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Research Article

Zener Viscoelastic Model Characterization Using the Atangana-Baleanu Fractional Derivative

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Abstract: The continuous search for new mathematical models that accurately represent real-world phenomena is a constant goal in the scientific community. To achieve this objective, the complexity of a large number of mathematical models has increased, such that certain considerations or restrictions and the use of numerical methods with greater computational requirements are required for their solution. For this, theories such as fractional calculus have been used, which have demonstrated an adequate characterization of physical phenomena, mainly in the area of biomechanics. However, there is no unique definition of the derivative concept, as in the case of integers, because it is a "new theory". In this article, the kernel of the Atangana-Baleanu fractional derivative is used, which satisfies the most common properties of the classical derivative, and addresses the existing problem when considering initial conditions of the model, which in biomechanics is associated with material memory.

***Keywords*:** Atangana-Baleanu, biomechanics, fractional calculus, viscoelasticity

MSC: 26A33, 92C10, 74L15, 33E12

1. Introduction

The fractional calculus theory has been shown to best describe physical behavior in various engineering topics and medical applications [1–4]. For mechanical characterization of viscoelastic materials [5–8], prediction of COVID transmission [9–11], geophysics [12, 13], and electrical circuits, the fractional element is called the memristor [14–17]. In [18], Richard Magin initiates a new set of applications in the field of biomechanics and mechanobiology, especially in modeling the mechanical behavior of soft tissues [19–22], since they cannot be characterized as solid (Hookean), but they can also not be treated as fluids using Newton's laws. Soft tissues behave in a way that "oscillates" between the Hookean and Newtonian material models. This phenomenon represents an analogy with the fractal dimension concept that arises when a numeric set does not fill the k-dimension but is larger than the $(k-1)$-dimension, in the sense of the Hausdorff dimension definition. For example, an artery is not really a solid or a fluid material, since it is essentially a combination of several biological materials and has a mechanical behavior that is a mixture of the classic materials models.

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On the other hand, since fractional calculus is an unfinished or developing theory, there are several observations about it. Some of these observations are related to the adaptation of models that involve derivatives of integer order to fractional derivatives without initially being given a physical interpretation that precedes this adaptation [23–25]. In addition, a special emphasis has been placed on the inability of the fractional calculus theory to recreate the memory effect that certain physical phenomena possess, as in the particular case of viscoelastic materials. Finally, there are some mathematical limitations of the theory itself, such as the series of restrictions that exist in some definitions of fractional derivatives and the non-fulfillment of basic properties that operators must satisfy; for example, the Caputo fractional integral does not recover the initial conditions [26, 27].

However, it must be taken into account that the development of mathematical models has been built on simplifying hypotheses, and these allow us to obtain useful representations of the real-world phenomenon under certain control conditions or considerations. Therefore, to obtain a model that reliably describes the physical phenomenon, it is proposed to apply the Atangana-Baleanu fractional derivative definition, which has been used to analyze the mechanical behavior of viscoelastic materials by applying different classical material models such as Zener, Burges, Kelvin, and other variants [28, 29] as well as in biomechanics, and especially in the mechanical behavior of soft tissues [30–32]. It is important to highlight that the Atangana-Baleanu definition satisfies the basic properties of the differential operators [33, 34].

Therefore, this paper compares the most promising fractional derivative definitions to determine whether the Atangana-Baleanu definition reproduces the memory effect present in viscoelastic materials. This will allow for better mechanical characterization of arteries and soft tissues in general, contributing to the development of tools that will help healthcare professionals determine the appropriate treatments or interventions.

2. Materials and methods

In this section, the two classical definitions of the fractional derivative, the Riemann-Liouville and Caputo-Liouville definitions, are first described, and then extended to the Atangana-Baleanu form. Finally, the viscoelastic fractional model that will be used to determine whether the Atangana-Baleanu definition adequately accounts for the nonzero initial conditions of the problem is defined.

2.1 *Fractional derivate background*

The local differential operator for the n-order derivative for an independent variable t, D_t^n is the inverse left operator of the non-local integral operator of the integral field aI_t^n where $a < t$. In fact, for any given continuous function with continuous derivative, i.e., $f(t) \in C^2$

$$D_t^n \circ_a I_t^n f(t) = f(t),\ t > a,$$

and

$${}_aI_t^n \circ D_t^n f(t) = f(t) - \sum_{k=0}^{\alpha-1} f^k(a^+) \frac{(t-a)^k}{k!},\ t > a.$$

where $f^{(k)}(a^+)$ is the k-order derivative of the function $f(t)$ evaluated to the right. As a consequence of this, if $a = 0$ it is required that the operator ${}_0D_t^n$ be defined as the inverse for the left operator ${}_0I_t^n$. For this purpose, the integer number m is introduced as:

$$m \in \mathbb{N},\ m-1 < \alpha < m$$

That is, the *Riemann-Liouville* definition of $\alpha > 0$:

$$ {}_0D_t^{\alpha} f(t) = D_t^m \circ {}_0I_t^{m-\alpha} f(t), \text{ where, } m-1 < \alpha < m, $$

then:

$$ {}_0D_t^{\alpha} f(t) = \frac{1}{\Gamma(m-\alpha)} \frac{\mathrm{d}^m}{\mathrm{d}t^m} \int_0^t \frac{f(\tau)\mathrm{d}\tau}{(t-\tau)^{\alpha+1-m}}, \tag{1} $$

when $m-1 < \alpha < m$ and $\frac{\mathrm{d}^m}{\mathrm{d}t^m}$, if $\alpha = m$.

To complement the derivative definition, it is necessary that ${}_0\mathrm{D}_t^0 = \mathrm{I}$. Where $\Gamma(x)$ is the Gamma Function that is used to determine the value of $x!$ when it is not a positive integer.

Changing the order in equation (1), given the *Caputo-Liouville* definition [35], of order $\alpha > 0$ defined as:

$$ {}_0^*\mathrm{D}_t^{\alpha} f(t) = {}_0\mathrm{I}_t^{m-\alpha} \circ \mathrm{D}_t^m \; f(t), \text{ con, } m-1 < \alpha < m, \tag{2} $$

then:

$$ {}_0^*\mathrm{D}_t^{\alpha} f(t) = \frac{1}{\Gamma(m-\alpha)} \int\limits_0^t \frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha-m+1}} \mathrm{d}\tau, \tag{3} $$

when $m-1 < \alpha < m$ and $\frac{\mathrm{d}^m}{\mathrm{d}t^m} f(t)$, if $\alpha = m$.

The Atangana-Baleanu fractional left derivative in the Caputo sense is redefined by the introduction of a normalization function $M(\alpha)$ [36, p. 1];

$$ {}^{ABC}{}_aD^{\alpha} f(t) = \frac{M(\alpha)}{1-\alpha} \int_a^t f'(s) E_{\alpha} \left(\frac{-\alpha}{1-\alpha} (t-s)^{\alpha} \right) \mathrm{d}s, \tag{4} $$

And in the same way, the left derivative in the Riemann-Lioville sense is given by:

$$ {}^{ABR}{}_aD^{\alpha} f(t) = \frac{M(\alpha)}{1-\alpha} \frac{\mathrm{d}}{\mathrm{d}t} \int_a^t f(s) E_{\alpha} \left(\frac{-\alpha}{1-\alpha} (t-s)^{\alpha} \right) \mathrm{d}s, \tag{5} $$

where $M(\alpha) > 0$ is a normalization function with $M(0) = M(1) = 1$, $\alpha \in [0,\ 1]$ and $E_{\alpha}(t)$ is the Mittag-Leffer function.

The Laplace transform will be used to solve the fractional Atangana-Baleanu model; for this reason, it is necessary to define the transform for the derivative operator in equations (4) and (5). This can be expressed as [37]:

$$ \mathscr{L}\left\{ {}^{ABC}{}_aD^{\alpha} f(t);\ s \right\} = \frac{M(\alpha)}{1-\alpha} \frac{s^{\alpha} F(s) - s^{\alpha-1} f(a)}{s^{\alpha} + \dfrac{\alpha}{1-\alpha}}, \tag{6} $$

and

$$\mathscr{L}\left\{{}^{ABR}{}_{a}D^{\alpha}f(t);\ s\right\} = \frac{M(\alpha)}{1-\alpha}\frac{s^{\alpha}F(s)}{s^{\alpha}+\dfrac{\alpha}{1-\alpha}} \tag{7}$$

From here, it is observed that the fractional derivative of Riemann-Liouville in the sense of Atangana-Baleanu definition does not consider initial conditions properly.

2.2 *Fractional viscoelastic models*

In the next step, the fractional generalization of the Standard Linear Solid (SLS), also known as the Zener model is considered, since it provides a better representation of the stress curve, an extremely accurate description of stress relaxation without the need to use more complex models and allows obtaining numerical solutions [38–42]. This model is shown in Figure 1. In order to obtain the fractional viscoelastic model, it is necessary to replace the first-order derivative with the fractional derivative of order $\alpha \in (0,\ 1)$ in the constitutive equations.

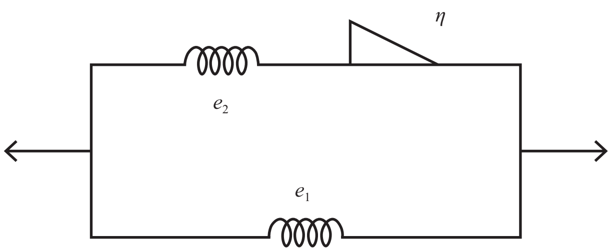

Figure 1. Zener Fractional Model (FZM), where the dash pot (first derivative order) was replacement for a fractional element known as spring-pot (fractional derivative order α)

The integer model for the Zener fractional model is [43];

$$\dot{\sigma}(t)+\frac{e_2}{\eta}\sigma(t) = (e_1+e_2)\,\dot{\varepsilon}(t)+\frac{e_1e_2}{\eta}\varepsilon(t) \tag{8}$$

where $\varepsilon(t)$ is the strain tensor, $\sigma(t)$ is the stress tensor, $e_1,\ e_2$ are spring constants and η its the dash pot viscosity constant.

Viscoelastic materials exhibit behavior that combines the elastic behavior found in rigid materials, such as silver, steel, and concrete, with the behavior of materials considered fluids, such as oil, water, blood, and others. However, materials such as those that make up arteries "oscillate" between elastic solids and fluids, so this effect is simulated by implementing the fractional derivative. Replacing the integer derivative associated with the dashpot, for the Atangana-Baleanu fractional derivative related to the fractional element called springpot, where the order of the derivative $\alpha \in (0, 1)$, in equation (8)

$${}^{ABC,\ R}{}_{a}D^{\alpha}\sigma(t)+\frac{e_2}{\eta}\sigma(t) = (e_1+e_2)\ {}^{ABC,\ R}{}_{a}D^{\alpha}\varepsilon(t)+\frac{e_1e_2}{\eta}\varepsilon(t) \tag{9}$$

the FZM is obtained in the Atangana-Baleanu sense.

2.3 *Aplication*

On the other hand, the mechanical behavior characterization of the soft tissues requires determining the mechanical parameters related to this are carried out through harmonic excitation tests [5], using the relationship between stress and strain through the stress relaxation modulus and creep, which is expressed in the form:

$$\sigma(t)=\int_{-\infty}^{t} G(t-\tau)\,{}^{ABC}{}_{a}D_{\tau}^{\alpha}\varepsilon(\tau)\,\mathrm{d}\tau \tag{10}$$

and

$$\varepsilon(t)=\int_{-\infty}^{t} J(t-\tau)\,{}^{ABC}{}_{a}D_{\tau}^{\alpha}\sigma(\tau)\,\mathrm{d}\tau \tag{11}$$

Therefore, it is also necessary to conclude the biomechanical analysis of the proposed model to determine whether the Atangana-Baleanu definition formally complies with the characterization of soft tissue biomechanics or if it is suitable to use some other theory, such as that proposed in [26, 44], since these definitions have the form of convolutions.

3. Results

First, the Atangana-Baleanu is replaced in the sense of Caputo fractional derivative in equation (9);

$${}^{ABC}{}_{a}D^{\alpha}\sigma(t)+\frac{e_2}{\eta}\sigma(t)=(e_1+e_2)\,{}^{ABC}{}_{a}D^{\alpha}\varepsilon(t)+\frac{e_1e_2}{\eta}\varepsilon(t) \tag{12}$$

Applying the Laplace transform to equation (6);

$$\frac{M(\alpha)}{1-\alpha}\frac{s^{\alpha}\bar{\sigma}(s)-s^{\alpha-1}\sigma(a_1)}{s^{\alpha}+\dfrac{\alpha}{1-\alpha}}+\frac{e_2}{\eta}\bar{\sigma}(s)=(e_1+e_2)\frac{M(\alpha)}{1-\alpha}\frac{s^{\alpha}\bar{\varepsilon}(s)-s^{\alpha-1}\varepsilon(a_2)}{s^{\alpha}+\dfrac{\alpha}{1-\alpha}}+\frac{e_1e_2}{\eta}\bar{\varepsilon}(s) \tag{13}$$

Expanding the equation, and solving for $\varepsilon(s)$

$$\bar{\sigma}(s)\left[\frac{\dfrac{M(\alpha)}{1-\alpha}s^{\alpha}+\dfrac{e_2}{\eta}s^{\alpha}+\dfrac{e_2\alpha}{\eta(1-\alpha)}}{s^{\alpha}+\dfrac{\alpha}{1-\alpha}}\right]+\zeta_1\frac{s^{\alpha-1}}{s^{\alpha}+\dfrac{\alpha}{1-\alpha}}=\bar{\varepsilon}(s)\left[\frac{(e_1+e_2)\dfrac{M(\alpha)}{1-\alpha}s^{\alpha}+\dfrac{e_1e_2}{\eta}s^{\alpha}+\dfrac{e_1e_2\alpha}{\eta(1-\alpha)}}{s^{\alpha}+\dfrac{\alpha}{1-\alpha}}\right] \tag{14}$$

then

$$\bar{\varepsilon}(s)=\left(\zeta_2\frac{s^{\alpha}}{s^{\alpha}+\psi}+\zeta_3\frac{1}{s^{\alpha}+\psi}\right)\bar{\sigma}(s)+\zeta_4\frac{s^{\alpha-1}}{s^{\alpha}+\dfrac{\alpha}{1-\alpha}} \tag{15}$$

using the inverse Laplace transform [45];

$$\mathscr{L}^{-1}\left\{\frac{s^{\alpha-\phi}}{s^{\alpha}+\omega};\ t\right\}=t^{\phi-1}E_{\alpha,\ \phi}(-\omega t^{\alpha}) \tag{16}$$

where $E_{\alpha,\ \phi}$ is the two variables Mittag-Leffer function [46, p.17]. Rewriting the equation (15) have;

$$\begin{aligned}\varepsilon(t)=&\left(\mathscr{L}^{-1}\{\zeta_2 s;\ t\}\cdot\mathscr{L}^{-1}\left\{\frac{s^{\alpha-1}}{s^{\alpha}+\psi};\ t\right\}+\mathscr{L}^{-1}\left\{\frac{\zeta_3\Gamma(\alpha-1)}{s^{\alpha-1}};\ t\right\}\cdot\mathscr{L}^{-1}\left\{\frac{s^{\alpha-1}}{s^{\alpha}+\psi};\ t\right\}\right)\sigma(t)\\&+\zeta_4\mathscr{L}^{-1}\left\{\frac{s^{\alpha-1}}{s^{\alpha}+\dfrac{\alpha}{1-\alpha}}\right\}\end{aligned} \tag{17}$$

then, using the Laplace convolution definition

$$\begin{aligned}\varepsilon(t)=&\left(\zeta_2\int_0^t\delta'(t-\tau)E_{\alpha}(-\phi\tau^{\alpha})\mathrm{d}\tau+\frac{\zeta_3}{\Gamma(\alpha-1)}\int_0^t(t-\tau)^{\alpha-2}E_{\alpha}(-\phi\tau^{\alpha})\mathrm{d}\tau\right)*\sigma(t)\\&+\zeta_4E_{\alpha,\ 1}\left(-\frac{\alpha}{1-\alpha}t^{\alpha}\right)\end{aligned} \tag{18}$$

an analytical solution of the model is obtained,

$$\varepsilon(t)=\left(\zeta_2\delta(t)+\frac{\zeta_2}{t}\sum_{n=1}^{\infty}\frac{(-\phi t^{\alpha})^n}{\Gamma(n\alpha)}+\zeta_3\sum_{n=0}^{\infty}\frac{(-\phi t^{\alpha})^n}{\Gamma(n\alpha+\alpha)}\right)*\delta(t)+\zeta_4\sum_{n=0}^{\infty}\frac{\left(\dfrac{\alpha}{\alpha-1}\right)}{\Gamma(\alpha n+1)} \tag{19}$$

where $\delta(t)$ is the Dirac delta function, and $*$ indicates the convolution.

In an analogous way, the solution can be obtained for the case of the Riemann-Liouville derivative in the sense of Atangana-Baleanu. But, as we mentioned before, this definition does not consider non-zero initial conditions.

$$\varepsilon(t)=\left(\zeta_2\delta(t)+\frac{\zeta_2}{t}\sum_{n=1}^{\infty}\frac{(-\phi t^{\alpha})^n}{\Gamma(n\alpha)}+\zeta_3\sum_{n=0}^{\infty}\frac{(-\phi t^{\alpha})^n}{\Gamma(n\alpha+\alpha)}\right)*\delta(t) \tag{20}$$

Once the solutions are obtained, a convergence analysis of the solution is carried out, considering an initial disturbance in the form of a unit step applied in a time interval of t from 0 to 5 seconds, in order to obtain a description of the behavior of temporary responses. Plotting the temporal response in Wolfram Mathematica V12, using values of the constants and the fractional order obtained in previous investigations [47], $e_1=0.68,\ e_2=0.39,\ \eta=2.14$, $M(\alpha)=\mathscr{U}(t)+\delta(t-1)$ and $\alpha=0.408$, Figure 2.

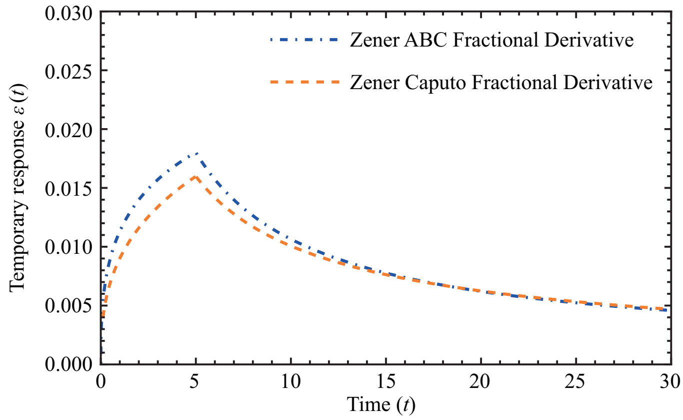

Figure 2. In this figure, is shown the classical behavior of the viscoelastic material

First, the classic behavior expected of a viscoelastic material is observed, such as the soft tissue from which the data were obtained, which corresponds to a segment of artery. For which there is a period of relaxation of prolonged efforts, which is to be expected.

The response of the Atangana-Baleanu model is obtained below for different values of α. In cases where α is greater than 0.4, material behavior similar to the integer-order model is observed, which was expected. Otherwise, the response effect expected by the viscoelastic model begins to disappear when it is in the elastic zone. It is worth highlighting the effect of the value of α, which seems to amplify the response of $\varepsilon(t)$, so it would be necessary to adjust the values of the model constants to fit the experimental curve. Figure 3.

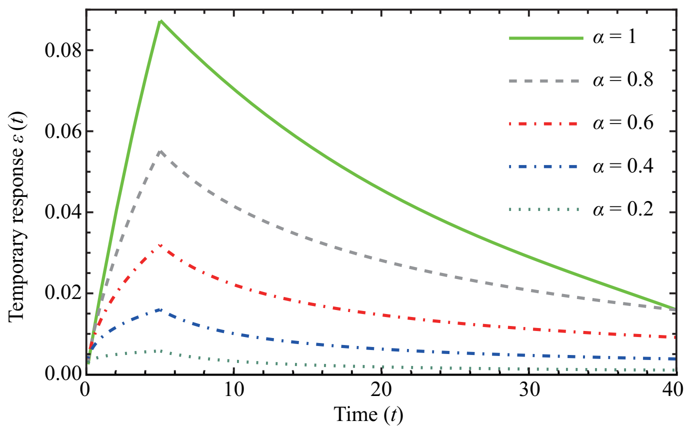

Figure 3. Here is shown the behavior for several values of α using the Atangana-Baleanu model

By varying the initial conditions in the solution, which are within the constant ζ_4, it is observed that the Atangana-Baleanu definition of fractional derivative adapts positively to non-zero initial conditions, as shown in Figure 4. However, the number of terms that must be considered in the transitory part of the solution is considerable. As long as the initial conditions are farther from zero, the number of terms in the summation that represent the transitory solution needs to be increased.

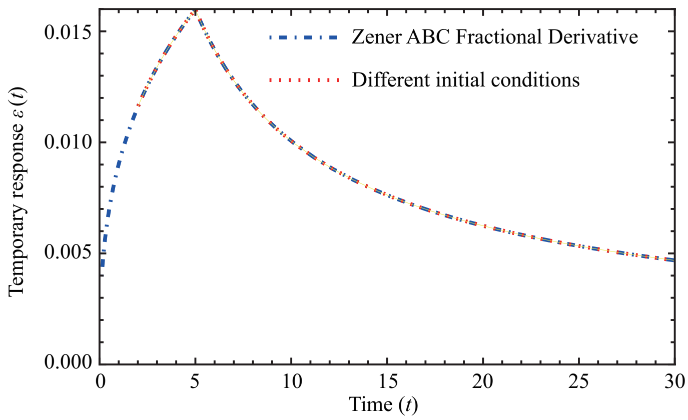

Figure 4. This graph shows that the Atangana-Baleanu derivative preserves the change in the initial conditions

In Figure 5, it can be seen how the behavior of the solution is preserved for a set of initial values other than zero, so it is concluded that the Atangana-Baleanu definition adequately considers the initial conditions in the case of generalization of the Caputo derivative form by the left.

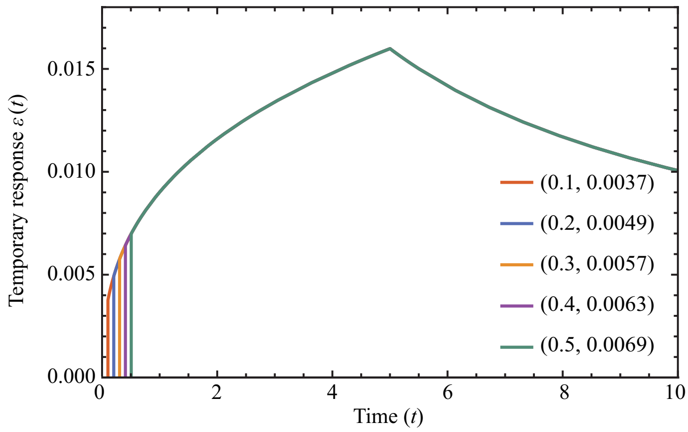

Figure 5. In this figure is shown that the Atangana-Baleanu derivative preserve the initials conditions difference

Once it has been proven that the Atangana-Baleanu fractional model adequately describes the mechanical behavior of the soft tissue. The equations (19) and (20) obtained are implemented using a geometric representation of an artery segment obtained through medical images, which were obtained using the Hounsfield scale in order to adequately identify the three layers of the artery [47]. Mechanical coefficients were obtained through oscillatory experiments [48], so these are introduced to the finite element software using the relaxation function and creep compliance. These values can be obtained using the complex module equation (21).

$$\Phi^*(s) = \frac{\bar{\sigma}^*(s)}{\bar{\varepsilon}^*(s)} \tag{21}$$

The complex modulus graph describes material behavior under the influence of an oscillatory load at different frequencies, Figure 6. The phase angle indicates the lag between the elastic and viscous responses, with values close to zero indicating elastic behavior; And on the other hand, values close to 90 degrees indicating fluid-like behavior.

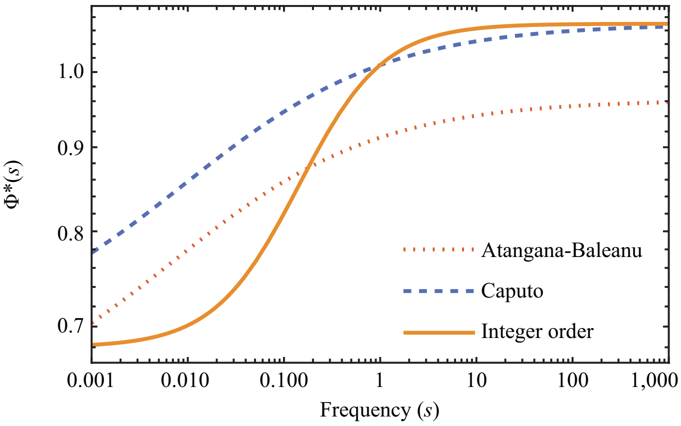

Figure 6. In this figure is shown the complex modulus for the integer case, Caputo and Atangana-Baleanu

Figure 7 shows the solution obtained once the complex modulus values are implemented in the finite element software. It shows small areas along the artery edge where high stress values are present. This is due to only considering a segment of the artery, which implies that its edges must be considered fixed elements, such that no energy dissipation occurs. In addition, the aforementioned effect is combined with the geometric area where a pronounced deformation occurs, so the values end up increasing. This generates a discrepancy with the results of previous investigations, where the results oscillate between 0.2 MPa and 0.4 MPa. However, this only occurs in the areas mentioned above; the rest of the artery shows values similar to those obtained in previous investigations.

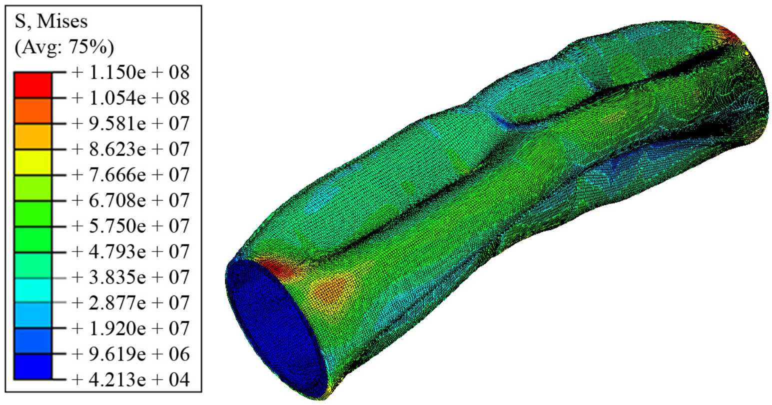

Figure 7. This figure shows that although the Atangana-Baleanu fractional model presents a deformation pattern similar to that generated using the fractional derivative of Caputo, it presents a set of very high values at the boundaries

Finally, a convergence analysis was carried out using various finite elements and different mesh refinements. No significant differences were found.

4. Conclusions

The definition of the Caputo fractional derivative in Atangana-Baleanu terms was shown to satisfy the shape of the stress-strain curve, showing its classical behavior. In this case, it is specifically considered that other values of the constants can cause the transient part of the solution to grow abruptly, dominating the behavior of the solution. In addition to showing adequate behavior, this definition satisfies the basic properties of the integer order differential operator, and, as can be seen, it does consider non-zero initial conditions and preserves the form of the solution. However, the number of terms in the transient part that must be considered is large and increases as we move away from the origin.

Future work should consider various models of viscoelastic behavior, as well as different fractional derivative models. Another approach is to incorporate subroutines into the finite element software that allow for adequate incorporation of the fractional model.

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Conflict of interest

The authors declare no competing financial interest.

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