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Research Article

On Comparison for Approximate Solutions of Modified Time Caputo Fractional Kawahara Equations in Shallow Water Theory by Using Some Techniques

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Abstract: In the theory of shallow water wave equations, the Kawahara equation and modified Kawahara equation are introduced to represent the solitary-wave propagation. In this paper, we use the residual power series method and Aboodh transform to provide a new technique, the Aboodh Residual Power Series Method (ARPSM). By this technique and the Caputo fractional operator, we calculate the coefficients of the power series of the modified Kawahara equation, which will serve as the approximate solution. For providing approximate analytical and numerical solutions of the modified Kawahara equation, we first consider the Modified Time Caputo Fractional Kawahara Equation (MTCFKE) and then use ARPSM in two cases: with the polynomial initial condition and with the perfect condition of MTCFKE. To show the capability, reliability, and efficiency of ARPSM, we describe ARPSM's approximate analytical solutions numerically and graphically and compare these solutions with other solutions obtained by using two methods: the homotopy analysis technique and the natural transform decomposition technique.

Keywords: Aboodh transform, Caputo derivative operator, residual power series, fractional partial differential equations

MSC: 5A20, 34A34, 33B15, 44A10

1. Introduction

Over the last years, the theory of partial fractional differential equations was considered one of the important subjects in mathematics, as it is considered a perfect representation for many systems and models in several scientific fields such as medicine, physics, engineering, groundwater problems, fluid mechanics, polymer science, and electrical networks [1, 2]. One of these fields is the theory of shallow water wave equations, which is widely applied to describe shock waves, the spread of storm floods, and tsunami waves. The Kawahara equation and its modified form are two of these equations [3], that were introduced in media to characterize solitary-wave propagation [4], which are defined by

$$\frac{\partial \varphi(x, y)}{\partial y} + \varphi(x, y)\frac{\partial \varphi(x, y)}{\partial x} + \frac{\partial^3 \varphi(x, y)}{\partial x^3} - \frac{\partial^5 \varphi(x, y)}{\partial x^5} = 0, \tag{1}$$

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and

$$\frac{\partial \varphi(x, y)}{\partial y}+\varphi^{2}(x, y) \frac{\partial \varphi(x, y)}{\partial x}+\beta \frac{\partial^{3} \varphi(x, y)}{\partial x^{3}}-\beta^{\prime} \frac{\partial^{5} \varphi(x, y)}{\partial x^{5}}=0. \tag{2}$$

The Kawahara equations have many applications [5–9]. The symmetry laws and generalized conservation of Kawahara equations were presented by [10].

The fractional calculus subject dates back to 1695. The fractional derivative and integral are the derivative and integral of arbitrary order. The Riemann-Liouville fractional differential operator and the Caputo fractional differential operator [11], are considered the oldest operators in this direction. Regarding these operators, some mathematical researchers introduced other fractional differential operators such as those in [12–14]. The fractional calculus subject has many applications in scientific fields such as electromagnetism, wave propagation, heat transfer, robotics system classification, physics, mechanics, and viscoelasticity. One of these applications is the Modified Time Caputo Fractional Kawahara Equation (MTCFKE) in the theory of shallow water waves, [15], which is defined by

$$\mathscr{D}_{0, y}^{\alpha} \varphi(x, y)+\varphi^{2}(x, y) \frac{\partial \varphi(x, y)}{\partial x}+\beta \frac{\partial^{3} \varphi(x, y)}{\partial x^{3}}-\beta^{\prime} \frac{\partial^{5} \varphi(x, y)}{\partial x^{5}}=0, \quad 0<\alpha \leq 1 \tag{3}$$

where $\mathscr{D}_{0, y}^{\alpha}$ is the Caputo fractional operator of order α, $\beta>0$ and $\beta^{\prime}<0$ are real numbers.

The numerical solutions of Partial Differential Equations (PDEs) play a crucial role in approximating their solutions. Methods such as the finite difference, finite element, and spectral techniques have been widely developed to discretize the domain and reduce PDEs into systems of algebraic equations that can be solved computationally. These numerical schemes provide flexibility in handling complex geometries, nonlinearities, and varying boundary conditions, making them indispensable in practical applications. Moreover, advancements in computational power and algorithms have further enhanced the accuracy and efficiency of numerical PDE solvers, enabling the simulation of large-scale and real-world problems with high precision. There are several mathematical methods and efficient techniques to solve fractional partial differential equations such as the Sine-Gordon expansion technique in solving Wu-Zhang system models [16], the fractional Newton method [17], the expansion method [18], the Laplace transform method [1], the monotone iterative technique in solving reaction-diffusion equations [19], the reproducing kernel Hilbert space method [20], the homotopy analysis method [21], the homotopy perturbation method with some equations [22], the modified Adams-Bashforth method [23], the modified expansion function method [24], etc. For the technique of Aboodh Residual Power Series Method (ARPSM), Liaqat et al. [25] solved the Black-Scholes differential equations, Noor et al. [26] solved some equations with one-dimensional nonlinear shock waves, Edalatpanah and Abdolmaleki [27] introduced some results of the N-Wh-S equation, and Yasmin and Almuqrin [28] used ARPSM to obtain some solutions. Several techniques and methods have been used in solving MTCFKE (3), such as the Homotopy Analysis Method (HAM) [29], an iterative Laplace transform method [30], the Laplace Adomian decomposition method [31], the residual power series method [32], the Natural Transform Decomposition Method (NTDM) [15], the septic B-spline collocation method [33], and the fixed-point theorem and homotopy analysis method [34].

The main motivation behind this study is to explore the effectiveness of the Caputo operator and the Aboodh transform combined with the residual power series technique in obtaining accurate approximate solutions for the modified Kawahara equation (1). The reliability of the derived solutions is evaluated by comparing them with those produced through other existing methods. In this paper, Section 2 reviews key definitions and establishes certain properties related to the Aboodh transform in combination with the Caputo fractional operator. Section 3 outlines the fundamental steps of the ARPSM technique. Section 4 highlights the application of ARPSM in deriving approximate analytical and numerical solutions of the MTCFKE (3) under two scenarios: with polynomial and with exact initial conditions. Furthermore, graphical representations of the obtained analytical approximations are provided, and these results are compared with approximate solutions derived using the NTDM method [15] and the HAM approach [29].

2. On Aboodh transform

Let φ be a map on $I \times [0, \infty)$, where I is an interval in $\mathbb{R}$. The Riemann-Liouville Fractional (R-LF) derivative operator [35] of $\varphi(x, y)$ of order $\alpha > 0$ is defined by

$$\mathscr{D}_{0,y}^{\alpha}\varphi(x, y) = \mathscr{D}_{y}^{n}\mathscr{I}_{0,y}^{\alpha}\varphi(x, y) \quad n-1 < \alpha < n \tag{4}$$

where $n \in \mathbb{N}$ and $\mathscr{I}_{0,y}^{\alpha}$ is the R-LF integral operator [35] of $\varphi(x, y)$ of order α defined by

$$\mathscr{I}_{0,y}^{\alpha}\varphi(x, y) = \begin{cases} \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_{0}^{y}(y-\nu)^{\alpha-1}\varphi(x, \nu)\,d\nu, & \alpha > 0, \\ \\ \varphi(x, y), & \alpha = 0. \end{cases} \tag{5}$$

The Caputo fractional derivative [36] of $\varphi(x, y)$ of order α is given by

$$\mathscr{D}_{0,y}^{\alpha}\varphi(x, y) = \begin{cases} \mathscr{I}_{0,y}^{n-\alpha}\left[\dfrac{\partial^{n}\varphi(x, y)}{\partial y^{n}}\right], & n-1 < \alpha < n \\ \\ \dfrac{\partial^{n}\varphi(x, y)}{\partial y^{n}}, & \alpha \in \mathbb{N}. \end{cases} \tag{6}$$

For $y \geq 0$ and $n-1 < \alpha < n$, by [36] we have $\mathscr{D}_{0,y}^{\alpha}\mathscr{I}_{0,y}^{\alpha}\varphi(x, y) = \varphi(x, y)$, $\mathscr{D}_{0,y}^{\alpha}y^{\gamma} = \frac{\Gamma(\alpha+2)}{\Gamma(\gamma-\alpha+2)}y^{\gamma-\alpha}$, and

$$\mathscr{I}_{0,y}^{\alpha}\mathscr{D}_{0,y}^{\alpha}\varphi(x, y) = \varphi(x, y) + \sum_{k=0}^{n-1}\frac{y^{k}}{k!}\frac{\partial^{k}\varphi(x, y)}{\partial y^{k}}\Big|_{y=0},$$

where $\gamma > -1$, $c \in \mathbb{R}$ is a real number, and $n \in \mathbb{N}$.

Let $\varphi(x, y)$ be continuous with exponential order ρ on $I \times [0, \infty)$. The Aboodh transform $\mathscr{A}_{y\rho}$ [26] of $\varphi(x, y)$ with respect to y is defined by

$$\varphi_{y}^{\mathscr{A}}(x, \rho) := \mathscr{A}_{y\rho}\varphi(x, y) = \frac{1}{\rho}\int_{0}^{\infty}e^{-y\rho}\varphi(x, y)dy \quad \rho_1 \leq \rho \leq \rho_2. \tag{7}$$

The inverse Aboodh transform $\mathscr{A}_{y\rho}^{-1}$ of $\varphi_{y}^{\mathscr{A}}(x, \rho)$ with respect to ρ is defined by

$$\varphi(x, y) := \mathscr{A}_{y\rho}^{-1}\varphi_{y}^{\mathscr{A}}(x, \rho) = \frac{1}{2\pi i}\int_{d-i\infty}^{d+i\infty}\rho e^{y\rho}\varphi_{y}^{\mathscr{A}}(x, \rho)d\rho. \tag{8}$$

Lemma 1 [25] Let φ, θ: $I \times [0, \infty) \to \mathbb{R}$ be a function on $I \times [0, \infty)$. Then
1. $\mathscr{A}_{y\rho}[\iota_1\varphi(x, y) + \iota_2\theta(x, y)] = \iota_1\mathscr{A}_{y\rho}\varphi(x, y) + \iota_2\mathscr{A}_{y\rho}\theta(x, y)$, where ι_1 and ι_2 are constants;
2. $\mathscr{A}_{y\rho}^{-1}[\iota_1\mathscr{A}_{y\rho}\varphi(x, y) + \iota_2\mathscr{A}_{y\rho}\theta(x, y)] = \iota_1\varphi(x, y) + \iota_2\theta(x, y)$, where ι_1 and ι_2 are constants;

3. $\mathscr{A}_{y\rho}\mathscr{I}_{0,y}^{\alpha}\varphi(x, y) = \rho^{-\alpha}\varphi_y^{\mathscr{A}}(x, \rho)$;

4. $\mathscr{A}_{y\rho}\left[\mathscr{D}_{0,y}^{\alpha}\varphi(x, y)\right] = \rho^{\alpha}\varphi_y^{\mathscr{A}}(x, \rho) - \sum_{j=0}^{n-1}\rho^{\alpha-j-2}\partial_y^j\varphi(x, y)\Big|_{y=0} \quad (n-1 < \alpha < n)$.

From [28], the power series form is defined by

$$\sum_{j=0}^{\infty}\varphi_j(x)(y-x_0')^{j\alpha} = \varphi_0(x)(y-x_0')^0 + \varphi_1(x)(y-x_0')^{\alpha} + \varphi_2(x)(y-x_0')^{2\alpha} + \cdots \tag{9}$$

where $x = (x_{11}, x_{12}, \cdots, x_{1n}) \in \mathbb{R}^n$, $n \in \mathbb{N}$. The MFPS is short for Multiple Fractional Power Series (MFPS), which means the series about $x_0' \in \mathbb{R}$ with series coefficients $\varphi_j(x)$, and $y \in \mathbb{R}$ is a variable.

Lemma 2 Let φ: $I \times [0, \infty) \to \mathbb{R}$ be continuous on $I \times [0, \infty)$. Then

$$\mathscr{A}_{y\rho}\left[\mathscr{D}_{0,y}^{k\alpha}\varphi(x, y)\right] = \rho^{k\alpha}\varphi_y^{\mathscr{A}}(x, \rho) - \sum_{j=0}^{k-1}\rho^{(k-j)\alpha-2}\mathscr{D}_{0,y}^{j\alpha}\varphi(x, y)\Big|_{y=0} (0 < \alpha \le 1). \tag{10}$$

Proof. Here, by induction, we will prove the relation (10). At $k = 1$, since $0 < \alpha \le 1$ in (10), that is, $n = 0$ in the lemma above by part (4). Hence

$$\mathscr{A}_{y\rho}\left[\mathscr{D}_{0,y}^{\alpha}\varphi(x, y)\right] = \rho^{\alpha}\varphi_y^{\mathscr{A}}(x, \rho) - \rho^{\alpha-2}\varphi(x, 0). \tag{11}$$

That is, (10) holds at $k = 1$. At $k = 2$, let $\theta(x, y) = \mathscr{D}_{0,y}^{\alpha}\varphi(x, y)$. By (11) above we have

$$\begin{aligned}
\mathscr{A}_{y\rho}\left[\mathscr{D}_{0,y}^{\alpha}\theta(x, y)\right] &= \rho^{\alpha}\theta_y^{\mathscr{A}}(x, \rho) - \rho^{\alpha-2}\theta(x, 0)\\
&= \rho^{\alpha}\mathscr{A}_{y\rho}\mathscr{D}_{0,y}^{\alpha}\varphi(x, y) - \rho^{\alpha-2}\,\mathscr{D}_{0,y}^{\alpha}\varphi(x, y)\Big|_{y=0}\\
&= \rho^{\alpha}\left[\rho^{\alpha}\varphi_y^{\mathscr{A}}(x, \rho) - \rho^{\alpha-2}\varphi(x, 0)\right] - \rho^{\alpha-2}\,\mathscr{D}_{0,y}^{\alpha}\varphi(x, y)\Big|_{y=0}\\
&= \rho^{2\alpha}\varphi_y^{\mathscr{A}}(x, \rho) - \rho^{2\alpha-2}\varphi(x, 0) - \rho^{\alpha-2}\,\mathscr{D}_{0,y}^{\alpha}\varphi(x, y)\Big|_{y=0}.
\end{aligned} \tag{12}$$

Hence (10) holds at $k = 2$. Let (10) be true at $k = m$, that is,

$$\mathscr{A}_{y\rho}\left[\mathscr{D}_{0,y}^{m\alpha}\varphi(x, y)\right] = \rho^{m\alpha}\varphi_y^{\mathscr{A}}(x, \rho) - \sum_{j=0}^{m-1}\rho^{(m-j)\alpha-2}\mathscr{D}_{0,y}^{j\alpha}\varphi(x, y)\Big|_{y=0}. \tag{13}$$

We show that (10) holds at $k = m+1$. Let $\theta(x, y) = \mathscr{D}_{0,y}^{m\alpha}\varphi(x, y)$. By (11) and (13)

$$
\begin{aligned}
\mathscr{A}_{y\rho}\left[\mathscr{D}_{0,y}^{k\alpha}\theta(x,y)\right] =& \mathscr{A}_{y\rho}\left[\mathscr{D}_{0,y}^{\alpha}\theta(x,y)\right]\\
=&\rho^{\alpha}\theta_{y}^{\mathscr{A}}(x,\rho)-\rho^{\alpha-2}\theta(x,0)\\
=&\rho^{\alpha}\mathscr{A}_{y\rho}\mathscr{D}_{0,y}^{m\alpha}\varphi(x,y)-\rho^{\alpha-2}\,\mathscr{D}_{0,y}^{m\alpha}\varphi(x,y)\Big|_{y=0}\\
=&\rho^{\alpha}\left[\rho^{m\alpha}\varphi_{y}^{\mathscr{A}}(x,\rho)-\sum_{j=0}^{m-1}\rho^{(m-j)\alpha-2}\,\mathscr{D}_{0,y}^{j\alpha}\varphi(x,y)\Big|_{y=0}\right]\\
&-\rho^{\alpha-2}\,\mathscr{D}_{0,y}^{m\alpha}\varphi(x,y)\Big|_{y=0}\\
=&\rho^{(m+1)\alpha}\varphi_{y}^{\mathscr{A}}(x,\rho)-\sum_{j=0}^{m-1}\rho^{(m+1-j)\alpha-2}\,\mathscr{D}_{0,y}^{j\alpha}\varphi(x,y)\Big|_{y=0}\\
&-\rho^{\alpha-2}\,\mathscr{D}_{0,y}^{\alpha}\varphi(x,y)\Big|_{y=0}\\
=&\rho^{(m+1)\alpha}\varphi_{y}^{\mathscr{A}}(x,\rho)-\sum_{j=0}^{m}\rho^{(m+1-j)\alpha-2}\,\mathscr{D}_{0,y}^{j\alpha}\varphi(x,y)\Big|_{y=0}\\
=&\rho^{k\alpha}\varphi_{y}^{\mathscr{A}}(x,\rho)-\sum_{j=0}^{k-1}\rho^{(k-j)\alpha-2}\,\mathscr{D}_{0,y}^{j\alpha}\varphi(x,y)\Big|_{y=0}.
\end{aligned}
\tag{14}
$$

Hence the relation (10) is true for all $k\in\mathbb{N}$. □

Lemma 3 Let φ: $\mathbb{R}^n\times[0,\infty)\rightarrow\mathbb{R}$ with an exponential order function. The MFPS notation for the Aboodh transform is defined by

$$
\varphi_{y}^{\mathscr{A}}(x,\rho)=\sum_{j=0}^{\infty}\frac{f_j(x)}{\rho^{j\alpha+2}}\quad \rho>0, \tag{15}
$$

where $x=(x_{11},x_{12},\cdots,x_{1n})\in\mathbb{R}^n$, $n\in\mathbb{N}$.

Proof. Use the Taylor series

$$
\varphi(x,y)=f_0(x)+f_1(x)\frac{y^{\alpha}}{\Gamma(\alpha+2)}+f_2(x)\frac{y^{2\alpha}}{\Gamma(2\alpha+2)}+\cdots \tag{16}
$$

Take the Aboodh transform of (15):

$$
\begin{aligned}
\mathscr{A}_{y\rho}\varphi(x, y) =& \mathscr{A}_{y\rho} f_0(x) + f_1(x)\frac{\mathscr{A}_{y\rho} y^{\alpha}}{\Gamma(\alpha+2)} + f_2(x)\frac{\mathscr{A}_{y\rho} y^{2\alpha}}{\Gamma(2\alpha+2)} + \cdots \\
=& \frac{f_0(x)}{\rho^2} + f_1(x)\frac{\Gamma(\alpha+2)}{\rho^{\alpha+2}\Gamma(\alpha+2)} + f_2(x)\frac{\Gamma(2\alpha+2)}{\rho^{2\alpha+2}\Gamma(2\alpha+2)} + \cdots \\
=& \sum_{j=0}^{\infty} \frac{f_j(x)}{\rho^{j\alpha+2}}.
\end{aligned} \tag{17}
$$

□

Lemma 4 Let φ: $\mathbb{R}^n \times [0, \infty) \to \mathbb{R}$ be continuous with exponential order. Then $\lim_{\rho\to\infty} \rho^2 \varphi_y^{\mathscr{A}}(x, \rho) = \varphi(x, 0)$ for all $x \in \mathbb{R}^n$.

Proof. From the lemma above, for all $x \in \mathbb{R}^n$,

$$
\begin{aligned}
\lim_{\rho\to\infty} \rho^2 \varphi_y^{\mathscr{A}}(x, \rho) &= \lim_{\rho\to\infty} \rho^2 \sum_{j=0}^{\infty} \frac{f_j(x)}{\rho^{j\alpha+2}} \\
&= \lim_{\rho\to\infty} \rho^2 \sum_{j=0}^{\infty} \frac{f_j(x)}{\rho^{j\alpha+2}} \\
&= \lim_{\rho\to\infty} \rho^2 \left[\frac{f_0(x)}{\rho^2} + \frac{f_1(x)}{\rho^{\alpha+2}} + \frac{f_2(x)}{\rho^{2\alpha+2}} + \frac{f_3(x)}{\rho^{3\alpha+2}} + \cdots \right] \\
&= f_0(x) = \varphi(x, 0).
\end{aligned}
$$

The proof is completed. □

Theorem 1 Let φ: $\mathbb{R}^n \times [0, \infty) \to \mathbb{R}$ be a continuous function with exponential order. Then

$$
\varphi_y^{\mathscr{A}}(x, \rho) = \sum_{j=0}^{\infty} \frac{f_j(x)}{\rho^{j\alpha+2}} \quad (0 < \alpha \leq 1) \tag{18}
$$

for all $x \in \mathbb{R}^n$ and $\rho > 0$, where $f_j(x) = \partial_y^j \varphi(x, y)\Big|_{y=0}$.

Proof. The new formula of Taylor's series is

$$
f_1(x) = \rho^{\alpha+2} \varphi_y^{\mathscr{A}}(x, \rho) - \rho^{\alpha} f_0(x) - \frac{1}{\rho^{\alpha}} f_2(x) - \frac{1}{\rho^{2\alpha}} f_3(x) - \cdots \tag{19}
$$

By taking the limit of (19) when $\rho \to \infty$,

$$f_1(x) = \lim_{\rho\to\infty}\left[\rho^{\alpha+2}\varphi_y^{\mathscr{A}}(x,\rho) - \rho^{\alpha} f_0(x)\right]. \tag{20}$$

By Lemma 2,

$$f_1(x) = \lim_{\rho\to\infty}\rho^2\mathscr{A}_{y\rho}\left[\partial_y^{\alpha}\varphi(x,y)\right]. \tag{21}$$

By Lemma 4, $f_1(x) = \partial_y^n\varphi(x,0)$. Similarly, the new formula of Taylor's series of f_2 is

$$f_2(x) = \rho^{2\alpha+2}\varphi_y^{\mathscr{A}}(x,\rho) - \rho^{2\alpha}f_0(x) - \rho^{\alpha}f_1(x) - \frac{1}{\rho^{\alpha}}f_3(x) - \frac{1}{\rho^{2\alpha}}f_4(x) - \cdots \tag{22}$$

By taking the limit of (22) when $\rho\to\infty$,

$$f_2(x) = \lim_{\rho\to\infty}\left[\rho^{2\alpha+2}\varphi_y^{\mathscr{A}}(x,\rho) - \rho^{2\alpha}f_0(x) - \rho^{\alpha}f_1(x)\right]. \tag{23}$$

By Lemmas 2 and 4, $f_2(x) = \partial_y^{2n}\varphi(x,0)$. By continuity, $f_j(x) = \partial_y^{jn}\varphi(x,0)$. □

By the theorem above,

$$\mathscr{A}_{y\rho}\left[\mathscr{D}_{0,y}^{\alpha}\varphi(x,y)\right] = \sum_{j=0}^{\infty}\frac{1}{\rho^{j\alpha+2}}\mathscr{D}_{0,y}^{j\alpha}\varphi(x,y)\Big|_{y=0} \quad (0<\alpha\le 1)$$

for all $x\in\mathbb{R}^n$, $\rho>0$, and the inverse Aboodh transform will be

$$\varphi(x,y) = \sum_{j=0}^{\infty}\frac{\mathscr{D}_{0,y}^{j\alpha}\varphi(x,y)\Big|_{y=0}}{\Gamma(j\alpha+2)}\,y^{j\alpha} \quad (0<\alpha\le 1)$$

for all $x\in\mathbb{R}^n$ and $y>0$.

Theorem 2 Let φ: $\mathbb{R}^n\times[0,\infty)\to\mathbb{R}$ be continuous with exponential order. If $\left|\rho^r\mathscr{A}_{y\rho}\left[\mathscr{D}_{0,y}^{(n+1)\alpha}\varphi(x,y)\right]\right|\le M$ for all $0<\rho\le q$ and $0<\alpha\le 1$, then the residual $\mathscr{R}es_n(x,\rho)$ of MFPS satisfies $\|\mathscr{R}es_n(x,\rho)\|\le\dfrac{M}{\rho^{(n+1)\alpha}}$.

Proof. By the new formula of Taylor's series,

$$\mathscr{R}es_n(x,\rho) = \varphi_y^{\mathscr{A}}(x,\rho) + \sum_{j=0}^{n}\frac{f_j(x)}{\rho^{j\alpha+2}}. \tag{24}$$

By Theorem 1,

$$\mathscr{R}es_n(x, \rho) = \varphi_y^{\mathscr{A}}(x, \rho) + \sum_{j=0}^{n} \frac{\mathscr{D}_{0,y}^{j\alpha}\varphi(x, 0)}{\rho^{j\alpha+2}}. \tag{25}$$

Multiply (25) by $\rho^{(n+1)\alpha}$:

$$\rho^{(n+1)\alpha}\mathscr{R}es_n(x, \rho) = \rho^{(n+1)\alpha}\varphi_y^{\mathscr{A}}(x, \rho) + \sum_{j=0}^{n} \rho^{(n+1-j)\alpha-2}\mathscr{D}_{0,y}^{j\alpha}\varphi(x, 0). \tag{26}$$

By Lemma 2,

$$\rho^{(n+1)\alpha}\mathscr{R}es_n(x, \rho) = \mathscr{A}_{y\rho}\left[\mathscr{D}_{0,y}^{(n+1)\alpha}\varphi(x, y)\right]. \tag{27}$$

Hence

$$\left|\rho^{(n+1)\alpha}\mathscr{R}es_n(x, \rho)\right| = \left|\mathscr{A}_{y\rho}\left[\mathscr{D}_{0,y}^{(n+1)\alpha}\varphi(x, y)\right]\right|.$$

That is, $|\mathscr{R}es_n(x, \rho)| \leq \dfrac{M}{\rho^{(n+1)\alpha}}$. □

3. The steps of the technique ARPSM

Here we will consider the following MTCFKE:

$$\begin{cases} \mathscr{D}_{0,y}^{\alpha}\varphi(x, y) = -\varphi(x, y)^2\mathscr{D}_x\varphi(x, y) - \beta\mathscr{D}_x^3\varphi(x, y) + \beta'\mathscr{D}_x^5\varphi(x, y), \quad 0 < \alpha \leq 1 \\ \varphi(x, 0) = h_0(x), \end{cases} \tag{28}$$

where $\mathscr{D}_x = \frac{\partial}{\partial x}$. Take $\mathscr{A}_{y\rho}$ of (28):

$$\begin{aligned} \mathscr{A}_{y\rho}\left[\mathscr{D}_{0,y}^{\alpha}[\varphi(x, y)]\right] = &- \mathscr{A}_{y\rho}\left[\varphi(x, y)^2\mathscr{D}_x\varphi(x, y)\right] - \beta\mathscr{A}_{y\rho}\left[\mathscr{D}_x^3\varphi(x, y)\right] \\ &+ \beta'\mathscr{A}_{y\rho}\left[\mathscr{D}_x^5\varphi(x, y)\right]. \end{aligned} \tag{29}$$

By Lemma 1,

$$\mathscr{A}_{y\rho}\left[\mathscr{D}_{0,y}^{\alpha}\varphi(x, y)\right] = \rho^{\alpha}\varphi_y^{\mathscr{A}}(x, \rho) - \rho^{\alpha-2}h_0(x). \tag{30}$$

Then, from (29) and (30) we get

$$\begin{aligned}\varphi_y^{\mathscr{A}}(x,\rho)=&\frac{1}{\rho^2}h_0(x)-\frac{1}{\rho^\alpha}\mathscr{A}_{y\rho}\left[\mathscr{A}_{y\rho}^{-1}(\varphi_y^{\mathscr{A}}(x,\rho))^2\mathscr{D}_x\mathscr{A}_{y\rho}^{-1}(\varphi_y^{\mathscr{A}}(x,\rho))\right]\\&-\frac{\beta}{\rho^\alpha}\mathscr{A}_{y\rho}\left[\mathscr{D}_x^3\mathscr{A}_{y\rho}^{-1}(\varphi_y^{\mathscr{A}}(x,\rho))\right]+\frac{\beta'}{\rho^\alpha}\mathscr{A}_{y\rho}\left[\mathscr{D}_x^5\mathscr{A}_{y\rho}^{-1}(\varphi_y^{\mathscr{A}}(x,\rho))\right].\end{aligned}\tag{31}$$

The analytical solution $\varphi_y^{\mathscr{A}}(x,\rho)$ for (29) is

$$\varphi_y^{\mathscr{A}}(x,\rho)=\sum_{j=0}^{\infty}\frac{h_j(x)}{\rho^{j\alpha+2}}.\tag{32}$$

By the initial condition in (28) with Lemma 1, we present the sequence of partial sums $\langle\varphi_{yn}^{\mathscr{A}}\rangle_{n\in\mathbb{N}\cup\{0\}}$ of (32) as

$$\varphi_{yn}^{\mathscr{A}}(x,\rho)=\sum_{j=0}^{n}\frac{h_j(x)}{\rho^{j\alpha+2}}.\tag{33}$$

The residual function of the Aboodh transform, $\mathscr{A}_{y\rho}\mathscr{R}es\ \varphi_{xy}$ for (31), is given by

$$\begin{aligned}\mathscr{A}_{y\rho}\mathscr{R}es\ \varphi_{xy}=&\varphi_y^{\mathscr{A}}(x,\rho)-\frac{1}{\rho^2}h_0(x)\\&+\frac{1}{\rho^\alpha}\mathscr{A}_{y\rho}\left[\mathscr{A}_{y\rho}^{-1}(\varphi_y^{\mathscr{A}}(x,\rho))^2\mathscr{D}_x\mathscr{A}_{y\rho}^{-1}(\varphi_y^{\mathscr{A}}(x,\rho))\right]\\&+\frac{\beta}{\rho^\alpha}\mathscr{A}_{y\rho}\left[\mathscr{D}_x^3\mathscr{A}_{y\rho}^{-1}(\varphi_y^{\mathscr{A}}(x,\rho))\right]-\frac{\beta'}{\rho^\alpha}\mathscr{A}_{y\rho}\left[\mathscr{D}_x^5\mathscr{A}_{y\rho}^{-1}(\varphi_y^{\mathscr{A}}(x,\rho))\right]\end{aligned}\tag{34}$$

and the n-th Aboodh residual function $\mathscr{A}_{y\rho}\mathscr{R}es_n\ \varphi_{xy}$ is

$$\begin{aligned}\mathscr{A}_{y\rho}\mathscr{R}es_n\ \varphi_{xy}=&\varphi_{yn}^{\mathscr{A}}(x,\rho)-\frac{1}{\rho^2}h_0(x)\\&+\frac{1}{\rho^\alpha}\mathscr{A}_{y\rho}\left[\mathscr{A}_{y\rho}^{-1}(\varphi_{yn}^{\mathscr{A}}(x,\rho))^2\mathscr{D}_x\mathscr{A}_{y\rho}^{-1}(\varphi_{yn}^{\mathscr{A}}(x,\rho))\right]\\&+\frac{\beta}{\rho^\alpha}\mathscr{A}_{y\rho}\left[\mathscr{D}_x^3\mathscr{A}_{y\rho}^{-1}(\varphi_{yn}^{\mathscr{A}}(x,\rho))\right]-\frac{\beta'}{\rho^\alpha}\mathscr{A}_{y\rho}\left[\mathscr{D}_x^5\mathscr{A}_{y\rho}^{-1}(\varphi_{yn}^{\mathscr{A}}(x,\rho))\right]\end{aligned}\tag{35}$$

It is clear that $\mathscr{A}_{y\rho}\mathscr{R}es\ \varphi_{xy}=0$ and $\lim_{n\to\infty}\mathscr{A}_{y\rho}\mathscr{R}es_n\ \varphi_{xy}=\mathscr{A}_{y\rho}\mathscr{R}es\ \varphi_{xy}$ for all $\rho>0$. If $\lim_{\rho\to\infty}\rho^2\mathscr{A}_{y\rho}\mathscr{R}es\ \varphi_{xy}=0$, then $\lim_{\rho\to\infty}\rho^2\mathscr{A}_{y\rho}\mathscr{R}es_n\ \varphi_{xy}=0$. In general, if

$$\lim_{\rho \to \infty} \rho^{n\alpha+2} \mathscr{A}_{y\rho} \mathscr{R}es\ \varphi_{xy} = 0$$

then

$$\lim_{\rho \to \infty} \rho^{n\alpha+2} \mathscr{A}_{y\rho} \mathscr{R}es_n\ \varphi_{xy} = 0$$

for $n \in \mathbb{N}$ and $0 < \alpha \leq 1$. To calculate $h_n(x)$, we will use the iterative technique in solving the following:

$$\lim_{\rho \to \infty} \rho^{n\alpha+2} \mathscr{A}_{y\rho} \mathscr{R}es_n\ \varphi_{xy} = 0 \tag{36}$$

for $n = 1, 2, 3, \cdots$. Put the values $h_n(x)$ in (33) to get the n-th solutions $\varphi_{yn}^{\mathscr{A}}(x, \rho)$ of (31) and then take the inverse Aboodh transform of the n-th solutions $\varphi_{yn}^{\mathscr{A}}(x, \rho)$ to get the n-th solutions $\varphi_y^n(x, y)$ of (28).

4. Some applications

In this section, we use ARPSM to obtain some solutions of MTCFKE (3) in two cases: with perfect and approximate initial conditions, where the approximate initial condition is a seventh-order Taylor approximation of the perfect one for MTCFKE (3). Consider the following MTCFKE

$$\begin{cases} \mathscr{D}_{0,y}^{\alpha} \varphi(x, y) + \varphi^2(x, y) \dfrac{\partial \varphi(x, y)}{\partial x} + \beta \dfrac{\partial^3 \varphi(x, y)}{\partial x^3} - \beta' \dfrac{\partial^5 \varphi(x, y)}{\partial x^5} = 0, \quad 0 < \alpha \leq 1 \\ \varphi(x, 0) = \dfrac{3\beta}{\sqrt{-10\beta'}} \operatorname{sech}^2 \left(\dfrac{1}{2} \sqrt{\dfrac{-\beta}{5\beta'}}\, x \right) \end{cases} \tag{37}$$

where $\beta > 0$ and $\beta' < 0$ are real numbers. The MTCFKE above arises in describing nonlinear waves in plasma, especially magneto-acoustic waves, where normal (quadratic) nonlinearity vanishes under certain plasma compositions, so a higher-order nonlinearity is required. In such cases, the mKE can model wave propagation, wave steepening, and dispersive effects together [7]. The MTCFKE is also used to model long waves in shallow water when surface tension effects are non-negligible [6]. By using [29], if $\alpha = 1$, then the perfect solution of (37) is

$$\varphi(x, y) = \frac{3\beta}{\sqrt{-10\beta'}} \operatorname{sech}^2 \left[\frac{1}{2} \sqrt{\frac{-\beta}{5\beta'}} \left(x - \frac{25\beta' - 4\beta^2}{25\beta'}\, y \right) \right]. \tag{38}$$

Take the Aboodh transforms on (37)

$$\varphi_{y}^{\mathscr{A}}(x,\rho)=\frac{3\beta}{\rho^{2}\sqrt{-10\beta'}}\operatorname{sech}^{2}\left(\frac{1}{2}\sqrt{\frac{-\beta}{5\beta'}}\,x\right)$$

$$-\frac{1}{\rho^{\alpha}}\mathscr{A}_{y}\left[\left[\mathscr{A}_{y}^{-1}(\varphi_{y}^{\mathscr{A}}(x,\rho))\right]^{2}\frac{\partial}{\partial x}[\mathscr{A}_{y}^{-1}(\varphi_{y}^{\mathscr{A}}(x,\rho))]\right] \tag{39}$$

$$-\frac{\beta}{\rho^{\alpha}}\mathscr{A}_{y}\left[\frac{\partial^{3}}{\partial x^{3}}[\mathscr{A}_{y}^{-1}(\varphi_{y}^{\mathscr{A}}(x,\rho))]\right]+\frac{\beta'}{\rho^{\alpha}}\mathscr{A}_{y}\left[\frac{\partial^{5}}{\partial x^{5}}[\mathscr{A}_{y}^{-1}(\varphi_{y}^{\mathscr{A}}(x,\rho))]\right]$$

with the sequence $\langle\varphi_{yn}^{\mathscr{A}}\rangle_{n\in\mathbb{N}\cup\{0\}}$

$$\varphi_{yn}^{\mathscr{A}}(x,\rho)=\frac{3\beta}{\rho^{2}\sqrt{-10\beta'}}\operatorname{sech}^{2}\left(\frac{1}{2}\sqrt{\frac{-\beta}{5\beta'}}\,x\right)+\sum_{j=1}^{n}\frac{h_{j}(x)}{\rho^{j\alpha+2}}. \tag{40}$$

The n-th structure for (39) is

$$\varphi_{yn}^{\mathscr{A}}(x,\rho)=\frac{3\beta}{\rho^{2}\sqrt{-10\beta'}}\operatorname{sech}^{2}\left(\frac{1}{2}\sqrt{\frac{-\beta}{5\beta'}}\,x\right)$$

$$-\frac{1}{\rho^{\alpha}}\mathscr{A}_{y}\left[\left[\mathscr{A}_{y}^{-1}(\varphi_{yn}^{\mathscr{A}}(x,\rho))\right]^{2}\frac{\partial}{\partial x}[\mathscr{A}_{y}^{-1}(\varphi_{yn}^{\mathscr{A}}(x,\rho))]\right] \tag{41}$$

$$-\frac{\beta}{\rho^{\alpha}}\mathscr{A}_{y}\left[\frac{\partial^{3}}{\partial x^{3}}[\mathscr{A}_{y}^{-1}(\varphi_{yn}^{\mathscr{A}}(x,\rho))]\right]+\frac{\beta'}{\rho^{\alpha}}\mathscr{A}_{y}\left[\frac{\partial^{5}}{\partial x^{5}}[\mathscr{A}_{y}^{-1}(\varphi_{yn}^{\mathscr{A}}(x,\rho))]\right].$$

The Aboodh residual function, $\mathscr{A}_{y}\mathscr{R}es\ \varphi(x,\rho)$ for (39) is given by

$$\mathscr{A}_{y}\mathscr{R}es\ \varphi(x,\rho)=\varphi_{y}^{\mathscr{A}}(x,\rho)-\frac{3\beta}{\rho^{2}\sqrt{-10\beta'}}\operatorname{sech}^{2}\left(\frac{1}{2}\sqrt{\frac{-\beta}{5\beta'}}\,x\right)$$

$$+\frac{1}{\rho^{\alpha}}\mathscr{A}_{y}\left[\left[\mathscr{A}_{y}^{-1}(\varphi_{y}^{\mathscr{A}}(x,\rho))\right]^{2}\frac{\partial}{\partial x}[\mathscr{A}_{y}^{-1}(\varphi_{y}^{\mathscr{A}}(x,\rho))]\right] \tag{42}$$

$$+\frac{\beta}{\rho^{\alpha}}\mathscr{A}_{y}\left[\frac{\partial^{3}}{\partial x^{3}}[\mathscr{A}_{y}^{-1}(\varphi_{y}^{\mathscr{A}}(x,\rho))]\right]-\frac{\beta'}{\rho^{\alpha}}\mathscr{A}_{y}\left[\frac{\partial^{5}}{\partial x^{5}}[\mathscr{A}_{y}^{-1}(\varphi_{y}^{\mathscr{A}}(x,\rho))]\right]$$

with the n-th structure

$$\mathscr{A}_y\mathscr{R}es_n\ \varphi(x,\rho)=\varphi_{yn}^{\mathscr{A}}(x,\rho)-\frac{3\beta}{\rho^2\sqrt{-10\beta'}}\operatorname{sech}^2\left(\frac{1}{2}\sqrt{\frac{-\beta}{5\beta'}}\,x\right)$$
$$+\frac{1}{\rho^\alpha}\mathscr{A}_y\left[\left[\mathscr{A}_y^{-1}(\varphi_{yn}^{\mathscr{A}}(x,\rho))\right]^2\frac{\partial}{\partial x}[\mathscr{A}_y^{-1}(\varphi_{yn}^{\mathscr{A}}(x,\rho))]\right] \tag{43}$$
$$+\frac{\beta}{\rho^\alpha}\mathscr{A}_y\left[\frac{\partial^3}{\partial x^3}[\mathscr{A}_y^{-1}(\varphi_{yn}^{\mathscr{A}}(x,\rho))]\right]-\frac{\beta'}{\rho^\alpha}\mathscr{A}_y\left[\frac{\partial^5}{\partial x^5}[\mathscr{A}_y^{-1}(\varphi_{yn}^{\mathscr{A}}(x,\rho))]\right]$$

for $n=1,2,3,\cdots$. The terms of $\langle h_n\rangle_{n\in\mathbb{N}\cup\{0\}}$ are calculating by the relation

$$\lim_{\rho\to\infty}\rho^{n\alpha+2}\mathscr{A}_y\mathscr{R}es_n\ \varphi(x,\rho)=0,\quad n=1,2,3,\cdots \tag{44}$$

At $n=1$, by (40) we have $\varphi_{y1}^{\mathscr{A}}(x,\rho)=\frac{3\beta}{\rho^2\sqrt{-10\beta'}}\operatorname{sech}^2\left(\frac{1}{2}\sqrt{\frac{-\beta}{5\beta'}}\,x\right)+\frac{h_1(x)}{\rho^{\alpha+2}}$ and (43) becomes

$$\mathscr{A}_y\mathscr{R}es_1\ \varphi(x,\rho)=\frac{h_1(x)}{\rho^{\alpha+2}}+\frac{1}{\rho^\alpha}\mathscr{A}_y\left[\left[\mathscr{A}_y^{-1}(\varphi_{y1}^{\mathscr{A}}(x,\rho))\right]^2\frac{\partial}{\partial x}[\mathscr{A}_y^{-1}(\varphi_{y1}^{\mathscr{A}}(x,\rho))]\right]$$
$$+\frac{\beta}{\rho^\alpha}\mathscr{A}_y\left[\frac{\partial^3}{\partial x^3}[\mathscr{A}_y^{-1}(\varphi_{y1}^{\mathscr{A}}(x,\rho))]\right]-\frac{\beta'}{\rho^\alpha}\mathscr{A}_y\left[\frac{\partial^5}{\partial x^5}[\mathscr{A}_y^{-1}(\varphi_{y1}^{\mathscr{A}}(x,\rho))]\right].$$

This implies

$$\mathscr{A}_y\mathscr{R}es_1\ \varphi(x,\rho)=\frac{1}{\rho^{\alpha+2}}\left[h_1(x)+\mathscr{H}^2(x)\frac{\partial}{\partial x}\mathscr{H}(x)+\beta\frac{\partial^3}{\partial x^3}\mathscr{H}(x)-\beta'\frac{\partial^5}{\partial x^5}\mathscr{H}(x)\right]$$
$$+\frac{1}{\rho^{2\alpha+2}}\left[\mathscr{H}^2(x)\frac{\partial}{\partial x}h_1(x)+2\mathscr{H}(x)h_1(x)\frac{\partial}{\partial x}\mathscr{H}(x)+\beta\frac{\partial^3}{\partial x^3}h_1(x)-\beta'\frac{\partial^5}{\partial x^5}h_1(x)\right]$$
$$+\frac{1}{\rho^{3\alpha+1}}\left[\frac{2\Gamma(2\alpha+1)}{(\Gamma(\alpha+1))^2}\mathscr{H}(x)h_1(x)\frac{\partial}{\partial x}h_1(x)+\frac{\Gamma(2\alpha+1)}{(\Gamma(\alpha+1))^2}h_1^2(x)\frac{\partial}{\partial x}\mathscr{H}(x)\right] \tag{45}$$
$$+\frac{1}{\rho^{4\alpha+1}}\frac{\Gamma(3\alpha+1)}{(\Gamma(\alpha+1))^3}h_1^2(x)\frac{\partial}{\partial x}h_1(x)$$

where $\mathscr{H}(x)=\frac{3\beta}{\sqrt{-10\beta'}}\operatorname{sech}^2\left(\frac{1}{2}\sqrt{\frac{-\beta}{5\beta'}}\,x\right)$. By multiplying (45) by $\rho^{\alpha+2}$ and by (44) we have

$$h_1(x) = \frac{27\beta^3\sqrt{\beta}}{5\beta'\sqrt{2}}\operatorname{sech}^6\left(\frac{1}{2}\sqrt{\frac{-\beta}{5\beta'}}x\right)\tanh\left(\frac{1}{2}\sqrt{\frac{-\beta}{5\beta'}}x\right)$$

$$+\frac{6\beta^3\sqrt{\beta}}{25\beta'^2\sqrt{2}}\operatorname{sech}^4\left(\frac{1}{2}\sqrt{\frac{-\beta}{5\beta'}}x\right)\tanh\left(\frac{1}{2}\sqrt{\frac{-\beta}{5\beta'}}x\right)-\frac{\beta}{5\beta'\sqrt{-10\beta'}}\tanh^2\left(\frac{1}{2}\sqrt{\frac{-\beta}{5\beta'}}x\right)$$

$$-\frac{39\beta^3\sqrt{\beta}}{125\beta'^3\sqrt{2}}\operatorname{sech}^4\left(\frac{1}{2}\sqrt{\frac{-\beta}{5\beta'}}x\right)\tanh^3\left(\frac{1}{2}\sqrt{\frac{-\beta}{5\beta'}}x\right) \tag{46}$$

$$+\frac{51\beta^3\sqrt{\beta}}{250\beta'^3\sqrt{2}}\operatorname{sech}^6\left(\frac{1}{2}\sqrt{\frac{-\beta}{5\beta'}}x\right)\tanh\left(\frac{1}{2}\sqrt{\frac{-\beta}{5\beta'}}x\right)$$

$$+\frac{3\beta^3\sqrt{\beta}}{125\beta'^3\sqrt{2}}\operatorname{sech}^2\left(\frac{1}{2}\sqrt{\frac{-\beta}{5\beta'}}x\right)\tanh^5\left(\frac{1}{2}\sqrt{\frac{-\beta}{5\beta'}}x\right).$$

At $n = 2$, by (40)

$$\varphi_{y2}^{\mathscr{A}}(x,\rho) = \frac{3\beta}{\rho^2\sqrt{-10\beta'}}\operatorname{sech}^2\left(\frac{1}{2}\sqrt{\frac{-\beta}{5\beta'}}\,x\right)+\frac{h_1(x)}{\rho^{\alpha+2}}+\frac{h_2(x)}{\rho^{2\alpha+2}}.$$

Then we have

$$\mathscr{A}_y\mathscr{R}es_2\,\varphi(x,\rho) = \frac{h_1(x)}{\rho^{\alpha+2}}+\frac{h_2(x)}{\rho^{2\alpha+2}}+\frac{1}{\rho^\alpha}\mathscr{A}_y\left[\left[\mathscr{A}_y^{-1}(\varphi_{y2}^{\mathscr{A}}(x,\rho))\right]^2\frac{\partial}{\partial x}[\mathscr{A}_y^{-1}(\varphi_{y2}^{\mathscr{A}}(x,\rho))]\right]$$

$$+\frac{\beta}{\rho^\alpha}\mathscr{A}_y\left[\frac{\partial^3}{\partial x^3}[\mathscr{A}_y^{-1}(\varphi_{y2}^{\mathscr{A}}(x,\rho))]\right]-\frac{\beta'}{\rho^\alpha}\mathscr{A}_y\left[\frac{\partial^5}{\partial x^5}[\mathscr{A}_y^{-1}(\varphi_{y2}^{\mathscr{A}}(x,\rho))]\right].$$

This implies

$$\mathscr{A}_y\mathscr{R}es_2\,\varphi(x,\rho) = \frac{1}{\rho^{2\alpha+2}}\Big[h_2(x)+\mathscr{H}^2(x)\frac{\partial}{\partial x}h_1(x)+2h_1(x)\mathscr{H}(x)\frac{\partial}{\partial x}\mathscr{H}(x)$$

$$+\beta\frac{\partial^3}{\partial x^3}h_1(x)+\beta'\frac{\partial^5}{\partial x^5}h_1(x)\Big]+\frac{1}{\rho^{3\alpha+2}}\Big[\mathscr{H}^2(x)h_2^2(x)$$

$$+\frac{2\Gamma(2\alpha+1)}{(\Gamma(\alpha+1))^2}\mathscr{H}(x)h_1(x)\frac{\partial}{\partial x}h_1(x)+2h_2(x)\mathscr{H}(x)\frac{\partial}{\partial x}\mathscr{H}(x)$$

$$+\frac{\Gamma(2\alpha+1)}{(\Gamma(\alpha+1))^2}h_1^2(x)\frac{\partial}{\partial x}\mathscr{H}(x)+\beta\frac{\partial^3}{\partial x^3}h_2(x)-\beta'\frac{\partial^5}{\partial x^5}h_2(x)\Big]$$

$$+\frac{1}{\rho^{4\alpha+1}}\Big[\frac{2\Gamma(3\alpha+1)}{(\Gamma(2\alpha+1))^2}\mathscr{H}(x)h_1(x)\frac{\partial}{\partial x}h_1(x)$$

$$+\frac{2\Gamma(3\alpha+1)}{\Gamma(\alpha+1)\Gamma(2\alpha+1)}h_2(x)\mathscr{H}(x)\frac{\partial}{\partial x}h_1(x)$$

$$+\frac{\Gamma(3\alpha+1)}{(\Gamma(\alpha+1))^3}h_1^2(x)\frac{\partial}{\partial x}h_1(x)+\frac{2\Gamma(3\alpha+1)}{\Gamma(\alpha+1)\Gamma(2\alpha+1)}h_1(x)h_2(x)\frac{\partial}{\partial x}\mathscr{H}(x)\Big]$$

$$+\frac{1}{\rho^{5\alpha+2}}\Big[\frac{2\Gamma(4\alpha+1)}{(\Gamma(\alpha+1))^2}\mathscr{H}(x)h_2(x)\frac{\partial}{\partial x}h_2(x)+\frac{\Gamma(4\alpha+1)}{\Gamma(\alpha+1)\Gamma(2\alpha+1)}h_1^2(x)\frac{\partial}{\partial x}h_2(x)$$

$$+\frac{2\Gamma(4\alpha+1)}{(\Gamma(\alpha+1))^2\Gamma(2\alpha+1)}h_1(x)h_2(x)\frac{\partial}{\partial x}h_1(x)$$

$$+\frac{\Gamma(4\alpha+1)}{(\Gamma(\alpha+1))^2\Gamma(2\alpha+1)}h_2^2(x)\frac{\partial}{\partial x}\mathscr{H}(x)\Big]$$

$$+\frac{1}{\rho^{6\alpha+2}}\Big[\frac{2\Gamma(5\alpha+1)}{(\Gamma(2\alpha+1))^2\Gamma(\alpha+1)}h_1(x)h_2(x)\frac{\partial}{\partial x}h_2(x)$$

$$+\frac{\Gamma(5\alpha+1)}{(\Gamma(2+1))^2\Gamma(+1)}h_2^2(x)\frac{\partial}{\partial x}h_1(x)\Big]+\frac{1}{\rho^{7\alpha+1}}\frac{\Gamma(6\alpha+1)}{(\Gamma(2\alpha+1))^3}h_2^2(x)\frac{\partial}{\partial x}h_2(x). \tag{47}$$

By multiplying (47) by $\rho^{\alpha+2}$ and by (44) we have

$$h_2(x)=-\mathscr{H}^2(x)\frac{\partial}{\partial x}h_1(x)-2h_1(x)\mathscr{H}(x)\frac{\partial}{\partial x}\mathscr{H}(x)-\beta\frac{\partial^3}{\partial x^3}h_1(x)+\beta'\frac{\partial^5}{\partial x^5}h_1(x). \tag{48}$$

Put the terms of $\langle h_n\rangle_{n\in\mathbb{N}\cup\{0\}}$ in (32)

$$\varphi_y^{\mathscr{A}}(x,\rho)=\sum_{j=0}^{\infty}\frac{h_j(x)}{\rho^{j\alpha+2}}=\frac{1}{\rho}h_0(x)+\frac{1}{\rho^{\alpha+2}}h_1(x)+\frac{1}{\rho^{2\alpha+2}}h_2(x)+\cdots$$

$$=\frac{3\beta}{\rho\sqrt{-10\beta'}}\operatorname{sech}^2\left(\frac{1}{2}\sqrt{\frac{-\beta}{5\beta'}}\,x\right)$$

$$+\frac{1}{\rho^{\alpha+2}}\Big\{\frac{27\beta^3\sqrt{\beta}}{5\beta'\sqrt{2}}\operatorname{sech}^6\left(\frac{1}{2}\sqrt{\frac{-\beta}{5\beta'}}x\right)\tanh\left(\frac{1}{2}\sqrt{\frac{-\beta}{5\beta'}}x\right)$$

$$+\frac{6\beta^3\sqrt{\beta}}{25\beta'^2\sqrt{2}}\operatorname{sech}^4\left(\frac{1}{2}\sqrt{\frac{-\beta}{5\beta'}}x\right)\tanh\left(\frac{1}{2}\sqrt{\frac{-\beta}{5\beta'}}x\right)-\frac{\beta}{5\beta'\sqrt{-10\beta'}}\tanh^2\left(\frac{1}{2}\sqrt{\frac{-\beta}{5\beta'}}x\right)$$

$$-\frac{39\beta^3\sqrt{\beta}}{125\beta'^3\sqrt{2}}\operatorname{sech}^4\left(\frac{1}{2}\sqrt{\frac{-\beta}{5\beta'}}x\right)\tanh^3\left(\frac{1}{2}\sqrt{\frac{-\beta}{5\beta'}}x\right)$$

$$+\frac{51\beta^3\sqrt{\beta}}{250\beta'^3\sqrt{2}}\operatorname{sech}^6\left(\frac{1}{2}\sqrt{\frac{-\beta}{5\beta'}}x\right)\tanh\left(\frac{1}{2}\sqrt{\frac{-\beta}{5\beta'}}x\right)$$

$$+\frac{3\beta^3\sqrt{\beta}}{125\beta'^3\sqrt{2}}\operatorname{sech}^2\left(\frac{1}{2}\sqrt{\frac{-\beta}{5\beta'}}x\right)\tanh^5\left(\frac{1}{2}\sqrt{\frac{-\beta}{5\beta'}}x\right)\Big\}-\frac{1}{\rho^{2\alpha+2}}\Big(\mathscr{H}^2(x)\frac{\partial}{\partial x}h_1(x)$$

$$+2h_1(x)\mathscr{H}(x)\frac{\partial}{\partial x}\mathscr{H}(x)\beta\frac{\partial^3}{\partial x^3}h_1(x)+\beta'\frac{\partial^5}{\partial x^5}h_1(x)\Big)\Big)+\cdots \tag{49}$$

Take the inverse Aboodh transform of (49),

$$\varphi(x, y)=\frac{3\beta}{\sqrt{-10\beta'}}\operatorname{sech}^2\left(\frac{1}{2}\sqrt{\frac{-\beta}{5\beta'}}\,x\right)$$

$$+\frac{y^\alpha}{\Gamma(\alpha+1)}\Big\{\frac{27\beta^3\sqrt{\beta}}{5\beta'\sqrt{2}}\operatorname{sech}^6\left(\frac{1}{2}\sqrt{\frac{-\beta}{5\beta'}}x\right)\tanh\left(\frac{1}{2}\sqrt{\frac{-\beta}{5\beta'}}x\right)$$

$$+\frac{6\beta^3\sqrt{\beta}}{25\beta'^2\sqrt{2}}\operatorname{sech}^4\left(\frac{1}{2}\sqrt{\frac{-\beta}{5\beta'}}x\right)\tanh\left(\frac{1}{2}\sqrt{\frac{-\beta}{5\beta'}}x\right)-\frac{\beta}{5\beta'\sqrt{-10\beta'}}\tanh^2\left(\frac{1}{2}\sqrt{\frac{-\beta}{5\beta'}}x\right)$$

$$-\frac{39\beta^3\sqrt{\beta}}{125\beta'^3\sqrt{2}}\operatorname{sech}^4\left(\frac{1}{2}\sqrt{\frac{-\beta}{5\beta'}}x\right)\tanh^3\left(\frac{1}{2}\sqrt{\frac{-\beta}{5\beta'}}x\right)$$

$$+\frac{51\beta^3\sqrt{\beta}}{250\beta'^3\sqrt{2}}\operatorname{sech}^6\left(\frac{1}{2}\sqrt{\frac{-\beta}{5\beta'}}x\right)\tanh\left(\frac{1}{2}\sqrt{\frac{-\beta}{5\beta'}}x\right)$$

$$+\frac{3\beta^3\sqrt{\beta}}{125\beta'^3\sqrt{2}}\operatorname{sech}^2\left(\frac{1}{2}\sqrt{\frac{-\beta}{5\beta'}}x\right)\tanh^5\left(\frac{1}{2}\sqrt{\frac{-\beta}{5\beta'}}x\right)\Bigg\}-\frac{y^{2\alpha}}{\Gamma(2\alpha+1)}\Big(\mathscr{H}^2(x)\frac{\partial}{\partial x}h_1(x)$$

$$+2h_1(x)\mathscr{H}(x)\frac{\partial}{\partial x}\mathscr{H}(x)\beta\frac{\partial^3}{\partial x^3}h_1(x)+\beta'\frac{\partial^5}{\partial x^5}h_1(x))\Big)+\cdots \tag{50}$$

Now we take MTCFKE (3) with the polynomial initial condition in MTCFKE (3). Consider the following MTCFKE

$$\begin{cases}\mathscr{D}^{\alpha}_{0,y}\varphi(x,y)+\varphi^2(x,y)\dfrac{\partial\varphi(x,y)}{\partial x}+\beta\dfrac{\partial^3\varphi(x,y)}{\partial x^3}-\beta'\dfrac{\partial^5\varphi(x,y)}{\partial x^5}=0, \quad 0<\alpha\le 1\\ \varphi(x,0)=0.9487\dfrac{\beta}{\sqrt{-\beta'}}+0.0474\dfrac{\beta^2}{\beta'\sqrt{-\beta'}}x^2+0.0021\dfrac{\beta^3}{\beta'^2\sqrt{-\beta'}}x^4+0.3584\dfrac{\beta^4}{\beta'^3\sqrt{-\beta'}}x^6.\end{cases} \tag{51}$$

Take Aboodh transforms of (51)

$$\begin{aligned}\varphi^{\mathscr{A}}_y(x,\rho)=&\frac{1}{\rho^2\sqrt{-\beta'}}[0.9487\beta+0.0474\frac{\beta^2}{\beta'}x^2+0.0021\frac{\beta^3}{\beta'^2}x^4+0.3584\frac{\beta^4}{\beta'^3}x^6]\\ &-\frac{1}{\rho^\alpha}\mathscr{A}_y\left[\left[\mathscr{A}^{-1}_y(\varphi^{\mathscr{A}}_y(x,\rho))\right]^2\frac{\partial}{\partial x}[\mathscr{A}^{-1}_y(\varphi^{\mathscr{A}}_y(x,\rho))]\right]\\ &-\frac{\beta}{\rho^\alpha}\mathscr{A}_y\left[\frac{\partial^3}{\partial x^3}[\mathscr{A}^{-1}_y(\varphi^{\mathscr{A}}_y(x,\rho))]\right]+\frac{\beta'}{\rho^\alpha}\mathscr{A}_y\left[\frac{\partial^5}{\partial x^5}[\mathscr{A}^{-1}_y(\varphi^{\mathscr{A}}_y(x,\rho))]\right].\end{aligned} \tag{52}$$

The sequence $\langle\varphi^{\mathscr{A}}_{yn}\rangle_{n\in\mathbb{N}\cup\{0\}}$ given by

$$\varphi^{\mathscr{A}}_{yn}(x,\rho)=\frac{1}{\rho^2\sqrt{-\beta'}}[0.9487\beta+0.0474\frac{\beta^2}{\beta'}x^2+0.0021\frac{\beta^3}{\beta'^2}x^4+0.3584\frac{\beta^4}{\beta'^3}x^6]+\sum_{j=1}^{n}\frac{h_j(x)}{\rho^{j\alpha+2}}. \tag{53}$$

The n-th structure for (52) is

$$\begin{aligned}\varphi^{\mathscr{A}}_{yn}(x,\rho)=&\frac{1}{\rho^2\sqrt{-\beta'}}[0.9487\beta+0.0474\frac{\beta^2}{\beta'}x^2+0.0021\frac{\beta^3}{\beta'^2}x^4+0.3584\frac{\beta^4}{\beta'^3}x^6]\\ &-\frac{1}{\rho^\alpha}\mathscr{A}_y\left[[\mathscr{A}^{-1}_y(\varphi^{\mathscr{A}}_{yn}(x,\rho))]^2\frac{\partial}{\partial x}[\mathscr{A}^{-1}_y(\varphi^{\mathscr{A}}_{yn}(x,\rho))]\right]\\ &-\frac{\beta}{\rho^\alpha}\mathscr{A}_y\left[\frac{\partial^3}{\partial x^3}[\mathscr{A}^{-1}_y(\varphi^{\mathscr{A}}_{yn}(x,\rho))]\right]+\frac{\beta'}{\rho^\alpha}\mathscr{A}_y\left[\frac{\partial^5}{\partial x^5}[\mathscr{A}^{-1}_y(\varphi^{\mathscr{A}}_{yn}(x,\rho))]\right].\end{aligned} \tag{54}$$

The function $\mathscr{A}_y\mathscr{R}es\ \varphi(x, \rho)$ for (52) is given by

$$\mathscr{A}_y\mathscr{R}es\ \varphi(x, \rho) = \varphi_y^{\mathscr{A}}(x, \rho) - \frac{1}{\rho^2\sqrt{-\beta'}}[0.9487\beta + 0.0474\frac{\beta^2}{\beta'}x^2 + 0.0021\frac{\beta^3}{\beta'^2}x^4 + 0.3584\frac{\beta^4}{\beta'^3}x^6] + \frac{1}{\rho^\alpha}\mathscr{A}_y\left[\left[\mathscr{A}_y^{-1}(\varphi_y^{\mathscr{A}}(x, \rho))\right]^2\frac{\partial}{\partial x}[\mathscr{A}_y^{-1}(\varphi_y^{\mathscr{A}}(x, \rho))]\right] + \frac{\beta}{\rho^\alpha}\mathscr{A}_y\left[\frac{\partial^3}{\partial x^3}[\mathscr{A}_y^{-1}(\varphi_y^{\mathscr{A}}(x, \rho))]\right] - \frac{\beta'}{\rho^\alpha}\mathscr{A}_y\left[\frac{\partial^5}{\partial x^5}[\mathscr{A}_y^{-1}(\varphi_y^{\mathscr{A}}(x, \rho))]\right] \tag{55}$$

with the n-th structure

$$\mathscr{A}_y\mathscr{R}es_n\ \varphi(x, \rho) = \varphi_{yn}^{\mathscr{A}}(x, \rho) - \frac{1}{\rho^2\sqrt{-\beta'}}[0.9487\beta + 0.0474\frac{\beta^2}{\beta'}x^2 + 0.0021\frac{\beta^3}{\beta'^2}x^4 + 0.3584\frac{\beta^4}{\beta'^3}x^6] + \frac{1}{\rho^\alpha}\mathscr{A}_y\left[[\mathscr{A}_y^{-1}(\varphi_{yn}^{\mathscr{A}}(x, \rho))]^2\frac{\partial}{\partial x}[\mathscr{A}_y^{-1}(\varphi_{yn}^{\mathscr{A}}(x, \rho))]\right] + \frac{\beta}{\rho^\alpha}\mathscr{A}_y\left[\frac{\partial^3}{\partial x^3}[\mathscr{A}_y^{-1}(\varphi_{yn}^{\mathscr{A}}(x, \rho))]\right] - \frac{\beta'}{\rho^\alpha}\mathscr{A}_y\left[\frac{\partial^5}{\partial x^5}[\mathscr{A}_y^{-1}(\varphi_{yn}^{\mathscr{A}}(x, \rho))]\right]. \tag{56}$$

We calculate the terms of $\langle h_n\rangle_{n\in\mathbb{N}\cup\{0\}}$ by

$$\lim_{\rho\to\infty}\rho^{n\alpha+2}\mathscr{A}_y\mathscr{R}es_n\ \varphi(x, \rho) = 0 \quad n = 1, 2, 3, \cdots \tag{57}$$

At $n = 1$, by (53)

$$\varphi_{y1}^{\mathscr{A}}(x, \rho) = \frac{1}{\rho\sqrt{-\beta'}}[0.9487\beta + 0.0474\frac{\beta^2}{\beta'}x^2 + 0.0021\frac{\beta^3}{\beta'^2}x^4 + 0.3584\frac{\beta^4}{\beta'^3}x^6] + \frac{h_1(x)}{\rho^{\alpha+2}}$$

and (56) will be

$$\mathscr{A}_y\mathscr{R}es_1\ \varphi(x, \rho) = \frac{h_1(x)}{\rho^{\alpha+2}} + \frac{1}{\rho^\alpha}\mathscr{A}_y\left[[\mathscr{A}_y^{-1}(\varphi_{y1}^{\mathscr{A}}(x, \rho))]^2\frac{\partial}{\partial x}[\mathscr{A}_y^{-1}(\varphi_{y1}^{\mathscr{A}}(x, \rho))]\right] + \frac{\beta}{\rho^\alpha}\mathscr{A}_y\left[\frac{\partial^3}{\partial x^3}[\mathscr{A}_y^{-1}(\varphi_{y1}^{\mathscr{A}}(x, \rho))]\right] - \frac{\beta'}{\rho^\alpha}\mathscr{A}_y\left[\frac{\partial^5}{\partial x^5}[\mathscr{A}_y^{-1}(\varphi_{y1}^{\mathscr{A}}(x, \rho))]\right].$$

This implies

$$\begin{aligned}\mathscr{A}_y\mathscr{R}es_1\ \varphi(x,\rho) =& \frac{1}{\rho^{\alpha+2}}\left[h_1(x)+\mathscr{E}^2(x)\frac{\partial}{\partial x}\mathscr{E}(x)+\beta\frac{\partial^3}{\partial x^3}\mathscr{E}(x)+\beta'\frac{\partial^5}{\partial x^5}\mathscr{E}(x)\right]\\ &+\frac{1}{\rho^{2\alpha+2}}\left[\mathscr{E}^2(x)\frac{\partial}{\partial x}h_1(x)+2\mathscr{E}(x)h_1(x)\frac{\partial}{\partial x}\mathscr{E}(x)+\beta\frac{\partial^3}{\partial x^3}h_1(x)-\beta'\frac{\partial^5}{\partial x^5}h_1(x)\right]\\ &+\frac{1}{\rho^{3\alpha+2}}\left[\frac{2\Gamma(2\alpha+1)}{(\Gamma(\alpha+1))^2}\mathscr{E}(x)h_1(x)\frac{\partial}{\partial x}h_1(x)+\frac{\Gamma(2\alpha+1)}{(\Gamma(\alpha+1))^2}h_1^2(x)\frac{\partial}{\partial x}\mathscr{E}(x)\right]\\ &+\frac{1}{\rho^{4\alpha+2}}\frac{\Gamma(3\alpha+1)}{(\Gamma(\alpha+1))^3}h_1^2(x)\frac{\partial}{\partial x}h_1(x)\end{aligned} \tag{58}$$

where $\mathscr{E}(x)=\frac{1}{\sqrt{-\beta'}}[0.9487\beta+0.0474\frac{\beta^2}{\beta'}x^2+0.0021\frac{\beta^3}{\beta'^2}x^4+0.3584\frac{\beta^4}{\beta'^3}x^6]$. By multiplying (58) by $\rho^{\alpha+1}$ and by (57) we get

$$\begin{aligned}h_1(x)=&\frac{0.0899\beta^3\sqrt{-\beta'}+25.80984\beta^4}{\beta'^2\sqrt{-\beta'}}x+\frac{0.0125\beta^3\sqrt{-\beta'}+43.008\beta^5}{\beta'^3\sqrt{-\beta'}}x^3\\ &+\frac{2.0406\beta^5}{\beta'^4}x^5+\frac{0.136\beta^6}{\beta'^5}x^7+\frac{0.008\beta^7}{\beta'^6}x^9+\frac{0.771\beta^8}{\beta'^7}x^{11}.\end{aligned} \tag{59}$$

At $n=2$, by (53)

$$\begin{aligned}\varphi_{y2}^{\mathscr{A}}(x,\rho)=&\frac{1}{\rho\sqrt{-\beta'}}\left[0.9487\beta+0.0474\frac{\beta^2}{\beta'}x^2+0.0021\frac{\beta^3}{\beta'^2}x^4+0.3584\frac{\beta^4}{\beta'^3}x^6\right]\\ &+\frac{h_1(x)}{\rho^{\alpha+2}}+\frac{h_2(x)}{\rho^{2\alpha+2}}\end{aligned}$$

and (56) becomes

$$\begin{aligned}\mathscr{A}_y\mathscr{R}es_2\ \varphi(x,\rho)=&\frac{h_1(x)}{\rho^{\alpha+2}}+\frac{h_2(x)}{\rho^{2\alpha+2}}+\frac{1}{\rho^{\alpha}}\mathscr{A}_y\left[[\mathscr{A}_y^{-1}(\varphi_{y2}^{\mathscr{A}}(x,\rho))]^2\frac{\partial}{\partial x}[\mathscr{A}_y^{-1}(\varphi_{y2}^{\mathscr{A}}(x,\rho))]\right]\\ &+\frac{\beta}{\rho^{\alpha}}\mathscr{A}_y\left[\frac{\partial^3}{\partial x^3}[\mathscr{A}_y^{-1}(\varphi_{y2}^{\mathscr{A}}(x,\rho))]\right]-\frac{\beta'}{\rho^{\alpha}}\mathscr{A}_y\left[\frac{\partial^5}{\partial x^5}[\mathscr{A}_y^{-1}(\varphi_{y2}^{\mathscr{A}}(x,\rho))]\right].\end{aligned}$$

This implies

$$
\begin{aligned}
\mathscr{A}_y \mathscr{R}es_2\ \varphi(x,\rho) = &\frac{1}{\rho^{2\alpha+2}}\Big[h_2(x) + 2\mathscr{E}^2(x)\frac{\partial}{\partial x}h_1(x) + 2h_1(x)\frac{\partial}{\partial x}\mathscr{E}(x) \\
&+ \beta\frac{\partial^3}{\partial x^3}h_1(x) - \beta'\frac{\partial^5}{\partial x^5}h_1(x)\Big] + \frac{1}{\rho^{3\alpha+2}}\Big[\mathscr{E}^2(x)\frac{\partial}{\partial x}h_2(x) \\
&+ \frac{2\Gamma(2\alpha+1)}{(\Gamma(\alpha+1))^2}\mathscr{E}(x)h_1(x)\frac{\partial}{\partial x}h_1(x) + 2h_2(x)\mathscr{E}(x)\frac{\partial}{\partial x}\mathscr{E}(x) \\
&+ h_1^2(x)\frac{\partial}{\partial x}\mathscr{E}(x) + \beta\frac{\partial^3}{\partial x^3}h_2(x) - \beta'\frac{\partial^5}{\partial x^5}h_2(x)\Big] \\
&+ \frac{1}{\rho^{4\alpha+2}}\Big[\frac{2\Gamma(3\alpha+1)}{\Gamma(\alpha+1)\Gamma(2\alpha+1)}\mathscr{E}(x)h_2(x)\frac{\partial}{\partial x}h_1(x) \\
&+ \frac{2\Gamma(3\alpha+1)}{\Gamma(2\alpha+1)}\mathscr{E}(x)h_1(x)\frac{\partial}{\partial x}h_2(x) + \frac{\Gamma(3\alpha+1)}{(\Gamma(\alpha+1))^3}h_1^2(x)\frac{\partial}{\partial x}h_1(x)\Big] \\
&+ \frac{1}{\rho^{5\alpha+2}}\Big[\frac{2\Gamma(4\alpha+1)}{(\Gamma(2\alpha+1))^2}\mathscr{E}(x)h_2(x)\frac{\partial}{\partial x}h_2(x) \\
&+ \frac{\Gamma(4\alpha+1)}{(\Gamma(\alpha+1))^2\Gamma(2\alpha+1)}h_1^2(x)\frac{\partial}{\partial x}h_2(x) \\
&+ \frac{2\Gamma(4\alpha+1)}{\Gamma(\alpha+1)(\Gamma(2\alpha+1))^2}h_1(x)h_2(x)\frac{\partial}{\partial x}\mathscr{E}(x) \\
&+ \frac{2\Gamma(4\alpha+1)}{(\Gamma(\alpha+1))^2\Gamma(2\alpha+1)}h_1(x)h_2(x)\frac{\partial}{\partial x}h_1(x) \\
&+ \frac{\Gamma(4\alpha+1)}{\Gamma(\alpha+1)(\Gamma(2\alpha+1))^2}h_2^2(x)\frac{\partial}{\partial x}h_1(x)\Big] \\
&+ \frac{1}{\rho^{6\alpha+1}}\Big[\frac{2\Gamma(5\alpha+1)}{\Gamma(\alpha+1)(\Gamma(2\alpha+1))^2}h_1(x)h_2(x)\frac{\partial}{\partial x}h_2(x) \\
&+ \frac{\Gamma(5\alpha+1)}{\Gamma(\alpha+1)(\Gamma(2\alpha+1))^2}h_2^2(x)\frac{\partial}{\partial x}h_1(x)\Big] \\
&+ \frac{1}{\rho^{7\alpha+1}}\frac{\Gamma(6\alpha+1)}{(\Gamma(2\alpha+1))^3}h_2^2(x)\frac{\partial}{\partial x}h_2(x)\Big].
\end{aligned} \tag{60}
$$

By multiplying (60) by $\rho^{\alpha+2}$ and by (57) we have

$$h_2(x) = -2\mathscr{E}^2(x)\frac{\partial}{\partial x}h_1(x) - 2h_1(x)\frac{\partial}{\partial x}\mathscr{E}(x) - \beta\frac{\partial^3}{\partial x^3}h_1(x) - \beta'\frac{\partial^5}{\partial x^5}h_1(x). \tag{61}$$

Put the terms of $\langle h_n\rangle_{n\in\mathbb{N}\cup\{0\}}$ in (32) to get

$$\begin{aligned}\varphi_y^{\mathscr{A}}(x, \rho) = &\sum_{j=0}^{\infty}\frac{h_j(x)}{\rho^{j\alpha+2}} = \frac{1}{\rho}h_0(x) + \frac{1}{\rho^{\alpha+2}}h_1(x) + \frac{1}{\rho^{2\alpha+2}}h_2(x) + \cdots \\
=&\frac{1}{\rho}\left[0.9487\frac{\beta}{\sqrt{-\beta'}} + 0.0474\frac{\beta^2}{\beta'\sqrt{-\beta'}}x^2 + 0.0021\frac{\beta^3}{\beta'^2\sqrt{-\beta'}}x^4 + 0.3584\frac{\beta^4}{\beta'^3\sqrt{-\beta'}}x^6\right] \\
&+ \frac{1}{\rho^{\alpha+2}}\left\{\frac{0.0899\beta^3\sqrt{-\beta'} + 25.80984\beta^4}{\beta'^2\sqrt{-\beta'}}x + \frac{0.0125\beta^3\sqrt{-\beta'} + 43.008\beta^5}{\beta'^3\sqrt{-\beta'}}x^3\right. \\
&\left. + \frac{2.0406\beta^5}{\beta'^4}x^5 + \frac{0.136\beta^6}{\beta'^5}x^7 + \frac{0.008\beta^7}{\beta'^6}x^9 + \frac{0.771\beta^8}{\beta'^7}x^{11}\right\} \\
&- \frac{1}{\rho^{2\alpha+2}}\left[2\mathscr{E}^2(x)\frac{\partial}{\partial x}h_1(x) + 2h_1(x)\frac{\partial}{\partial x}\mathscr{E}(x) + \beta\frac{\partial^3}{\partial x^3}h_1(x) + \right. \\
&\left.\beta'\frac{\partial^5}{\partial x^5}h_1(x)\right] + \cdots\end{aligned} \tag{62}$$

Now take the inverse Aboodh transform of (62)

$$\begin{aligned}\varphi(x, y) =&0.9487\frac{\beta}{\sqrt{-\beta'}} + 0.0474\frac{\beta^2}{\beta'\sqrt{-\beta'}}x^2 + 0.0021\frac{\beta^3}{\beta'^2\sqrt{-\beta'}}x^4 + 0.3584\frac{\beta^4}{\beta'^3\sqrt{-\beta'}}x^6 \\
&+ \frac{y^\alpha}{\Gamma(\alpha+1)}\left\{\frac{0.0899\beta^3\sqrt{-\beta'} + 25.80984\beta^4}{\beta'^2\sqrt{-\beta'}}x + \frac{0.0125\beta^3\sqrt{-\beta'} + 43.008\beta^5}{\beta'^3\sqrt{-\beta'}}x^3\right. \\
&\left. + \frac{2.0406\beta^5}{\beta'^4}x^5 + \frac{0.136\beta^6}{\beta'^5}x^7 + \frac{0.008\beta^7}{\beta'^6}x^9 + \frac{0.771\beta^8}{\beta'^7}x^{11}\right\} \\
&- \frac{y^{2\alpha}}{\Gamma(2\alpha+1)}\left[2\mathscr{E}^2(x)\frac{\partial}{\partial x}h_1(x) + 2h_1(x)\frac{\partial}{\partial x}\mathscr{E}(x) + \beta\frac{\partial^3}{\partial x^3}h_1(x) + \right. \\
&\left.\beta'\frac{\partial^5}{\partial x^5}h_1(x)\right] + \cdots\end{aligned} \tag{63}$$

5. Numerical discussion

This section illustrates the approximate solutions for MTCFKE (51) corresponding to different fractional orders α. The results are displayed in Tables 1 and 2, while Figures 1-5, obtained through numerical simulations, highlight the dynamical behavior of these approximate solutions. Specifically, Table (1) reports a comparison of the Absolute Error (AE) between the exact and approximate solutions of TCFKE (51) when $\beta = 0.001$, $\beta' = -1$, $\alpha = 1$, and $x = 10$, together with some solutions of (51) obtained by the RPSM [32] and HAM [29]. In Table (2), we compare the ARPSM approximate solution of (51) at $\beta = 0.001$, $\beta' = -1$, $x = 10$, and various values of α and y with solutions obtained by RPSM [32] and HAM [29]. Figure (1) presents sample curves of the approximate solutions of (51) with $\beta = 0.001$, $\beta' = -1$, $y = 0$, $y = 2$, $y = 4$, and several values of α. Figure (2) shows surface plots of ARPSM solutions with $\beta = 0.001$, $\beta' = -1$, $\alpha = 1.00$, and different values of y. Figure (3) shows surface plots of ARPSM solutions with $\beta = 0.001$, $\beta' = -1$, $\alpha = 0.75$, and different values of y. Figure (4) shows surface plots of ARPSM solutions with $\beta = 0.001$, $\beta' = -1$, $\alpha = 0.50$, and different values of y. Figure (5) shows surface plots of ARPSM solutions with $\beta = 0.001$, $\beta' = -1$, $\alpha = 0.25$, and different values of y. The graphs and tables highlight the accuracy and applicability of ARPSM. In particular, the tables provide a comparison of the proposed method with existing techniques for different fractional orders, while the figures illustrate the similarity and symmetry observed in the graphical patterns of the three derivatives. From our results, we note that ARPSM yields approximate solutions that show excellent agreement with exact and numerical solutions, demonstrating its reliability. Regarding its systematic and simple procedure, the residual power series method does not require linearization, discretization, or perturbation techniques, making it straightforward to implement. For convergent series solutions, ARPSM generates rapidly convergent series, which ensures stable and accurate approximations. In terms of flexibility, ARPSM can be applied to both linear and nonlinear fractional differential equations with various fractional orders. Concerning computational cost, compared with many classical numerical methods, the presented approach requires fewer computations to achieve a comparable level of accuracy. Finally, regarding the capability to handle fractional operators, ARPSM effectively incorporates fractional derivatives and integral operators, making it suitable for modern fractional models. Moreover, the obtained series solutions allow the accuracy to be adjusted by considering more terms, providing a balance between efficiency and precision.

Table 1. ARPSM absolute errors with other techniques in solving (51) at $\alpha = 1$, $\beta = 0.001$, $\beta' = -1$, $x = 10$ and some values of y

y	AE (ARPSM)	AE (NTDM) [15]	AE (RPSM) [32]
0.1	$1.41551E^{-15}$	$1.41553E^{-15}$	$1.41553E^{-15}$
0.2	$4.68058E^{-14}$	$4.68063E^{-14}$	$4.68063E^{-14}$
0.3	$3.63906E^{-13}$	$3.63910E^{-13}$	$3.63910E^{-13}$
0.4	$1.56880E^{-12}$	$1.56886E^{-12}$	$1.56886E^{-12}$
0.5	$4.89612E^{-12}$	$4.89617E^{-12}$	$4.89617E^{-12}$
0.6	$1.24531E^{-11}$	$1.24542E^{-11}$	$1.24542E^{-11}$
0.7	$2.75063E^{-11}$	$2.75069E^{-11}$	$2.75069E^{-11}$
0.8	$5.47822E^{-11}$	$5.47829E^{-11}$	$5.47829E^{-11}$
0.9	$1.00801E^{-10}$	$1.00810E^{-10}$	$1.00810E^{-10}$
1.0	$1.74272E^{-10}$	$1.74280E^{-10}$	$1.74280E^{-10}$

Table 2. ARPSM solutions of (51) at $\beta = 0.001$, $\beta' = -1$, $\alpha = 0.25$ and $\alpha = 0.5$ with some values of x and y

		$\alpha = 0.25$			$\alpha = 0.50$		
y	x	ARPSM	NTDM [15]	HAM [29]	ARPSM	NTDM [15]	HAM [29]
20	0.2	$9.2994E^{-4}$	$9.2996E^{-4}$	$9.299E^{-4}$	$9.2992E^{-4}$	$9.2996E^{-4}$	$9.299E^{-4}$
	0.4	$9.2993E^{-4}$	$9.2996E^{-4}$	$9.299E^{-4}$	$9.2992E^{-4}$	$9.2996E^{-4}$	$9.299E^{-4}$
	0.6	$9.2995E^{-4}$	$9.2996E^{-4}$	$9.299E^{-4}$	$9.2992E^{-4}$	$9.2996E^{-4}$	$9.299E^{-4}$
	0.8	$9.2992E^{-4}$	$9.2996E^{-4}$	$9.299E^{-4}$	$9.2992E^{-4}$	$9.2996E^{-4}$	$9.299E^{-4}$
	1.0	$9.2993E^{-4}$	$9.2996E^{-4}$	$9.299E^{-4}$	$9.2992E^{-4}$	$9.2996E^{-4}$	$9.299E^{-4}$
10	0.2	$9.4392E^{-4}$	$9.4396E^{-4}$	$9.439E^{-4}$	$9.4391E^{-4}$	$9.4396E^{-4}$	$9.439E^{-4}$
	0.4	$9.4391E^{-4}$	$9.4396E^{-4}$	$9.439E^{-4}$	$9.4391E^{-4}$	$9.4396E^{-4}$	$9.439E^{-4}$
	0.6	$9.4393E^{-4}$	$9.4396E^{-4}$	$9.439E^{-4}$	$9.4391E^{-4}$	$9.4396E^{-4}$	$9.439E^{-4}$
	0.8	$9.4392E^{-4}$	$9.4396E^{-4}$	$9.439E^{-4}$	$9.4391E^{-4}$	$9.4396E^{-4}$	$9.439E^{-4}$
	1.0	$9.4394E^{-4}$	$9.4396E^{-4}$	$9.439E^{-4}$	$9.4391E^{-4}$	$9.4396E^{-4}$	$9.439E^{-4}$
0	0.2	$9.4863E^{-4}$	$9.4868E^{-4}$	$9.486E^{-4}$	$9.4862E^{-4}$	$9.4868E^{-4}$	$9.486E^{-4}$
	0.4	$9.4865E^{-4}$	$9.4868E^{-4}$	$9.486E^{-4}$	$9.4862E^{-4}$	$9.4868E^{-4}$	$9.486E^{-4}$
	0.6	$9.4866E^{-4}$	$9.4868E^{-4}$	$9.486E^{-4}$	$9.4862E^{-4}$	$9.4868E^{-4}$	$9.486E^{-4}$
	0.8	$9.4866E^{-4}$	$9.4868E^{-4}$	$9.486E^{-4}$	$9.4862E^{-4}$	$9.4868E^{-4}$	$9.486E^{-4}$
	1.0	$9.4867E^{-4}$	$9.4868E^{-4}$	$9.486E^{-4}$	$9.4862E^{-4}$	$9.4868E^{-4}$	$9.486E^{-4}$
−10	0.2	$9.4392E^{-4}$	$9.4396E^{-4}$	$9.439E^{-4}$	$9.4391E^{-4}$	$9.4396E^{-4}$	$9.439E^{-4}$
	0.4	$9.4393E^{-4}$	$9.4396E^{-4}$	$9.439E^{-4}$	$9.4391E^{-4}$	$9.4396E^{-4}$	$9.439E^{-4}$
	0.6	$9.4861E^{-4}$	$9.4868E^{-4}$	$9.486E^{-4}$	$9.4862E^{-4}$	$9.4868E^{-4}$	$9.486E^{-4}$
	0.8	$9.4394E^{-4}$	$9.4396E^{-4}$	$9.439E^{-4}$	$9.4391E^{-4}$	$9.4396E^{-4}$	$9.439E^{-4}$
	1.0	$9.4393E^{-4}$	$9.4396E^{-4}$	$9.439E^{-4}$	$9.4391E^{-4}$	$9.4396E^{-4}$	$9.439E^{-4}$
−20	0.2	$9.2992E^{-4}$	$9.2996E^{-4}$	$9.299E^{-4}$	$9.2991E^{-4}$	$9.2996E^{-4}$	$9.299E^{-4}$
	0.4	$9.2991E^{-4}$	$9.2996E^{-4}$	$9.299E^{-4}$	$9.2991E^{-4}$	$9.2996E^{-4}$	$9.299E^{-4}$
	0.6	$9.2994E^{-4}$	$9.2996E^{-4}$	$9.299E^{-4}$	$9.2991E^{-4}$	$9.2996E^{-4}$	$9.299E^{-4}$
	0.8	$9.2993E^{-4}$	$9.2996E^{-4}$	$9.299E^{-4}$	$9.2991E^{-4}$	$9.2996E^{-4}$	$9.299E^{-4}$
	1.0	$9.2995E^{-4}$	$9.2996E^{-4}$	$9.299E^{-4}$	$9.2991E^{-4}$	$9.2996E^{-4}$	$9.299E^{-4}$

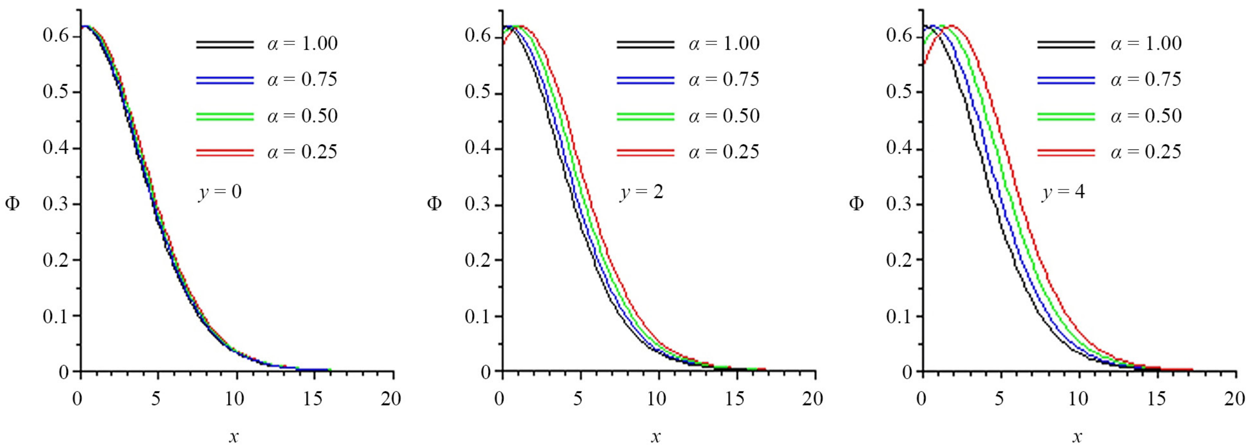

Figure 1. ARPSM solutions of (51) with $\beta = 0.001$, $\beta' = -1$, $y = 0$, $y = 2$, $y = 4$ and some values of α

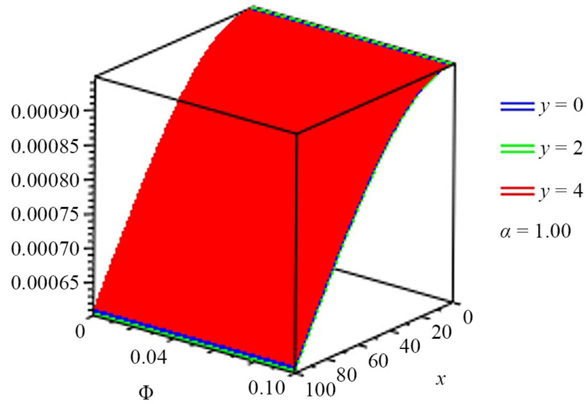

Figure 2. Some plots of ARPSM solutions with $\beta = 0.001$, $\beta' = -1$, $\alpha = 1.00$ and some values of y

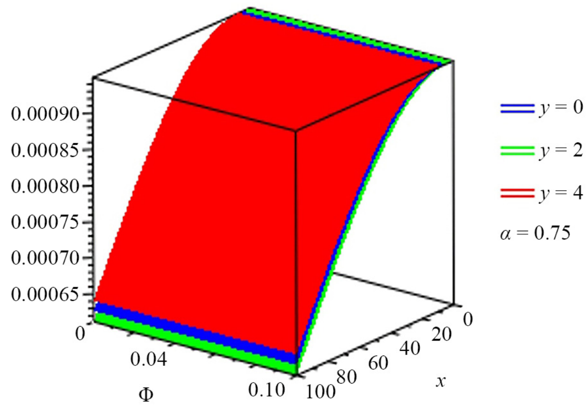

Figure 3. Some plots of ARPSM solutions with $\beta = 0.001$, $\beta' = -1$, $\alpha = 0.75$ and values of y

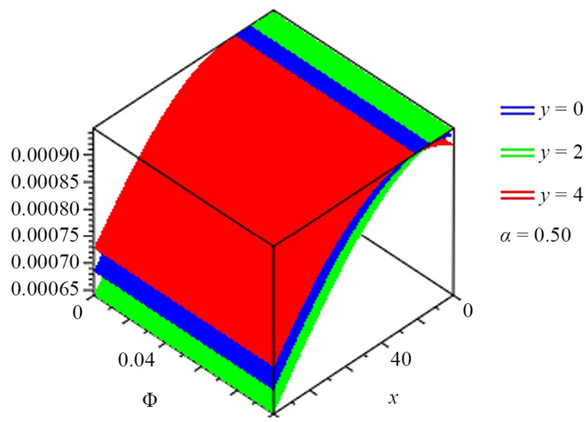

Figure 4. Some plots of ARPSM solutions with $\beta = 0.001$, $\beta' = -1$, $\alpha = 0.50$ and values of y

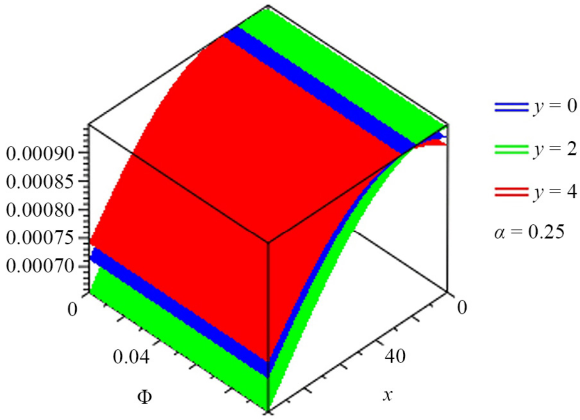

Figure 5. Some plots of ARPSM solutions with $\beta = 0.001$, $\beta' = -1$, $\alpha = 0.25$ and values of y

6. Conclusions

The limitations of this study are primarily linked to its scope, underlying assumptions, and chosen methodology. Our focus was restricted to solving fractional partial differential equations, particularly the MTCFKE, by employing the residual power series method in conjunction with the Aboodh transform. While the combination of these techniques has demonstrated effectiveness in deriving both approximate and exact solutions of time-modified fractional Kawahara equations, the results are constrained by the specific class of equations considered and the methodological framework adopted. The ARPSM generated approximate solutions in the form of a convergent series, which showed strong agreement with numerical simulations. Owing to its systematic and efficient structure, the method provided reliable approximations that were validated through comprehensive comparisons presented in the tables and figures. These analyses confirmed the accuracy and robustness of the approach. Moreover, the results highlighted the suitability of ARPSM for solving problems in mathematical physics, biological models, and related scientific fields. An additional contribution of this study is the demonstration of ARPSM as a valuable tool for future investigations of water-wave equations, as well as for advancing research in fractional calculus and fractional differential equations.

Conflict of interest

The authors declare no competing financial interest.

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