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Research Article

A Note on Approximations of Bi-continuous Cosine Families

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Abstract: We investigate Trotter-Kato approximation results for the class of bi-continuous cosine families. After a concise overview of bi-continuous cosine families, we introduce uniformly bi-continuous cosine families and prove, under natural resolvent-convergence assumptions, a Trotter-Kato approximation theorem. This result both generalizes and refines existing approximation theorems for strongly continuous cosine families. To illustrate the power of our approach, we construct an explicit mollification procedure on $\mathrm{C_b}(\mathbb{R})$, yielding a practical approximation of solutions of the wave equation. The techniques developed here open new directions for rigorous numerical analysis of evolution equations that lack strong continuity and provide a framework functional analytic approaches tp partial differential equations.

***Keywords*:** cosine families, non-strongly continuous cosine families, bi-continuous semigroups, Trotter-Kato, approximation

MSC: 47D09, 35A35

1. Introduction

Recently the author launched the theory of bi-continuous cosine families [1]. This theory is related to the one of bi-continuous semigroups that was introduced by Kühnemund [2, 3] and further developed by Farkas [4–6], the author [7–9] as well as several others [10–12]. The main motivation for bi-continuous semigroup is that certain first-order abstract Cauchy problems arise from stochastic differential equations, Ornstein-Uhlenbeck processes or Feller processes, that give rise to transition semigroups which are in general not strongly continuous, see for example [13]. The theory of bi-continuous semigroups has been very promising and in the last decades, even though the handling of both the norm and locally convex topology can be challenging.

Solutions to second-order abstract Cauchy problems, are given by cosine families. The original work on operator cosine families is due to Sova [14]. Generalizations of strongly continuous cosine families on Banach spaces to the locally convex framework have for example been studied by Fattorini [15–17], Konishi [18] and others [19, 20]. The theory of bi-continuous cosine families [1] yields a uniform framework for non-strongly continuous cosine families.

This present paper deals with a Trotter-Kato approximation type theorem for bi-continuous cosine families. For C_0-semigroups the original results are due to Kato [21, Section 3.3] and Trotter [22]. A modified version has been presented

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by Ito and Kappel [23, Theorem 2.1]. For bi-continuous semigroups Trotter-Kato approximation type results have also already attracted attention [24, 25]. For strongly continuous cosine families, approximation results have been proven and applied by several authors, see for example the following references [26–31], also showing the importance of this topic.

Beside the pure theoretical results on cosine families, there is also an interest in applications of approximation of cosine families, see for example [29, 32–34]. Generally speaking, approximation theorems for operator cosine families are foundational in establishing the convergence and stability of solutions to quasilinear parabolic equations. Such results are alspo critical in numerical methods for Partial Differential Equations (PDEs), where approximations of operator cosine families enable the discretization of time-dependent systems while preserving convergence properties.

The paper is structured as follows: The first section consists of preliminaries regarding bi-continuous semigroups. Section 2 contains the main result, which is Theorem 1, as well as its proof. In the last section, we consider an application to the one-dimensional wave equation.

2. Preliminaries

As usual, the general idea is to equip the Banach space we are working on with another additional locally convex topology τ which is subjected to some general assumption. Those assumptions were introduced by Kühnemund [2, Assumption 1] and also appear in a similar fashion in the theory of Saks spaces [35].

Assumption 1 Let $(X, \|\cdot\|)$ be a Banach space and τ a locally convex topology on X such that

(i) τ is a Hausdorff topology and is coarser than the norm-topology on X, i.e., the identity map $(X, \|\cdot\|) \to (X, \tau)$ is continuous.

(ii) τ is sequentially complete on $\|\cdot\|$-bounded sets, i.e., every $\|\cdot\|$-bounded τ-Cauchy sequence is τ-convergent.

(iii) The dual space of (X, τ) is norming for X, i.e.,

$$\|x\| = \sup_{\substack{\varphi \in (X, \tau)' \\ \|\varphi\| \leq 1}} |\varphi(x)|.$$

We call the triple $(X, \|\cdot\|, \tau)$ a *bi-admissible space*. Whenever we want to stress the system of seminorms that generate the locally convex topology τ we denote it by $\mathscr{P}_\tau$.

The definition of bi-continuous semigroups was introduced by the author [1] and relates to that of bi-continuous semigroups [2, Definition 3]. The study of strongly continuous cosine families on Banach spaces goes back to Sova [14, Definition 2.2] and is included as a special case in the definition of bi-continuous cosine families.

Definition 1 Let $(X, \|\cdot\|, \tau)$ be a bi-admissible space. We call a family of bounded linear operators $(C(t))_{t\geq 0}$ a *bi-continuous cosine family* if the following holds:

(i) $2C(t)C(s) = C(t+s) + C(t-s)$ and $C(0) = I$ for all $t \geq s \geq 0$.

(ii) $(C(t))_{t\geq 0}$ is strongly τ-continuous.

(iii) There exist $M \geq 1$ and $\omega \in \mathbb{R}$ such that $\|C(t)\| \leq M\mathrm{e}^{\omega t}$ for each $t \geq 0$.

(iv) $(C(t))_{t\geq 0}$ is *locally-bi-equicontinuous*, i.e., if $(x_n)_{n\in\mathbb{N}}$ is a norm-bounded sequence in X which is τ-convergent to 0, then also $(C(s)x_n)_{n\in\mathbb{N}}$ is τ-convergent to 0 uniformly for $s \in [0, t_0]$ for each fixed $t_0 \geq 0$.

The generator of a bi-continuous semigroups combines the definition for strongly continuous cosine families [14, Definition 2.12] and for bi-continuous semigroups [2, p. 214].

Definition 2 Let $(C(t))_{t\geq 0}$ be a bi-continuous cosine family on a bi-admissible space $(X, \|\cdot\|, \tau)$. The *generator* $(A, \mathrm{D}(A))$ of $(C(t))_{t\geq 0}$ is the linear operator on X defined by

$$Ax := \tau\lim_{t\to 0}\frac{2}{t^2}(C(t)x-x),$$

$$\mathrm{D}(A) := \left\{x\in X:\ \tau\lim_{t\to 0}\frac{2}{t^2}(C(t)x-x) \text{ exists in } X\right\}.$$

For properties of generators of bi-continuous cosine families we refer to [1, Section 2 & 3]. One important property, we nonetheless want to mention here as we will use this in the paper. In fact, if $(C(t))_{t\geq 0}$ is a bi-continuous cosine family satisfying $\|C(t)\|\leq M\mathrm{e}^{\omega t}$ for all $t\geq 0$, and if $(A,\mathrm{D}(A))$ is its generator according to Definition 2, then $(\omega^2,\ \infty)\subseteq\rho(A)$ and

$$\lambda R(\lambda^2,\ A)x = \int_0^\infty \mathrm{e}^{-\lambda t}C(t)x\ \mathrm{d}t, \qquad x\in X,$$

where $R(\lambda^2,\ A) := (\lambda^2 - A)^{-1}$ abbreviates the resolvent operator. In particular, we resolvent operator of the generator $(A,\ \mathrm{D}(A))$ can be expressed as Laplace transform of the cosine family, cf. [1, Proposition 3.2].

3. A first Trotter-Kato approximation type theorem for bi-continuous cosine families

The following definition is inspired by several other references such as [24 (Definition 3.1), 25 (Definition 11), 36 (Theorem 4.8)].

Definition 3 Let $\{(C_n(t))_{t\geq 0}:\ n\in\mathbb{N}\}$ be a sequence of bi-continuous cosine families on X. We say that they are *uniformly bi-continuous* if the following hold:

(i) There exist $M\geq 0$ and $\omega\in\mathbb{R}$ such that $\|C_n(t)\|\leq M\mathrm{e}^{\omega t}$ for all $t\geq 0$ and all $n\in\mathbb{N}$.

(ii) The bi-continuous cosine families $(C_n(t))_{t\geq 0}$ are locally bi-equicontinuous uniformly in $n\in\mathbb{N}$, i.e., if $(x_n)_{n\in\mathbb{N}}$ is a $\|\cdot\|$-bounded τ-null sequence, then $C_n(t)\to 0$ with respect to τ uniformly for $t\in[0,\ T]$ and $n\in\mathbb{N}$.

Let us now formulate the approximation theorem for bi-continuous cosine families.

Theorem 1 Let $(C(t))_{t\geq 0}$ and $(C_n(t))_{t\geq 0}$, $n\in\mathbb{N}$, be uniformly bi-continuous cosine families on a bi-admissible space $(X,\ \|\cdot\|,\ \tau)$, according to Definition 3. Let us denote the generators by $(A,\ \mathrm{D}(A))$ and $(A_n,\ \mathrm{D}(A_n))$, respectively. Assume that $\tau\lim_{n\to\infty}R(\lambda^2,\ A_n)x = R(\lambda^2,\ A)x$ for all $x\in X$. Then $\tau\lim_{n\to\infty}C_n(t)x = C(t)$ for all $t\in[0,\ T]$ and all $x\in X$

Proof. Our proof is inspired by the work of Konishi [18, Section 4] about cosine families on locally convex spaces. We need to show that $\tau\lim_{n\to\infty}C_n(t)x = C(t)$ for all $t\in[0,\ T]$ and all $x\in X$. Therefore, let $x\in X$ be arbitrary and $\mathscr{P}_\tau$ the family of seminorms that generates the locally convex topology τ of the given bi-admissible space $(X,\ \|\cdot\|,\ \tau)$. We observe that for $p\in\mathscr{P}_\tau$ one has that

$$\begin{aligned} p(C_n(t)x - C(t)x) \leq{}& p(C_n(t)x - k^2C_n(t)R(k^2,\ A_n)x) + p(k^2C_n(t)R(k^2,\ A_n)x - k^2C(t)R(k^2,\ A)x) \\ &+ p(k^2C(t)R(k^2,\ A)x - C(t)x). \end{aligned} \tag{1}$$

Let us consider sequences $\left(x_k^{(n)}\right)_{k\in\mathbb{N}}$ and $(y_k)_{k\in\mathbb{N}}$ of elements in X defined by

$$x_k^{(n)} := k^2R(k^2,\ A_n)x \quad \text{and} \quad y_k := k^2R(k^2,\ A)x,$$

for $k,\ n\in\mathbb{N}$. Then (1) simplifies to the following in terms of these sequences

$$p(C_n(t)x-C(t)x)\leq\underbrace{p(C_n(t)x-C_n(t)x_k^{(n)})}_{=:\alpha_n(t)}+\underbrace{p(C_n(t)x_k-C(t)y_k)}_{=:\beta_n(t)}+\underbrace{p(C(t)x_k^{(n)}-C(t)x)}_{=:\gamma_n(t)}.$$

For two of those terms above, we want to make use of Definition 3. Indeed, we observe that

$$\left\|x_k^{(n)}\right\|\leq k\int_0^\infty \mathrm{e}^{-kt}\|C_n(t)x\|\,\mathrm{d}t\leq Mk\|x\|\int_0^\infty \mathrm{e}^{(\omega-k)t}\,\mathrm{d}t=\frac{Mk\|x\|}{k-\omega},$$

showing that the sequences $\left(x_k^{(n)}\right)_{k\in\mathbb{N}}$, $n\in\mathbb{N}$ (and similarly $(y_k)_{k\in\mathbb{N}}$ with the same arguments), are $\|\cdot\|$-bounded. Moreover, by [1, Lemma 4.3] we have that $x_k^{(n)}\to x$ and $y_k\to x$ with respect to τ. In particular, we can now conclude from Definition 3 that $\alpha_n(t)\to 0$ and $\gamma_n(t)\to 0$ for $n\to\infty$ uniformly on compact intervals. In order to prove our result, we have to estimate the second term and show that $\beta_n(t)\to 0$ for $n\to\infty$ on compact intervals.

We observe that from [1 (Proposition 3.2), 3 (Appendix A), 15 (p. 42), 37 (Theorem 2.3.4)] one obtains that the operator of the cosine family can be represented in terms of the resolvent operator by means of the inverse Laplace transform, i.e.,

$$C(t)x=\frac{1}{2\pi i}\int_{\mu-i\infty}^{\mu+i\infty}\frac{\mathrm{e}^{\lambda t}}{\lambda}R(\lambda^2,\ A)x\,\mathrm{d}\lambda,$$

for all $x\in \mathrm{D}(A)$ and $t\geq 0$, where the convergence of the integral (seen al τ-integral) is uniformly on compact intervals. By using the resolvent identity, we finally obtain for $x\in\mathrm{D}(A)$ that

$$C(t)x=x+\frac{1}{2\pi i}\int_{\mu-i\infty}^{\mu+i\infty}\frac{\mathrm{e}^{\lambda t}}{\lambda^2}R(\lambda^2,\ A)Ax\,\mathrm{d}\lambda,$$

uniformly on compact intervals. Hence, we obtain

$$C_n(t)x_k^{(n)}-C(t)y_k=(x_k^{(n)}-y_k)+\frac{1}{2\pi i}\int_{\mu-i\infty}^{\mu+i\infty}\frac{\mathrm{e}^{\lambda t}}{\lambda^2}\left(R(\lambda^2,\ A_n)A_nx_k^{(n)}-R(\lambda^2,\ A)Ay_k\right)\,\mathrm{d}\lambda.$$

Due to the construction of the sequences $\left(x_k^{(n)}\right)_{k\in\mathbb{N}}$ and $(y_k)_{k\in\mathbb{N}}$ as well as the assumption that $\tau\lim_{n\to\infty}R(\lambda^2,\ A_n)x=R(\lambda^2,\ A)x$ we conclude that $C_n(t)x_k^{(n)}-C(t)y_k$ and hence $\beta_n(t)$ tend to zero for $n\to\infty$ on compact intervals. □

Remark 1 We observe that the converse implication of Theorem 1 is always true as well. Indeed, for $p\in\mathscr{P}_\tau$ and $x\in X$ we obtain by means of [1, Proposition 3.2 & Proposition 3.4] that

$$p\left(R(\lambda^2,\ A_n)x-R(\lambda^2,\ A)x\right)\leq\frac{1}{\lambda}\int_0^\infty \mathrm{e}^{-\lambda t}p(C_n(t)x-C(t)x)\,\mathrm{d}t,$$

The desired convergence is now a consequence of Lebesgue's dominated convergence theorem.

4. Example: Approximation of one-dimensional wave equation

Let us consider the space $X := \mathrm{C_b}(\mathbb{R})$ of bounded continuous functions and equip it with both the supremum norm $\|\cdot\|_\infty$ and the compact-open topology τ_{co}. We recall, that this locally convex topology is generated by the seminorms of the form

$$p_K(f) := \sup_{x\in K}|f(x)|, \quad K\subseteq\mathbb{R},\ f\in\mathrm{C_b}(\mathbb{R}),$$

for every compact set $K\subset\mathbb{R}$. This space is known to be bi-admissible, see for example [2, p. 206]. On this bi-admissible space $(\mathrm{C_b}(\mathbb{R}),\ \|\cdot\|_\infty,\ \tau_{\mathrm{co}})$, we consider the family of operators $(C(t))_{t\geq 0}$ defined by

$$(C(t)f)(x) := \frac{1}{2}\left(f(x+t)+f(x-t)\right), \quad f\in\mathrm{C_b}(\mathbb{R}),\ x\in\mathbb{R}.$$

We know from [1, Example 1.7] that $(C(t))_{t\geq 0}$ defines a bi-continuous cosine family on the bi-admissible space $(\mathrm{C_b}(\mathbb{R}),\ \|\cdot\|_\infty,\ \tau_{\mathrm{co}})$. Moreover, its generator is known to be the operator $(A,\ \mathrm{D}(A))$ given by

$$Af := f'', \quad \mathrm{D}(A) := \{f\in\mathrm{C_b}(\mathbb{R}):\ f''\in\mathrm{C_b}(\mathbb{R})\} = \mathrm{C_b^2}(\mathbb{R}),$$

see for example [1, Example 2.11]. To approximate $C(t)$, we introduce a family of smoothing (mollification) operators. For each $n\in\mathbb{N}$, choose a mollifier $\varphi_n\in\mathrm{C_c^\infty}(\mathbb{R})$ satisfying:

(i) $\varphi_n\geq 0$,

(ii) $\mathrm{supp}(\varphi_n)\subset\left[-\frac{1}{n},\ \frac{1}{n}\right]$, and

(iii) $\int_{\mathbb{R}}\varphi_n(x)\,\mathrm{d}x = 1$.

The existence of such functions is pointed out for example here, [38, p. 108]. With this in hand, we define for $n\in\mathbb{N}$ the operator P_n on $\mathrm{C_b}(\mathbb{R})$ by

$$(P_nf)(x) = (f*\varphi_n)(x) = \int_{\mathbb{R}} f(x-y)\varphi_n(y)\,\mathrm{d}y, \quad f\in\mathrm{C_b}(\mathbb{R}).$$

From [38, Proposition 4.21] we obtain that for $f\in\mathrm{C_b}(\mathbb{R})$ one has that $P_nf\to f$ for $n\to\infty$ with respect to τ_{co}. Moreover, by Young's convolution inequality we obtain that $P_n\in\mathscr{L}(\mathrm{C_b}(\mathbb{R}))$ with $\|P_n\|\leq 1$. Hence, we can define families of operators $(C_n(t))_{t\geq 0}$, $n\in\mathbb{N}$, by

$$C_n(t) = C(t)P_n, \quad t\geq 0.$$

Then the families $(C_n(t))_{t\geq 0}$, $n\in\mathbb{N}$, are uniformly bi-continuous according to Definition 3 as they inherit the bi-continuity of the original bi-continuous cosine family $(C(t))_{t\geq 0}$ by construction. Let $(A,\ \mathrm{D}(A))$ and $(A_n,\ \mathrm{D}(A_n))$, $n\in\mathbb{N}$, be the generators of $(C(t))_{t\geq 0}$ and $(C_n(t))_{t\geq 0}$, $n\in\mathbb{N}$, respectively. Let us now show that $R(\lambda^2,\ A_n)f\to R(\lambda^2,\ A)f$ for all $f\in\mathrm{C_b}(\mathbb{R})$ with respect to τ_{co} whenever $n\to\infty$. To do so, we first determine $R(\lambda^2,\ A)$ explicitly. In particular, let $g\in\mathrm{C_b}(\mathbb{R})$ be arbitrary and consider the equation $(\lambda^2-A)f = g$. This is equivalent to the equation $\lambda^2 f - f'' = g$. This linear second-order ordinary differential equation gives rise to the solution for f given by

$$f(x)=\mathrm{e}^{-\lambda x}\int_1^x \frac{\mathrm{e}^{\lambda\xi}}{2\lambda}g(\xi)\,\mathrm{d}\xi-\mathrm{e}^{\lambda x}\int_1^x \frac{\mathrm{e}^{-\lambda\xi}}{2\lambda}g(\xi)\,\mathrm{d}\xi,$$

so that we can conclude

$$\left(R(\lambda^2,\,A)g\right)(x)=\mathrm{e}^{-\lambda x}\int_1^x \frac{\mathrm{e}^{\lambda\xi}}{2\lambda}g(\xi)\,\mathrm{d}\xi-\mathrm{e}^{\lambda x}\int_1^x \frac{\mathrm{e}^{-\lambda\xi}}{2\lambda}g(\xi)\,\mathrm{d}\xi.$$

Moreover, we have

$$\begin{aligned}\left(R(\lambda^2,\,A_n)g\right)(x)&=\frac{1}{\lambda}\int_0^\infty \mathrm{e}^{-\lambda t}(C_n(t)g)(x)\,\mathrm{d}t=\frac{1}{\lambda}\int_0^\infty \mathrm{e}^{-\lambda t}(C(t)P_ng)(x)\,\mathrm{d}t\\
&=\frac{1}{\lambda}\int_0^\infty \mathrm{e}^{-\lambda t}(C(t)(g*\varphi_n))(x)\,\mathrm{d}t\\
&=\frac{1}{2\lambda}\int_0^\infty \mathrm{e}^{-\lambda t}(g*\varphi_n)(x+t)\,\mathrm{d}t+\frac{1}{2\lambda}\int_0^\infty (g*\varphi_n)(x-t)\,\mathrm{d}t\\
&=\frac{1}{2\lambda}\int_0^\infty\int_{-\infty}^\infty \mathrm{e}^{-\lambda t}g(x+t-y)\varphi_n(y)\,\mathrm{d}y\,\mathrm{d}t+\frac{1}{2\lambda}\int_0^\infty\int_{-\infty}^\infty \mathrm{e}^{-\lambda t}g(x-t-y)\varphi_n(y)\,\mathrm{d}y\,\mathrm{d}t\\
&=\frac{1}{2\lambda}\int_{-\infty}^\infty \varphi_n(y)\left(\int_0^\infty \mathrm{e}^{-\lambda t}g(x+t-y)\,\mathrm{d}t\right)\mathrm{d}y+\frac{1}{2\lambda}\int_{-\infty}^\infty \varphi_n(y)\left(\int_0^\infty \mathrm{e}^{-\lambda t}g(x-t-y)\,\mathrm{d}t\right)\mathrm{d}y\\
&=\mathrm{e}^{-\lambda x}\int_1^x \frac{\mathrm{e}^{\lambda\xi}}{2\lambda}(g*\varphi_n)(\xi)\,\mathrm{d}\xi-\mathrm{e}^{\lambda x}\int_1^x \frac{\mathrm{e}^{-\lambda\xi}}{2\lambda}(g*\varphi_n)(\xi)\,\mathrm{d}\xi=(R(\lambda^2,\,A)(P_ng))(x),\end{aligned}$$

where the second last equality follows from evaluating the inner integrals using substitutions. This shows indeed that $R(\lambda^2,\,A_n)f\to R(\lambda^2,\,A)f$ for all $f\in \mathrm{C_b}(\mathbb{R})$ with respect to τ_{co} whenever $n\to\infty$ again due to [38, Proposition 4.21] and the continuity of the resolvents. Hence, we can also conclude by Theorem 1 that the corresponding bi-continuous cosine familie converge. For illustrative purposes the paper also include a specific example of an approximation, see Figure 1 below.

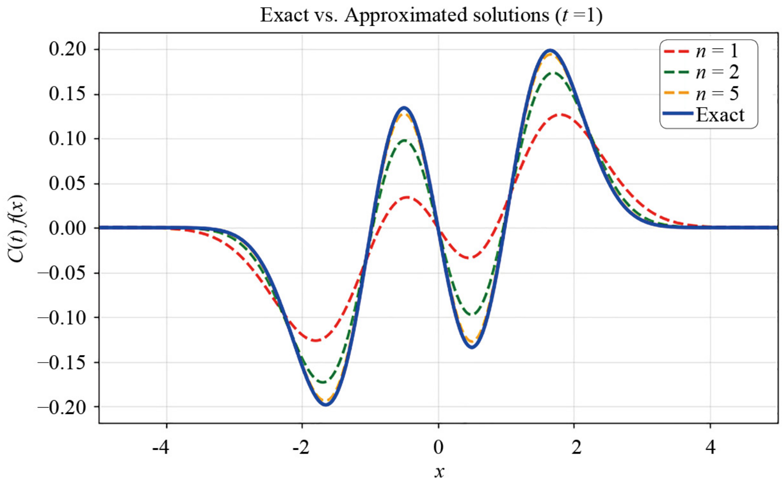

Figure 1. Exact solution $C(t)f(x) = \frac{1}{2}(f(x+t)+f(x-t))$ (solid blue curve) and mollified approximations $C_n(t)f(x)$ (dashed curves) for $f(x) = \sin(x)e^{-x^2} \in \mathrm{C_b}(\mathbb{R})$, with $t = 1$. The mollifiers $\varphi_n(x) = n\varphi(nx)$, where $\varphi \in \mathrm{C}_c^\infty(\mathbb{R})$ satisfies $\operatorname{supp}(\varphi) \subset [-1,\ 1]$ and $\int_{\mathbb{R}} \varphi = 1$, generate the smoothing operators $P_n f = f * \varphi_n$. As n increases, the approximations $C_n(t)f = C(t)P_n f$ (red/green/orange) converge to $C(t)f$ on on compact subsets, illustrating the τ_{co}-convergence (compact-open topology) of Theorem 2.2

5. Conclusion

This work establishes a Trotter-Kato approximation type theorem for bi-continuous cosine families, generalizing classical approximation results to settings where there is a lack of strong continuity. The relevance of the work is demonstrated through approximations of the wave equation on $\mathrm{C_b}(\mathbb{R})$ via mollifiers, enabling rigorous convergence analysis for non-strongly continuous solutions. Future directions include extensions to nonlinear or non-autonomous evolution equations, numerical implementations for stochastic PDEs as well as applications to infinite-dimensional control systems. This work advances tools for analyzing evolution equations in non-standard settings, with possible implications for stochastic analysis, mathematical physics and numerical PDEs.

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Conflict of interest

The author declares no competing financial interest.

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