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Research Article

Four-Leaf Inequalities: A q-Series Approach

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Abstract: This research delves into several inequalities of complex functions, which are key to understanding their behavior. Such inequalities are vital in complex analysis for setting bounds, demonstrating convergence, and uncovering analytical and geometrical characteristics of regular q-functions. Our investigation focuses on a new collection of regular and univalent q-functions, denoted by $\Xi_{q,r}^{\eta,\phi}(4\ell)$ and defined within the unit disk: $|z|<1$. The definition and study of functions in $\Xi_{q,r}^{\eta,\phi}(4\ell)$ encompass several established subsets of regular q-functions, including q-bounded turning functions, Carathéodory functions, and four-leaf-type function. The study also incorporates several foundational mathematical ideas: a fresh Mathieu q-series expansion, subordination and convolution principles, q-operator theory, transformation of complex functions, and a few domain mappings. In particular, we explore the properties of two regular functions of the forms:

$$\mathfrak{g}(z)=z+\sum_{c\geqslant 2}\mathfrak{a}_c z^c,\ \ \mathfrak{f}_j(z)=z+\sum_{c\geqslant 1}\mathfrak{b}_{(cj+1)}z^{(cj+1)}\in\Xi_{q,r}^{\eta,\phi}(4\ell).$$

Here, $\mathfrak{f}$ is the jth-root transform of $\mathfrak{g}$. Our findings include estimates for some prominent inequalities such as Fekete-Szegö, Zalcman, Krushkal, and Ma inequalities. We also include numerous corollaries to highlight the connection between $\mathfrak{g}$ and $\mathfrak{f}$.

***Keywords*:** regular function, Mathieu q-series, four-leaf-shaped domain, q-derivative operator, subordination, Zalcman inequality, Krushkal inequality

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1. Background to the study

A set of complex-valued functions $\mathfrak{g}$ that are regular (i.e. holomorphic or analytic) and defined in the open unit disk: $|z| < 1$, is connoted by $\mathbf{A}$ in this work. Also, let $\mathbf{S}$ connote the subset of $\mathbf{A}$, a set consisting of functions having Maclaurin-Taylor's series type

$$\mathfrak{g}(z) = z + \mathfrak{a}_2 z^2 + \mathfrak{a}_3 z^3 + \cdots + \mathfrak{a}_c z^c + \cdots = z + \sum_{c \geqslant 2} \mathfrak{a}_c z^c, \tag{1}$$

normalized such that $\mathfrak{g}(0) = 0 = \mathfrak{g}'(0) - 1$ for all $|z| < 1$. The set $\mathbf{S}$ is famously refers to as the set of normalized regular and univalent functions defined for all $|z| < 1$. The various analytic and geometric properties of $\mathbf{S}$ and its subsets, such as the sets of starlike functions (where $\mathfrak{Re}\,(z\mathfrak{g}'/\mathfrak{g}) > 0$), convex functions (where $\mathfrak{Re}\,[(z\mathfrak{g}''/\mathfrak{g}') + 1] > 0$), bounded turning functions (where $\mathfrak{Re}\,(\mathfrak{g}') > 0$), and many others, have drawn the attention of numerous researchers for further explorations on functions $\mathfrak{g} \in \mathbf{S}$. For more details on set $\mathbf{S}$ and its subsets, see [1–3].

Many researchers have shown significant interest in Geometric Function Theory (GFT), a subfield of Complex Analysis. This is primarily due to the geometric and analytic properties of these regular functions, and their applications in engineering design [4], conformal mappings [5, p.242–249], mathematical physics [5, p.280–318], complex dynamics [6], and special functions [7]. One can find more application fields in the work of Zill and Shanahan [8]. Intriguingly, GFT combines analysis with geometry to focus on how complex functions transform shapes and domains in the complex plane. A crucial mathematical problem that led to the advent of GFT is a conjecture of 1916. This conjecture states that if $\mathfrak{g} \in \mathbf{S}$ with the Maclaurin-Taylor series (1), then $|\mathfrak{a}_c| \leqslant c$ for all $c \geqslant 2$. Bieberbach, the conjecturer, ascertained the conjecture for $c = 2$, while subsequent researchers, including Löwner, Garabedian and Schiffer, Pederson and Schiffer, and Pederson, confirmed it for $c = 3$, 4, 5, and 6, respectively. Nonetheless, settling the conjecture for $c \geqslant 7$ elusively lingered until 1985 when Branges [9] engaged the idea of hypergeometric functions in GFT to prove the inequality $|\mathfrak{a}_c| \leqslant c$ for all $c \geqslant 2$. For more historic details, see the canonical texts in [1–3, 10].

Sometime in the 1960s, Zalcman, see [3, p.127], conjectured the inequality

$$\left|\mathfrak{a}_c^2 - \mathfrak{a}_{2c-1}\right| \leqslant (c-1)^2 \quad (c \geqslant 2,\ \mathfrak{g} \in \mathbf{S}),$$

while Krushkal [11] ascertained the inequality for $c = 3$. Presumably, the inequality is still an open problem for $c > 3$. More so, the inequality

$$\mathbf{Z}_c(\lambda, \mathfrak{g}) = \left|\lambda \mathfrak{a}_c^2 - \mathfrak{a}_{2c-1}\right| \leqslant \lambda c^2 - 2c + 1 \quad (\lambda \geqslant 0,\ c \geqslant 2,\ \mathfrak{g} \in \mathbf{S}), \tag{2}$$

was explored in [12–14] for different subsets of $\mathbf{S}$. In particular, the celebrated Fekete-Szegö [15] coefficient inequality

$$\mathbf{F}(\lambda, \mathfrak{g}) = \left|\lambda \mathfrak{a}_2^2 - \mathfrak{a}_3\right| \leqslant 4\lambda - 3 \quad (\lambda \geqslant 0,\ \mathfrak{g} \in \mathbf{S}), \tag{3}$$

holds for $c = 2$ in (2). Furthermore, Pfluger [16] extended the work of Fekete-Szegö by proving it for complex parameter λ. In the process of solving the Zalcman conjecture, Krushkal [17] explored the inequality

$$\mathbf{K}_{c,k}(\mathfrak{g}) = \left|\mathfrak{a}_c^k - \mathfrak{a}_2^{k(c-1)}\right| \leqslant 2^{k(c-1)} - c^k \quad (c, k \geqslant 2,\ \mathfrak{g} \in \mathbf{S}), \tag{4}$$

while Ma [14] explored the inequality

$$\mathbf{M}_{c,\,d}(\mathfrak{g}) = |\mathfrak{a}_c\mathfrak{a}_d - \mathfrak{a}_{c+d-1}| \leqslant (c-1)(d-1) \quad (c, d \geqslant 2,\ \mathfrak{g} \in \mathbf{S}). \tag{5}$$

The above-mentioned inequalities are all sharp for the Koebe function

$$\mathfrak{K}(z) = z(1-z)^{-2} = \sum_{c\geqslant 1} cz^c \quad (|z| < 1),$$

or one of its various rotations

$$e^{-i\theta}\mathfrak{K}(e^{i\theta}z) = \sum_{c\geqslant 1} ce^{-i(c-1)\theta}z^c \quad (0 \leqslant \theta < 2\pi,\ |z| < 1).$$

Two fundamental principles in GFT that we will be needing in this work are the subordination and convolution principles. The subordination principle implies that if we have two regular functions $\mathfrak{g}$ in (1) and

$$\mathscr{G}(z) = z + \sum_{c\geqslant 2} \mathfrak{b}_c z^c \quad (|z| < 1), \tag{6}$$

then $\mathfrak{g}$ 'is subordinate to' $\mathbf{G}$ (usually expressed in notation $\mathfrak{g} \prec \mathbf{G}$), if and only if, there is a third regular function

$$\xi(z) = \sum_{c\geqslant 1} \varepsilon_c z^c \quad (|\xi(z)| < 1,\ \xi(0) = 0,\ |z| < 1), \tag{7}$$

such that

$$\mathfrak{g}(z) = \mathbf{G} \circ \xi(z) = \mathbf{G}(\xi(z)) \quad (|z| < 1).$$

Suppose $\mathbf{G}$ is also univalent in $|z| < 1$, then the definition improves to say that

$$\mathfrak{g} \prec \mathbf{G} \iff \mathfrak{g}(0) = \mathbf{G}(0) \quad \text{and} \quad \mathfrak{g}(|z| < 1) \subset \mathbf{G}(|z| < 1).$$

For more properties on subordination, see [18–22]. Further, the Hadamard product (sometimes called convolution) of two functions $\mathfrak{g}$ and $\mathbf{G}$ (usually expressed in notation $\mathfrak{g} \star \mathbf{G}$) is defined as:

$$(\mathfrak{g} \star \mathbf{G})(z) := z + \sum_{c\geqslant 2} (\mathfrak{a}_c \times \mathfrak{b}_c) z^c = z + \sum_{c\geqslant 2} (\mathfrak{b}_c \times \mathfrak{a}_c) z^c =: (\mathbf{G} \star \mathfrak{g})(z).$$

Let the set of regular functions of Taylor's series type

$$\mathfrak{h}(z) = 1 + \rho_1 z + \rho_2 z^2 + \rho_3 z^3 + \cdots = 1 + \sum_{c \geqslant 1} \rho_c z^c, \tag{8}$$

be represented by $\mathbf{H}$ such that $\mathfrak{Re}\,\mathfrak{h}(z) > 0$ and $\mathfrak{h}(0) = 1$ for all $|z| < 1$. The function $\mathfrak{h}$ is called a regular function with positive real part in $|z| < 1$. However, it is well-known that functions ξ in (7) and $\mathfrak{h}$ in (8) are interconnected by the relation

$$\xi(z) = \frac{\mathfrak{h}(z)-1}{\mathfrak{h}(z)+1} = \frac{1}{2}\rho_1 z + \left(\frac{1}{2}\rho_2 - \frac{1}{4}\rho_1^2\right) z^2 + \left(\frac{1}{8}\rho_1^3 - \frac{1}{2}\rho_1\rho_2 + \frac{1}{2}\rho_3\right) z^3$$

$$+ \left(\frac{1}{2}\rho_4 - \frac{1}{2}\rho_1\rho_3 - \frac{1}{4}\rho_2^2 - \frac{1}{16}\rho_1^4 + \frac{3}{8}\rho_1^2\rho_2\right) z^4 + \cdots. \tag{9}$$

The series (9) is famous as a tool for solving many coefficient problems involving function-type $\mathfrak{g}$ in (1).

1.1 *The n-leaf-shaped domain*

A n-leaf-shaped domain is a term primarily used in mathematical contexts, particularly in the study of complex functions that map the unit disk onto a domain bounded by an epicycloid with n cusps. In 2020, Gandhi [23] studied some analytical properties of a certain set of starlike functions satisfying the condition

$$\frac{z\mathfrak{g}'(z)}{\mathfrak{g}(z)} \prec 3\ell(z) = 1 + \frac{4}{5}z + \frac{1}{5}z^4 \quad (|z| < 1).$$

Function $3\ell(z)$ was said to map the unit disk $|z| < 1$ onto a three-leaf-shaped domain. Later, Gandhi et al. [24] generalized the three-leaf-shaped domain and introduced the function

$$n\ell(z) = 1 + \frac{N}{N+1}z + \frac{1}{N+1}z^N \quad (N = 4, 5, 6, \ldots,\ n = N-1,\ |z| < 1),$$

where the function $n\ell(z)$ maps the unit disk $|z| < 1$ onto a n-leaf-shaped domain, or in other words, we say that function $n\ell(z)$ maps the unit disk $|z| < 1$ onto a domain bounded by an epicycloid with $n = (N-1)$ cusps. Studies in the same paper were on the set $\mathbf{ST}(n\ell(z))$ of starlike functions satisfying the condition

$$\frac{z\mathfrak{g}'(z)}{\mathfrak{g}(z)} \prec n\ell(z).$$

Explorations in the paper included some radii problems, several inclusion properties, and some sharp estimations for a few early coefficients of function $\mathfrak{g} \in \mathbf{ST}(n\ell(z))$. Furthermore, Gandhi et al. [24] provided some analytical properties and descriptive diagrams of function $\mathfrak{g} \in \mathbf{ST}(n\ell(z))$ for some specific values of n. For instance, some subsets of $\mathbf{S}$ in association with the four-leaf-type function

$$4\ell(z) = 1 + \frac{5}{6}z + \frac{1}{6}z^5 \quad (|z| < 1), \tag{10}$$

were studied by authors in [25–31] for various geometric conditions, diagrams, definitions and examples.

This study's impetus comes from the aforementioned sets and their results. So, in view of (9) and (10),

$$4\ell(z)\circ\xi(z)=4\ell(\xi(z))=1+\frac{5}{6}\xi(z)+\frac{1}{6}[\xi(z)]^5\quad(|z|<1),$$

and by simple expansion we have

$$4\ell(\xi(z))=1+\frac{5}{12}\rho_1 z+\frac{5}{12}\left(\rho_2-\frac{1}{2}\rho_1^2\right)z^2+\frac{5}{12}\left(\frac{1}{4}\rho_1^3-\rho_1\rho_2+\rho_3\right)z^3$$

$$+\frac{5}{4}\left(-\frac{1}{24}\rho_1^4+\frac{1}{4}\rho_1^2\rho_2-\frac{1}{3}\rho_1\rho_3-\frac{1}{6}\rho_2^2+\frac{1}{3}\rho_4\right)z^4+\cdots. \tag{11}$$

1.2 *The Jackson's differentiation*

Foremost, let $q\in(0,1)$ and define the q-number c by

$$[c]_q=\begin{cases}0 & \text{when}\quad c=0,\\ 1 & \text{when}\quad c=1,\\ 1+q & \text{when}\quad c=2,\\ \dfrac{1-q^c}{1-q}=\displaystyle\sum_{j=0}^{c-1}q^j & \text{when}\quad c\in\mathbb{R}.\end{cases} \tag{12}$$

We can deduce, by L'Hôpital's rule, that $\lim\limits_{q\uparrow1}[c]_q=c$. As a result of the q-number system, the quantum derivative (q-difference, q-derivative or Jackson's derivative) for function $\mathfrak{g}$ was defined by Jackson [32] as

$$\left.\begin{aligned}&\mathfrak{D}_q\mathfrak{g}(0)=\mathfrak{g}'(0)=1,\\ &\mathfrak{D}_q\mathfrak{g}(z)=\frac{\mathfrak{g}(z)-\mathfrak{g}(qz)}{(1-q)z}=1+\sum_{c\geqslant2}[c]_q\mathfrak{a}_c z^{c-1}\quad(z\neq0),\\ &\mathfrak{D}_q^2\mathfrak{g}(z)=\mathfrak{D}_q(\mathfrak{D}_q\mathfrak{g}(z))=\sum_{c\geqslant2}[c]_q[c-1]_q\mathfrak{a}_c z^{c-2},\end{aligned}\right\} \tag{13}$$

and clearly,

$$\lim_{q\uparrow1}\mathfrak{D}_q\mathfrak{g}(z)=\lim_{q\uparrow1}\frac{\mathfrak{g}(z)-\mathfrak{g}(qz)}{(1-q)z}=\mathfrak{g}'(z),$$

where $\mathfrak{g}'(z)$ is the classical differentiation of function $\mathfrak{g}$. For more details, see [32–40].

1.3 *A novel set of regular q-functions*

The special infinite series type

$$\mu(r)=\sum_{c\geqslant 1}\frac{2c}{(c^2+r^2)^2}\quad (r>0),$$

was introduced by Mathieu [41] in 1890. The series has been extensively explored in mathematical analysis for establishing inequalities, approximations, as well as establishing some connections with special functions such as the Riemann zeta function and Gamma functions. Applications of the Mathieu series also extend to mathematical physics, including heat conduction and wave propagation problems in cylindrical or spherical geometries. In 2019, Bansal and Sokół [42] (see also [43]), presented the normalized form of series $\mu(r)$ as

$$\mu(r,z)=\frac{(1+r^2)^2}{2}\sum_{c\geqslant 1}\frac{2c}{(c^2+r^2)^2}z^c=z+\sum_{c\geqslant 2}\frac{c(1+r^2)^2}{(c^2+r^2)^2}z^c\quad (r>0,\ |z|<1),$$

where several geometric properties of function $\mathfrak{g}$, defined in connection with $\mu(r,z)$, were reported.

Definition 1.1 Let $q\in(0,1)$ and $r>0$, then using the q-number system in (12), we introduce the Mathieu q-series $\mu_q(r,z)$ as

$$\mu_q(r,z)=\frac{(1+r^2)^2}{2}\sum_{c\geqslant 1}\frac{2[c]_q}{([c]_q^2+r^2)^2}z^c=z+\sum_{c\geqslant 2}\frac{[c]_q(1+r^2)^2}{([c]_q^2+r^2)^2}z^c\quad (|z|<1),$$

and observe that $\lim_{q\uparrow 1}\mu_q(r,z)=\mu(r,z)$.

Definition 1.2 Define the q-function $\gamma_q(r,z)$ as

$$\gamma_q(r,z)=\mathfrak{g}(z)\star\mu_q(r,z)=z+\sum_{c\geqslant 2}\frac{[c]_q(1+r^2)^2}{([c]_q^2+r^2)^2}\mathfrak{a}_c z^c\quad (q\in(0,1),\ r>0,\ |z|<1),$$

and using (13) means

$$\mathfrak{D}_q(\gamma_q(r,z))=1+\sum_{c\geqslant 2}\frac{[c]_q^2(1+r^2)^2}{([c]_q^2+r^2)^2}\mathfrak{a}_c z^{c-1}\quad (q\in(0,1),\ r>0,\ |z|<1).\tag{14}$$

Definition 1.3 Let $\eta\in\mathbb{C}\setminus\{0\}$, $q\in(0,1)$, $r>0$, and $\phi\in(-\pi,\pi]$. Then function $\mathfrak{g}\in\mathbf{A}$ of the series type (1) is said to be in the set $\Xi_{q,r}^{\eta,\phi}(4\ell)$, if and only if, the subordination condition

$$1+\frac{1}{\eta}\left(\mathfrak{D}_q(\gamma_q(r,z))+\left(\frac{1+e^{i\phi}}{2}\right)z\mathfrak{D}_q^2(\gamma_q(r,z))-1\right)\prec 4\ell(z),\tag{15}$$

holds for $4\ell(z)$ in (10) and $|z|<1$.

Remark 1.4 Enumerated below are the subordination conditions of the sets of functions that are closely related to the set $\Xi_{q,r}^{\eta,\phi}(4\ell)$. Thus, setting

1. $\eta = 1$ and $\phi = \pi$ leads to the condition

$$\mathfrak{D}_q(\gamma_q(r,z)) \prec 4\ell(z);$$

2. $\eta = 1$, $\phi = \pi$, and $q \uparrow 1$ leads to the condition

$$(\gamma_q(r,z))' \prec 4\ell(z);$$

3. $\eta = 1$ leads to the condition

$$\mathfrak{D}_q(\gamma_q(r,z)) + \left(\frac{1+e^{i\phi}}{2}\right) z\mathfrak{D}_q^2(\gamma_q(r,z)) \prec 4\ell(z);$$

4. $\eta = 1$ and $q \uparrow 1$ leads to the condition

$$(\gamma_q(r,z))' + \left(\frac{1+e^{i\phi}}{2}\right) z(\gamma_q(r,z))'' \prec 4\ell(z);$$

5. $\eta = 1$ and $\phi = 0$ leads to the condition

$$\mathfrak{D}_q(\gamma_q(r,z)) + z\mathfrak{D}_q^2(\gamma_q(r,z)) \prec 4\ell(z);$$

6. $\eta = 1$, $\phi = 0$, and $q \uparrow 1$ leads to the condition

$$(\gamma_q(r,z))' + z(\gamma_q(r,z))'' \prec 4\ell(z).$$

Example 1.5 Observe that some examples of functions $\mathfrak{g}$ in (1) are:

$$\left.\begin{aligned}
&\mathfrak{g}_1(z) = z,\\
&\mathfrak{g}_2(z) = \frac{z}{1-z} = z+z^2+z^3+z^4+\cdots,\\
&\mathfrak{g}_3(z) = \frac{z}{1+z+z^2} = z-z^2+z^4-z^5+\cdots,\\
&\mathfrak{g}_4(z) = -[z+\ln(1-z)^2] = z+z^2+\tfrac{2}{3}z^3+\frac{1}{2}z^4+\cdots,\\
&\mathfrak{g}_5(z) = e^{-i\vartheta}\frac{e^{i\vartheta}z}{(1-e^{i\vartheta}z)^2} = z+2e^{i\vartheta}z^2+3e^{2i\vartheta}z^3+4e^{3i\vartheta}z^4+\cdots,
\end{aligned}\right\} \in \mathbf{S}$$

to get the novel functions:

$$\gamma_{1,q}(r,z) = z,$$

$$\gamma_{2,q}(r,z) = z + \frac{[2]_q(1+r^2)^2}{([2]_q^2+r^2)^2}z^2 + \frac{[3]_q(1+r^2)^2}{([3]_q^2+r^2)^2}z^3 + \frac{[4]_q(1+r^2)^2}{([4]_q^2+r^2)^2}z^4 + \cdots,$$

$$\gamma_{3,q}(r,z) = z - \frac{[2]_q(1+r^2)^2}{([2]_q^2+r^2)^2}z^2 + \frac{[4]_q(1+r^2)^2}{([4]_q^2+r^2)^2}z^4 - \frac{[5]_q(1+r^2)^2}{([5]_q^2+r^2)^2}z^5 + \cdots,$$

$$\gamma_{4,q}(r,z) = z + \frac{[2]_q(1+r^2)^2}{([2]_q^2+r^2)^2}z^2 + \frac{2[3]_q(1+r^2)^2}{3([3]_q^2+r^2)^2}z^3 + \frac{[4]_q(1+r^2)^2}{2([4]_q^2+r^2)^2}z^4 + \cdots,$$

$$\gamma_{5,q}(r,z) = z + 2e^{i\vartheta}\frac{[2]_q(1+r^2)^2}{([2]_q^2+r^2)^2}z^2 + 3e^{2i\vartheta}\frac{[3]_q(1+r^2)^2}{([3]_q^2+r^2)^2}z^3 + 4e^{3i\vartheta}\frac{[4]_q(1+r^2)^2}{([4]_q^2+r^2)^2}z^4 + \cdots,$$

respectively.

Indeed, the aforementioned subsets, the novel Mathieu q-series, and the robust nature of condition (15) are the motivations for this exploration. Thus, in this exploration, we provided some upper estimates for the initial coefficients of function $\mathfrak{g} \in \Xi_{q,r}^{\eta,\phi}(4\ell)$. After that, we investigated the jth-root transformations of the function $\mathfrak{g}$ and derived several upper estimates for a number of functionals.

From now onwards, all parameters are assumed to be as stated in Definition 1.3 unless otherwise specified. The findings are presented in Section 3.

2. Lemmas and propositions

The following lemmas and propositions will be required to support the findings. Now, let **H** be the set of functions $\mathfrak{h}$ in (8), then the following lemmas and propositions hold.

Lemma 2.1 [3, p.25] If $\mathfrak{h} \in \mathbf{H}$ and $c \in \mathbb{N}$, then

$$|\rho_c| \leqslant 2. \tag{16}$$

Lemma 2.2 [44, Lemma 2.3] If $\mathfrak{h} \in \mathbf{H}$ and $c, d \in \mathbb{N}$, then

$$|\rho_{c+d} - \beta\rho_c\rho_d| \leqslant \begin{cases} 2 & \text{when } 0 \leqslant \beta \leqslant 1, \\ 2|2\beta - 1| & \text{elsewhere.} \end{cases}$$

Lemma 2.3 [45, Lemma 2.2] If $\mathfrak{h} \in \mathbf{H}$, then

$$|t_1\rho_1^3 - t_2\rho_1\rho_2 + t_3\rho_3| \leqslant 2|t_1| + 2|t_2 - 2t_1| + 2|t_1 - t_2 + t_3|.$$

Lemma 2.4 [46, Corollary 2.5] If $\mathfrak{h} \in \mathbf{H}$ and $\beta \in \mathbb{C}$, then

$$\left|\rho_2 - \beta\frac{\rho_1^2}{2}\right| \leqslant 2\max\left\{1, |\beta - 1|\right\}.$$

Proposition 2.5 If $\mathfrak{h} \in \mathbf{H}$, then
(i) Lemma 2.2 implies

$$\left|\rho_2 - \frac{1}{2}\rho_1^2\right| \leqslant 2; \tag{17}$$

(ii) Lemma 2.3 implies

$$\left|\frac{1}{4}\rho_1^3 - \rho_1\rho_2 + \rho_3\right| \leqslant 2; \tag{18}$$

and
(iii) Lemmas 2.1, 2.2, and 2.3 implies

$$\begin{aligned}
&\left|-\frac{1}{24}\rho_1^4 + \frac{1}{4}\rho_1^2\rho_2 - \frac{1}{3}\rho_1\rho_3 - \frac{1}{6}\rho_2^2 + \frac{1}{3}\rho_4\right| \\
&= \left|-\rho_1\left\{\frac{1}{24}\rho_1^3 - \frac{1}{4}\rho_1\rho_2 + \frac{1}{3}\rho_3\right\} + \frac{1}{3}\left\{\rho_4 - \frac{1}{2}\rho_2^2\right\}\right| \\
&\leqslant |\rho_1|\left|\frac{1}{24}\rho_1^3 - \frac{1}{4}\rho_1\rho_2 + \frac{1}{3}\rho_3\right| + \frac{1}{3}\left|\rho_4 - \frac{1}{2}\rho_2^2\right| \leqslant 2.
\end{aligned} \tag{19}$$

Proposition 2.6 If $q \in (0, 1)$, $\phi \in (-\pi, \pi]$, $x \in \mathbb{N}$, and

$$Q_x = 2 + [x]_q(1 + e^{i\phi}), \tag{20}$$

then

$$|Q_x| = \sqrt{4 + 2[x]_q(2 + [x]_q)(1 + \cos\phi)}. \tag{21}$$

3. Main results

3.1 *Coefficient inequalities for* $\mathfrak{g} \in \Xi_{q,r}^{\eta,\phi}(4\ell)$

Theorem 3.1 If $\mathfrak{g} \in \Xi_{q,r}^{\eta,\phi}(4\ell)$, then

$$|\mathfrak{a}_2| \leqslant \frac{5|\eta|([2]_q^2+r^2)^2}{3[2]_q^2(1+r^2)^2\sqrt{10+6\cos\phi}}, \tag{22}$$

$$|\mathfrak{a}_3| \leqslant \frac{5|\eta|([3]_q^2+r^2)^2}{3[3]_q^2(1+r^2)^2\sqrt{4+2[2]_q(2+[2]_q)(1+\cos\phi)}}, \tag{23}$$

$$|\mathfrak{a}_4| \leqslant \frac{5|\eta|([4]_q^2+r^2)^2}{3[4]_q^2(1+r^2)^2\sqrt{4+2[3]_q(2+[3]_q)(1+\cos\phi)}}, \tag{24}$$

and

$$|\mathfrak{a}_5| \leqslant \frac{5|\eta|([5]_q^2+r^2)^2}{[5]_q^2(1+r^2)^2\sqrt{4+2[4]_q(2+[4]_q)(1+\cos\phi)}}. \tag{25}$$

Proof. If $\mathfrak{g} \in \Xi_{q,r}^{\eta,\phi}(4\ell)$, then by the principle of subordination, we can write (15) as

$$1+\frac{1}{\eta}\left(\mathfrak{D}_q(\gamma_q(r,z))+\left(\frac{1+e^{i\phi}}{2}\right)z\mathfrak{D}_q^2(\gamma_q(r,z))-1\right)=4\ell(\xi(z)), \tag{26}$$

where the use of (14) and (20) in (26) implies

$$\begin{aligned}
&1+\frac{1}{\eta}\left(\mathfrak{D}_q(\gamma_q(r,z))+\left(\frac{1+e^{i\phi}}{2}\right)z\mathfrak{D}_q^2(\gamma_q(r,z))-1\right)\\
&=1+\frac{[2]_q^2(1+r^2)^2Q_1}{2\eta([2]_q^2+r^2)^2}\mathfrak{a}_2z+\frac{[3]_q^2(1+r^2)^2Q_2}{2\eta([3]_q^2+r^2)^2}\mathfrak{a}_3z^2+\frac{[4]_q^2(1+r^2)^2Q_3}{\eta([4]_q^2+r^2)^2}\mathfrak{a}_4z^3\\
&+\frac{[5]_q^2(1+r^2)^2Q_4}{\eta([5]_q^2+r^2)^2}\mathfrak{a}_5z^4+\cdots=4\ell(\xi(z)).
\end{aligned} \tag{27}$$

A systematic comparison of terms in (27) and (11) shows that

$$\mathfrak{a}_2=\frac{2\eta([2]_q^2+r^2)^2}{[2]_q^2(1+r^2)^2Q_1}\times\frac{5}{12}\rho_1 \implies |\mathfrak{a}_2|=\frac{5|\eta|([2]_q^2+r^2)^2}{6[2]_q^2(1+r^2)^2|Q_1|}|\rho_1|, \tag{28}$$

and the application of (16) with (21) shows that inequality (22) holds. Similarly, we have

$$\mathfrak{a}_3=\frac{2\eta([3]_q^2+r^2)^2}{[3]_q^2(1+r^2)^2Q_2}\times\frac{5}{12}\left(\rho_2-\frac{1}{2}\rho_1^2\right) \implies |\mathfrak{a}_3|=\frac{5|\eta|([3]_q^2+r^2)^2}{6[3]_q^2(1+r^2)^2|Q_2|}\left|\rho_2-\frac{1}{2}\rho_1^2\right|, \tag{29}$$

and the application of (17) with (21) shows that inequality (23) holds. Furthermore, we have

$$\mathfrak{a}_4 = \frac{2\eta([4]_q^2+r^2)^2}{[4]_q^2(1+r^2)^2Q_3} \times \frac{5}{12}\left(\frac{1}{4}\rho_1^3 - \rho_1\rho_2 + \rho_3\right) \implies |\mathfrak{a}_4| = \frac{5|\eta|([4]_q^2+r^2)^2}{6[4]_q^2(1+r^2)^2\,|Q_3|}\left|\frac{1}{4}\rho_1^3 - \rho_1\rho_2 + \rho_3\right|, \tag{30}$$

and the application of (18) with (21) shows that inequality (24) holds. And finally,

$$\mathfrak{a}_5 = \frac{2\eta([5]_q^2+r^2)^2}{[5]_q^2(1+r^2)^2Q_4} \times \frac{5}{4}\left(-\frac{1}{24}\rho_1^4 + \frac{1}{4}\rho_1^2\rho_2 - \frac{1}{3}\rho_1\rho_3 - \frac{1}{6}\rho_2^2 + \frac{1}{3}\rho_4\right), \tag{31}$$

from where

$$\mathfrak{a}_5 = \frac{5\eta([5]_q^2+r^2)^2}{2[5]_q^2(1+r^2)^2Q_4}\left(-\rho_1\left\{\frac{1}{24}\rho_1^3 - \frac{1}{4}\rho_1\rho_2 + \frac{1}{3}\rho_3\right\} + \frac{1}{3}\left\{\rho_4 - \frac{1}{2}\rho_2^2\right\}\right),$$

so,

$$|\mathfrak{a}_5| \leqslant \frac{5|\eta|([5]_q^2+r^2)^2}{2[5]_q^2(1+r^2)^2\,|Q_4|}\left(|\rho_1|\left|\frac{1}{24}\rho_1^3 - \frac{1}{4}\rho_1\rho_2 + \frac{1}{3}\rho_3\right| + \frac{1}{3}\left|\rho_4 - \frac{1}{2}\rho_2^2\right|\right),$$

and the successive application of (19) with (21) shows that inequality (25) holds. □

3.2 *The jth-root transform of* $\mathfrak{g} \in \mathbf{S}$

The jth-root transformation, see [2, p.18], is one of the basic transformations that preserves the properties of functions $\mathfrak{g} \in \mathbf{S}$. This transformation is expressed in series type:

$$\begin{aligned}
\mathfrak{f} \equiv \mathfrak{f}_j(z) &= \sqrt[j]{\mathfrak{g}(z^j)} \\
&= z + \delta_1\mathfrak{a}_2 z^{(j+1)} + \left[\delta_1\mathfrak{a}_3 - \delta_2\mathfrak{a}_2^2\right] z^{(2j+1)} + \left[\delta_1\mathfrak{a}_4 - 2\delta_2\mathfrak{a}_2\mathfrak{a}_3 + \delta_3\mathfrak{a}_2^3\right] z^{(3j+1)} \\
&\quad + \left[\delta_1\mathfrak{a}_5 - \delta_2(2\mathfrak{a}_2\mathfrak{a}_4 + \mathfrak{a}_3^2) + 3\delta_3\mathfrak{a}_2^2\mathfrak{a}_3 - \delta_4\mathfrak{a}_2^4\right] z^{(4j+1)} + \cdots \\
&\equiv z + \sum_{c\geqslant 1} \mathfrak{b}_{(cj+1)} z^{(cj+1)},
\end{aligned} \tag{32}$$

where we let $\mathbf{S}^{(j)}$ connote the set of functions $\mathfrak{f}$, $j \in \mathbb{N}$, and the δ-function

$$\delta_c := \begin{cases} \dfrac{1}{j} & \text{for} \quad c = 1, \\ \\ \dfrac{1}{c!}\dfrac{1}{j}\displaystyle\prod_{k=1}^{c-1}\left(k-\frac{1}{j}\right) & \text{for} \quad c \geqslant 2. \end{cases} \tag{33}$$

applies. It can easily be observed that for some j in (32), we have

1. $\mathfrak{f}_1(z) = \mathfrak{g}(z) \in \mathbf{S}^{(1)} =: \mathbf{S}$ in (1);
2. $\mathfrak{f}_2(z) = \sqrt{\mathfrak{g}(z^2)} = z + \mathfrak{b}_3 z^3 + \mathfrak{b}_5 z^5 + \cdots \in \mathbf{S}^{(2)}$;
3. $\mathfrak{f}_3(z) = \sqrt[3]{\mathfrak{g}(z^3)} = z + \mathfrak{b}_4 z^4 + b_7 z^7 + \cdots \in \mathbf{S}^{(3)}$;
4. etc.

Theorem 3.2 If $\mathfrak{f} \in \Xi_{q,r}^{\eta;\,\phi}(4\ell)$, then

$$|\mathfrak{b}_{j+1}| \leqslant \frac{5\delta_1|\eta|([2]_q^2 + r^2)^2}{3[2]_q^2(1+r^2)^2\sqrt{10+6\cos\phi}},$$

$$|\mathfrak{b}_{2j+1}| \leqslant \frac{5\delta_1|\eta|([3]_q^2 + r^2)^2}{3[3]_q^2(1+r^2)^2\sqrt{4+2[2]_q(2+[2]_q)(1+\cos\phi)}} + \frac{25\delta_2|\eta|^2([2]_q^2 + r^2)^4}{9[2]_q^4(1+r^2)^4(10+6\cos\phi)},$$

$$\begin{aligned} |\mathfrak{b}_{3j+1}| \leqslant{}& \frac{5\delta_1|\eta|([4]_q^2 + r^2)^2}{3[4]_q^2(1+r^2)^2\sqrt{4+2[3]_q(2+[3]_q)(1+\cos\phi)}} \\ &+ \frac{50\delta_2|\eta|^2([2]_q^2 + r^2)^2([3]_q^2 + r^2)^2}{9[2]_q^2[3]_q^2(1+r^2)^4\sqrt{10+6\cos\phi}\sqrt{4+2[2]_q(2+[2]_q)(1+\cos\phi)}} \\ &+ \frac{125\delta_3|\eta|^3([2]_q^2 + r^2)^6}{27[2]_q^6(1+r^2)^6(\sqrt{10+6\cos\phi})^3}, \end{aligned}$$

and

$$\begin{aligned} |\mathfrak{b}_{4j+1}| \leqslant{}& \frac{5\delta_1|\eta|([5]_q^2 + r^2)^2}{[5]_q^2(1+r^2)^2\sqrt{4+2[4]_q(2+[4]_q)(1+\cos\phi)}} \\ &+ \frac{50\delta_2|\eta|^2([2]_q^2 + r^2)^2([4]_q^2 + r^2)^2}{9[2]_q^2[4]_q^2(1+r^2)^4\sqrt{10+6\cos\phi}\sqrt{4+2[3]_q(2+[3]_q)(1+\cos\phi)}} \\ &+ \frac{25\delta_2|\eta|^2([3]_q^2 + r^2)^4}{9[3]_q^4(1+r^2)^4[4+2[2]_q(2+[2]_q)(1+\cos\phi)]} \end{aligned}$$

$$+\frac{125\delta_3|\eta|^3([2]_q^2+r^2)^4([3]_q^2+r^2)^2}{9[2]_q^4[3]_q^2(1+r^2)^6(10+6\cos\phi)\sqrt{4+2[2]_q(2+[2]_q)(1+\cos\phi)}}$$

$$+\frac{625\delta_4|\eta|^4([2]_q^2+r^2)^8}{81[2]_q^8(1+r^2)^8(10+6\cos\phi)^2}.$$

Note that setting $j=1$ leads to the results in Theorem 3.1.

Proof. Let $\mathfrak{f}\in\Xi_{q,r}^{\eta,\phi}(4\ell)$, then putting (28)–(31) in (32) and using (16)–(21) complete the proof. □

The functional (3) is modified for $\mathfrak{f}\in\Xi_{q,r}^{\eta,\phi}(4\ell)$ as

$$\mathbf{F}_j(\lambda,\mathfrak{f})=|\lambda\mathfrak{b}_{j+1}^2-\mathfrak{b}_{2j+1}|\leqslant 4\lambda-3\quad(j\in\mathbb{N},\ \lambda\geqslant 0,\ \mathfrak{f}\in\mathbf{S}^{(j)}),\tag{34}$$

and we shall let $\mathbf{F}_1(\lambda,\mathfrak{f})=\mathbf{F}(\lambda,\mathfrak{g})$. Moreover, see [47, 48] for some related works on functional (34).

Theorem 3.3 (Fekete-Szegö Inequality) If $\mathfrak{f}\in\Xi_{q,r}^{\eta,\phi}(4\ell)$, then for $\lambda\geqslant 0$,

$$\mathbf{F}_j(\lambda,\mathfrak{f})\leqslant\frac{5|\eta|([3]_q^2+r^2)^2}{3j[3]_q^2(1+r^2)^2\sqrt{4+2[2]_q(2+[2]_q)(1+\cos\phi)}}\max\{1,\chi\},$$

where

$$\chi:=|\beta-1|=\frac{5|\eta|(2\lambda+j-1)[3]_q^2([2]_q^2+r^2)^4\sqrt{4+2[2]_q(2+[2]_q)(1+\cos\phi)}}{6j[2]_q^4(1+r^2)^2([3]_q^2+r^2)^2(10+6\cos\phi)}.$$

Proof. Let $\mathfrak{f}\in\Xi_{q,r}^{\eta,\phi}(4\ell)$, then putting (32) into (34) means

$$\lambda\mathfrak{b}_{j+1}^2-\mathfrak{b}_{2j+1}=\lambda\left(\frac{1}{j}\mathfrak{a}_2\right)^2-\left(\frac{1}{j}\mathfrak{a}_3-\frac{(j-1)}{2j^2}\mathfrak{a}_2^2\right)=\mathfrak{a}_2^2\left(\frac{2\lambda+j-1}{2j^2}\right)-\frac{1}{j}\mathfrak{a}_3,$$

where using (28) and (29) gives

$$\lambda\mathfrak{b}_{j+1}^2-\mathfrak{b}_{2j+1}=\left(\frac{2\lambda+j-1}{2j^2}\right)\left\{\frac{5\eta([2]_q^2+r^2)^2}{6[2]_q^2(1+r^2)^2Q_1}\rho_1\right\}^2-\frac{1}{j}\left\{\frac{5\eta([3]_q^2+r^2)^2}{6[3]_q^2(1+r^2)^2Q_2}\left(\rho_2-\frac{1}{2}\rho_1^2\right)\right\}.$$

Some simplifications give

$$\lambda\mathfrak{b}_{j+1}^2-\mathfrak{b}_{2j+1}=-\frac{5\eta([3]_q^2+r^2)^2}{6j[3]_q^2(1+r^2)^2Q_2}\left\{\rho_2-\left(1+\frac{5\eta(2\lambda+j-1)[3]_q^2([2]_q^2+r^2)^4Q_2}{6j[2]_q^4(1+r^2)^2([3]_q^2+r^2)^2Q_1^2}\right)\frac{\rho_1^2}{2}\right\}.$$

So,

$$|\lambda \mathfrak{b}_{j+1}^2 - \mathfrak{b}_{2j+1}| \leqslant \frac{5|\eta|([3]_q^2+r^2)^2}{6j[3]_q^2(1+r^2)^2|Q_2|}\left|\rho_2 - \left(1+\frac{5\eta(2\lambda+j-1)[3]_q^2([2]_q^2+r^2)^4 Q_2}{6j[2]_q^4(1+r^2)^2([3]_q^2+r^2)^2 Q_1^2}\right)\frac{\rho_1^2}{2}\right|,$$

or for short,

$$\left|\lambda \mathfrak{b}_{j+1}^2 - \mathfrak{b}_{2j+1}\right| \leqslant \frac{5|\eta|([3]_q^2+r^2)^2}{6j[3]_q^2(1+r^2)^2|Q_2|}\left|\rho_2 - \beta\frac{\rho_1^2}{2}\right|,$$

where

$$\beta = 1+\frac{5\eta(2\lambda+j-1)[3]_q^2([2]_q^2+r^2)^4 Q_2}{6j[2]_q^4(1+r^2)^2([3]_q^2+r^2)^2 Q_1^2}.$$

Now, the application of Lemma 2.4 with the use of (21) gives the result. □

Setting $j=1$ leads to:

Corollary 3.4 If $\mathfrak{g} \in \Xi_{q,r}^{\eta,\phi}(4\ell)$, then for $\lambda \geqslant 0$,

$$\mathbf{F}(\lambda, \mathfrak{g}) \leqslant \frac{5|\eta|([3]_q^2+r^2)^2}{3[3]_q^2(1+r^2)^2\sqrt{4+2[2]_q(2+[2]_q)(1+\cos\phi)}}\max\{1, \chi\},$$

where

$$\chi = \frac{10\lambda|\eta|[3]_q^2([2]_q^2+r^2)^4\sqrt{4+2[2]_q(2+[2]_q)(1+\cos\phi)}}{6[2]_q^4(1+r^2)^2([3]_q^2+r^2)^2(10+6\cos\phi)}.$$

The functional (2) is modified for $\mathfrak{f} \in \Xi_{q,r}^{\eta,\phi}(4\ell)$ as

$$\mathbf{Z}_c(\lambda, \mathfrak{f}) = |\lambda \mathfrak{b}_c^2 - \mathfrak{b}_{2c-1}| \leqslant \lambda c^2 - 2c + 1 \quad (\lambda \geqslant 0,\ j \in \mathbb{N},\ c \geqslant j+1,\ \mathfrak{f} \in \mathbf{S}^{(j)}). \tag{35}$$

Theorem 3.5 (Zalcman Inequality) If $\mathfrak{f} \in \Xi_{q,r}^{\eta,\phi}(4\ell)$, then
(i) For $c = j+1$, $\mathbf{Z}_{j+1}(\lambda, \mathfrak{f}) = \mathbf{F}_j(\lambda, \mathfrak{f})$ in Theorem 3.3, and
(ii) For $c = 2j+1$ we get

$$\mathbf{Z}_{2j+1}(\lambda, \mathfrak{f})$$

$$= \frac{625(\lambda\delta_2^2+\delta_4)|\eta|^4([2]_q^2+r^2)^8}{81[2]_q^8(1+r^2)^8(10+6\cos\phi)^2}$$

$$+\frac{125(2\lambda\delta_1\delta_2+3\delta_3)|\eta|^3([2]_q^2+r^2)^4([3]_q^2+r^2)^2}{27[2]_q^4[3]_q^2(1+r^2)^6(10+6\cos\phi)\sqrt{4+2[2]_q(2+[2]_q)(1+\cos\phi)}}$$

$$+\frac{25(\lambda\delta_1^2+\delta_2)|\eta|^2([3]_q^2+r^2)^4}{9[3]_q^4(1+r^2)^4[4+2[2]_q(2+[2]_q)(1+\cos\phi)]}$$

$$+\frac{50\delta_2|\eta|^2([2]_q^2+r^2)^2([4]_q^2+r^2)^2}{9[2]_q^2[4]_q^2(1+r^2)^4\sqrt{10+6\cos\phi}\sqrt{4+2[3]_q(2+[3]_q)(1+\cos\phi)}}$$

$$+\frac{5\delta_1|\eta|([5]_q^2+r^2)^2}{[5]_q^2(1+r^2)^2\sqrt{4+2[4]_q(2+[4]_q)(1+\cos\phi)}}.$$

Proof. Let $\mathfrak{f}\in\Xi_{q,r}^{\eta,\phi}(4\ell)$, then putting (32) into (35) means

$$\lambda\mathfrak{b}_{2j+1}^2-\mathfrak{b}_{4j+1}=\lambda\left(\delta_1\mathfrak{a}_3-\delta_2\mathfrak{a}_2^2\right)^2-\left[\delta_1\mathfrak{a}_5-\delta_2(2\mathfrak{a}_2\mathfrak{a}_4+\mathfrak{a}_3^2)+3\delta_3\mathfrak{a}_2^2\mathfrak{a}_3-\delta_4\mathfrak{a}_2^4\right]$$

$$=(\lambda\delta_2^2+\delta_4)\mathfrak{a}_2^4-(2\lambda\delta_1\delta_2+3\delta_3)\mathfrak{a}_2^2\mathfrak{a}_3+(\lambda\delta_1^2+\delta_2)\mathfrak{a}_3^2+2\delta_2\mathfrak{a}_2\mathfrak{a}_4-\delta_1\mathfrak{a}_5. \tag{36}$$

Now, putting (28)–(31) in (36) simplifies to

$$\lambda\mathfrak{b}_{2j+1}^2-\mathfrak{b}_{4j+1}=\frac{625(\lambda\delta_2^2+\delta_4)\eta^4([2]_q^2+r^2)^8}{1{,}296[2]_q^8(1+r^2)^8Q_1^4}\rho_1^4$$

$$-\frac{125(2\lambda\delta_1\delta_2+3\delta_3)\eta^3([2]_q^2+r^2)^4([3]_q^2+r^2)^2}{216[2]_q^4[3]_q^2(1+r^2)^6Q_1^2Q_2}\left(\rho_2-\frac{1}{2}\rho_1^2\right)\rho_1^2$$

$$+\frac{25(\lambda\delta_1^2+\delta_2)\eta^2([3]_q^2+r^2)^4}{36[3]_q^4(1+r^2)^4Q_2^2}\left(\rho_2-\frac{1}{2}\rho_1^2\right)^2$$

$$+\frac{25\delta_2\eta^2([2]_q^2+r^2)^2([4]_q^2+r^2)^2}{18[2]_q^2[4]_q^2(1+r^2)^4Q_1Q_3}\left(\frac{1}{4}\rho_1^3-\rho_1\rho_2+\rho_3\right)\rho_1$$

$$-\frac{5\delta_1\eta([5]_q^2+r^2)^2}{2[5]_q^2(1+r^2)^2Q_4}\left(-\frac{1}{24}\rho_1^4+\frac{1}{4}\rho_1^2\rho_2-\frac{1}{3}\rho_1\rho_3-\frac{1}{6}\rho_2^2+\frac{1}{3}\rho_4\right).$$

So,

$$|\lambda \mathfrak{b}_{2j+1}^2 - \mathfrak{b}_{4j+1}| \leqslant \frac{625(\lambda\delta_2^2+\delta_4)|\eta|^4([2]_q^2+r^2)^8}{1{,}296[2]_q^8(1+r^2)^8|Q_1|^4}|\rho_1|^4$$

$$+\frac{125(2\lambda\delta_1\delta_2+3\delta_3)|\eta|^3([2]_q^2+r^2)^4([3]_q^2+r^2)^2}{216[2]_q^4[3]_q^2(1+r^2)^6|Q_1|^2|Q_2|}\left|\rho_2-\frac{1}{2}\rho_1^2\right||\rho_1|^2$$

$$+\frac{25(\lambda\delta_1^2+\delta_2)|\eta|^2([3]_q^2+r^2)^4}{36[3]_q^4(1+r^2)^4|Q_2|^2}\left|\rho_2-\frac{1}{2}\rho_1^2\right|^2$$

$$+\frac{25\delta_2|\eta|^2([2]_q^2+r^2)^2([4]_q^2+r^2)^2}{18[2]_q^2[4]_q^2(1+r^2)^4|Q_1||Q_3|}\left|\frac{1}{4}\rho_1^3-\rho_1\rho_2+\rho_3\right||\rho_1|$$

$$+\frac{5\delta_1|\eta|([5]_q^2+r^2)^2}{2[5]_q^2(1+r^2)^2|Q_4|}\left|-\frac{1}{24}\rho_1^4+\frac{1}{4}\rho_1^2\rho_2-\frac{1}{3}\rho_1\rho_3-\frac{1}{6}\rho_2^2+\frac{1}{3}\rho_4\right|,$$

and we can systematically implement (16)–(21) to arrive at the result. □

Setting $j=1$ leads to:

Corollary 3.6 If $\mathfrak{g}\in\Xi_{q,r}^{\eta,\phi}(4\ell)$, then

(i) $\mathbf{Z}_2(\lambda,\mathfrak{g})=\mathbf{F}(\lambda,\mathfrak{g})$ in Corollary 3.4, and

(ii) $\mathbf{Z}_3(\lambda,\mathfrak{g})\leqslant\dfrac{25\lambda|\eta|^2([3]_q^2+r^2)^4}{9[3]_q^4(1+r^2)^4[4+2[2]_q(\kappa)(1+\cos\phi)]}+\dfrac{5|\eta|([5]_q^2+r^2)^2}{[5]_q^2(1+r^2)^2\sqrt{4+2[4]_q(2+[4]_q)(1+\cos\phi)}}.$

The functional (4) is modified for $\mathfrak{f}\in\Xi_{q,r}^{\eta,\phi}(4\ell)$ as

$$\left.\begin{array}{l}\mathbf{K}_{c,k}^m(\mathfrak{f})=|\mathfrak{b}_c^k-\mathfrak{a}_2^{k(m-1)}|\leqslant 2^{k(m-1)}-m^k\\ (m,k\geqslant 2,\ \ j\in\mathbb{N},\ \ c\geqslant j+1,\ \ \mathfrak{f}\in\mathbf{S}^{(j)}).\end{array}\right\}\tag{37}$$

Theorem 3.7 (Krushkal Inequality) If $\mathfrak{f}\in\Xi_{q,r}^{\eta,\phi}(4\ell)$, then

(i) For $m=3$, $k=2$, and $c=2j+1$,

$$\mathbf{K}_{2j+1,2}^3(\mathfrak{f})\leqslant\frac{625(1-\delta_2^2)|\eta|^4([2]_q^2+r^2)^8}{81[2]_q^8(1+r^2)^8(10+6\cos\phi)^2}$$

$$+\frac{250\delta_1\delta_2|\eta|^3([2]_q^2+r^2)^4([3]_q^2+r^2)^2}{27[2]_q^4[3]_q^2(1+r^2)^6(10+6\cos\phi)\sqrt{4+2[2]_q(2+[2]_q)(1+\cos\phi)}}$$

$$+\frac{25\delta_1^2|\eta|^2([3]_q^2+r^2)^4}{9[3]_q^4(1+r^2)^4[4+2[2]_q(2+[2]_q)(1+\cos\phi)]},$$

and

(ii) For $m=3=k$, and $c=2j+1$

$$\mathbf{K}^3_{2j+1,3}(\mathfrak{f}) \leqslant \frac{15{,}625(1+\delta_2^3)|\eta|^6([2]_q^2+r^2)^{12}}{729[2]_q^{12}(1+r^2)^{12}(10+6\cos\phi)^3}$$

$$+\frac{3{,}125\delta_1\delta_2^2|\eta|^5([2]_q^2+r^2)^8([3]_q^2+r^2)^2}{81[2]_q^8[3]_q^2(1+r^2)^{10}(10+6\cos\phi)^2\sqrt{4+2[2]_q(2+[2]_q)(1+\cos\phi)}}$$

$$+\frac{625\delta_1^2\delta_2|\eta|^4([2]_q^2+r^2)^4([3]_q^2+r^2)^4}{27[2]_q^4[3]_q^4(1+r^2)^8(10+6\cos\phi)[4+2[2]_q(2+[2]_q)(1+\cos\phi)]}$$

$$+\frac{125\delta_1^3|\eta|^3([3]_q^2+r^2)^6}{27[3]_q^6(1+r^2)^6(\sqrt{4+2[2]_q(2+[2]_q)(1+\cos\phi)})^3}.$$

Proof. Let $\mathfrak{f}\in\Xi^{\eta;\,\phi}_{q,\,r}(4\ell)$, then
(i) Putting (32) into (37) means

$$\mathfrak{b}_{2j+1}^2-\mathfrak{a}_2^4=\left(\delta_1\mathfrak{a}_3-\delta_2\mathfrak{a}_2^2\right)^2-\mathfrak{a}_2^4=-(1-\delta_2^2)\mathfrak{a}_2^4-2\delta_1\delta_2\mathfrak{a}_2^2\mathfrak{a}_3+\delta_1^2\mathfrak{a}_3^2. \tag{38}$$

Now, putting (28) and (29) in (38) simplifies to

$$\mathfrak{b}_{2j+1}^2-\mathfrak{a}_2^4=-\frac{625(1-\delta_2^2)\eta^4([2]_q^2+r^2)^8}{1{,}296[2]_q^8(1+r^2)^8Q_1^4}\rho_1^4$$

$$-\frac{125\delta_1\delta_2\eta^3([2]_q^2+r^2)^4([3]_q^2+r^2)^2}{108[2]_q^4[3]_q^2(1+r^2)^6Q_1^2Q_2}\left(\rho_2-\frac{1}{2}\rho_1^2\right)\rho_1^2$$

$$+\frac{25\delta_1^2\eta^2([3]_q^2+r^2)^4}{36[3]_q^4(1+r^2)^4Q_2^2}\left(\rho_2-\frac{1}{2}\rho_1^2\right)^2.$$

So,

$$|\mathfrak{b}_{2j+1}^2-\mathfrak{a}_2^4|\leqslant\frac{625(1-\delta_2^2)|\eta|^4([2]_q^2+r^2)^8}{1{,}296[2]_q^8(1+r^2)^8|Q_1|^4}|\rho_1|^4$$

$$+\frac{125\delta_1\delta_2|\eta|^3([2]_q^2+r^2)^4([3]_q^2+r^2)^2}{108[2]_q^4[3]_q^2(1+r^2)^6|Q_1|^2|Q_2|}\left|\rho_2-\frac{1}{2}\rho_1^2\right||\rho_1|^2$$

$$+\frac{25\delta_1^2|\eta|^2([3]_q^2+r^2)^4}{36[3]_q^4(1+r^2)^4|Q_2|^2}\left|\rho_2-\frac{1}{2}\rho_1^2\right|^2,$$

and we can systematically implement (16), (17), and (21) to arrive at the result;

(ii) Putting (32) into (37) means

$$\mathfrak{b}_{2j+1}^3 - \mathfrak{a}_2^6 = \left(\delta_1\mathfrak{a}_3 - \delta_2\mathfrak{a}_2^2\right)^3 - \mathfrak{a}_2^6 = -(1+\delta_2^3)\mathfrak{a}_2^6 + 3\delta_1\delta_2^2\mathfrak{a}_2^4\mathfrak{a}_3 - 3\delta_1^2\delta_2\mathfrak{a}_2^2\mathfrak{a}_3^2 + \delta_1^3\mathfrak{a}_3^3. \tag{39}$$

Now, putting (28) and (29) into (39) simplifies to

$$\begin{aligned}
\mathfrak{b}_{2j+1}^3 - \mathfrak{a}_2^6 = & -\frac{15{,}625(1+\delta_2^3)\eta^6([2]_q^2+r^2)^{12}}{46{,}656[2]_q^{12}(1+r^2)^{12}Q_1^6}\rho_1^6 \\
& + \frac{3{,}125\delta_1\delta_2^2\eta^5([2]_q^2+r^2)^8([3]_q^2+r^2)^2}{2{,}592[2]_q^8[3]_q^2(1+r^2)^{10}Q_1^4Q_2}\left(\rho_2 - \frac{1}{2}\rho_1^2\right)\rho_1^4 \\
& - \frac{625\delta_1^2\delta_2\eta^4([2]_q^2+r^2)^4([3]_q^2+r^2)^4}{432[2]_q^4[3]_q^4(1+r^2)^8Q_1^2Q_2^2}\left(\rho_2 - \frac{1}{2}\rho_1^2\right)^2\rho_1^2 \\
& + \frac{125\delta_1^3\eta^3([3]_q^2+r^2)^6}{216[3]_q^6(1+r^2)^6Q_2^3}\left(\rho_2 - \frac{1}{2}\rho_1^2\right)^3.
\end{aligned}$$

So,

$$\begin{aligned}
\left|\mathfrak{b}_{2j+1}^3 - \mathfrak{a}_2^6\right| \leqslant & \frac{15{,}625(1+\delta_2^3)|\eta|^6([2]_q^2+r^2)^{12}}{46{,}656[2]_q^{12}(1+r^2)^{12}|Q_1|^6}|\rho_1|^6 \\
& + \frac{3{,}125\delta_1\delta_2^2|\eta|^5([2]_q^2+r^2)^8([3]_q^2+r^2)^2}{2{,}592[2]_q^8[3]_q^2(1+r^2)^{10}|Q_1|^4|Q_2|}\left|\rho_2 - \frac{1}{2}\rho_1^2\right||\rho_1|^4 \\
& + \frac{625\delta_1^2\delta_2|\eta|^4([2]_q^2+r^2)^4([3]_q^2+r^2)^4}{432[2]_q^4[3]_q^4(1+r^2)^8|Q_1|^2|Q_2|^2}\left|\rho_2 - \frac{1}{2}\rho_1^2\right|^2|\rho_1|^2 \\
& + \frac{125\delta_1^3|\eta|^3([3]_q^2+r^2)^6}{216[3]_q^6(1+r^2)^6|Q_2|^3}\left|\rho_2 - \frac{1}{2}\rho_1^2\right|^3.
\end{aligned}$$

Hence, a careful usage of (16), (17), and (21) completes the proof. □

Setting $j = 1$ leads to:

Corollary 3.8 If $\mathfrak{g} \in \Xi_{q,r}^{\eta,\phi}(4\ell)$, then

(i) For $m = 3 = c$, and $k = 2$,

$$\mathbf{K}_{3,2}^3(\mathfrak{g}) \leqslant \frac{625|\eta|^4([2]_q^2+r^2)^8}{81[2]_q^8(1+r^2)^8(10+6\cos\phi)^2} + \frac{25|\eta|^2([3]_q^2+r^2)^4}{9[3]_q^4(1+r^2)^4[4+2[2]_q(2+[2]_q)(1+\cos\phi)]},$$

and

(ii) For $m = c = k = 3$,

$$\mathbf{K}^{3}_{3,3}(\mathfrak{g}) \leqslant \frac{15{,}625|\eta|^6([2]_q^2+r^2)^{12}}{729[2]_q^{12}(1+r^2)^{12}(10+6\cos\phi)^3} + \frac{125|\eta|^3([3]_q^2+r^2)^6}{27[3]_q^6(1+r^2)^6(\sqrt{4+2[2]_q(2+[2]_q)}(1+\cos\phi))^3}.$$

The functional (5) is modified for $\mathfrak{f} \in \Xi_{q,r}^{\eta,\phi}(4\ell)$ as

$$\mathbf{M}_{c,d}(\mathfrak{f}) = |\mathfrak{a}_c\mathfrak{a}_d - \mathfrak{a}_{c+d-1}| \leqslant (c-1)(d-1) \quad (c, d \geqslant (j+1);\ \mathfrak{f} \in \mathbf{S}^{(j)}). \tag{40}$$

Theorem 3.9 (Ma Inequality) If $\mathfrak{f} \in \Xi_{q,r}^{\eta,\phi}(4\ell)$, then
(i) For $c = (j+1) = d$, $\mathbf{M}_{j+1,j+1}(\mathfrak{f}) = \mathbf{F}_j(1, \mathfrak{f})$ in Theorem 3.3;
(ii) For $c = j+1$ and $d = 2j+1$, or $c = 2j+1$ and $d = j+1$,

$$\begin{aligned}\mathbf{M}_{j+1,2j+1}(\mathfrak{f}) \leqslant{}& \frac{125(\delta_1\delta_2+\delta_3)|\eta|^3([2]_q^2+r^2)^6}{27[2]_q^6(1+r^2)^6(\sqrt{10+6\cos\phi})^3} \\ &+ \frac{25(\delta_1^2+2\delta_2)|\eta|^2([2]_q^2+r^2)^2([3]_q^2+r^2)^2}{9[2]_q^2[3]_q^2(1+r^2)^4\sqrt{10+6\cos\phi}\sqrt{4+2[2]_q(2+[2]_q)(1+\cos\phi)}} \\ &+ \frac{5\delta_1|\eta|([4]_q^2+r^2)^2}{3[4]_q^2(1+r^2)^2\sqrt{4+2[3]_q(2+[3]_q)(1+\cos\phi)}};\end{aligned}$$

and

(iii) For $c = j+1$ and $d = 3j+1$, or $c = 3j+1$ and $d = j+1$,

$$\begin{aligned}&\mathbf{M}_{j+1,3j+1}(\mathfrak{f}) \\ \leqslant{}& \frac{625(\delta_1\delta_3+\delta_4)|\eta|^4([2]_q^2+r^2)^8}{81[2]_q^8(1+r^2)^8(10+6\cos\phi)^2} \\ &+ \frac{25(\delta_1^2+2\delta_2)|\eta|^2([2]_q^2+r^2)^2([4]_q^2+r^2)^2}{9[2]_q^2[4]_q^2(1+r^2)^4\sqrt{10+6\cos\phi}\sqrt{4+2[3]_q(2+[3]_q)(1+\cos\phi)}} \\ &+ \frac{125(2\delta_1\delta_2+3\delta_3)|\eta|^3([2]_q^2+r^2)^4([3]_q^2+r^2)^2}{27[2]_q^4[3]_q^2(1+r^2)^6(10+6\cos\phi)\sqrt{4+2[2]_q(2+[2]_q)(1+\cos\phi)}} \\ &+ \frac{25\delta_2|\eta|^2([3]_q^2+r^2)^4}{9[3]_q^4(1+r^2)^4[4+2[2]_q(2+[2]_q)(1+\cos\phi)]} + \frac{5\delta_1|\eta|([5]_q^2+r^2)^2}{[5]_q^2(1+r^2)^2\sqrt{4+2[4]_q(2+[4]_q)(1+\cos\phi)}}.\end{aligned}$$

Proof. Let $\mathfrak{f} \in \Xi_{q,r}^{\eta;\phi}(4\ell)$, then
(i) $\mathbf{M}_{j+1,\,j+1}(\mathfrak{f}) = |\mathfrak{b}_{j+1}^2 - \mathfrak{b}_{2j+1}| = \mathbf{F}_j(1,\mathfrak{f})$ in Theorem 3.3;
(ii) Putting (32) into (40) means

$$\mathfrak{b}_{j+1}\mathfrak{b}_{2j+1} - \mathfrak{b}_{3j+1} = (\delta_1\mathfrak{a}_2)\left(\delta_1\mathfrak{a}_3 - \delta_2\mathfrak{a}_2^2\right) - (\delta_1\mathfrak{a}_4 - 2\delta_2\mathfrak{a}_2\mathfrak{a}_3 + \delta_3\mathfrak{a}_2^3)$$

$$= -(\delta_1\delta_2 + \delta_3)\mathfrak{a}_2^3 + (\delta_1^2 + 2\delta_2)\mathfrak{a}_2\mathfrak{a}_3 - \delta_1\mathfrak{a}_4. \tag{41}$$

Now, putting (28), (29), and (30) into (41) simplifies to

$$\begin{aligned}\mathfrak{b}_{j+1}\mathfrak{b}_{2j+1} - \mathfrak{b}_{3j+1} = &-\frac{125(\delta_1\delta_2+\delta_3)\eta^3([2]_q^2+r^2)^6}{216[2]_q^6(1+r^2)^6Q_1^3}\rho_1^3\\ &+\frac{25(\delta_1^2+2\delta_2)\eta^2([2]_q^2+r^2)^2([3]_q^2+r^2)^2}{36[2]_q^2[3]_q^2(1+r^2)^4Q_1Q_2}\left(\rho_2-\frac{1}{2}\rho_1^2\right)\rho_1\\ &-\frac{5\delta_1\eta([4]_q^2+r^2)^2}{6[4]_q^2(1+r^2)^2Q_3}\left(\frac{1}{4}\rho_1^3-\rho_1\rho_2+\rho_3\right).\end{aligned}$$

So,

$$\begin{aligned}|\mathfrak{b}_{j+1}\mathfrak{b}_{2j+1} - \mathfrak{b}_{3j+1}| \leqslant &\frac{125(\delta_1\delta_2+\delta_3)|\eta|^3([2]_q^2+r^2)^6}{216[2]_q^6(1+r^2)^6|Q_1|^3}|\rho_1|^3\\ &+\frac{25(\delta_1^2+2\delta_2)|\eta|^2([2]_q^2+r^2)^2([3]_q^2+r^2)^2}{36[2]_q^2[3]_q^2(1+r^2)^4|Q_1||Q_2|}\left|\rho_2-\frac{1}{2}\rho_1^2\right||\rho_1|\\ &+\frac{5\delta_1|\eta|([4]_q^2+r^2)^2}{6[4]_q^2(1+r^2)^2|Q_3|}\left|\frac{1}{4}\rho_1^3-\rho_1\rho_2+\rho_3\right|,\end{aligned}$$

and we can systematically implement (16), (17), and (21) to arrive at the result; and
(iii) Putting (32) into (40) means

$$\mathfrak{b}_{j+1}\mathfrak{b}_{3j+1} - \mathfrak{b}_{4j+1} = (\delta_1\mathfrak{a}_2)\left(\delta_1\mathfrak{a}_4 - 2\delta_2\mathfrak{a}_2\mathfrak{a}_3 + \delta_3\mathfrak{a}_2^3\right) - \left[\delta_1\mathfrak{a}_5 - \delta_2(2\mathfrak{a}_2\mathfrak{a}_4 + \mathfrak{a}_3^2) + 3\delta_3\mathfrak{a}_2^2\mathfrak{a}_3 - \delta_4\mathfrak{a}_2^4\right]$$

$$= (\delta_1\delta_3 + \delta_4)\mathfrak{a}_2^4 + (\delta_1^2 + 2\delta_2)\mathfrak{a}_2\mathfrak{a}_4 - (2\delta_1\delta_2 + 3\delta_3)\mathfrak{a}_2^2\mathfrak{a}_3 + \delta_2\mathfrak{a}_3^2 - \delta_1\mathfrak{a}_5. \tag{42}$$

Now, putting (28)–(31) into (42) simplifies to

$$\mathfrak{b}_{j+1}\mathfrak{b}_{3j+1}-\mathfrak{b}_{4j+1}=\frac{625(\delta_1\delta_3+\delta_4)\eta^4([2]_q^2+r^2)^8}{1{,}296[2]_q^8(1+r^2)^8Q_1^4}\rho_1^4$$

$$+\frac{25(\delta_1^2+2\delta_2)\eta^2([2]_q^2+r^2)^2([4]_q^2+r^2)^2}{36[2]_q^2[4]_q^2(1+r^2)^4Q_1Q_3}\left(\frac{1}{4}\rho_1^3-\rho_1\rho_2+\rho_3\right)\rho_1$$

$$-\frac{125(2\delta_1\delta_2+3\delta_3)\eta^3([2]_q^2+r^2)^4([3]_q^2+r^2)^2}{216[2]_q^4[3]_q^2(1+r^2)^6Q_1^2Q_2}\left(\rho_2-\frac{1}{2}\rho_1^2\right)\rho_1^2$$

$$+\frac{25\delta_2\eta^2([3]_q^2+r^2)^4}{36[3]_q^4(1+r^2)^4Q_2^2}\left(\rho_2-\frac{1}{2}\rho_1^2\right)^2$$

$$-\frac{5\delta_1\eta([5]_q^2+r^2)^2}{2[5]_q^2(1+r^2)^2Q_4}\left(-\frac{1}{24}\rho_1^4+\frac{1}{4}\rho_1^2\rho_2-\frac{1}{3}\rho_1\rho_3-\frac{1}{6}\rho_2^2+\frac{1}{2}\rho_4\right).$$

So,

$$|\mathfrak{b}_{j+1}\mathfrak{b}_{3j+1}-\mathfrak{b}_{4j+1}|\leqslant\frac{625(\delta_1\delta_3+\delta_4)|\eta|^4([2]_q^2+r^2)^8}{1{,}296[2]_q^8(1+r^2)^8|Q_1|^4}|\rho_1|^4$$

$$+\frac{25(\delta_1^2+2\delta_2)|\eta|^2([2]_q^2+r^2)^2([4]_q^2+r^2)^2}{36[2]_q^2[4]_q^2(1+r^2)^4|Q_1||Q_3|}\left|\frac{1}{4}\rho_1^3-\rho_1\rho_2+\rho_3\right||\rho_1|$$

$$+\frac{125(2\delta_1\delta_2+3\delta_3)|\eta|^3([2]_q^2+r^2)^4([3]_q^2+r^2)^2}{216[2]_q^4[3]_q^2(1+r^2)^6|Q_1|^2|Q_2|}\left|\rho_2-\frac{1}{2}\rho_1^2\right||\rho_1|^2$$

$$+\frac{25\delta_2|\eta|^2([3]_q^2+r^2)^4}{36[3]_q^4(1+r^2)^4|Q_2|^2}\left|\rho_2-\frac{1}{2}\rho_1^2\right|^2$$

$$+\frac{5\delta_1|\eta|([5]_q^2+r^2)^2}{2[5]_q^2(1+r^2)^2|Q_4|}\left|-\frac{1}{24}\rho_1^4+\frac{1}{4}\rho_1^2\rho_2-\frac{1}{3}\rho_1\rho_3-\frac{1}{6}\rho_2^2+\frac{1}{2}\rho_4\right|.$$

Hence, a careful application of (16)–(21) complete the proof. □

4. Conclusion

The importance of seeking the upper bounds of inequalities especially in the study of coefficient properties of real and complex functions can not be overlooked any more. In this work, we defined a novel regular q-function $\mu_q(r,z)$ which is the quantum analogue of the well-known Mathieu special function. So, using the q-function $\mu_q(r,z)$, q-differentiation, and

subordination principle, we introduced a novel set of regular q-functions that contained many known and new subsets of regular and univalent functions. Among the results obtained are the estimates for some initial coefficient bounds of the regular function $\mathfrak{g}(z) = z + \sum_{c \geqslant 2} \mathfrak{a}_c z^c$. Further, we presented some estimates for some functionals whose entries are the coefficients of the regular function-type $\mathfrak{f}_j(z) = z + \sum_{c \geqslant 1} \mathfrak{b}_{(cj+1)} z^{(cj+1)}$, where $\mathfrak{b}_{(cj+1)}$ are the coefficients of the jth-root transform of $\mathfrak{g}$. Some of the explored functionals are the Zalcman, Krushkal, Fekete-Szegö, and Ma functionals. These functionals are important in geometric function theory in order to understand and solve some coefficient problems of regular functions. The inequalities from the results are vital in complex analysis for setting bounds, demonstrating convergence, and uncovering analytical and geometrical characteristics for certain regular functions.

A situation where the geometric condition in (15) is generalized as

$$1 + \frac{1}{\eta}\left(\mathfrak{D}_q(\gamma_q(r,z)) + \left(\frac{1+e^{i\phi}}{2}\right) z\mathfrak{D}_q^2(\gamma_q(r,z)) - 1\right) \prec n\ell(z) \quad (n \geqslant 3),$$

may be considered for future study. More so, further study may be geared towards investigating some Toeplitz and Hankel determinants for the set $\Xi_{q,r}^{\eta,\phi}(4\ell)$.

Conflict of interest

The authors declare no competing financial interest.

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