1}^{\varphi\cdot-1}\rho_{\varsigma\cdot}}f(u)du$$

By adding $\frac{1-\alpha}{B(\alpha)}f\left(\sum\limits_{\varsigma\cdot=1}^{\varphi\cdot}\omega_{\varsigma\cdot}-\sum\limits_{\varsigma\cdot=1}^{\varphi\cdot-1}\rho_{\varsigma\cdot}\right)+\frac{1-\alpha}{B(\alpha)}f\left(\sum\limits_{\varsigma\cdot=1}^{\varphi\cdot}\omega_{\varsigma\cdot}-\sum\limits_{\varsigma\cdot=1}^{\varphi\cdot-1}\varrho_{\varsigma\cdot}\right)$ to both sides of (11), we get

$$\frac{2\alpha\sum\limits_{\varsigma\cdot=1}^{\varphi\cdot-1}\left(\varrho_{\varsigma\cdot}-\rho_{\varsigma\cdot}\right)}{B(\alpha)}f\left(\sum_{\varsigma\cdot=1}^{\varphi\cdot}\omega_{\varsigma\cdot}-\sum_{\varsigma\cdot=1}^{\varphi\cdot-1}\left(\frac{\rho_{\varsigma\cdot}+\varrho_{\varsigma\cdot}}{2}\right)\right)+\frac{1-\alpha}{B(\alpha)}f\left(\sum_{\varsigma\cdot=1}^{\varphi\cdot}\omega_{\varsigma\cdot}-\sum_{\varsigma\cdot=1}^{\varphi\cdot-1}\rho_{\varsigma\cdot}\right)$$

$$+\frac{1-\alpha}{B(\alpha)}f\left(\sum_{\varsigma\cdot=1}^{\varphi\cdot}\omega_{\varsigma\cdot}-\sum_{\varsigma\cdot=1}^{\varphi\cdot-1}\varrho_{\varsigma\cdot}\right)$$

$$\leq\frac{1-\alpha}{B(\alpha)}f\left(\sum_{\varsigma\cdot=1}^{\varphi\cdot}\omega_{\varsigma\cdot}-\sum_{\varsigma\cdot=1}^{\varphi\cdot-1}\rho_{\varsigma\cdot}\right)+\frac{\alpha}{B(\alpha)}\int_{\sum\limits_{\varsigma\cdot=1}^{\varphi\cdot}\omega_{\varsigma\cdot}-\sum\limits_{\varsigma\cdot=1}^{\varphi\cdot-1}\varrho_{\varsigma\cdot}}^{\sum\limits_{\varsigma\cdot=1}^{\varphi\cdot}\omega_{\varsigma\cdot}-\sum\limits_{\varsigma\cdot=1}^{\varphi\cdot-1}\rho_{\varsigma\cdot}}f(u)du \tag{12}$$

$$+\frac{1-\alpha}{B(\alpha)}f\left(\sum_{\varsigma\cdot=1}^{\varphi\cdot}\omega_{\varsigma\cdot}-\sum_{\varsigma\cdot=1}^{\varphi\cdot-1}\varrho_{\varsigma\cdot}\right)+\frac{\alpha}{B(\alpha)}\int_{\sum\limits_{\varsigma\cdot=1}^{\varphi\cdot}\omega_{\varsigma\cdot}-\sum\limits_{\varsigma\cdot=1}^{\varphi\cdot-1}\varrho_{\varsigma\cdot}}^{\sum\limits_{\varsigma\cdot=1}^{\varphi\cdot}\omega_{\varsigma\cdot}-\sum\limits_{\varsigma\cdot=1}^{\varphi\cdot-1}\rho_{\varsigma\cdot}}f(u)du$$

To proceed with the application of Caputo-Fabrizio fractional operators in (12), first we show that

$$\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}} \omega_{\varsigma^{\cdot}} - \sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1} \varrho_{\varsigma^{\cdot}} < \sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}} \omega_{\varsigma^{\cdot}} - \sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1} \rho_{\varsigma^{\cdot}}.$$

Since $\rho \prec \boldsymbol{\omega}$ and $\varrho \prec \boldsymbol{\omega}$. Therefore $\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1} \varrho_{\varsigma^{\cdot}} - \sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1} \rho_{\varsigma^{\cdot}} = \rho_{\varphi^{\cdot}} - \varrho_{\varphi^{\cdot}}$. By using the assumption $\rho_{\varphi^{\cdot}} > \varrho_{\varphi^{\cdot}} \Rightarrow \rho_{\varphi^{\cdot}} - \varrho_{\varphi^{\cdot}} > 0$, we get $-\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1} \varrho_{\varsigma^{\cdot}} < -\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1} \rho_{\varsigma^{\cdot}}$. Now, adding $\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}} \omega_{\varsigma^{\cdot}}$ to both sides, we get

$$\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}} \omega_{\varsigma^{\cdot}} - \sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1} \varrho_{\varsigma^{\cdot}} < \sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}} \omega_{\varsigma^{\cdot}} - \sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1} \rho_{\varsigma^{\cdot}}.$$

Now, (12) implies

$$\begin{aligned} & f\left(\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}} \omega_{\varsigma^{\cdot}} - \sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1} \left(\frac{\rho_{\varsigma^{\cdot}} + \varrho_{\varsigma^{\cdot}}}{2}\right)\right) \\ \leq & \frac{B(\alpha)}{2\alpha \sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1} \left(\varrho_{\varsigma^{\cdot}} - \rho_{\varsigma^{\cdot}}\right)} \left[{}^{CF}_{\left(\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}} \omega_{\varsigma^{\cdot}} - \sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1} \varrho_{\varsigma^{\cdot}}\right)} I^{\alpha} f\left(\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}} \omega_{\varsigma^{\cdot}} - \sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1} \rho_{\varsigma^{\cdot}}\right)\right. \\ & + {}^{CF} I^{\alpha}_{\left(\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}} \omega_{\varsigma^{\cdot}} - \sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1} \rho_{\varsigma^{\cdot}}\right)} f\left(\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}} \omega_{\varsigma^{\cdot}} - \sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1} \varrho_{\varsigma^{\cdot}}\right) \\ & \left. - \frac{1-\alpha}{B(\alpha)} \left\{ f\left(\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}} \omega_{\varsigma^{\cdot}} - \sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1} \rho_{\varsigma^{\cdot}}\right) + f\left(\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}} \omega_{\varsigma^{\cdot}} - \sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1} \varrho_{\varsigma^{\cdot}}\right)\right\}\right] \end{aligned} \tag{13}$$

Hence, the first part of the inequality (8) is established. The derivation of the remaining part of this inequality is an outcome ensured whenever f exhibits convexity as given below:

$$\begin{aligned} & f\left(t\left(\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}} \omega_{\varsigma^{\cdot}} - \sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1} \rho_{\varsigma^{\cdot}}\right) + (1-t)\left(\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}} \omega_{\varsigma^{\cdot}} - \sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1} \varrho_{\varsigma^{\cdot}}\right)\right) \\ \leq & t f\left(\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}} \omega_{\varsigma^{\cdot}} - \sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1} \rho_{\varsigma^{\cdot}}\right) + (1-t) f\left(\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}} \omega_{\varsigma^{\cdot}} - \sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1} \varrho_{\varsigma^{\cdot}}\right) \end{aligned} \tag{14}$$

and

$$f\left(t\left(\sum_{\varsigma\cdot=1}^{\varphi\cdot}\omega_{\varsigma\cdot}-\sum_{\varsigma\cdot=1}^{\varphi\cdot-1}\varrho_{\varsigma\cdot}\right)+(1-t)\left(\sum_{\varsigma\cdot=1}^{\varphi\cdot}\omega_{\varsigma\cdot}-\sum_{\varsigma\cdot=1}^{\varphi\cdot-1}\rho_{\varsigma\cdot}\right)\right)$$
$$\leq tf\left(\sum_{\varsigma\cdot=1}^{\varphi\cdot}\omega_{\varsigma\cdot}-\sum_{\varsigma\cdot=1}^{\varphi\cdot-1}\varrho_{\varsigma\cdot}\right)+(1-t)f\left(\sum_{\varsigma\cdot=1}^{\varphi\cdot}\omega_{\varsigma\cdot}-\sum_{\varsigma\cdot=1}^{\varphi\cdot-1}\rho_{\varsigma\cdot}\right). \tag{15}$$

By adding (14) and (15) and by setting $\sigma_1 = 1$ and $n = 1$ in Theorem 1, it follows that

$$f\left(t\left(\sum_{\varsigma\cdot=1}^{\varphi\cdot}\omega_{\varsigma\cdot}-\sum_{\varsigma\cdot=1}^{\varphi\cdot-1}\rho_{\varsigma\cdot}\right)+(1-t)\left(\sum_{\varsigma\cdot=1}^{\varphi\cdot}\omega_{\varsigma\cdot}-\sum_{\varsigma\cdot=1}^{\varphi\cdot-1}\varrho_{\varsigma\cdot}\right)\right)$$
$$+f\left(t\left(\sum_{\varsigma\cdot=1}^{\varphi\cdot}\omega_{\varsigma\cdot}-\sum_{\varsigma\cdot=1}^{\varphi\cdot-1}\varrho_{\varsigma\cdot}\right)+(1-t)\left(\sum_{\varsigma\cdot=1}^{\varphi\cdot}\omega_{\varsigma\cdot}-\sum_{\varsigma\cdot=1}^{\varphi\cdot-1}\rho_{\varsigma\cdot}\right)\right)$$
$$\leq f\left(\sum_{\varsigma\cdot=1}^{\varphi\cdot}\omega_{\varsigma\cdot}-\sum_{\varsigma\cdot=1}^{\varphi\cdot-1}\rho_{\varsigma\cdot}\right)+f\left(\sum_{\varsigma\cdot=1}^{\varphi\cdot}\omega_{\varsigma\cdot}-\sum_{\varsigma\cdot=1}^{\varphi\cdot-1}\varrho_{\varsigma\cdot}\right)$$
$$\leq 2\sum_{\varsigma\cdot=1}^{\varphi\cdot}f(\omega_{\varsigma\cdot})-\left(\sum_{\varsigma\cdot=1}^{\varphi\cdot-1}f(\rho_{\varsigma\cdot})+\sum_{\varsigma\cdot=1}^{\varphi\cdot-1}f(\varrho_{\varsigma\cdot})\right).$$

Integrating over $t \in [0, 1]$, and adopting the above mentioned procedure, we get

$${}^{CF}_{\left(\sum_{\varsigma\cdot=1}^{\varphi\cdot}\omega_{\varsigma\cdot}-\sum_{\varsigma\cdot=1}^{\varphi\cdot-1}\varrho_{\varsigma\cdot}\right)}I^{\alpha}f\left(\sum_{\varsigma\cdot=1}^{\varphi\cdot}\omega_{\varsigma\cdot}-\sum_{\varsigma\cdot=1}^{\varphi\cdot-1}\rho_{\varsigma\cdot}\right)+{}^{CF}I^{\alpha}_{\left(\sum_{\varsigma\cdot=1}^{\varphi\cdot}\omega_{\varsigma\cdot}-\sum_{\varsigma\cdot=1}^{\varphi\cdot-1}\rho_{\varsigma\cdot}\right)}f\left(\sum_{\varsigma\cdot=1}^{\varphi\cdot}\omega_{\varsigma\cdot}-\sum_{\varsigma\cdot=1}^{\varphi\cdot-1}\varrho_{\varsigma\cdot}\right)$$
$$\leq\frac{\alpha\sum_{\varsigma\cdot=1}^{\varphi\cdot-1}\left(\varrho_{\varsigma\cdot}-\rho_{\varsigma\cdot}\right)}{B(\alpha)}\left(f\left(\sum_{\varsigma\cdot=1}^{\varphi\cdot}\omega_{\varsigma\cdot}-\sum_{\varsigma\cdot=1}^{\varphi\cdot-1}\rho_{\varsigma\cdot}\right)+f\left(\sum_{\varsigma\cdot=1}^{\varphi\cdot}\omega_{\varsigma\cdot}-\sum_{\varsigma\cdot=1}^{\varphi\cdot-1}\varrho_{\varsigma\cdot}\right)\right)$$
$$+\frac{1-\alpha}{B(\alpha)}f\left(\sum_{\varsigma\cdot=1}^{\varphi\cdot}\omega_{\varsigma\cdot}-\sum_{\varsigma\cdot=1}^{\varphi\cdot-1}\rho_{\varsigma\cdot}\right)+\frac{1-\alpha}{B(\alpha)}f\left(\sum_{\varsigma\cdot=1}^{\varphi\cdot}\omega_{\varsigma\cdot}-\sum_{\varsigma\cdot=1}^{\varphi\cdot-1}\varrho_{\varsigma\cdot}\right)$$
$$\leq\frac{\alpha\sum_{\varsigma\cdot=1}^{\varphi\cdot-1}\left(\varrho_{\varsigma\cdot}-\rho_{\varsigma\cdot}\right)}{B(\alpha)}\left(2\sum_{\varsigma\cdot=1}^{\varphi\cdot}f(\omega_{\varsigma\cdot})-\left(\sum_{\varsigma\cdot=1}^{\varphi\cdot-1}f(\rho_{\varsigma\cdot})+\sum_{\varsigma\cdot=1}^{\varphi\cdot-1}f(\varrho_{\varsigma\cdot})\right)\right)$$

$$+\frac{1-\alpha}{B(\alpha)} f\left(\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}} \omega_{\varsigma^{\cdot}}-\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1} \rho_{\varsigma^{\cdot}}\right)+\frac{1-\alpha}{B(\alpha)} f\left(\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}} \omega_{\varsigma^{\cdot}}-\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1} \varrho_{\varsigma^{\cdot}}\right).$$

Re-arrangement of the above inequality, gives

$$\begin{aligned}
&\frac{B(\alpha)}{2\alpha \sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1}\left(\varrho_{\varsigma^{\cdot}}-\rho_{\varsigma^{\cdot}}\right)}\left[{}^{CF}_{\left(\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}} \omega_{\varsigma^{\cdot}}-\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1} \varrho_{\varsigma^{\cdot}}\right)} I^{\alpha} f\left(\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}} \omega_{\varsigma^{\cdot}}-\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1} \rho_{\varsigma^{\cdot}}\right)\right. \\
&+{}^{CF} I^{\alpha}_{\left(\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}} \omega_{\varsigma^{\cdot}}-\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1} \rho_{\varsigma^{\cdot}}\right)} f\left(\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}} \omega_{\varsigma^{\cdot}}-\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1} \varrho_{\varsigma^{\cdot}}\right) \\
&\left.-\frac{1-\alpha}{B(\alpha)}\left\{f\left(\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}} \omega_{\varsigma^{\cdot}}-\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1} \rho_{\varsigma^{\cdot}}\right)+f\left(\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}} \omega_{\varsigma^{\cdot}}-\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1} \varrho_{\varsigma^{\cdot}}\right)\right\}\right] \qquad (16)\\
&\leq \frac{1}{2}\left[f\left(\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}} \omega_{\varsigma^{\cdot}}-\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1} \rho_{\varsigma^{\cdot}}\right)+f\left(\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}} \omega_{\varsigma^{\cdot}}-\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1} \varrho_{\varsigma^{\cdot}}\right)\right] \\
&\leq \sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}} f\left(\omega_{\varsigma^{\cdot}}\right)-\frac{1}{2}\left(\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1} f\left(\rho_{\varsigma^{\cdot}}\right)+\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1} f\left(\varrho_{\varsigma^{\cdot}}\right)\right).
\end{aligned}$$

Now, by combining (13) and (16), we get complete result of the inequality (8). □

Remark 2

(i) For $\varphi^{\cdot}=2$, in (8), we get the following inequality, which is a new addition to the literature.

$$\begin{aligned}
& f\left(\omega_1+\omega_2-\frac{\rho_1+\varrho_1}{2}\right) \\
& \leq \frac{B(\alpha)}{2\alpha \sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1}\left(\varrho_1-\rho_1\right)} \times\left[{}^{CF}_{\left(\omega_1+\omega_2-\rho_1\right)} I^{\alpha} f\left(\omega_1+\omega_2-\rho_1\right)+{}^{CF} I^{\alpha}_{\left(\omega_1+\omega_2-\rho_1\right)} f\left(\omega_1+\omega_2-\varrho_1\right)\right. \\
& \left.-\frac{1-\alpha}{B(\alpha)}\left\{f\left(\omega_1+\omega_2-\rho_1\right)+f\left(\omega_1+\omega_2-\varrho_1\right)\right\}\right] \\
& \leq \frac{1}{2}\left[f\left(\omega_1+\omega_2-\rho_1\right)+f\left(\omega_1+\omega_2-\varrho_1\right)\right]
\end{aligned}$$

$$\leq f(\omega_1)+f(\omega_2)-\frac{1}{2}\big(f(\rho_1)+f(\varrho_1)\big).$$

(ii) For $\varphi^{\cdot}=2$, $\rho_1=\omega_1$, and $\varrho_1=\omega_2$ in (8), we get the following inequality which is also a new addition to the literature.

$$f\left(\frac{\omega_1+\omega_2}{2}\right)$$

$$\leq \frac{B(\alpha)}{2\alpha\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1}(\omega_2-\omega_1)}\left[{}^{CF}_{\omega_1}I^{\alpha}f(\omega_2)+{}^{CF}I^{\alpha}_{\omega_2}f(\omega_1)-\frac{1-\alpha}{B(\alpha)}\{f(\omega_1)+f(\omega_2)\}\right]$$

$$\leq \frac{1}{2}\left[f(\omega_1)+f(\omega_2)\right].$$

4. Weighted extentions of the principal results

Theorem 4 Let $f\colon I\to\mathbb{R}$ denote a function throughout the interval I, where f is assumed to be convex. Let the tuples $\boldsymbol{\omega}$, ρ, ϱ, $\boldsymbol{p}\in I^{\varphi^{\cdot}}$ be defined by $\boldsymbol{\omega}=(\omega_1,\omega_2,\ldots,\omega_{\varphi^{\cdot}})$, $\rho=(\rho_1,\rho_2,\ldots,\rho_{\varphi^{\cdot}})$, $\varrho=(\varrho_1,\varrho_2,\ldots,\varrho_{\varphi^{\cdot}})$, and $\boldsymbol{p}=(p_1,p_2\ldots,p_{\varphi^{\cdot}})$ with $\rho_{\varphi^{\cdot}}>\varrho_{\varphi^{\cdot}}$ and $\omega_{\varsigma^{\cdot}},\rho_{\varsigma^{\cdot}},\varrho_{\varsigma^{\cdot}},p_{\varsigma^{\cdot}}\in I$, $p_{\varsigma^{\cdot}}\geq 0$, $p_{\varphi^{\cdot}}\neq 0$ for all $\varsigma^{\cdot}=1,2,\ldots,\varphi^{\cdot}$, $\eta=\frac{1}{p_{\varphi^{\cdot}}}$. Suppose further that $0\leq\alpha\leq 1$, $B(\alpha)$ denotes normalization mapping, and the tuples ρ, ϱ are decreasing which satisfy

$$\sum_{\varsigma^{\cdot}=1}^{s}p_{\varsigma^{\cdot}}\rho_{\varsigma^{\cdot}}\leq\sum_{\varsigma^{\cdot}=1}^{s}p_{\varsigma^{\cdot}}\omega_{\varsigma^{\cdot}},\ \sum_{\varsigma^{\cdot}=1}^{s}p_{\varsigma^{\cdot}}\varrho_{\varsigma^{\cdot}}\leq\sum_{\varsigma^{\cdot}=1}^{s}p_{\varsigma^{\cdot}}\omega_{\varsigma^{\cdot}},\ (1\leq s\leq\varphi^{\cdot}-1),$$

and

$$\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}}p_{\varsigma^{\cdot}}\omega_{\varsigma^{\cdot}}=\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}}p_{\varsigma^{\cdot}}\rho_{\varsigma^{\cdot}},\ \sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}}p_{\varsigma^{\cdot}}\omega_{\varsigma^{\cdot}}=\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}}p_{\varsigma^{\cdot}}\varrho_{\varsigma^{\cdot}}.$$

Then,

$$f\left(\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}}\eta p_{\varsigma^{\cdot}}\omega_{\varsigma^{\cdot}}-\eta\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1}\left(\frac{p_{\varsigma^{\cdot}}\rho_{\varsigma^{\cdot}}+p_{\varsigma^{\cdot}}\varrho_{\varsigma^{\cdot}}}{2}\right)\right)$$

$$\leq\frac{B(\alpha)}{\alpha\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1}(\eta p_{\varsigma^{\cdot}}\varrho_{\varsigma^{\cdot}}-\eta p_{\varsigma^{\cdot}}\rho_{\varsigma^{\cdot}})}\left[{}^{CF}I^{\alpha}_{\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}}\eta p_{\varsigma^{\cdot}}\omega_{\varsigma^{\cdot}}-\eta\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1}\left(\frac{p_{\varsigma^{\cdot}}\rho_{\varsigma^{\cdot}}+p_{\varsigma^{\cdot}}\varrho_{\varsigma^{\cdot}}}{2}\right)}f\left(\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}}\eta p_{\varsigma^{\cdot}}\omega_{\varsigma^{\cdot}}-\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1}\eta p_{\varsigma^{\cdot}}\varrho_{\varsigma^{\cdot}}\right)\right.$$

$$+{}^{CF}_{\sum_{\varsigma^\cdot=1}^{\varphi^\cdot}\eta p_{\varsigma^\cdot}\omega_{\varsigma^\cdot}-\eta\sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1}\left(\frac{p_{\varsigma^\cdot}\rho_{\varsigma^\cdot}+p_{\varsigma^\cdot}\varrho_{\varsigma^\cdot}}{2}\right)} I^{\alpha} f\left(\sum_{\varsigma^\cdot=1}^{\varphi^\cdot}\eta p_{\varsigma^\cdot}\omega_{\varsigma^\cdot}-\sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1}\eta p_{\varsigma^\cdot}\rho_{\varsigma^\cdot}\right)$$

$$-\frac{1-\alpha}{B(\alpha)}\left\{f\left(\sum_{\varsigma^\cdot=1}^{\varphi^\cdot}\eta p_{\varsigma^\cdot}\omega_{\varsigma^\cdot}-\sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1}\eta p_{\varsigma^\cdot}\rho_{\varsigma^\cdot}\right)+f\left(\sum_{\varsigma^\cdot=1}^{\varphi^\cdot}\eta p_{\varsigma^\cdot}\omega_{\varsigma^\cdot}-\sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1}\eta p_{\varsigma^\cdot}\varrho_{\varsigma^\cdot}\right)\right\}\Bigg]$$

$$\leq\sum_{\varsigma^\cdot=1}^{\varphi^\cdot}\eta p_{\varsigma^\cdot}f(\omega_{\varsigma^\cdot})-\frac{1}{2}\left[\sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1}\eta p_{\varsigma^\cdot}f(\rho_{\varsigma^\cdot})+\sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1}\eta p_{\varsigma^\cdot}f(\varrho_{\varsigma^\cdot})\right]. \tag{17}$$

Proof. Let $0\leq t\leq 1$. The establishment of the desired outcome is carried out in the following manner:

$$\begin{aligned}
&f\left(\sum_{\varsigma^\cdot=1}^{\varphi^\cdot}\eta p_{\varsigma^\cdot}\omega_{\varsigma^\cdot}-\eta\sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1}\left(\frac{p_{\varsigma^\cdot}\rho_{\varsigma^\cdot}+p_{\varsigma^\cdot}\varrho_{\varsigma^\cdot}}{2}\right)\right)\\
&=f\left\{\frac{1}{2}\left(\sum_{\varsigma^\cdot=1}^{\varphi^\cdot}\eta p_{\varsigma^\cdot}\omega_{\varsigma^\cdot}-\sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1}\eta p_{\varsigma^\cdot}\rho_{\varsigma^\cdot}+\sum_{\varsigma^\cdot=1}^{\varphi^\cdot}\eta p_{\varsigma^\cdot}\omega_{\varsigma^\cdot}-\sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1}\eta p_{\varsigma^\cdot}\varrho_{\varsigma^\cdot}\right)\right\}\\
&=f\left\{\frac{1}{2}\left(\sum_{\varsigma^\cdot=1}^{\varphi^\cdot}\eta p_{\varsigma^\cdot}\omega_{\varsigma^\cdot}-\left(\frac{t}{2}\sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1}\eta p_{\varsigma^\cdot}\rho_{\varsigma^\cdot}+\frac{2-t}{2}\sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1}\eta p_{\varsigma^\cdot}\varrho_{\varsigma^\cdot}\right)\right.\right.\\
&\quad\left.\left.+\sum_{\varsigma^\cdot=1}^{\varphi^\cdot}\eta p_{\varsigma^\cdot}\omega_{\varsigma^\cdot}-\left(\frac{t}{2}\sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1}\eta p_{\varsigma^\cdot}\varrho_{\varsigma^\cdot}+\frac{2-t}{2}\sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1}\eta p_{\varsigma^\cdot}\rho_{\varsigma^\cdot}\right)\right)\right\}.
\end{aligned} \tag{18}$$

The convexity of f ensures that (18) results in the below inequality

$$\begin{aligned}
&f\left(\sum_{\varsigma^\cdot=1}^{\varphi^\cdot}\eta p_{\varsigma^\cdot}\omega_{\varsigma^\cdot}-\eta\sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1}\left(\frac{p_{\varsigma^\cdot}\rho_{\varsigma^\cdot}+p_{\varsigma^\cdot}\varrho_{\varsigma^\cdot}}{2}\right)\right)\\
&\leq\frac{1}{2}\left\{f\left(\sum_{\varsigma^\cdot=1}^{\varphi^\cdot}\eta p_{\varsigma^\cdot}\omega_{\varsigma^\cdot}-\left(\frac{t}{2}\sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1}\eta p_{\varsigma^\cdot}\rho_{\varsigma^\cdot}+\frac{2-t}{2}\sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1}\eta p_{\varsigma^\cdot}\varrho_{\varsigma^\cdot}\right)\right)\right.\\
&\quad\left.+f\left(\sum_{\varsigma^\cdot=1}^{\varphi^\cdot}\eta p_{\varsigma^\cdot}\omega_{\varsigma^\cdot}-\left(\frac{t}{2}\sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1}\eta p_{\varsigma^\cdot}\varrho_{\varsigma^\cdot}+\frac{2-t}{2}\sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1}\eta p_{\varsigma^\cdot}\rho_{\varsigma^\cdot}\right)\right)\right\}.
\end{aligned}$$

By integrating over $t\in[0, 1]$, we obtain

$$f\left(\sum_{\varsigma^\cdot=1}^{\varphi^\cdot}\eta p_{\varsigma^\cdot}\omega_{\varsigma^\cdot}-\eta\sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1}\left(\frac{p_{\varsigma^\cdot}\rho_{\varsigma^\cdot}+p_{\varsigma^\cdot}\varrho_{\varsigma^\cdot}}{2}\right)\right)$$

$$\leq\frac{1}{\sum\limits_{\varsigma^\cdot=1}^{\varphi^\cdot-1}\left(\eta p_{\varsigma^\cdot}\varrho_{\varsigma^\cdot}-\eta p_{\varsigma^\cdot}\rho_{\varsigma^\cdot}\right)} \tag{19}$$

$$\times\left[\int_{\sum\limits_{\varsigma^\cdot=1}^{\varphi^\cdot}\eta p_{\varsigma^\cdot}\omega_{\varsigma^\cdot}-\sum\limits_{\varsigma^\cdot=1}^{\varphi^\cdot-1}\eta p_{\varsigma^\cdot}\varrho_{\varsigma^\cdot}}^{\sum\limits_{\varsigma^\cdot=1}^{\varphi^\cdot}\eta p_{\varsigma^\cdot}\omega_{\varsigma^\cdot}-\eta\sum\limits_{\varsigma^\cdot=1}^{\varphi^\cdot-1}\left(\frac{p_{\varsigma^\cdot}\rho_{\varsigma^\cdot}+p_{\varsigma^\cdot}\varrho_{\varsigma^\cdot}}{2}\right)}f(u)du+\int_{\sum\limits_{\varsigma^\cdot=1}^{\varphi^\cdot}\eta p_{\varsigma^\cdot}\omega_{\varsigma^\cdot}-\eta\sum\limits_{\varsigma^\cdot=1}^{\varphi^\cdot-1}\left(\frac{p_{\varsigma^\cdot}\rho_{\varsigma^\cdot}+p_{\varsigma^\cdot}\varrho_{\varsigma^\cdot}}{2}\right)}^{\sum\limits_{\varsigma^\cdot=1}^{\varphi^\cdot}\eta p_{\varsigma^\cdot}\omega_{\varsigma^\cdot}-\sum\limits_{\varsigma^\cdot=1}^{\varphi^\cdot-1}\eta p_{\varsigma^\cdot}\rho_{\varsigma^\cdot}}f(u)du\right].$$

The fact that $\sum_{\varsigma^\cdot=1}^{\varphi^\cdot}p_{\varsigma^\cdot}\omega_{\varsigma^\cdot}=\sum_{\varsigma^\cdot=1}^{\varphi^\cdot}p_{\varsigma^\cdot}\rho_{\varsigma^\cdot}$, $\sum_{\varsigma^\cdot=1}^{\varphi^\cdot}p_{\varsigma^\cdot}\omega_{\varsigma^\cdot}=\sum_{\varsigma^\cdot=1}^{\varphi^\cdot}p_{\varsigma^\cdot}\varrho_{\varsigma^\cdot}$, leads to $\sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1}p_{\varsigma^\cdot}\varrho_{\varsigma^\cdot}-\sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1}p_{\varsigma^\cdot}\rho_{\varsigma^\cdot}=p_{\varphi^\cdot}\rho_{\varphi^\cdot}-p_{\varphi^\cdot}\varrho_{\varphi^\cdot}$. Also, taking into account $p_{\varphi^\cdot}\neq 0$ and $\rho_{\varphi^\cdot}>\varrho_{\varphi^\cdot}$ and doing simple calculation, we can prove that $\sum_{\varsigma^\cdot=1}^{\varphi^\cdot}\eta p_{\varsigma^\cdot}\omega_{\varsigma^\cdot}-\eta\sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1}\left(\frac{p_{\varsigma^\cdot}\rho_{\varsigma^\cdot}+p_{\varsigma^\cdot}\varrho_{\varsigma^\cdot}}{2}\right)>\sum_{\varsigma^\cdot=1}^{\varphi^\cdot}\eta p_{\varsigma^\cdot}\omega_{\varsigma^\cdot}-\sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1}\eta p_{\varsigma^\cdot}\varrho_{\varsigma^\cdot}$ and $\sum_{\varsigma^\cdot=1}^{\varphi^\cdot}\eta p_{\varsigma^\cdot}\omega_{\varsigma^\cdot}-\eta\sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1}\left(\frac{p_{\varsigma^\cdot}\rho_{\varsigma^\cdot}+p_{\varsigma^\cdot}\varrho_{\varsigma^\cdot}}{2}\right)<\sum_{\varsigma^\cdot=1}^{\varphi^\cdot}\eta p_{\varsigma^\cdot}\omega_{\varsigma^\cdot}-\sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1}\eta p_{\varsigma^\cdot}\rho_{\varsigma^\cdot}$. This enables us to apply the definition of Caputo-Fabrizio fractional integral operator.

Now, by multiplying (19) with $\frac{\alpha\sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1}\left(\eta p_{\varsigma^\cdot}\varrho_{\varsigma^\cdot}-\eta p_{\varsigma^\cdot}\rho_{\varsigma^\cdot}\right)}{B(\alpha)}$ and then adding $\frac{1-\alpha}{B(\alpha)}f\left(\sum_{\varsigma^\cdot=1}^{\varphi^\cdot}\eta p_{\varsigma^\cdot}\omega_{\varsigma^\cdot}-\sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1}\eta p_{\varsigma^\cdot}\rho_{\varsigma^\cdot}\right)+\frac{1-\alpha}{B(\alpha)}f\left(\sum_{\varsigma^\cdot=1}^{\varphi^\cdot}\eta p_{\varsigma^\cdot}\omega_{\varsigma^\cdot}-\sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1}\eta p_{\varsigma^\cdot}\varrho_{\varsigma^\cdot}\right)$, we have

$$\frac{\alpha\sum\limits_{\varsigma^\cdot=1}^{\varphi^\cdot-1}\left(\eta p_{\varsigma^\cdot}\varrho_{\varsigma^\cdot}-\eta p_{\varsigma^\cdot}\rho_{\varsigma^\cdot}\right)}{B(\alpha)}f\left(\sum_{\varsigma^\cdot=1}^{\varphi^\cdot}\eta p_{\varsigma^\cdot}\omega_{\varsigma^\cdot}-\eta\sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1}\left(\frac{p_{\varsigma^\cdot}\rho_{\varsigma^\cdot}+p_{\varsigma^\cdot}\varrho_{\varsigma^\cdot}}{2}\right)\right)$$

$$+\frac{1-\alpha}{B(\alpha)}f\left(\sum_{\varsigma^\cdot=1}^{\varphi^\cdot}\eta p_{\varsigma^\cdot}\omega_{\varsigma^\cdot}-\sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1}\eta p_{\varsigma^\cdot}\rho_{\varsigma^\cdot}\right)+\frac{1-\alpha}{B(\alpha)}f\left(\sum_{\varsigma^\cdot=1}^{\varphi^\cdot}\eta p_{\varsigma^\cdot}\omega_{\varsigma^\cdot}-\sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1}\eta p_{\varsigma^\cdot}\varrho_{\varsigma^\cdot}\right)$$

$$\leq\frac{1-\alpha}{B(\alpha)}f\left(\sum_{\varsigma^\cdot=1}^{\varphi^\cdot}\eta p_{\varsigma^\cdot}\omega_{\varsigma^\cdot}-\sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1}\eta p_{\varsigma^\cdot}\varrho_{\varsigma^\cdot}\right)+\frac{\alpha}{B(\alpha)}\int_{\sum\limits_{\varsigma^\cdot=1}^{\varphi^\cdot}\eta p_{\varsigma^\cdot}\omega_{\varsigma^\cdot}-\sum\limits_{\varsigma^\cdot=1}^{\varphi^\cdot-1}\eta p_{\varsigma^\cdot}\varrho_{\varsigma^\cdot}}^{\sum\limits_{\varsigma^\cdot=1}^{\varphi^\cdot}\eta p_{\varsigma^\cdot}\omega_{\varsigma^\cdot}-\eta\sum\limits_{\varsigma^\cdot=1}^{\varphi^\cdot-1}\left(\frac{p_{\varsigma^\cdot}\rho_{\varsigma^\cdot}+p_{\varsigma^\cdot}\varrho_{\varsigma^\cdot}}{2}\right)}f(u)du$$

$$+\frac{1-\alpha}{B(\alpha)} f\left(\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}} \eta p_{\varsigma^{\cdot}} \omega_{\varsigma^{\cdot}}-\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1} \eta p_{\varsigma^{\cdot}} \rho_{\varsigma^{\cdot}}\right)+\frac{\alpha}{B(\alpha)} \int_{\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}} \eta p_{\varsigma^{\cdot}} \omega_{\varsigma^{\cdot}}-\eta \sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1}\left(\frac{p_{\varsigma^{\cdot}} \rho_{\varsigma^{\cdot}}+p_{\varsigma^{\cdot}} \varrho_{\varsigma^{\cdot}}}{2}\right)}^{\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}} \eta p_{\varsigma^{\cdot}} \omega_{\varsigma^{\cdot}}-\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1} \eta p_{\varsigma^{\cdot}} \rho_{\varsigma^{\cdot}}} f(u) d u .$$

$$={}^{CF}I^{\alpha}_{\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}} \eta p_{\varsigma^{\cdot}} \omega_{\varsigma^{\cdot}}-\eta \sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1}\left(\frac{p_{\varsigma^{\cdot}} \rho_{\varsigma^{\cdot}}+p_{\varsigma^{\cdot}} \varrho_{\varsigma^{\cdot}}}{2}\right)} f\left(\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}} \eta p_{\varsigma^{\cdot}} \omega_{\varsigma^{\cdot}}-\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1} \eta p_{\varsigma^{\cdot}} \varrho_{\varsigma^{\cdot}}\right)$$

$$+{}^{CF}_{\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}} \eta p_{\varsigma^{\cdot}} \omega_{\varsigma^{\cdot}}-\eta \sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1}\left(\frac{p_{\varsigma^{\cdot}} \rho_{\varsigma^{\cdot}}+p_{\varsigma^{\cdot}} \varrho_{\varsigma^{\cdot}}}{2}\right)} I^{\alpha} f\left(\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}} \eta p_{\varsigma^{\cdot}} \omega_{\varsigma^{\cdot}}-\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1} \eta p_{\varsigma^{\cdot}} \rho_{\varsigma^{\cdot}}\right)$$

Re-arrangement of the above inequality yields

$$\begin{aligned}
& f\left(\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}} \eta p_{\varsigma^{\cdot}} \omega_{\varsigma^{\cdot}}-\eta \sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1}\left(\frac{p_{\varsigma^{\cdot}} \rho_{\varsigma^{\cdot}}+p_{\varsigma^{\cdot}} \varrho_{\varsigma^{\cdot}}}{2}\right)\right) \\
& \leq \frac{B(\alpha)}{\alpha \sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1}\left(\eta p_{\varsigma^{\cdot}} \varrho_{\varsigma^{\cdot}}-\eta p_{\varsigma^{\cdot}} \rho_{\varsigma^{\cdot}}\right)}\left[{}^{CF}I^{\alpha}_{\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}} \eta p_{\varsigma^{\cdot}} \omega_{\varsigma^{\cdot}}-\eta \sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1}\left(\frac{p_{\varsigma^{\cdot}} \rho_{\varsigma^{\cdot}}+p_{\varsigma^{\cdot}} \varrho_{\varsigma^{\cdot}}}{2}\right)} f\left(\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}} \eta p_{\varsigma^{\cdot}} \omega_{\varsigma^{\cdot}}-\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1} \eta p_{\varsigma^{\cdot}} \varrho_{\varsigma^{\cdot}}\right)\right. \\
& +{}^{CF}_{\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}} \eta p_{\varsigma^{\cdot}} \omega_{\varsigma^{\cdot}}-\eta \sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1}\left(\frac{p_{\varsigma^{\cdot}} \rho_{\varsigma^{\cdot}}+p_{\varsigma^{\cdot}} \varrho_{\varsigma^{\cdot}}}{2}\right)} I^{\alpha} f\left(\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}} \eta p_{\varsigma^{\cdot}} \omega_{\varsigma^{\cdot}}-\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1} \eta p_{\varsigma^{\cdot}} \rho_{\varsigma^{\cdot}}\right) \\
& \left.-\frac{1-\alpha}{B(\alpha)}\left\{f\left(\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}} \eta p_{\varsigma^{\cdot}} \omega_{\varsigma^{\cdot}}-\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1} \eta p_{\varsigma^{\cdot}} \rho_{\varsigma^{\cdot}}\right)+f\left(\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}} \eta p_{\varsigma^{\cdot}} \omega_{\varsigma^{\cdot}}-\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1} \eta p_{\varsigma^{\cdot}} \varrho_{\varsigma^{\cdot}}\right)\right\}\right].
\end{aligned} \tag{20}$$

This ends the first part of (17). Next, by using Lemma 1 with the choices $\sigma_1=\frac{t}{2}$, $n=2$, and $\sigma_2=\frac{2-t}{2}$, we derive the second portion of the inequality (17) in the subsequent way:

$$\begin{aligned}
& f\left(\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}} \eta p_{\varsigma^{\cdot}} \omega_{\varsigma^{\cdot}}-\left(\frac{t}{2} \sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1} \eta p_{\varsigma^{\cdot}} \rho_{\varsigma^{\cdot}}+\frac{2-t}{2} \sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1} \eta p_{\varsigma^{\cdot}} \varrho_{\varsigma^{\cdot}}\right)\right) \\
& \leq \sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}} \eta p_{\varsigma^{\cdot}} f(\omega_{\varsigma^{\cdot}})-\left(\frac{t}{2} \sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1} \eta p_{\varsigma^{\cdot}} f(\rho_{\varsigma^{\cdot}})+\frac{2-t}{2} \sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1} \eta p_{\varsigma^{\cdot}} f(\varrho_{\varsigma^{\cdot}})\right)
\end{aligned} \tag{21}$$

and

$$f\left(\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}}\eta p_{\varsigma^{\cdot}}\omega_{\varsigma^{\cdot}}-\left(\frac{t}{2}\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1}\eta p_{\varsigma^{\cdot}}\varrho_{\varsigma^{\cdot}}+\frac{2-t}{2}\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1}\eta p_{\varsigma^{\cdot}}\rho_{\varsigma^{\cdot}}\right)\right)$$

$$\leq\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}}\eta p_{\varsigma^{\cdot}}f(\omega_{\varsigma^{\cdot}})-\left(\frac{t}{2}\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1}\eta p_{\varsigma^{\cdot}}f(\varrho_{\varsigma^{\cdot}})+\frac{2-t}{2}\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1}\eta p_{\varsigma^{\cdot}}f(\rho_{\varsigma^{\cdot}})\right). \tag{22}$$

Adding (21) and (22), we get

$$f\left(\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}}\eta p_{\varsigma^{\cdot}}\omega_{\varsigma^{\cdot}}-\left(\frac{t}{2}\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1}\eta p_{\varsigma^{\cdot}}\rho_{\varsigma^{\cdot}}+\frac{2-t}{2}\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1}\eta p_{\varsigma^{\cdot}}\varrho_{\varsigma^{\cdot}}\right)\right)$$

$$+f\left(\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}}\eta p_{\varsigma^{\cdot}}\omega_{\varsigma^{\cdot}}-\left(\frac{t}{2}\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1}\eta p_{\varsigma^{\cdot}}\varrho_{\varsigma^{\cdot}}+\frac{2-t}{2}\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1}\eta p_{\varsigma^{\cdot}}\rho_{\varsigma^{\cdot}}\right)\right)$$

$$\leq 2\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}}\eta p_{\varsigma^{\cdot}}f(\omega_{\varsigma^{\cdot}})-\left(\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1}\eta p_{\varsigma^{\cdot}}f(\rho_{\varsigma^{\cdot}})+\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1}\eta p_{\varsigma^{\cdot}}f(\varrho_{\varsigma^{\cdot}})\right).$$

Integrating over $t \in [0, 1]$, and adopting the above mentioned procedure, we get

$${}^{CF}I^{\alpha}_{\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}}\eta p_{\varsigma^{\cdot}}\omega_{\varsigma^{\cdot}}-\eta\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1}\left(\frac{p_{\varsigma^{\cdot}}\rho_{\varsigma^{\cdot}}+p_{\varsigma^{\cdot}}\varrho_{\varsigma^{\cdot}}}{2}\right)}f\left(\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}}\eta p_{\varsigma^{\cdot}}\omega_{\varsigma^{\cdot}}-\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1}\eta p_{\varsigma^{\cdot}}\varrho_{\varsigma^{\cdot}}\right)$$

$$+{}^{CF}_{\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}}\eta p_{\varsigma^{\cdot}}\omega_{\varsigma^{\cdot}}-\eta\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1}\left(\frac{p_{\varsigma^{\cdot}}\rho_{\varsigma^{\cdot}}+p_{\varsigma^{\cdot}}\varrho_{\varsigma^{\cdot}}}{2}\right)}I^{\alpha}f\left(\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}}\eta p_{\varsigma^{\cdot}}\omega_{\varsigma^{\cdot}}-\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1}\eta p_{\varsigma^{\cdot}}\rho_{\varsigma^{\cdot}}\right)$$

$$\leq\frac{\alpha\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1}\left(\eta p_{\varsigma^{\cdot}}\varrho_{\varsigma^{\cdot}}-\eta p_{\varsigma^{\cdot}}\rho_{\varsigma^{\cdot}}\right)}{B(\alpha)}\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}}\eta p_{\varsigma^{\cdot}}f(\omega_{\varsigma^{\cdot}}) \tag{23}$$

$$-\frac{\alpha\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1}\left(\eta p_{\varsigma^{\cdot}}\varrho_{\varsigma^{\cdot}}-\eta p_{\varsigma^{\cdot}}\rho_{\varsigma^{\cdot}}\right)}{B(\alpha)}\left(\frac{\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1}\eta p_{\varsigma^{\cdot}}f(\rho_{\varsigma^{\cdot}})+\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1}\eta p_{\varsigma^{\cdot}}f(\varrho_{\varsigma^{\cdot}})}{2}\right)$$

$$+\frac{1-\alpha}{B(\alpha)}\left[f\left(\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}}\eta p_{\varsigma^{\cdot}}\omega_{\varsigma^{\cdot}}-\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1}\eta p_{\varsigma^{\cdot}}\rho_{\varsigma^{\cdot}}\right)+f\left(\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}}\eta p_{\varsigma^{\cdot}}\omega_{\varsigma^{\cdot}}-\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1}\eta p_{\varsigma^{\cdot}}\varrho_{\varsigma^{\cdot}}\right)\right].$$

Now, by re-arranging (23) and then joining with (20), we get complete result of (17). □

Using Lemma 2, we deduce the following result.

Theorem 5 Let $f: I \to \mathbb{R}$ denote a function throughout the interval I, where f is assumed to be convex. Let the tuples $\boldsymbol{\omega}, \rho, \varrho, \boldsymbol{p} \in I^{\varphi^{\cdot}}$ be defined by $\boldsymbol{\omega} = (\omega_1, \omega_2, \ldots, \omega_{\varphi^{\cdot}})$, $\rho = (\rho_1, \rho_2, \ldots, \rho_{\varphi^{\cdot}})$, $\varrho = (\varrho_1, \varrho_2, \ldots, \varrho_{\varphi^{\cdot}})$, and $\boldsymbol{p} = (p_1, p_2 \ldots, p_{\varphi^{\cdot}})$ with $\rho_{\varphi^{\cdot}} > \varrho_{\varphi^{\cdot}}$ and $\omega_{\varsigma^{\cdot}}, \rho_{\varsigma^{\cdot}}, \varrho_{\varsigma^{\cdot}}, p_{\varsigma^{\cdot}} \in I$, $p_{\varsigma^{\cdot}} \geq 0$, $p_{\varphi^{\cdot}} \neq 0$ for all $\varsigma^{\cdot} = 1, 2, \ldots, \varphi^{\cdot}$, $\eta = \frac{1}{p_{\varphi^{\cdot}}}$. Suppose further that $0 \leq \alpha \leq 1$, $B(\alpha)$ represents normalization mapping, and the tuples ρ, $\boldsymbol{\omega} - \rho$, ϱ and $\boldsymbol{\omega} - \varrho$ exhibit identical monotonic behavior and satisfying

$$\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}} p_{\varsigma^{\cdot}} \omega_{\varsigma^{\cdot}} = \sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}} p_{\varsigma^{\cdot}} \rho_{\varsigma^{\cdot}}, \quad \sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}} p_{\varsigma^{\cdot}} \omega_{\varsigma^{\cdot}} = \sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}} p_{\varsigma^{\cdot}} \varrho_{\varsigma^{\cdot}}.$$

Then, inequality (17) holds.

Proof. Lemma 2, when applied with the procedure utilized in Theorem 4, one can establishes (17). □

Remark 3 We can also obtain weighted extensions of Theorem 3 by utilizing Lemma 1 and Lemma 2.

5. Key integral identity and bounds for the gap of the major results

5.1 *Formulation of the core integral identity*

Lemma 3 Let $f: I \to \mathbb{R}$ represents a differentiable function whose domain is in the interval I. Let the tuples $\boldsymbol{\omega}, \rho, \varrho \in I^{\varphi^{\cdot}}$ be defined by $\boldsymbol{\omega} = (\omega_1, \omega_2, \ldots, \omega_{\varphi^{\cdot}})$, $\rho = (\rho_1, \rho_2, \ldots, \rho_{\varphi^{\cdot}})$ and $\varrho = (\varrho_1, \varrho_2, \ldots, \varrho_{\varphi^{\cdot}})$ with $\rho_{\varphi^{\cdot}} > \varrho_{\varphi^{\cdot}}$ and $\omega_{\varsigma^{\cdot}}, \rho_{\varsigma^{\cdot}}, \varrho_{\varsigma^{\cdot}} \in I$ for all $\varsigma^{\cdot} = 1, 2, \ldots, \varphi^{\cdot}$. Suppose further that $0 \leq \alpha \leq 1$, $B(\alpha)$ represents normalization mapping and $f' \in L(I)$. Then

$$\begin{aligned}\zeta(\omega_{\varsigma^{\cdot}}, \rho_{\varsigma^{\cdot}}, \varrho_{\varsigma^{\cdot}}, f) = & \frac{\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1} (\varrho_{\varsigma^{\cdot}} - \rho_{\varsigma^{\cdot}})}{4} \left[\int_0^1 t f' \left(\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}} \omega_{\varsigma^{\cdot}} - \left(\frac{t}{2} \sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1} \varrho_{\varsigma^{\cdot}} + \frac{2-t}{2} \sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1} \rho_{\varsigma^{\cdot}} \right) \right) dt \right. \\ & \left. - \int_0^1 t f' \left(\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}} \omega_{\varsigma^{\cdot}} - \left(\frac{t}{2} \sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1} \rho_{\varsigma^{\cdot}} + \frac{2-t}{2} \sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1} \varrho_{\varsigma^{\cdot}} \right) \right) dt \right],\end{aligned} \tag{24}$$

where

$$\begin{aligned}\zeta(\omega_{\varsigma^{\cdot}}, \rho_{\varsigma^{\cdot}}, \varrho_{\varsigma^{\cdot}}, f) = & \frac{B(\alpha)}{\alpha \sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1} (\varrho_{\varsigma^{\cdot}} - \rho_{\varsigma^{\cdot}})} \left[{}^{CF}I^{\alpha}_{\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}} \omega_{\varsigma^{\cdot}} - \sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1} \left(\frac{\rho_{\varsigma^{\cdot}} + \varrho_{\varsigma^{\cdot}}}{2} \right)} f \left(\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}} \omega_{\varsigma^{\cdot}} - \sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1} \varrho_{\varsigma^{\cdot}} \right) \right. \\ & + {}^{CF}_{\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}} \omega_{\varsigma^{\cdot}} - \sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1} \left(\frac{\rho_{\varsigma^{\cdot}} + \varrho_{\varsigma^{\cdot}}}{2} \right)} I^{\alpha} f \left(\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}} \omega_{\varsigma^{\cdot}} - \sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1} \rho_{\varsigma^{\cdot}} \right) \\ & \left. - \frac{1-\alpha}{B(\alpha)} \left\{ f \left(\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}} \omega_{\varsigma^{\cdot}} - \sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1} \rho_{\varsigma^{\cdot}} \right) + f \left(\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}} \omega_{\varsigma^{\cdot}} - \sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1} \varrho_{\varsigma^{\cdot}} \right) \right\} \right] - f \left(\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}} \omega_{\varsigma^{\cdot}} - \sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1} \left(\frac{\rho_{\varsigma^{\cdot}} + \varrho_{\varsigma^{\cdot}}}{2} \right) \right).\end{aligned}$$

Proof. With the goal of achieving the desired outcome, we begin by assuming that

$$K = \int_0^1 t f'\left(\sum_{\varsigma^\cdot=1}^{\varphi^\cdot} \omega_{\varsigma^\cdot} - \left(\frac{t}{2}\sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1} \varrho_{\varsigma^\cdot} + \frac{2-t}{2}\sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1} \rho_{\varsigma^\cdot}\right)\right)dt$$

$$- \int_0^1 t f'\left(\sum_{\varsigma^\cdot=1}^{\varphi^\cdot} \omega_{\varsigma^\cdot} - \left(\frac{t}{2}\sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1} \rho_{\varsigma^\cdot} + \frac{2-t}{2}\sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1} \varrho_{\varsigma^\cdot}\right)\right)dt \tag{25}$$

$$= K_1 - K_2$$

By using Integration by parts, we determine K_1 and K_2 as follows:

$$K_1 = \int_0^1 t f'\left(\sum_{\varsigma^\cdot=1}^{\varphi^\cdot} \omega_{\varsigma^\cdot} - \left(\frac{t}{2}\sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1} \varrho_{\varsigma^\cdot} + \frac{2-t}{2}\sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1} \rho_{\varsigma^\cdot}\right)\right)dt$$

$$= t\frac{f\left(\sum_{\varsigma^\cdot=1}^{\varphi^\cdot} \omega_{\varsigma^\cdot} - \left(\frac{t}{2}\sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1} \varrho_{\varsigma^\cdot} + \frac{2-t}{2}\sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1} \rho_{\varsigma^\cdot}\right)\right)}{-\frac{1}{2}\sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1}\left(\varrho_{\varsigma^\cdot} - \rho_{\varsigma^\cdot}\right)}\Bigg|_0^1 + \frac{2}{\sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1}\left(\varrho_{\varsigma^\cdot} - \rho_{\varsigma^\cdot}\right)}$$

$$\times \int_0^1 f\left(\sum_{\varsigma^\cdot=1}^{\varphi^\cdot} \omega_{\varsigma^\cdot} - \left(\frac{t}{2}\sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1} \rho_{\varsigma^\cdot} + \frac{2-t}{2}\sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1} \varrho_{\varsigma^\cdot}\right)\right)dt$$

$$K_1 = \frac{-2}{\sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1}\left(\varrho_{\varsigma^\cdot} - \rho_{\varsigma^\cdot}\right)} f\left(\sum_{\varsigma^\cdot=1}^{\varphi^\cdot} \omega_{\varsigma^\cdot} - \sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1}\left(\frac{\rho_{\varsigma^\cdot} + \varrho_{\varsigma^\cdot}}{2}\right)\right)$$

$$+ \frac{4}{\left(\sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1}\left(\varrho_{\varsigma^\cdot} - \rho_{\varsigma^\cdot}\right)\right)^2} \int_{\sum_{\varsigma^\cdot=1}^{\varphi^\cdot} \omega_{\varsigma^\cdot} - \sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1}\left(\frac{\rho_{\varsigma^\cdot}+\varrho_{\varsigma^\cdot}}{2}\right)}^{\sum_{\varsigma^\cdot=1}^{\varphi^\cdot} \omega_{\varsigma^\cdot} - \sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1} \rho_{\varsigma^\cdot}} f(u)du \tag{26}$$

And

$$K_2 = \int_0^1 t f'\left(\sum_{\varsigma^\cdot=1}^{\varphi^\cdot} \omega_{\varsigma^\cdot} - \left(\frac{t}{2}\sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1} \rho_{\varsigma^\cdot} + \frac{2-t}{2}\sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1} \varrho_{\varsigma^\cdot}\right)\right)dt$$

$$= t\frac{f\left(\sum_{\varsigma^\cdot=1}^{\varphi^\cdot} \omega_{\varsigma^\cdot} - \left(\frac{t}{2}\sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1} \rho_{\varsigma^\cdot} + \frac{2-t}{2}\sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1} \varrho_{\varsigma^\cdot}\right)\right)}{\frac{1}{2}\sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1}\left(\varrho_{\varsigma^\cdot} - \rho_{\varsigma^\cdot}\right)}\Bigg|_0^1 - \frac{2}{\sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1}\left(\varrho_{\varsigma^\cdot} - \rho_{\varsigma^\cdot}\right)}$$

$$\times \int_0^1 f\left(\sum_{\varsigma^\cdot=1}^{\varphi^\cdot} \omega_{\varsigma^\cdot} - \left(\frac{t}{2}\sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1} \rho_{\varsigma^\cdot} + \frac{2-t}{2}\sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1} \varrho_{\varsigma^\cdot}\right)\right)dt$$

$$K_2 = \frac{2}{\sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1}\left(\varrho_{\varsigma^\cdot} - \rho_{\varsigma^\cdot}\right)} f\left(\sum_{\varsigma^\cdot=1}^{\varphi^\cdot} \omega_{\varsigma^\cdot} - \sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1}\left(\frac{\rho_{\varsigma^\cdot} + \varrho_{\varsigma^\cdot}}{2}\right)\right)$$

$$- \frac{4}{\left(\sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1}\left(\varrho_{\varsigma^\cdot} - \rho_{\varsigma^\cdot}\right)\right)^2} \int_{\sum_{\varsigma^\cdot=1}^{\varphi^\cdot} \omega_{\varsigma^\cdot} - \sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1} \varrho_{\varsigma^\cdot}}^{\sum_{\varsigma^\cdot=1}^{\varphi^\cdot} \omega_{\varsigma^\cdot} - \sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1}\left(\frac{\rho_{\varsigma^\cdot} + \varrho_{\varsigma^\cdot}}{2}\right)} f(u)du \tag{27}$$

By subtracting (27) from (26), we have

$$K_1 - K_2 = \frac{4}{\left(\sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1}\left(\varrho_{\varsigma^\cdot} - \rho_{\varsigma^\cdot}\right)\right)^2}\left[\int_{\sum_{\varsigma^\cdot=1}^{\varphi^\cdot} \omega_{\varsigma^\cdot} - \sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1} \varrho_{\varsigma^\cdot}}^{\sum_{\varsigma^\cdot=1}^{\varphi^\cdot} \omega_{\varsigma^\cdot} - \sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1}\left(\frac{\rho_{\varsigma^\cdot} + \varrho_{\varsigma^\cdot}}{2}\right)} f(u)du + \int_{\sum_{\varsigma^\cdot=1}^{\varphi^\cdot} \omega_{\varsigma^\cdot} - \sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1}\left(\frac{\rho_{\varsigma^\cdot} + \varrho_{\varsigma^\cdot}}{2}\right)}^{\sum_{\varsigma^\cdot=1}^{\varphi^\cdot} \omega_{\varsigma^\cdot} - \sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1} \rho_{\varsigma^\cdot}} f(u)du\right]$$

$$- \frac{4}{\sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1}\left(\varrho_{\varsigma^\cdot} - \rho_{\varsigma^\cdot}\right)} f\left(\sum_{\varsigma^\cdot=1}^{\varphi^\cdot} \omega_{\varsigma^\cdot} - \sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1}\left(\frac{\rho_{\varsigma^\cdot} + \varrho_{\varsigma^\cdot}}{2}\right)\right) \tag{28}$$

On multiplying (28) with $\frac{\sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1}\left(\varrho_{\varsigma^\cdot} - \rho_{\varsigma^\cdot}\right)}{4}$, we deduce

$$\frac{\sum_{\varsigma\cdot=1}^{\varphi\cdot-1}\left(\varrho_{\varsigma\cdot}-\rho_{\varsigma\cdot}\right)}{4}(K_1-K_2)$$

$$=\frac{1}{\sum_{\varsigma\cdot=1}^{\varphi\cdot-1}\left(\varrho_{\varsigma\cdot}-\rho_{\varsigma\cdot}\right)}\left[\int_{\sum_{\varsigma\cdot=1}^{\varphi\cdot}\omega_{\varsigma\cdot}-\sum_{\varsigma\cdot=1}^{\varphi\cdot-1}\varrho_{\varsigma\cdot}}^{\sum_{\varsigma\cdot=1}^{\varphi\cdot}\omega_{\varsigma\cdot}-\sum_{\varsigma\cdot=1}^{\varphi\cdot-1}\left(\frac{\rho_{\varsigma\cdot}+\varrho_{\varsigma\cdot}}{2}\right)} f(u)du+\int_{\sum_{\varsigma\cdot=1}^{\varphi\cdot}\omega_{\varsigma\cdot}-\sum_{\varsigma\cdot=1}^{\varphi\cdot-1}\left(\frac{\rho_{\varsigma\cdot}+\varrho_{\varsigma\cdot}}{2}\right)}^{\sum_{\varsigma\cdot=1}^{\varphi\cdot}\omega_{\varsigma\cdot}-\sum_{\varsigma\cdot=1}^{\varphi\cdot-1}\rho_{\varsigma\cdot}} f(u)du\right] \tag{29}$$

$$-f\left(\sum_{\varsigma\cdot=1}^{\varphi\cdot}\omega_{\varsigma\cdot}-\sum_{\varsigma\cdot=1}^{\varphi\cdot-1}\left(\frac{\rho_{\varsigma\cdot}+\varrho_{\varsigma\cdot}}{2}\right)\right)$$

By multiplying (29) with $\frac{\alpha\sum_{\varsigma\cdot=1}^{\varphi\cdot-1}\left(\varrho_{\varsigma\cdot}-\rho_{\varsigma\cdot}\right)}{B(\alpha)}$ and then adding $\frac{1-\alpha}{B(\alpha)}f\left(\sum_{\varsigma\cdot=1}^{\varphi\cdot}\omega_{\varsigma\cdot}-\sum_{\varsigma\cdot=1}^{\varphi\cdot-1}\rho_{\varsigma\cdot}\right)+\frac{1-\alpha}{B(\alpha)}f\left(\sum_{\varsigma\cdot=1}^{\varphi\cdot}\omega_{\varsigma\cdot}-\sum_{\varsigma\cdot=1}^{\varphi\cdot-1}\varrho_{\varsigma\cdot}\right)$, we get

$$\frac{\alpha\left(\sum_{\varsigma\cdot=1}^{\varphi\cdot-1}\left(\varrho_{\varsigma\cdot}-\rho_{\varsigma\cdot}\right)\right)^2}{4B(\alpha)}(K_1-K_2)+\frac{1-\alpha}{B(\alpha)}\left\{f\left(\sum_{\varsigma\cdot=1}^{\varphi\cdot}\omega_{\varsigma\cdot}-\sum_{\varsigma\cdot=1}^{\varphi\cdot-1}\rho_{\varsigma\cdot}\right)+f\left(\sum_{\varsigma\cdot=1}^{\varphi\cdot}\omega_{\varsigma\cdot}-\sum_{\varsigma\cdot=1}^{\varphi\cdot-1}\varrho_{\varsigma\cdot}\right)\right\}$$

$$\begin{aligned}&={}^{CF}I^{\alpha}_{\sum_{\varsigma\cdot=1}^{\varphi\cdot}\omega_{\varsigma\cdot}-\sum_{\varsigma\cdot=1}^{\varphi\cdot-1}\left(\frac{\rho_{\varsigma\cdot}+\varrho_{\varsigma\cdot}}{2}\right)}f\left(\sum_{\varsigma\cdot=1}^{\varphi\cdot}\omega_{\varsigma\cdot}-\sum_{\varsigma\cdot=1}^{\varphi\cdot-1}\varrho_{\varsigma\cdot}\right)\\&+{}^{CF}_{\sum_{\varsigma\cdot=1}^{\varphi\cdot}\omega_{\varsigma\cdot}-\sum_{\varsigma\cdot=1}^{\varphi\cdot-1}\left(\frac{\rho_{\varsigma\cdot}+\varrho_{\varsigma\cdot}}{2}\right)}I^{\alpha}f\left(\sum_{\varsigma\cdot=1}^{\varphi\cdot}\omega_{\varsigma\cdot}-\sum_{\varsigma\cdot=1}^{\varphi\cdot-1}\rho_{\varsigma\cdot}\right)\\&-\frac{\alpha\sum_{\varsigma\cdot=1}^{\varphi\cdot-1}\left(\varrho_{\varsigma\cdot}-\rho_{\varsigma\cdot}\right)}{B(\alpha)}f\left(\sum_{\varsigma\cdot=1}^{\varphi\cdot}\omega_{\varsigma\cdot}-\sum_{\varsigma\cdot=1}^{\varphi\cdot-1}\left(\frac{\rho_{\varsigma\cdot}+\varrho_{\varsigma\cdot}}{2}\right)\right)\end{aligned} \tag{30}$$

Through multiplication of (30) with $\frac{B(\alpha)}{\alpha\sum_{\varsigma\cdot=1}^{\varphi\cdot-1}\left(\varrho_{\varsigma\cdot}-\rho_{\varsigma\cdot}\right)}$, we achieve

$$
\frac{\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1}\left(\varrho_{\varsigma^{\cdot}}-\rho_{\varsigma^{\cdot}}\right)}{4}(K_1-K_2)
$$

$$
\begin{aligned}
&= \frac{B(\alpha)}{\alpha\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1}\left(\varrho_{\varsigma^{\cdot}}-\rho_{\varsigma^{\cdot}}\right)}\left[{}^{CF}I^{\alpha}_{\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}}\omega_{\varsigma^{\cdot}}-\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1}\left(\frac{\rho_{\varsigma^{\cdot}}+\varrho_{\varsigma^{\cdot}}}{2}\right)} f\left(\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}}\omega_{\varsigma^{\cdot}}-\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1}\varrho_{\varsigma^{\cdot}}\right)\right.\\
&\quad + {}^{CF}_{\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}}\omega_{\varsigma^{\cdot}}-\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1}\left(\frac{\rho_{\varsigma^{\cdot}}+\varrho_{\varsigma^{\cdot}}}{2}\right)}I^{\alpha} f\left(\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}}\omega_{\varsigma^{\cdot}}-\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1}\rho_{\varsigma^{\cdot}}\right)\\
&\quad \left.-\frac{1-\alpha}{B(\alpha)}\left\{f\left(\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}}\omega_{\varsigma^{\cdot}}-\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1}\rho_{\varsigma^{\cdot}}\right)+f\left(\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}}\omega_{\varsigma^{\cdot}}-\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1}\varrho_{\varsigma^{\cdot}}\right)\right\}\right]\\
&\quad - f\left(\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}}\omega_{\varsigma^{\cdot}}-\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1}\left(\frac{\rho_{\varsigma^{\cdot}}+\varrho_{\varsigma^{\cdot}}}{2}\right)\right)
\end{aligned}
\tag{31}
$$

Now, by inserting (31) in (25), we acquire (24). □

Remark 4 (i) If we substitute $\varphi^{\cdot}=2$ in (24), then we get the following equality of [59].

$$
\begin{aligned}
&\frac{B(\alpha)}{\alpha(\varrho_1-\rho_1)}\left[{}^{CF}I^{\alpha}_{\omega_1+\omega_2-\frac{\rho_1+\varrho_1}{2}} f(\omega_1+\omega_2-\varrho_1)+{}^{CF}_{\omega_1+\omega_2-\frac{\rho_1+\varrho_1}{2}}I^{\alpha} f(\omega_1+\omega_2-\rho_1)\right.\\
&\left.-\frac{1-\alpha}{B(\alpha)}\left[f(\omega_1+\omega_2-\rho_1)+f(\omega_1+\omega_2-\varrho_1)\right]\right]-f\left(\omega_1+\omega_2-\frac{\rho_1+\varrho_1}{2}\right)\\
&=\frac{\varrho_1-\rho_1}{4}\left[\int_0^1 tf'\left(\omega_1+\omega_2-\left(\frac{t}{2}\varrho_1+\frac{2-t}{2}\rho_1\right)\right)dt\right.\\
&\left.-\int_0^1 tf'\left(\omega_1+\omega_2-\left(\frac{t}{2}\rho_1+\frac{2-t}{2}\varrho_1\right)\right)dt\right]
\end{aligned}
$$

(ii) If we substitute $\varphi^{\cdot}=2$, $\rho_1=\omega_1$, $\varrho_1=\omega_2$ in (24), then we get the following equality of [60].

$$
\begin{aligned}
&\frac{B(\alpha)}{\alpha(\omega_2-\omega_1)}\left[{}^{CF}I^{\alpha}_{\frac{\omega_1+\omega_2}{2}} f(\omega_1)+{}^{CF}_{\frac{\omega_1+\omega_2}{2}}I^{\alpha} f(\omega_2)-\frac{1-\alpha}{B(\alpha)}\left[f(\omega_2)+f(\omega_1)\right]\right]-f\left(\frac{\omega_1+\omega_2}{2}\right)\\
&=\frac{\omega_2-\omega_1}{4}\left[\int_0^1 tf'\left(\frac{t}{2}\omega_1+\frac{2-t}{2}\omega_2\right)dt-\int_0^1 tf'\left(\frac{t}{2}\omega_2+\frac{2-t}{2}\omega_1\right)dt\right]
\end{aligned}
$$

5.2 *Estimates based on the integral identity*

Now, employing Lemma 3, we obtain the subsequent findings:

Theorem 6 Let $f\colon I \to \mathbb{R}$ represents a differentiable function whose domain is in the interval I. Let the tuples $\boldsymbol{\omega}$, ρ, $\varrho \in I^{\varphi^{\cdot}}$ be defined by $\boldsymbol{\omega} = (\omega_1, \omega_2, \ldots, \omega_{\varphi^{\cdot}})$, $\rho = (\rho_1, \rho_2, \ldots, \rho_{\varphi^{\cdot}})$ and $\varrho = (\varrho_1, \varrho_2, \ldots, \varrho_{\varphi^{\cdot}})$ with $\rho_{\varphi^{\cdot}} > \varrho_{\varphi^{\cdot}}$ and $\omega_{\varsigma^{\cdot}}, \rho_{\varsigma^{\cdot}}, \varrho_{\varsigma^{\cdot}} \in I$ for all $\varsigma^{\cdot} = 1, 2, \ldots, \varphi^{\cdot}$. Suppose further that $\rho \prec \boldsymbol{\omega}$ and $\varrho \prec \boldsymbol{\omega}$, $|f'|$ is convex function, $0 \le \alpha \le 1$, and $B(\alpha)$ denotes normalization mapping. Then

$$\left|\zeta(\omega_{\varsigma^{\cdot}}, \rho_{\varsigma^{\cdot}}, \varrho_{\varsigma^{\cdot}}, f)\right| \le \frac{\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1} |\varrho_{\varsigma^{\cdot}} - \rho_{\varsigma^{\cdot}}|}{4} \left[\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}} |f'(\omega_{\varsigma^{\cdot}})| - \frac{1}{2}\left\{\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1} |f'(\rho_{\varsigma^{\cdot}})| + \sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1} |f'(\varrho_{\varsigma^{\cdot}})|\right\}\right]. \tag{32}$$

Proof. Applying the modulus to Lemma 3, we obtain

$$\begin{aligned} \left|\zeta(\omega_{\varsigma^{\cdot}}, \rho_{\varsigma^{\cdot}}, \varrho_{\varsigma^{\cdot}}, f)\right| = &\left|\frac{\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1} (\varrho_{\varsigma^{\cdot}} - \rho_{\varsigma^{\cdot}})}{4}\left[\int_0^1 t f'\left(\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}} \omega_{\varsigma^{\cdot}} - \left(\frac{2-t}{2}\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1} \rho_{\varsigma^{\cdot}} + \frac{t}{2}\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1} \varrho_{\varsigma^{\cdot}}\right)\right)dt\right.\right. \\ &\left.\left. - \int_0^1 t f'\left(\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}} \omega_{\varsigma^{\cdot}} - \left(\frac{2-t}{2}\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1} \varrho_{\varsigma^{\cdot}} + \frac{t}{2}\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1} \rho_{\varsigma^{\cdot}}\right)\right)dt\right]\right|. \end{aligned} \tag{33}$$

By employing property of modulus in (33), we get

$$\begin{aligned} \left|\zeta(\omega_{\varsigma^{\cdot}}, \rho_{\varsigma^{\cdot}}, \varrho_{\varsigma^{\cdot}}, f)\right| \le &\frac{\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1} |\varrho_{\varsigma^{\cdot}} - \rho_{\varsigma^{\cdot}}|}{4}\left[\int_0^1 t\left|f'\left(\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}} \omega_{\varsigma^{\cdot}} - \left(\frac{2-t}{2}\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1} \rho_{\varsigma^{\cdot}} + \frac{t}{2}\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1} \varrho_{\varsigma^{\cdot}}\right)\right)\right|dt\right. \\ &\left. + \int_0^1 t\left|f'\left(\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}} \omega_{\varsigma^{\cdot}} - \left(\frac{2-t}{2}\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1} \varrho_{\varsigma^{\cdot}} + \frac{t}{2}\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1} \rho_{\varsigma^{\cdot}}\right)\right)\right|dt\right] \end{aligned} \tag{34}$$

Using Theorem 1 for the choices $\sigma_1 = \frac{2-t}{2}$, $n = 2$ and $\sigma_2 = \frac{t}{2}$ in (34), we get

$$\begin{aligned} &\left|\zeta(\omega_{\varsigma^{\cdot}}, \rho_{\varsigma^{\cdot}}, \varrho_{\varsigma^{\cdot}}, f)\right| \\ &\le \frac{\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1} |\varrho_{\varsigma^{\cdot}} - \rho_{\varsigma^{\cdot}}|}{4}\left[\int_0^1 t\left(\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}} |f'(\omega_{\varsigma^{\cdot}})| - \left(\frac{2-t}{2}\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1} |f'(\rho_{\varsigma^{\cdot}})| + \frac{t}{2}\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1} |f'(\varrho_{\varsigma^{\cdot}})|\right)\right)dt\right. \\ &\left. + \int_0^1 t\left(\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}} |f'(\omega_{\varsigma^{\cdot}})| - \left(\frac{2-t}{2}\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1} |f'(\varrho_{\varsigma^{\cdot}})| + \frac{t}{2}\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1} |f'(\rho_{\varsigma^{\cdot}})|\right)\right)dt\right] \end{aligned}$$

$$=\frac{\sum\limits_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1}|\varrho_{\varsigma^{\cdot}}-\rho_{\varsigma^{\cdot}}|}{4}$$

$$\times\left[\left(\int_0^1 tdt\right)\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}}|f'(\omega_{\varsigma^{\cdot}})|-\left(\int_0^1\left(\frac{t-t^2}{2}\right)dt\right)\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1}|f'(\rho_{\varsigma^{\cdot}})|-\left(\int_0^1\frac{t^2}{2}dt\right)\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1}|f'(\varrho_{\varsigma^{\cdot}})|\right.$$

$$\left.+\left(\int_0^1 tdt\right)\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}}|f'(\omega_{\varsigma^{\cdot}})|-\left(\int_0^1\left(\frac{t-t^2}{2}\right)dt\right)\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1}|f'(\varrho_{\varsigma^{\cdot}})|-\left(\int_0^1\frac{t^2}{2}dt\right)\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1}|f'(\rho_{\varsigma^{\cdot}})|\right] \quad (35)$$

Since,

$$\int_0^1 tdt=\frac{1}{2},\ \int_0^1\left(\frac{t-t^2}{2}\right)dt=\frac{1}{3},\ \int_0^1\frac{t^2}{2}dt=\frac{1}{6}.$$

Therefore, (35) implies

$$\left|\zeta(\omega_{\varsigma^{\cdot}},\rho_{\varsigma^{\cdot}},\varrho_{\varsigma^{\cdot}},f)\right|\leq\frac{\sum\limits_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1}|\varrho_{\varsigma^{\cdot}}-\rho_{\varsigma^{\cdot}}|}{4}\left[\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}}|f'(\omega_{\varsigma^{\cdot}})|-\frac{1}{2}\left\{\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1}|f'(\rho_{\varsigma^{\cdot}})|+\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1}|f'(\varrho_{\varsigma^{\cdot}})|\right\}\right].$$

This finishes the proof. □

Remark 5

(i) If we substitute $\varphi^{\cdot}=2$ in (32), then we get Theorem 4.1 of [59].

(ii) If we substitute $\varphi^{\cdot}=2$, $\rho_1=\omega_1$, $\varrho_1=\omega_2$ in (32), then we get Theorem 5 of [60].

Theorem 7 Let $f\colon I\to\mathbb{R}$ represents a differentiable function whose domain is in the interval I. Let the tuples $\boldsymbol{\omega}$, ρ, $\varrho\in I^{\varphi^{\cdot}}$ be defined by $\boldsymbol{\omega}=(\omega_1,\omega_2,\ldots,\omega_{\varphi^{\cdot}})$, $\rho=(\rho_1,\rho_2,\ldots,\rho_{\varphi^{\cdot}})$ and $\varrho=(\varrho_1,\varrho_2,\ldots,\varrho_{\varphi^{\cdot}})$ with $\rho_{\varphi^{\cdot}}>\varrho_{\varphi^{\cdot}}$ and $\omega_{\varsigma^{\cdot}},\rho_{\varsigma^{\cdot}},\varrho_{\varsigma^{\cdot}}\in I$ for all $\varsigma^{\cdot}=1,2,\ldots,\varphi^{\cdot}$. Suppose further that $\rho\prec\boldsymbol{\omega}$ and $\varrho\prec\boldsymbol{\omega}$, $q>1$ such that $\frac{1}{p}+\frac{1}{q}=1$, $|f'|^q$ is convex function, $0\leq\alpha\leq1$, and $B(\alpha)$ represents normalization mapping. Then

$$\left|\zeta(\omega_{\varsigma^{\cdot}},\rho_{\varsigma^{\cdot}},\varrho_{\varsigma^{\cdot}},f)\right|$$

$$\leq\frac{\sum\limits_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1}|\varrho_{\varsigma^{\cdot}}-\rho_{\varsigma^{\cdot}}|}{4(p+1)^{\frac{1}{p}}}\left[\left(\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}}|f'(\omega_{\varsigma^{\cdot}})|^q-\frac{1}{4}\left(3\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1}|f'(\rho_{\varsigma^{\cdot}})|^q+\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1}|f'(\varrho_{\varsigma^{\cdot}})|^q\right)\right)^{\frac{1}{q}}\right. \quad (36)$$

$$\left.+\left(\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}}|f'(\omega_{\varsigma^{\cdot}})|^q-\frac{1}{4}\left(3\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1}|f'(\varrho_{\varsigma^{\cdot}})|^q+\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1}|f'(\rho_{\varsigma^{\cdot}})|^q\right)\right)^{\frac{1}{q}}\right].$$

Proof. Lemma 3, when used together with the modulus property, gives

$$\left|\zeta(\omega_{\varsigma^{\cdot}}, \rho_{\varsigma^{\cdot}}, \varrho_{\varsigma^{\cdot}}, f)\right| \leq \frac{\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1} |\varrho_{\varsigma^{\cdot}} - \rho_{\varsigma^{\cdot}}|}{4} \left[\int_0^1 t \left| f'\left(\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}} \omega_{\varsigma^{\cdot}} - \left(\frac{2-t}{2} \sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1} \rho_{\varsigma^{\cdot}} + \frac{t}{2} \sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1} \varrho_{\varsigma^{\cdot}}\right)\right)\right| dt\right.$$

$$\left. + \int_0^1 t \left| f'\left(\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}} \omega_{\varsigma^{\cdot}} - \left(\frac{2-t}{2} \sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1} \varrho_{\varsigma^{\cdot}} + \frac{t}{2} \sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1} \rho_{\varsigma^{\cdot}}\right)\right)\right| dt\right]$$

By employing Hölder inequality, we get

$$\begin{aligned}
&\left|\zeta(\omega_{\varsigma^{\cdot}}, \rho_{\varsigma^{\cdot}}, \varrho_{\varsigma^{\cdot}}, f)\right| \\
&\leq \frac{\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1} |\varrho_{\varsigma^{\cdot}} - \rho_{\varsigma^{\cdot}}|}{4} \left[\left(\int_0^1 t^p dt\right)^{\frac{1}{p}} \left(\int_0^1 \left| f'\left(\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}} \omega_{\varsigma^{\cdot}} - \left(\frac{2-t}{2} \sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1} \rho_{\varsigma^{\cdot}} + \frac{t}{2} \sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1} \varrho_{\varsigma^{\cdot}}\right)\right)\right|^q dt\right)^{\frac{1}{q}}\right. \\
&\left. + \left(\int_0^1 t^p dt\right)^{\frac{1}{p}} \left(\int_0^1 \left| f'\left(\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}} \omega_{\varsigma^{\cdot}} - \left(\frac{2-t}{2} \sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1} \varrho_{\varsigma^{\cdot}} + \frac{t}{2} \sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1} \rho_{\varsigma^{\cdot}}\right)\right)\right|^q dt\right)^{\frac{1}{q}}\right]
\end{aligned} \tag{37}$$

Utilizing Theorem 1 for the case $n = 2$ with $\sigma_1 = \frac{2-t}{2}$ and $\sigma_2 = \frac{t}{2}$ in (37) in light of the convexity of $|f'|^q$, we achieve

$$\begin{aligned}
&\left|\zeta(\omega_{\varsigma^{\cdot}}, \rho_{\varsigma^{\cdot}}, \varrho_{\varsigma^{\cdot}}, f)\right| \\
&\leq \frac{\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1} |\varrho_{\varsigma^{\cdot}} - \rho_{\varsigma^{\cdot}}|}{4} \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \\
&\times \left[\left(\int_0^1 \left(\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}} |f'(\omega_{\varsigma^{\cdot}})|^q - \frac{2-t}{2} \sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1} |f'(\rho_{\varsigma^{\cdot}})|^q - \frac{t}{2} \sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1} |f'(\varrho_{\varsigma^{\cdot}})|^q\right) dt\right)^{\frac{1}{q}}\right. \\
&\left. + \left(\int_0^1 \left(\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}} |f'(\omega_{\varsigma^{\cdot}})|^q - \frac{2-t}{2} \sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1} |f'(\varrho_{\varsigma^{\cdot}})|^q - \frac{t}{2} \sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1} |f'(\rho_{\varsigma^{\cdot}})|^q\right) dt\right)^{\frac{1}{q}}\right].
\end{aligned} \tag{38}$$

Since,

$$\int_0^1 \left(\frac{2-t}{2}\right) dt = \frac{3}{4}, \int_0^1 \frac{t}{2} dt = \frac{1}{4}.$$

Therefore, (38) implies

$$\left|\zeta(\omega_{\varsigma^{\cdot}}, \rho_{\varsigma^{\cdot}}, \varrho_{\varsigma^{\cdot}}, f)\right|$$

$$\leq \frac{\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1}|\varrho_{\varsigma^{\cdot}}-\rho_{\varsigma^{\cdot}}|}{4(p+1)^{\frac{1}{p}}}\left[\left(\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}}|f'(\omega_{\varsigma^{\cdot}})|^{q}-\frac{1}{4}\left(3\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1}|f'(\rho_{\varsigma^{\cdot}})|^{q}+\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1}|f'(\varrho_{\varsigma^{\cdot}})|^{q}\right)\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}}|f'(\omega_{\varsigma^{\cdot}})|^{q}-\frac{1}{4}\left(3\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1}|f'(\varrho_{\varsigma^{\cdot}})|^{q}+\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1}|f'(\rho_{\varsigma^{\cdot}})|^{q}\right)\right)^{\frac{1}{q}}\right].$$

Thus, the desired result has been proven. □

Remark 6

(i) If we substitute $\varphi^{\cdot}=2$ in (36), then we get Theorem 4.2 of [59].

(ii) If we substitute $\varphi^{\cdot}=2$, $\rho_1=\omega_1$, $\varrho_1=\omega_2$ in (36), then we get Theorem 6 of [60].

Theorem 8 Let $f\colon I\to\mathbb{R}$ represents a differentiable function whose domain is in the interval I. Let the tuples $\boldsymbol{\omega}$, ρ, $\varrho\in I^{\varphi^{\cdot}}$ be defined by $\boldsymbol{\omega}=(\omega_1, \omega_2, \ldots, \omega_{\varphi^{\cdot}})$, $\rho=(\rho_1, \rho_2, \ldots, \rho_{\varphi^{\cdot}})$ and $\varrho=(\varrho_1, \varrho_2, \ldots, \varrho_{\varphi^{\cdot}})$ with $\rho_{\varphi^{\cdot}}>\varrho_{\varphi^{\cdot}}$ and $\omega_{\varsigma^{\cdot}}, \rho_{\varsigma^{\cdot}}, \varrho_{\varsigma^{\cdot}}\in I$ for all $\varsigma^{\cdot}=1, 2, \ldots, \varphi^{\cdot}$. Suppose further that $\rho\prec\boldsymbol{\omega}$ and $\varrho\prec\boldsymbol{\omega}$, $|f'|^{q}, (1<q)$ is convex function, $0\leq\alpha\leq 1$, and $B(\alpha)$ denotes normalization mapping. Then

$$\left|\zeta(\omega_{\varsigma^{\cdot}}, \rho_{\varsigma^{\cdot}}, \varrho_{\varsigma^{\cdot}}, f)\right|$$

$$\leq \frac{\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1}|\varrho_{\varsigma^{\cdot}}-\rho_{\varsigma^{\cdot}}|}{4}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left[\left(\frac{1}{2}\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}}|f'(\omega_{\varsigma^{\cdot}})|^{q}-\frac{1}{6}\left(\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1}|f'(\varrho_{\varsigma^{\cdot}})|^{q}+2\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1}|f'(\rho_{\varsigma^{\cdot}})|^{q}\right)\right)^{\frac{1}{q}}\right. \tag{39}$$

$$\left.+\left(\frac{1}{2}\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}}|f'(\omega_{\varsigma^{\cdot}})|^{q}-\frac{1}{6}\left(\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1}|f'(\rho_{\varsigma^{\cdot}})|^{q}+2\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1}|f'(\varrho_{\varsigma^{\cdot}})|^{q}\right)\right)^{\frac{1}{q}}\right].$$

Proof. Lemma 3, when used together with the modulus property, gives

$$\left|\zeta(\omega_{\varsigma^{\cdot}}, \rho_{\varsigma^{\cdot}}, \varrho_{\varsigma^{\cdot}}, f)\right|\leq\frac{\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1}|\varrho_{\varsigma^{\cdot}}-\rho_{\varsigma^{\cdot}}|}{4}\left[\int_0^1 t\left|f'\left(\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}}\omega_{\varsigma^{\cdot}}-\left(\frac{2-t}{2}\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1}\rho_{\varsigma^{\cdot}}+\frac{t}{2}\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1}\varrho_{\varsigma^{\cdot}}\right)\right)\right|dt\right.$$

$$\left.+\int_0^1 t\left|f'\left(\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}}\omega_{\varsigma^{\cdot}}-\left(\frac{2-t}{2}\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1}\varrho_{\varsigma^{\cdot}}+\frac{t}{2}\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1}\rho_{\varsigma^{\cdot}}\right)\right)\right|dt\right]$$

By employing power mean inequality in the above integral, we get

$$\left|\zeta(\omega_{\varsigma^\cdot}, \rho_{\varsigma^\cdot}, \varrho_{\varsigma^\cdot}, f)\right|$$

$$\leq \frac{\sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1}|\varrho_{\varsigma^\cdot}-\rho_{\varsigma^\cdot}|}{4}\left[\left(\int_0^1 t dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t\left|f'\left(\sum_{\varsigma^\cdot=1}^{\varphi^\cdot}\omega_{\varsigma^\cdot}-\left(\frac{2-t}{2}\sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1}\rho_{\varsigma^\cdot}+\frac{t}{2}\sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1}\varrho_{\varsigma^\cdot}\right)\right)\right|^q dt\right)^{\frac{1}{q}}\right. \tag{40}$$

$$\left.+\left(\int_0^1 t dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t\left|f'\left(\sum_{\varsigma^\cdot=1}^{\varphi^\cdot}\omega_{\varsigma^\cdot}-\left(\frac{2-t}{2}\sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1}\varrho_{\varsigma^\cdot}+\frac{t}{2}\sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1}\rho_{\varsigma^\cdot}\right)\right)\right|^q dt\right)^{\frac{1}{q}}\right]$$

Utilizing Theorem 1 for the case $n=2$ with $\sigma_1=\frac{2-t}{2}$ and $\sigma_2=\frac{t}{2}$ in (40) in light of the convexity of $|f'|^q$, we achieve

$$\left|\zeta(\omega_{\varsigma^\cdot}, \rho_{\varsigma^\cdot}, \varrho_{\varsigma^\cdot}, f)\right|$$

$$\leq \frac{\sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1}|\varrho_{\varsigma^\cdot}-\rho_{\varsigma^\cdot}|}{4}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}$$

$$\times\left[\left(\int_0^1 t\left(\sum_{\varsigma^\cdot=1}^{\varphi^\cdot}|f'(\omega_{\varsigma^\cdot})|^q-\frac{2-t}{2}\sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1}|f'(\rho_{\varsigma^\cdot})|^q-\frac{t}{2}\sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1}|f'(\varrho_{\varsigma^\cdot})|^q\right)dt\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_0^1 t\left(\sum_{\varsigma^\cdot=1}^{\varphi^\cdot}|f'(\omega_{\varsigma^\cdot})|^q-\frac{2-t}{2}\sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1}|f'(\varrho_{\varsigma^\cdot})|^q-\frac{t}{2}\sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1}|f'(\rho_{\varsigma^\cdot})|^q\right)dt\right)^{\frac{1}{q}}\right] \tag{41}$$

$$=\frac{\sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1}|\varrho_{\varsigma^\cdot}-\rho_{\varsigma^\cdot}|}{4}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left[\left(\sum_{\varsigma^\cdot=1}^{\varphi^\cdot}|f'(\omega_{\varsigma^\cdot})|^q\int_0^1 t dt-\int_0^1\left(t\frac{2-t}{2}\right)dt\sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1}|f'(\rho_{\varsigma^\cdot})|^q\right.\right.$$

$$\left.-\int_0^1\frac{t^2}{2}dt\sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1}|f'(\varrho_{\varsigma^\cdot})|^q dt\right)^{\frac{1}{q}}+\left(\int_0^1 t dt\sum_{\varsigma^\cdot=1}^{\varphi^\cdot}|f'(\omega_{\varsigma^\cdot})|^q\right.$$

$$\left.\left.-\int_0^1\left(t\frac{2-t}{2}\right)dt\sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1}|f'(\varrho_{\varsigma^\cdot})|^q-\int_0^1\frac{t^2}{2}\sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1}|f'(\rho_{\varsigma^\cdot})|^q dt\right)^{\frac{1}{q}}\right].$$

Since,

$$\int_0^1 t dt = \frac{1}{2}, \int_0^1 \left(t\frac{2-t}{2}\right) dt = \frac{1}{3}, \int_0^1 \frac{t^2}{2} dt = \frac{1}{6}.$$

Therefore, (41) implies

$$\begin{aligned}
&\left|\zeta(\omega_{\varsigma^\cdot}, \rho_{\varsigma^\cdot}, \varrho_{\varsigma^\cdot}, f)\right| \\
&\leq \frac{\sum\limits_{\varsigma^\cdot=1}^{\varphi^\cdot-1} |\varrho_{\varsigma^\cdot} - \rho_{\varsigma^\cdot}|}{4} \left(\frac{1}{2}\right)^{1-\frac{1}{q}} \\
&\times \left[\left(\frac{1}{2} \sum_{\varsigma^\cdot=1}^{\varphi^\cdot} |f'(\omega_{\varsigma^\cdot})|^q - \frac{1}{6} \left(\sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1} |f'(\varrho_{\varsigma^\cdot})|^q + 2 \sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1} |f'(\rho_{\varsigma^\cdot})|^q \right) \right)^{\frac{1}{q}} \right. \\
&\left. + \left(\frac{1}{2} \sum_{\varsigma^\cdot=1}^{\varphi^\cdot} |f'(\omega_{\varsigma^\cdot})|^q - \frac{1}{6} \left(\sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1} |f'(\rho_{\varsigma^\cdot})|^q + 2 \sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1} |f'(\varrho_{\varsigma^\cdot})|^q \right) \right)^{\frac{1}{q}} \right].
\end{aligned}$$

Thus, the desired result has been proven. □

Remark 7

(i) If we substitute $\varphi^\cdot = 2$ in (39), then we get Theorem 4.3 of [59].

(ii) If we substitute $\varphi^\cdot = 2$, $\rho_1 = \omega_1$, $\varrho_1 = \omega_2$ in (39), then we get Theorem 7 of [60].

Theorem 9 Let $f\colon I \to \mathbb{R}$ represents a differentiable function whose domain is in the interval I. Let the tuples $\boldsymbol{\omega}$, ρ, $\varrho \in I^{\varphi^\cdot}$ be defined by $\boldsymbol{\omega} = (\omega_1, \omega_2, \ldots, \omega_{\varphi^\cdot})$, $\rho = (\rho_1, \rho_2, \ldots, \rho_{\varphi^\cdot})$ and $\varrho = (\varrho_1, \varrho_2, \ldots, \varrho_{\varphi^\cdot})$ with $\rho_{\varphi^\cdot} > \varrho_{\varphi^\cdot}$ and $\omega_{\varsigma^\cdot}, \rho_{\varsigma^\cdot}, \varrho_{\varsigma^\cdot} \in I$ for all $\varsigma^\cdot = 1, 2, \ldots, \varphi^\cdot$. Suppose further that $\rho \prec \boldsymbol{\omega}$ and $\varrho \prec \boldsymbol{\omega}$, $q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, $|f'|^q$ is convex function, $0 \leq \alpha \leq 1$, and $B(\alpha)$ denotes normalization mapping. Then

$$\begin{aligned}
\left|\zeta(\omega_{\varsigma^\cdot}, \rho_{\varsigma^\cdot}, \varrho_{\varsigma^\cdot}, f)\right| &\leq \frac{\sum\limits_{\varsigma^\cdot=1}^{\varphi^\cdot-1} |\varrho_{\varsigma^\cdot} - \rho_{\varsigma^\cdot}|}{4} \\
&\times \left[\frac{2}{p(p+1)} + \frac{2\sum\limits_{\varsigma^\cdot=1}^{\varphi^\cdot} |f'(\omega_{\varsigma^\cdot})|^q}{q} - \frac{\sum\limits_{\varsigma^\cdot=1}^{\varphi^\cdot-1} |f'(\varrho_{\varsigma^\cdot})|^q + \sum\limits_{\varsigma^\cdot=1}^{\varphi^\cdot-1} |f'(\rho_{\varsigma^\cdot})|^q}{q} \right].
\end{aligned} \tag{42}$$

Proof. Employing Lemma 3, and property of modulus, we obtain

$$\left|\zeta(\omega_{\varsigma^\cdot},\rho_{\varsigma^\cdot},\varrho_{\varsigma^\cdot},f)\right|\leq\frac{\sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1}|\varrho_{\varsigma^\cdot}-\rho_{\varsigma^\cdot}|}{4}\left[\int_0^1 t\left|f'\left(\sum_{\varsigma^\cdot=1}^{\varphi^\cdot}\omega_{\varsigma^\cdot}-\left(\frac{2-t}{2}\sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1}\rho_{\varsigma^\cdot}+\frac{t}{2}\sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1}\varrho_{\varsigma^\cdot}\right)\right)\right|dt\right.$$

$$\left.+\int_0^1 t\left|f'\left(\sum_{\varsigma^\cdot=1}^{\varphi^\cdot}\omega_{\varsigma^\cdot}-\left(\frac{2-t}{2}\sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1}\varrho_{\varsigma^\cdot}+\frac{t}{2}\sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1}\rho_{\varsigma^\cdot}\right)\right)\right|dt\right]$$

By employing Young's inequality in the above integral, we get

$$\left|\zeta(\omega_{\varsigma^\cdot},\rho_{\varsigma^\cdot},\varrho_{\varsigma^\cdot},f)\right|\leq\frac{\sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1}|\varrho_{\varsigma^\cdot}-\rho_{\varsigma^\cdot}|}{4}$$

$$\times\left[\left(\frac{1}{p}\int_0^1 t^p dt+\frac{1}{q}\int_0^1\left|f'\left(\sum_{\varsigma^\cdot=1}^{\varphi^\cdot}\omega_{\varsigma^\cdot}-\left(\frac{2-t}{2}\sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1}\rho_{\varsigma^\cdot}+\frac{t}{2}\sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1}\varrho_{\varsigma^\cdot}\right)\right)\right|^q dt\right)\right. \tag{43}$$

$$\left.+\left(\frac{1}{p}\int_0^1 t^p dt+\frac{1}{q}\int_0^1\left|f'\left(\sum_{\varsigma^\cdot=1}^{\varphi^\cdot}\omega_{\varsigma^\cdot}-\left(\frac{2-t}{2}\sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1}\varrho_{\varsigma^\cdot}+\frac{t}{2}\sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1}\rho_{\varsigma^\cdot}\right)\right)\right|^q dt\right)\right]$$

Utilizing Theorem 1 for the case $n=2$ with $\sigma_1=\frac{2-t}{2}$ and $\sigma_2=\frac{t}{2}$ in (43) in light of the convexity of $|f'|^q$, we achieve

$$\left|\zeta(\omega_{\varsigma^\cdot},\rho_{\varsigma^\cdot},\varrho_{\varsigma^\cdot},f)\right|\leq\frac{\sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1}|\varrho_{\varsigma^\cdot}-\rho_{\varsigma^\cdot}|}{4}$$

$$\times\left[\left(\frac{1}{p(p+1)}+\frac{1}{q}\left(\sum_{\varsigma^\cdot=1}^{\varphi^\cdot}|f'(\omega_{\varsigma^\cdot})|^q-\int_0^1\left(\frac{2-t}{2}\right)dt\sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1}|f'(\rho_{\varsigma^\cdot})|^q\right.\right.\right.$$

$$\left.\left.-\int_0^1\frac{t}{2}dt\sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1}|f'(\varrho_{\varsigma^\cdot})|^q\right)\right)+\left(\frac{1}{p(p+1)}+\frac{1}{q}\left(\sum_{\varsigma^\cdot=1}^{\varphi^\cdot}|f'(\omega_{\varsigma^\cdot})|^q\right.\right. \tag{44}$$

$$\left.\left.\left.-\int_0^1\left(\frac{2-t}{2}\right)dt\sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1}|f'(\varrho_{\varsigma^\cdot})|^q-\int_0^1\frac{t}{2}dt\sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1}|f'(\rho_{\varsigma^\cdot})|^q\right)\right)\right]$$

Since,

$$\int_0^1\left(\frac{2-t}{2}\right)dt=\frac{3}{4},\ \int_0^1\frac{t}{2}dt=\frac{1}{4}.$$

Therefore, (44) implies

$$\left|\zeta(\omega_{\varsigma^\cdot}, \rho_{\varsigma^\cdot}, \varrho_{\varsigma^\cdot}, f)\right|$$

$$\leq \frac{\sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1}|\varrho_{\varsigma^\cdot}-\rho_{\varsigma^\cdot}|}{4}\left[\frac{2}{p(p+1)}+\frac{2\sum_{\varsigma^\cdot=1}^{\varphi^\cdot}|f'(\omega_{\varsigma^\cdot})|^q}{q}-\frac{\sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1}|f'(\varrho_{\varsigma^\cdot})|^q+\sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1}|f'(\rho_{\varsigma^\cdot})|^q}{q}\right].$$

The desired result has been successfully proved. □

Remark 8

(i) If we substitute $\varphi^\cdot = 2$ in (42), then we get Theorem 4.4 of [59].

(ii) If we substitute $\varphi^\cdot = 2$, $\rho_1 = \omega_1$, $\varrho_1 = \omega_2$ in (42), then we get Theorem 8 of [60].

6. Applications of the main results

6.1 *Modified bessel functions*

Modified Bessel functions arise naturally in various physical and engineering contexts, especially in solving differential equations involving cylindrical or spherical symmetry. These functions, known for their smooth behavior and exponential growth, serve as essential tools in applied mathematics, including problems related to heat conduction, wave propagation, and statistical distributions. Their intrinsic connection with convexity properties makes them a fertile ground for analytical investigations through inequalities. Motivated by this, we now demonstrate the applicability of our newly established results, which are derived through the connection between convex functions and the concept of majorization. These results provide broader and more refined inequalities that yield meaningful estimates for special functions. In particular, we illustrate how our generalized framework can be applied to the modified Bessel functions, demonstrating both the versatility and analytical power of our approach.

Watson [61] defined the function $J_\nu\colon \mathbb{R} \to \infty$ as

$$J_\nu(x) = 2^x\Gamma(\nu+1)x^{-\nu}I_\nu(x), \quad x \in \mathbb{R},$$

and the modified Bessel function of the first kind as

$$I_\nu(x) = \sum_{j=0}^{\infty}\frac{(x/2)^{\nu+2j}}{j!\Gamma(\nu+j+1)}.$$

Employing the above two functions, one can obtain the following:

$$J'_\nu(x) = \frac{x}{2(\nu+1)}J_{\nu+1}(x),$$

$$J''_\nu(x) = \frac{1}{4(\nu+1)}\left[\frac{x^2}{\nu+2}J_{\nu+2}(x)+2J_{\nu+1}(x)\right].$$

In what follows, the derivatives of first and second order obtained earlier will be applied to deduce the results:

Example 1 By choosing $f(x) = J'_{\nu}(x)$ and substituting the two preceding identities into Theorem 6 while using the assumptions $\alpha = 1$, and $B(\alpha) = 1$, we establish the following outcome.

$$\left|\frac{1}{\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1}(\varrho_{\varsigma^{\cdot}} - \rho_{\varsigma^{\cdot}})}\left[\left\{J_{\nu}\left(\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}}\omega_{\varsigma^{\cdot}} - \sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1}\rho_{\varsigma^{\cdot}}\right) - J_{\nu}\left(\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}}\omega_{\varsigma^{\cdot}} - \sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1}\varrho_{\varsigma^{\cdot}}\right)\right\}\right.\right.$$

$$\left.\left. - \frac{\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}}\omega_{\varsigma^{\cdot}} - \sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1}\left(\frac{\rho_{\varsigma^{\cdot}}+\varrho_{\varsigma^{\cdot}}}{2}\right)}{2(\nu+1)} J_{\nu}\left(\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}}\omega_{\varsigma^{\cdot}} - \sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1}\left(\frac{\rho_{\varsigma^{\cdot}}+\varrho_{\varsigma^{\cdot}}}{2}\right)\right)\right]\right|$$

$$\leq \frac{\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1}(\varrho_{\varsigma^{\cdot}} - \rho_{\varsigma^{\cdot}})}{4}\left[\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}}\left|\frac{1}{4(\nu+1)}\left(\frac{\omega_{\varsigma^{\cdot}}^2}{\nu+2}J_{\nu+2}(\omega_{\varsigma^{\cdot}}) + 2J_{\nu+1}(\omega_{\varsigma^{\cdot}})\right)\right|\right.$$

$$-\frac{1}{2}\left\{\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1}\left|\frac{1}{4(\nu+1)}\left(\frac{\rho_{\varsigma^{\cdot}}^2}{\nu+2}J_{\nu+2}(\rho_{\varsigma^{\cdot}}) + 2J_{\nu+1}(\rho_{\varsigma^{\cdot}})\right)\right|\right.$$

$$\left.\left.+\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1}\left|\frac{1}{4(\nu+1)}\left(\frac{\varrho_{\varsigma^{\cdot}}^2}{\nu+2}J_{\nu+2}(\varrho_{\varsigma^{\cdot}}) + 2J_{\nu+1}(\varrho_{\varsigma^{\cdot}})\right)\right|\right\}\right].$$

Example 2 By choosing $f(x) = J'_{\nu}(x)$ and substituting the two preceding identities into Theorem 7 while using the assumptions $\alpha = 1$, and $B(\alpha) = 1$, we obtain the subsequent result.

$$\left|\frac{1}{\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1}(\varrho_{\varsigma^{\cdot}} - \rho_{\varsigma^{\cdot}})}\left[\left\{J_{\nu}\left(\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}}\omega_{\varsigma^{\cdot}} - \sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1}\rho_{\varsigma^{\cdot}}\right) - J_{\nu}\left(\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}}\omega_{\varsigma^{\cdot}} - \sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1}\varrho_{\varsigma^{\cdot}}\right)\right\}\right.\right.$$

$$\left.\left. - \frac{\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}}\omega_{\varsigma^{\cdot}} - \sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1}\left(\frac{\rho_{\varsigma^{\cdot}}+\varrho_{\varsigma^{\cdot}}}{2}\right)}{2(\nu+1)} J_{\nu}\left(\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}}\omega_{\varsigma^{\cdot}} - \sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1}\left(\frac{\rho_{\varsigma^{\cdot}}+\varrho_{\varsigma^{\cdot}}}{2}\right)\right)\right]\right|$$

$$\leq \frac{\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1}(\varrho_{\varsigma^{\cdot}} - \rho_{\varsigma^{\cdot}})}{4(p+1)^{\frac{1}{p}}}\left[A^{\frac{1}{q}} + B^{\frac{1}{q}}\right],$$

where,

$$A=\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}}\left|\frac{1}{4(\nu+1)}\left(\frac{\omega_{\varsigma^{\cdot}}^{2}}{\nu+2}J_{\nu+2}(\omega_{\varsigma^{\cdot}})+2J_{\nu+1}(\omega_{\varsigma^{\cdot}})\right)\right|^{q}$$

$$-\left(\frac{3}{4}\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1}\left|\frac{1}{4(\nu+1)}\left(\frac{\rho_{\varsigma^{\cdot}}^{2}}{\nu+2}J_{\nu+2}(\rho_{\varsigma^{\cdot}})+2J_{\nu+1}(\rho_{\varsigma^{\cdot}})\right)\right|^{q}\right.$$

$$\left.+\frac{1}{4}\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1}\left|\frac{1}{4(\nu+1)}\left(\frac{\varrho_{\varsigma^{\cdot}}^{2}}{\nu+2}J_{\nu+2}(\varrho_{\varsigma^{\cdot}})+2J_{\nu+1}(\varrho_{\varsigma^{\cdot}})\right)\right|^{q}\right),$$

and

$$B=\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}}\left|\frac{1}{4(\nu+1)}\left(\frac{\omega_{\varsigma^{\cdot}}^{2}}{\nu+2}J_{\nu+2}(\omega_{\varsigma^{\cdot}})+2J_{\nu+1}(\omega_{\varsigma^{\cdot}})\right)\right|^{q}$$

$$-\left(\frac{1}{4}\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1}\left|\frac{1}{4(\nu+1)}\left(\frac{\rho_{\varsigma^{\cdot}}^{2}}{\nu+2}J_{\nu+2}(\rho_{\varsigma^{\cdot}})+2J_{\nu+1}(\rho_{\varsigma^{\cdot}})\right)\right|^{q}\right.$$

$$\left.+\frac{3}{4}\sum_{\varsigma^{\cdot}=1}^{\varphi^{\cdot}-1}\left|\frac{1}{4(\nu+1)}\left(\frac{\varrho_{\varsigma^{\cdot}}^{2}}{\nu+2}J_{\nu+2}(\varrho_{\varsigma^{\cdot}})+2J_{\nu+1}(\varrho_{\varsigma^{\cdot}})\right)\right|^{q}\right).$$

The presented examples illustrate how the inequalities established in Theorem 6 and Theorem 7 can be specialized to yield concrete bounds for the Bessel function of the first kind. In particular, by selecting $f(x)=J'\nu(x)$, the imposed differentiability and convexity conditions are satisfied, thereby enabling a direct application of the general results. The derived inequalities furnish explicit upper bounds for differences of Bessel functions in terms of their higher-order counterparts $J_{\nu+1}$ and $J_{\nu+2}$. These findings demonstrate the applicability of our theoretical framework to special functions that arise frequently in applied mathematics, especially in contexts such as wave propagation and signal analysis.

6.2 *Applications to special means*

Arithmetic mean:

$$A(\zeta_1,\,\zeta_2)=\frac{\zeta_1+\zeta_2}{2},\quad \zeta_1,\,\zeta_2\in\mathbb{R},$$

Logarithmic mean:

$$\bar{L}(\zeta_1,\,\zeta_2)=\frac{\zeta_2-\zeta_1}{\ln|\zeta_2|-\ln|\zeta_1|},\qquad \zeta_1,\,\zeta_2\in\mathbb{R}\setminus\{0\}$$

Generalized log-mean:

$$L_n(\zeta_1, \zeta_2) = \left(\frac{\zeta_2^{n+1} - \zeta_1^{n+1}}{(n+1)(\zeta_2 - \zeta_1)}\right)^{\frac{1}{n}}, \qquad n \in \mathbb{R}\backslash\{-1, 0\}, \zeta_1 < \zeta_2.$$

Proposition 1 Suppose that the requirements of Theorem 6, are met then

$$\left| A^n\left(\sum_{\varsigma\cdot=1}^{\varphi\cdot} \omega_{\varsigma\cdot} - \sum_{\varsigma\cdot=1}^{\varphi\cdot-1} \rho_{\varsigma\cdot}, \sum_{\varsigma\cdot=1}^{\varphi\cdot} \omega_{\varsigma\cdot} - \sum_{\varsigma\cdot=1}^{\varphi\cdot-1} \varrho_{\varsigma\cdot}\right)\right.$$

$$\left. -L_n^n\left(\sum_{\varsigma\cdot=1}^{\varphi\cdot} \omega_{\varsigma\cdot} - \sum_{\varsigma\cdot=1}^{\varphi\cdot-1} \rho_{\varsigma\cdot}, \sum_{\varsigma\cdot=1}^{\varphi\cdot} \omega_{\varsigma\cdot} - \sum_{\varsigma\cdot=1}^{\varphi\cdot-1} \varrho_{\varsigma\cdot}\right)\right|$$

$$\leq \frac{n\sum_{\varsigma\cdot=1}^{\varphi\cdot-1} |\varrho_{\varsigma\cdot} - \rho_{\varsigma\cdot}|}{4} A\left(\sum_{\varsigma\cdot=1}^{\varphi\cdot} |\omega_{\varsigma\cdot}|^{n-1} - \sum_{\varsigma\cdot=1}^{\varphi\cdot-1} |\rho_{\varsigma\cdot}|^{n-1}, \sum_{\varsigma\cdot=1}^{\varphi\cdot} |\omega_{\varsigma\cdot}|^{n-1} - \sum_{\varsigma\cdot=1}^{\varphi\cdot-1} |\varrho_{\varsigma\cdot}|^{n-1}\right)$$

Proof. The result follows immediately from Theorem 6 by choosing $\alpha = 1$, $B(\alpha) = 1$, and considering $f(x) = x^n$ for $x \in \mathbb{R}$. □

Proposition 2 Suppose that the requirements of Theorem 7, are met then

$$\left| A^n\left(\sum_{\varsigma\cdot=1}^{\varphi\cdot} \omega_{\varsigma\cdot} - \sum_{\varsigma\cdot=1}^{\varphi\cdot-1} \rho_{\varsigma\cdot}, \sum_{\varsigma\cdot=1}^{\varphi\cdot} \omega_{\varsigma\cdot} - \sum_{\varsigma\cdot=1}^{\varphi\cdot-1} \varrho_{\varsigma\cdot}\right)\right.$$

$$\left. -L_n^n\left(\sum_{\varsigma\cdot=1}^{\varphi\cdot} \omega_{\varsigma\cdot} - \sum_{\varsigma\cdot=1}^{\varphi\cdot-1} \rho_{\varsigma\cdot}, \sum_{\varsigma\cdot=1}^{\varphi\cdot} \omega_{\varsigma\cdot} - \sum_{\varsigma\cdot=1}^{\varphi\cdot-1} \varrho_{\varsigma\cdot}\right)\right|$$

$$\leq \frac{n\sum_{\varsigma\cdot=1}^{\varphi\cdot-1} |\varrho_{\varsigma\cdot} - \rho_{\varsigma\cdot}|}{4(p+1)^{\frac{1}{p}}} \left[\left(n\sum_{\varsigma\cdot=1}^{\varphi\cdot} |\omega_{\varsigma\cdot}|^{(n-1)q}\right.\right.$$

$$\left. - \frac{n}{2}A\left(3\sum_{\varsigma\cdot=1}^{\varphi\cdot-1} |\rho_{\varsigma\cdot}|^{(n-1)q}, \sum_{\varsigma\cdot=1}^{\varphi\cdot-1} |\varrho_{\varsigma\cdot}|^{(n-1)q}\right)\right)^{\frac{1}{q}} + \left(n\sum_{\varsigma\cdot=1}^{\varphi\cdot} |\omega_{\varsigma\cdot}|^{(n-1)q}\right.$$

$$\left.\left. - \frac{n}{2}A\left(\sum_{\varsigma\cdot=1}^{\varphi\cdot-1} |\rho_{\varsigma\cdot}|^{(n-1)q}, 3\sum_{\varsigma\cdot=1}^{\varphi\cdot-1} |\varrho_{\varsigma\cdot}|^{(n-1)q}\right)\right)^{\frac{1}{q}}\right].$$

Proof. The result follows immediately from Theorem 7 by choosing $\alpha = 1$, $B(\alpha) = 1$, and considering $f(x) = x^n$ for $x \in \mathbb{R}$. □

Proposition 3 Suppose that the requirements of Theorem 8, are met then

$$\left|A^n\left(\sum_{\varsigma^\cdot=1}^{\varphi^\cdot}\omega_{\varsigma^\cdot}-\sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1}\rho_{\varsigma^\cdot},\ \sum_{\varsigma^\cdot=1}^{\varphi^\cdot}\omega_{\varsigma^\cdot}-\sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1}\varrho_{\varsigma^\cdot}\right)-L_n^n\left(\sum_{\varsigma^\cdot=1}^{\varphi^\cdot}\omega_{\varsigma^\cdot}-\sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1}\rho_{\varsigma^\cdot},\ \sum_{\varsigma^\cdot=1}^{\varphi^\cdot}\omega_{\varsigma^\cdot}-\sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1}\varrho_{\varsigma^\cdot}\right)\right|$$

$$\leq\frac{n\sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1}|\varrho_{\varsigma^\cdot}-\rho_{\varsigma^\cdot}|}{4}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\times\left[\left(\frac{n}{2}\sum_{\varsigma^\cdot=1}^{\varphi^\cdot}|\omega_{\varsigma^\cdot}|^{(n-1)q}-\frac{n}{3}A\left(2\sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1}|\rho_{\varsigma^\cdot}|^{(n-1)q},\ \sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1}|\varrho_{\varsigma^\cdot}|^{(n-1)q}\right)\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\frac{n}{2}\sum_{\varsigma^\cdot=1}^{\varphi^\cdot}|\omega_{\varsigma^\cdot}|^{(n-1)q}-\frac{n}{3}A\left(\sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1}|\rho_{\varsigma^\cdot}|^{(n-1)q},\ 2\sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1}|\varrho_{\varsigma^\cdot}|^{(n-1)q}\right)\right)^{\frac{1}{q}}\right].$$

Proof. The result follows immediately from Theorem 8 by choosing $\alpha=1$, $B(\alpha)=1$, and considering $f(x)=x^n$ for $x\in\mathbb{R}$. □

Proposition 4 Suppose that the requirements of Theorem 6, are met then

$$\left|A^{-1}\left(\sum_{\varsigma^\cdot=1}^{\varphi^\cdot}\omega_{\varsigma^\cdot}-\sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1}\rho_{\varsigma^\cdot},\ \sum_{\varsigma^\cdot=1}^{\varphi^\cdot}\omega_{\varsigma^\cdot}-\sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1}\varrho_{\varsigma^\cdot}\right)-\bar{L}^{-1}\left(\sum_{\varsigma^\cdot=1}^{\varphi^\cdot}\omega_{\varsigma^\cdot}-\sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1}\rho_{\varsigma^\cdot},\ \sum_{\varsigma^\cdot=1}^{\varphi^\cdot}\omega_{\varsigma^\cdot}-\sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1}\varrho_{\varsigma^\cdot}\right)\right|$$

$$\leq\frac{\sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1}|\varrho_{\varsigma^\cdot}-\rho_{\varsigma^\cdot}|}{4}A\left(\sum_{\varsigma^\cdot=1}^{\varphi^\cdot}|\omega_{\varsigma^\cdot}|^{-2}-\sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1}|\rho_{\varsigma^\cdot}|^{-2},\ \sum_{\varsigma^\cdot=1}^{\varphi^\cdot}|\omega_{\varsigma^\cdot}|^{-2}-\sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1}|\varrho_{\varsigma^\cdot}|^{-2}\right)$$

Proof. The result follows immediately from Theorem 6 by choosing $\alpha=1$, $B(\alpha)=1$, and considering $f(x)=\frac{1}{x}$, $x\neq 0$, $x\in\mathbb{R}$. □

Proposition 5 Suppose that the requirements of Theorem 7, are met then

$$\left|A^{-1}\left(\sum_{\varsigma^\cdot=1}^{\varphi^\cdot}\omega_{\varsigma^\cdot}-\sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1}\rho_{\varsigma^\cdot},\ \sum_{\varsigma^\cdot=1}^{\varphi^\cdot}\omega_{\varsigma^\cdot}-\sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1}\varrho_{\varsigma^\cdot}\right)-\bar{L}^{-1}\left(\sum_{\varsigma^\cdot=1}^{\varphi^\cdot}\omega_{\varsigma^\cdot}-\sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1}\rho_{\varsigma^\cdot},\ \sum_{\varsigma^\cdot=1}^{\varphi^\cdot}\omega_{\varsigma^\cdot}-\sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1}\varrho_{\varsigma^\cdot}\right)\right|$$

$$\leq\frac{\sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1}|\varrho_{\varsigma^\cdot}-\rho_{\varsigma^\cdot}|}{4(p+1)^{\frac{1}{p}}}\left[\left(\sum_{\varsigma^\cdot=1}^{\varphi^\cdot}|\omega_{\varsigma^\cdot}|^{-2q}-\frac{1}{2}A\left(3\sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1}|\rho_{\varsigma^\cdot}|^{-2q},\ \sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1}|\varrho_{\varsigma^\cdot}|^{-2q}\right)\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\sum_{\varsigma^\cdot=1}^{\varphi^\cdot}|\omega_{\varsigma^\cdot}|^{-2q}-\frac{1}{2}A\left(\sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1}|\rho_{\varsigma^\cdot}|^{-2q},\ 3\sum_{\varsigma^\cdot=1}^{\varphi^\cdot-1}|\varrho_{\varsigma^\cdot}|^{-2q}\right)\right)^{\frac{1}{q}}\right].$$

Proof. The result follows immediately from Theorem 7 by choosing $\alpha=1$, $B(\alpha)=1$, and considering $f(x)=\frac{1}{x}$, $x\neq 0$, $x\in\mathbb{R}$. □

Proposition 6 Suppose that the requirements of Theorem 8, are met then

$$\left|A^{-1}\left(\sum_{\varsigma\cdot=1}^{\varphi\cdot}\omega_{\varsigma\cdot}-\sum_{\varsigma\cdot=1}^{\varphi\cdot-1}\rho_{\varsigma\cdot},\ \sum_{\varsigma\cdot=1}^{\varphi\cdot}\omega_{\varsigma\cdot}-\sum_{\varsigma\cdot=1}^{\varphi\cdot-1}\varrho_{\varsigma\cdot}\right)\right.$$

$$\left.-\bar{L}^{-1}\left(\sum_{\varsigma\cdot=1}^{\varphi\cdot}\omega_{\varsigma\cdot}-\sum_{\varsigma\cdot=1}^{\varphi\cdot-1}\rho_{\varsigma\cdot},\ \sum_{\varsigma\cdot=1}^{\varphi\cdot}\omega_{\varsigma\cdot}-\sum_{\varsigma\cdot=1}^{\varphi\cdot-1}\varrho_{\varsigma\cdot}\right)\right|$$

$$\leq\frac{\sum_{\varsigma\cdot=1}^{\varphi\cdot-1}|\varrho_{\varsigma\cdot}-\rho_{\varsigma\cdot}|}{4}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left[\left(\frac{1}{2}\sum_{\varsigma\cdot=1}^{\varphi\cdot}|\omega_{\varsigma\cdot}|^{-2q}-\frac{1}{3}A\left(2\sum_{\varsigma\cdot=1}^{\varphi\cdot-1}|\rho_{\varsigma\cdot}|^{-2q},\ \sum_{\varsigma\cdot=1}^{\varphi\cdot-1}|\varrho_{\varsigma\cdot}|^{-2q}\right)\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\frac{1}{2}\sum_{\varsigma\cdot=1}^{\varphi\cdot}|\omega_{\varsigma\cdot}|^{-2q}-\frac{1}{3}A\left(\sum_{\varsigma\cdot=1}^{\varphi\cdot-1}|\rho_{\varsigma\cdot}|^{-2q},\ 2\sum_{\varsigma\cdot=1}^{\varphi\cdot-1}|\varrho_{\varsigma\cdot}|^{-2q}\right)\right)^{\frac{1}{q}}\right].$$

Proof. The result follows immediately from Theorem 8 by choosing $\alpha = 1$, $B(\alpha) = 1$, and considering $f(x) = \frac{1}{x}$, $x \neq 0$, $x \in \mathbb{R}$. □

7. Conclusion

The Hermite-Hadamard inequality has become a key focus in recent mathematical studies. It ensures integrability and provides approximations for convex functions. Over time, it has been extended to the functions belonging to s-convexity, η-convexity, coordinate convexity, and strong convexity. These extensions, together with related integral identities, have produced many refined inequalities.

This study has advanced the theory of Hermite-Hadamard type inequalities by formulating Mercer-type versions in conticrete fractional settings. The methodology relied on the joining of majorization, convexity, and Caputo-Fabrizio fractional operators, which allowed the derivation of several new and generalized results. Weighted forms were also obtained by employing certain monotonic tuples together with weighted majorized Jensen-Mercer inequalities. An integral identity for differentiable functions was developed, which served as a foundation for establishing sharp bounds for the discrepancy of terms in the main inequalities. These estimates were derived by employing the convexity of $|f'|^q$, $(1 < q)$ and $|f'|$, alongside classical inequalities such as the power mean, Hölder, and Young's. The applications presented to modified Bessel functions and special means further highlighted the versatility of the findings. The outcomes not only produce several existing results as special cases but also generate entirely new inequalities. Thus, this work contributes both to the theoretical enrichment of fractional inequalities and to their potential applications in broader areas of mathematical analysis.

Conflict of interest

The author declares no competing financial interest.

References

[1] Khan S, Khan MA, Chu YM. Converses of the Jensen inequality derived from the Green functions with applications in information theory. *Mathematical Methods in the Applied Sciences*. 2020; 43: 2577-2587.

[2] Mohammed PO, Abdeljawad T. Modification of certain fractional integral inequalities for convex functions. *Advances in Difference Equations*. 2020; 2020: 69.

[3] Cloud MJ, Drachman BC, Lebedev LP. *Inequalities with Applications to Engineering*. Springer; 2014.

[4] Lin Q. Jensen inequality for superlinear expectations. *Statistics and Probability Letters*. 2019; 151: 79-83.

[5] Lakshmikantham V, Vatsala AS. *Theory of Differential and Integral Inequalities with Initial Time Difference and Applications, Analytic and Geometric Inequalities and Applications*. Springer; 1999.

[6] Niculescu CP, Persson LE. *Convex Functions and Their Applications*. Springer; 2006.

[7] Özdemir ME, Yıldız C, Akdemir AO, Set E. On some inequalities for s-convex functions and applications. *Journal of Inequalities and Applications*. 2013; 2013: 1-11.

[8] Iscan I. Ostrowski type inequalities for p-convex functions. *New Trends in Mathematical Sciences*. 2016; 4(3): 140-150.

[9] Zhang XM, Chu YM, Zhang XH. The Hermite-Hadamard type inequality of GA-convex functions and its application. *Journal of Inequalities and Applications*. 2010; 2010: 1-11.

[10] Srivastava HM, Tariq M, Mohammed PO, Alrweili H, Al-Sarairah E, Sen MDL. On modified integral inequalities for a generalized class of convexity and applications. *Axioms*. 2023; 12(2): 162.

[11] Dragomir SS, Torebek BT. Some Hermite-Hadamard type inequalities in the class of hyperbolic p-convex functions. *Journal of the Royal Spanish Academy of Exact, Physical and Natural Sciences Series A Mathematics*. 2019; 113(4): 3413-3423.

[12] Fahad A, Butt SI, Pecaric J, Praljak M. Generalized Taylor's formula and Steffensen's inequality. *Mathematics*. 2023; 11(16): 3570.

[13] Khan MA, Mohammad N, Nwaeze ER, Chu YM. Quantum Hermite-Hadamard inequality by means of a Green function. *Advances in Difference Equations*. 2020; 2020: 1-20.

[14] Özdemir ME, Latif MA, Akdemir AO. On some Hadamard-type inequalities for product of two s-convex functions on the co-ordinates. *Journal of Inequalities and Applications*. 2012; 2012: 1-13.

[15] Deng Y, Ullah H, Khan MA, Iqbal S, Wu S. Refinements of Jensen's inequality via majorization results with applications in the information theory. *Journal of Mathematics*. 2021; 2021: 1-12.

[16] Sahar S, Khan MA, Khan S, Altoumb SH, Nwaeze ER. A refined form of the generalized Jensen inequality through an arbitrary function. *Journal of Mathematics and Computer Science*. 2025; 39: 376-385.

[17] Fejér L. About the Fourier series. *Mathematical and Scientific Reports from Hungary*. 1906; 24: 369-390.

[18] You X, Khan MA, Ullah H, Saeed T. Improvements of Slater's inequality by means of 4-convexity and its applications. *Mathematics*. 2022; 10(8): 1274.

[19] Sohail A, Khan MA, Ullah H, Alnowibet KA, Li YX, Chu YM. Improvements to Slater's inequality and their applications via functions whose fourth-order derivatives are convex. *Applied Mathematics in Science and Engineering*. 2025; 33(1): 2468935.

[20] Niculescu CP. Old and new on the Hermite-Hadamard inequality. *Real Analysis Exchange*. 2003; 29(2): 663-685.

[21] Ekinci A, Özdemir ME, Set E. New integral inequalities of Ostrowski type for quasi-convex functions with applications. *Turkish Journal of Science*. 2020; 5(3): 290-304.

[22] Marshall AW, Olkin I. *Theory of Majorization and Its Applications*. Academic Press; 1979.

[23] Guce IK. On fractional derivatives: the non-integer order of the derivative. *International Journal of Science and Engineering Research*. 2013; 4(3): 1-5.

[24] Almeida R, Tavares D, Torres DF. *The Variable-Order Fractional Calculus of Variations*. Springer; 2019.

[25] Iaffaldano G, Caputo M, Martino S. Experimental and theoretical memory diffusion of water in the sand. *Hydrology and Earth System Sciences*. 2006; 10: 93-100.

[26] Cesarone F, Caputo M, Cametti C. Memory formalism in the passive diffusion across a biological membrane. *Journal of Membrane Science*. 2004; 250: 79-84.

[27] Wang MK, Hong MY, Xu YF, Shen ZH, Chu YM. Inequalities for generalized trigonometric and hyperbolic functions with one parameter. *Journal of Mathematical Inequalities*. 2020; 14(1): 1-21.

[28] Sun H, Zhang Y, Baleanu D, Chen W, Chen Y. A new collection of real world applications of fractional calculus in science and engineering. *Communications in Nonlinear Science and Numerical Simulation*. 2018; 64: 213-231.

[29] Matusu R. Application of fractional order calculus to control theory. *International Journal of Mathematical Models and Methods in Applied Sciences*. 2011; 5(7): 1162-1169.

[30] El-Shahed M. Fractional calculus model of the semilunar heart valve vibrations. In: *International Design Engineering Technical Conferences and Computers and Information in Engineering Conference*. Chicago, Illinois, USA; 2003. p.711-714.

[31] Banerjee S, Easwaramoorthy D, Gowrisankar A. *Fractal Functions, Dimensions and Signal Analysis*. Springer; 2021.

[32] Sheng H, Chen Y, Qiu T. *Fractional Processes and Fractional-Order Signal Processing: Techniques and Applications*. Springer Science & Business Media; 2011.

[33] Chen SB, Rashid S, Noor MA, Ashraf R, Chu YM. A new approach on fractional calculus and probability density function. *AIMS Mathematics*. 2020; 5(6): 7041-7054.

[34] Koca I, Akçetin E, Yarakdal P. Numerical approximation for the spread of SIQR model with Caputo fractional order derivative. *Turkish Journal of Science*. 2020; 5(2): 124-139.

[35] Hadamard J. Essay on the study of given functions by their Taylor series expansion. *Journal of Pure and Applied Mathematics*. 1892; 4(8): 101-186.

[36] Zhao S, Butt SI, Nazeer W, Nasir J, Umar M, Liu Y. Some Hermite-Jensen-Mercer-type inequalities for k-Caputo-fractional derivatives and related results. *Advances in Differential Equations*. 2020; 2020: 1-17.

[37] Abdeljawad T, Baleanu D. On fractional derivatives with exponential kernel and their discrete versions. *Reports on Mathematical Physics*. 2017; 80: 11-27.

[38] Abdeljawad T. Fractional operators with exponential kernels and a Lyapunov type inequality. *Advances in Difference Equations*. 2017; 2017: Article 313.

[39] Katugampola UN. New approach to a generalized fractional integral. *Applied Mathematics and Computation*. 2011; 218(3): 860-865.

[40] Atangana A, Baleanu D. New fractional derivatives with non-local and non-singular kernel. *Thermal Science*. 2016; 20(2): 763-769.

[41] Rockafellar RT. *Convex Analysis*. Princeton University Press; 1997.

[42] Hiriart-Urruty JB, Lemarechal C. *Fundamentals of Convex Analysis*. Springer Science & Business Media; 2004.

[43] Rashid S, Abouelmagd EI, Sultana S, Chu YM. New developments in weighted n-fold type inequalities via discrete generalized $\hat{h}$-proportional fractional operators. *Fractals*. 2022; 30(2): 2240056.

[44] Rashid S, Abouelmagd EI, Khalid A, Faroo FB. Some recent developments on dynamical $\hat{h}$-discrete fractional type inequalities in the frame of nonsingular and nonlocal kernels. *Fractals*. 2022; 30(2): 2240110.

[45] Srivastava HM, Mishra AK. Applications of fractional calculus to parabolic starlike and uniformly convex functions. *Computers and Mathematics with Applications*. 2000; 39(4): 57-69.

[46] Zhou T, Yuan Z, Du T. On the fractional integral inclusions having exponential kernels for interval-valued convex functions. *Mathematical Sciences*. 2023; 17(2): 107-120.

[47] Faisal S, Khan MA, Khan TU, Saeed T, Alshehri AM, Nwaeze ER. New "Conticrete" Hermite-Hadamard-Jensen-Mercer fractional inequalities. *Symmetry*. 2022; 14: 294.

[48] Wu S, Khan MA, Faisal S, Saeed T, Nwaeze ER. Derivation of Hermite-Hadamard-Jensen-Mercer conticrete inequalities for Atangana-Baleanu fractional integrals by means of majorization. *Demonstratio Mathematica*. 2024; 57(1): 20240024.

[49] Saeed T, Khan MA, Faisal S, Alsulami HH, Alhodaly MSH. New conticrete inequalities of the Hermite-Hadamard-Jensen-Mercer-type in terms of generalized conformable fractional operators via majorization. *Demonstratio Mathematica*. 2023; 56: 20220225.

[50] Bhatia R. *Matrix Analysis*. Springer; 1997.

[51] Horn RA, Johnson CR. *Matrix Analysis*. Cambridge University Press; 1990.

[52] Jorswieck EA, Boche H. Majorization and matrix-monotone functions in wireless communications. *Foundations and Trends in Communications and Information Theory*. 2006; 3: 553-701.

[53] Palomar DP, Jiang Y. MIMO transceiver design via majorization theory. *Foundations and Trends in Communications and Information Theory*. 2006; 3: 331-551.

[54] Mercer AM. A variant of Jensen inequality. *Journal of Inequalities in Pure and Applied Mathematics*. 2003; 4(4): 73.

[55] Khan MA, Bradanović MSI, Mahmoud HA. New improvements of the Jensen-Mercer inequality for strongly convex functions with applications. *Axioms*. 2024; 13(8): 553.

[56] Dragomir SS, Agarwal RP. Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula. *Applied Mathematics Letters*. 1998; 11: 91-95.

[57] Wang BY. *Foundations of Majorization Inequalities*. Beijing Normal University Press; 1990.

[58] Niezgoda M. A generalization of Mercer's result on convex functions. *Nonlinear Analysis*. 2009; 71: 2771-2779.

[59] Sahoo SK, Hamed YS, Mohammed PO, Kodamasingh B, Nonlaopon K. New midpoint type Hermite-Hadamard-Mercer inequalities pertaining to Caputo-Fabrizio fractional operators. *Alexandria Engineering Journal*. 2023; 65: 689-698.

[60] Sahoo SK, Mohammed PO, Kodamasingh B, Tariq M, Hamed YS. New fractional integral inequalities for convex functions pertaining to Caputo-Fabrizio operator. *Fractal and Fractional*. 2022; 6(3): 171.

[61] Watson GN. *A Treatise on the Theory of Bessel Functions*. Cambridge University Press; 1995.