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Research Article

Pseudo-Spectral Method for Finite and Infinite Time Synchronization Problems

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Abstract: This paper presents a unified pseudo-spectral framework for solving both finite-time and infinite-time synchronization problems in nonlinear dynamical systems. The core of our approach is the formulation of two novel optimal control problems, where the achievement of synchronization is established as the optimal solution. We rigorously prove that the solution to these optimal control problems guarantees synchronization. The proposed method leverages a Legendre pseudo-spectral technique to transcribe the continuous-time optimal control problems into nonlinear programming problems. The unknown coefficients of the approximating polynomials are then efficiently determined by solving the resulting algebraic system derived from the Karush-Kuhn-Tucker optimality conditions. The efficacy and superiority of the method are demonstrated through several numerical examples, where it is compared against other well-known synchronization techniques. The results confirm that our method offers significant advantages in accuracy and convergence. Finally, we discuss the potential for extending this approach to a broader class of complex synchronization problems.

Keywords: synchronization problems, lagrange interpolating polynomial, non-linear programming problems, pseudo-spectral method

MSC: 49M37, 90C30, 93C10, 34D06

1. Introduction

Synchronization phenomena represent one of the most fundamental and pervasive behaviors in natural and engineered systems, with applications spanning from biological rhythms to power grid stability and secure communications. The systematic study of synchronization dates back to Huygens' seminal observations of coupled pendulum clocks [1], while modern research has been profoundly influenced by foundational works [2, 3] on oscillator populations. A pivotal advancement came from Pecora et al. [4], who demonstrated that chaotic systems could synchronize, opening new avenues for research and applications.

The landscape of synchronization methodologies has evolved considerably, with several distinct approaches emerging to address different challenges. Generalized synchronization frameworks, pioneered by Pecora et al. [5] and Abarbane et al. [6], extended synchronization concepts to systems with functional relationships between their states.

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Lyapunov-Based (LB) methods, particularly the finite-time stability theory developed by Bhat et al. [7], provided rigorous mathematical foundations for synchronization analysis and controller design. More recently, intelligent control approaches have gained prominence such as the Output-Feedback Controller Based Projective Lag-Synchronization (OFCBPLS) [8], Adaptive Fuzzy Control (AFC) [9] for practical fixed-time synchronization, Practical Finite-Time Fuzzy Synchronization (PFTFS) [10], and an Intelligent Fuzzy Controller (IFC) [11] for chaos synchronization.

The advent of network science revolutionized synchronization research, with comprehensive reviews by Boccaletti et al. [12] and Arenas et al. [13] establishing fundamental relationships between network topology and collective dynamics. This framework has enabled critical applications in: power systems [14–16], biological rhythms [17] and secure communications [18]. Recent advances have extended to time-varying networks [19], multi-layer systems, and sophisticated analysis using topological data analysis [20].

The complexity of synchronization problems in nonlinear systems has driven the development of advanced computational techniques. Among these, Pseudo-Spectral (PS) methods have emerged as particularly powerful tools due to their exponential convergence properties for smooth problems [21, 22]. The theoretical foundation for PS methods in optimal control was established through the covector mapping theorem [23], with convergence properties rigorously analyzed by Kang et al. [24]. The versatility of PS methods is demonstrated by their successful application to increasingly challenging problems: fractional-order systems [25, 26], delay differential equations [27, 28], nonsmooth dynamical systems [29], and stabilization and adaptive control design [30].

Despite these advancements, a significant gap persists between high-precision computational methods and their direct application to synchronization problems. Existing control strategies often face limitations in: (i) The need for intricate Lyapunov function construction, (ii) Computational efficiency for high-accuracy requirements (iii) Unified treatment of finite-time and infinite-time synchronization. This paper bridges this gap by introducing a unified pseudo-spectral framework for both finite-time and infinite-time synchronization problems. Our principal contributions are

1. Theoretical reformulation: We propose novel optimal control problems for both Finite-Time Synchronization (FTS) and Infinite-Time Synchronization (ITS) problems and rigorously prove that their solutions guarantee synchronization objectives.

2. Computational framework: We develop a Legendre PS methodology using Legendre-Gauss-Radau (LGR) points for ITS problem and Legendre-Gauss-Lobatto (LGL) points for FTS problem, providing exponential convergence rates.

3. Implementation advantage: Our approach eliminates the need for complex Lyapunov function construction, instead solving the resulting algebraic systems via Karush-Kuhn-Tucker (KKT) conditions and the Levenberg-Marquardt algorithm (see Bazaraa et al. [31]).

4. Comprehensive validation: We demonstrate superior performance against five established methods including LB, IFC, AFC, PFTFS and OFCBPLS.

The remainder of this paper is organized as follows: Section 2 formally defines the FTS and ITS problems. Sections 3 and 4 detail the PS implementation for ITS and FTS problems respectively. Section 5 describes the solution of algebraic systems. Numerical examples and comparative analysis are presented in Section 6, with conclusions and suggestions in Section 7.

2. Presenting synchronization problems

Consider the following nonlinear control system

$$\begin{cases} \dot{\phi} = f\big(\omega,\ \phi(\omega),\ \theta(\omega),\ u(\omega)\big),\ \omega \in I, \\ \dot{\theta} = g\big(\omega,\ \theta(\omega),\ v(\omega)\big),\ \omega \in I, \\ \phi(0) = \alpha,\ \theta(0) = \beta. \end{cases} \tag{1}$$

In this system $u(.)$ and $v(.)$ are the control variables, $\phi(.)$ and $\theta(.)$ are the state variables where $\phi: I \to A \subseteq \mathbb{R}^n$, $\theta: I \to B \subseteq \mathbb{R}^n$, $u: I \to C \subseteq \mathbb{R}^m$ and $v: I \to D \subseteq \mathbb{R}^p$. Also $f: I \times A \times B \times \mathbb{R}^m \to \mathbb{R}^n$ and $g: I \times B \times \mathbb{R}^p \to \mathbb{R}^n$ are given functions. Here for all $\omega \geq 0$ we have $f(\omega) = (f_1(\omega),\ f_2(\omega),\ \ldots,\ f_n(\omega))$ and $g(\omega) = (g_1(\omega),\ g_2(\omega),\ \ldots,\ g_n(\omega))$ where $f_i: I \to \mathbb{R}$ and $g_i: I \to \mathbb{R}$ are given continuous functions. Moreover for any function $\chi: [0,\ +\infty) \to \mathbb{R}^n$ in $L_2(I)$, we use the following norms

$$\|\chi(\cdot)\|_2 = \sqrt{\int_I \|\chi(\omega)\|_2^2 d\omega},\ \ \|\chi(\omega)\|_2 = \sqrt{\sum_{i=1}^{n} |\chi_i(\omega)|^2},\ \omega \in I,$$

where $\chi_i(\omega): I \to \mathbb{R}$ and $\chi(\omega) = \big(\chi_1(\omega),\ \chi_2(\omega),\ \ldots,\ \chi_n(\omega)\big)$. We suppose that $\phi(\cdot) \in L_2(I)$ and $\theta(\cdot) \in L_2(I)$ which means $\|\phi(\cdot)\|_2 < \infty$ and $\|\theta(\cdot)\|_2 < \infty$. Moreover, $\phi(\cdot)$ and $\theta(\cdot)$ are differentiable almost everywhere on the interval I.

ITS problem: Find a control pair $\big(u(\cdot),\ v(\cdot)\big)$ for the system (1) over the infinite interval $I = [0, \infty)$ such that the corresponding state pair $\big(\phi(\cdot),\ \theta(\cdot)\big)$ satisfies $\lim_{\omega\to\infty}\|\phi(\omega) - \theta(\omega)\| = 0$ or there exists $\omega_1 > 0$ such that $\phi(\omega) = \theta(\omega)$ for all $\omega \geq \omega_1$.

FTS problem: Find a control pair $\big(u(\cdot),\ v(\cdot)\big)$ for the system (1) over the finite interval $I = [0,\ T]$ such that for the corresponding state pair $\big(\phi(\cdot),\ \theta(\cdot)\big)$ there exists $\omega_1 \in [0,\ T)$ such that $\phi(\omega) = \theta(\omega)$ for all $\omega \in [\omega_1,\ T]$.

3. Implementing method for ITS problem

Note that in the ITS problem, a smaller value of ω_1 indicates a faster convergence rate. We assume that the ITS problem admits a solution in the space $L_2(I)$. In this section, we first formulate an Infinite-Time Optimal Control (ITOC) problem as a means to solve the ITS problem. We then introduce a PS method to approximate the solution, which leads to an Nonlinear Programming (NLP) formulation. Furthermore, we apply the KKT optimality conditions to solve the resulting NLP problem and thereby obtain a solution to the original ITS problem.

3.1 *Proposing an ITOC problem*

Here, an ITOC problem is proposed to solve ITS problem as follows

$$\text{Minimize} \quad J(\phi,\ \theta) = \int_0^{\infty} \|\phi(\omega) - \theta(\omega)\|_2^2\, d\omega \tag{2}$$

$$\text{subject to} \begin{cases} \dot{\phi} = f\big(\omega,\ \phi(\omega),\ \theta(\omega),\ u(\omega)\big)\ , \omega \geq 0, \\ \\ \dot{\theta} = g\big(\omega,\ \theta(\omega),\ v(\omega)\big)\ , \omega \geq 0, \\ \\ \phi(0) = \alpha,\ \theta(0) = \beta. \end{cases} \tag{3}$$

Theorem 3.1 If $\big(\phi^*(\cdot),\ \theta^*(\cdot),\ u^*(\cdot),\ v^*(\cdot)\big)$ is an optimal solution of ITOC problem (2)-(3) then $\lim_{\omega\to\infty}\phi^*(\omega) = \theta^*(\omega)$ which means $\big(\phi^*(\cdot),\ \theta^*(\cdot),\ u^*(\cdot),\ v^*(\cdot)\big)$ is a solution for ITS problem.

Proof. Assume that $\big(\bar{\phi}(\cdot),\ \bar{\theta}(\cdot),\ \bar{u}(\cdot),\ \bar{v}(\cdot)\big)$ is a solution of ITS problem. Then we have $\lim_{\omega\to\infty}\bar{\phi}(\omega) = \bar{\theta}(\omega)$ or equivalently $\lim_{\omega\to\infty}\|\bar{\phi}(\omega) - \bar{\theta}(\omega)\| = 0$. Also $\bar{\phi}(\cdot) \in L_2([0,\ +\infty))$ and $\bar{\theta}(\cdot) \in L_2([0,\ +\infty))$. So we get

$$J(\bar{\phi},\ \bar{\theta}) = \int_0^{\infty} \|\bar{\phi}(\omega) - \bar{\theta}(\omega)\|_2^2\, d\omega < \infty. \tag{4}$$

Moreover $(\bar{\phi}(\cdot),\ \bar{\theta}(\cdot))$ is a feasible solution for the ITOC problem (2)-(3), so it satisfies constraints (3). Now since $\big(\phi^*(\cdot),\ \theta^*(\cdot),\ u^*(\cdot),\ v^*(\cdot)\big)$ is an optimal solution of ITOC problem so it satisfies system (3) and

$$0 \le J(\phi^*,\ \theta^*) = \int_0^{\infty} \|(\phi^*(\omega) - \theta^*(\omega)\|_2^2\, d\omega \le J(\bar{\phi},\ \bar{\theta}) < \infty. \tag{5}$$

Now with contradiction, assume that $\lim_{\omega\to\infty}\big(\phi^*(\omega) - \theta^*(\omega)\big) = k \neq 0$ where $k \in \mathbb{R}^n$. Then $\lim_{\omega\to\infty}\|\phi^*(\omega) - \theta^*(\omega)\|_2 = \|k\|_2 > 0$. So with relation (5), $\|k\|_2$ must be finite. Hence there are $n_1 \in \mathbb{N}$ and $\omega_1 \ge 0$ such that for all $\omega \ge \omega_1$ we have $\|\phi^*(\omega) - \theta^*(\omega)\| \ge \|k\|_2 - \dfrac{1}{n_1}$. Therefore we get

$$J(\phi^*, \theta^*) = \int_0^{\infty} \|(\phi^*(\cdot) - \theta^*(\cdot)\|_2^2\, d\omega = \int_0^{\omega_1} \|(\phi^*(\cdot) - \theta^*(\cdot)\|_2^2\, d\omega + \int_{\omega_1}^{\infty} \|(\phi^*(\cdot) - \theta^*(\cdot)\|_2^2\, d\omega$$

$$\ge \int_0^{\omega_1} \|(\phi^*(\cdot) - \theta^*(\cdot)\|_2^2\, d\omega + \left(\|k\|_2 - \frac{1}{n_1}\right) \int_{\omega_1}^{\infty} d\omega.$$

By relation (4) and right hand side of the above inequality, the value of $J\big(\phi^*,\ \theta^*\big)$ cannot be finite and this contradicts with relation (5). Therefore $\lim_{\omega\to\infty}\|\phi^*(\omega) - \theta^*(\omega)\| = 0$ that means $\lim_{\omega\to\infty}\phi^*(\omega) = \theta^*(\omega)$. □

3.2 *Proposing a PS method*

In this paper, because the ITOC problem (2)-(3) is defined in the interval $[0,\ \infty)$, with the help of variable change $\omega = \delta(\tau) = \dfrac{1+\tau}{1-\tau}$, we can transfer the interval $[0,\ \infty)$ to $[-1,\ 1)$. We define $\gamma(\tau) = \dot{\delta}(\tau) = \dfrac{2}{(1-\tau)^2}$ and $\Phi(\tau) = \phi(\delta(\tau))$ and $\Theta(\tau) = \theta(\delta(\tau))$. Therefore $\dot{\Phi}(\tau) = \gamma(\tau)\dot{\phi}(\tau)$ and $\dot{\Theta}(\tau) = \gamma(\tau)\dot{\theta}(\tau)$. Hence the problem (2)-(3) can be written as the following equivalent form

$$\text{Minimize}\quad J(\Phi,\ \Theta) = \int_{-1}^{1} \gamma(\tau)\|\Phi(\tau) - \Theta(\tau)\|_2^2\, d\tau \tag{6}$$

$$\text{subject to}\begin{cases} \dot{\Phi}(\tau) = \gamma(\tau) f\big(\delta(\tau),\ \Phi(\tau),\ \Theta(\tau),\ U(\tau)\big),\ -1 \le \tau < 1, \\ \dot{\Theta}(\tau) = \gamma(\tau) g\big(\delta(\tau),\ \Theta(\tau),\ V(\tau)\big),\ -1 \le \tau < 1, \\ \Phi(-1) = \alpha,\ \Theta(-1) = \beta, \end{cases} \tag{7}$$

where $U(\tau) = u(\delta(\tau))$ and $V(\tau) = v(\delta(\tau))$. Here $(U(.),\ V(.))$ and $(\Phi(.),\ \Theta(.))$ are the new control and state variables, respectively. Now we solve the problem (6)-(7) by using a PS method. Suppose $P_j(\cdot)$ for $j = 0,\ 1,\ 2,\ \ldots,\ N$, are Legendre polynomials in the interval $[-1,\ 1)$ which satisfies recurrence relation

$$\begin{cases} P_{j+1}(\tau) = \dfrac{2j+1}{j+1} P_j(\tau) - \dfrac{j}{j+1} P_{j-1}(\tau)\ ,\ j = 1,\ 2,\ \ldots \\ \\ P_0(\tau) = 1,\ P_1(\tau) = \tau\ ,\ \tau \in [-1,\ 1)\,. \end{cases}$$

Also suppose that $\tau_0 = -1 < \tau_1 < \cdots < \tau_N < 1$ are LGR in the interval $[-1,\ 1)$, which are the roots of polynomials $P(\cdot) = P_N(\cdot) + P_{N-1}(\cdot)$. In this paper, we suggest the following approximations for the state and control variables in the problem (6)-(7)

$$\Phi(\tau) \simeq \sum_{j=0}^{N} \bar{\phi}_j L_j(\tau),\ \Theta(\tau) \simeq \sum_{j=0}^{N} \bar{\theta}_j L_j(\tau),\ U(\tau) \simeq \sum_{j=0}^{N} \bar{u}_j L_j(\tau),\ V(\tau) \simeq \sum_{j=0}^{N} \bar{v}_j L_j(\tau). \tag{8}$$

In the relations (8), $(\bar{\phi}_j,\ \bar{\theta}_j,\ \bar{u}_j,\ \bar{v}_j)$ for $j = 0,\ 1,\ 2,\ \ldots,\ N$, are unknown coefficients and $L_j(\tau)$ for $j = 0,\ 1,\ 2,\ \ldots,\ N$, are the Lagrange polynomials, defined by

$$Z_j(\tau) = \prod_{\substack{i=1 \\ i \neq j}}^{N} \frac{\tau - \tau_i}{\tau_j - \tau_i},\ \ j = 0,\ 1,\ 2,\ \ldots,\ N.$$

Now according to the Kronecker delta property of Lagrange polynomials, we have

$$\Phi(\tau_k) = \bar{\phi}_k,\ \Theta(\tau_k) = \bar{\theta}_k,\ U(\tau_k) = \bar{u}_k,\ V(\tau_k) = \bar{v}_k,\ \ k = 0,\ 1,\ 2,\ \ldots,\ N. \tag{9}$$

Also with (8), we get

$$\dot{\Phi}(\tau_k) \simeq \sum_{j=0}^{N} \bar{\phi}_j Q_{kj},\ \dot{\Theta}(\tau_k) \simeq \sum_{j=0}^{N} \bar{\theta}_j Q_{kj},\ k = 0,\ 1,\ 2,\ \ldots,\ N, \tag{10}$$

where $Q = \left(Q_{kj}\right)_{(N+1)\times(N+1)}$ is the derivative matrix in this PS method and is defined as

$$Q_{kj} = \dot{L}_j(\tau_k) = \begin{cases} -\dfrac{N(N+2)}{4}, & j = k = 0, \\ \\ \dfrac{\tau_k}{1-\tau_k^2} + \dfrac{(N+1)P_N(\tau_k)}{(1-\tau_k^2)\left(P'_N(\tau_k) + P'_{N+1}(\tau_k)\right)}, & 1 \leq j = k \leq N, \\ \\ \dfrac{P'_N(\tau_k) + P'_{N+1}(\tau_k)}{(\tau_k - \tau_j)\left(P'_N(\tau_j) + P'_{N+1}(\tau_j)\right)}, & k \neq j. \end{cases} \tag{11}$$

Lemma 3.1 Assume $\tau_0 = -1 < \tau_1 < \cdots < \tau_N < 1$ are LGR points in interval $[-1,\ 1)$. There exists a unique set of weights $\{w_k\}_{k=0}^{N}$ such that for every polynomial $q(\cdot)$ of degree $2N$ or less, we have

$$\int_{-1}^{1} q(t)dt = \sum_{k=0}^{N} w_k q(\tau_k),$$

where the weights come from the following relation

$$w_k = \frac{1-\tau_k}{(N+1)^2 \left(P_N(\tau_k)\right)^2},\ k = 0,\ 1,\ 2,\ 3,\ \ldots,\ N.$$

Now, for better display of indices, formulas and method, we assume $n = 1$. By Lemma 3.1, we approximate the integral of the objective functional in the problem (6)-(7) as

$$\int_{-1}^{1} \gamma(\tau_k)\left(\Phi(\tau) - \Theta(\tau)\right)^2 d\tau \simeq \sum_{k=0}^{N} w_k \gamma(\tau_k)\left(\bar{\phi}_k - \bar{\theta}_k\right)^2. \tag{12}$$

Now with the help of equations (9), (10) and (12) the problem (6)-(7) becomes a NLP problem as

$$\text{Minimize} \quad J^N(\bar{\phi},\ \bar{\theta}) = \sum_{k=0}^{N} w_k \gamma(\tau_k)(\bar{\phi}_k - \bar{\theta}_k)^2 \tag{13}$$

$$\text{subject to} \begin{cases} \sum_{j=0}^{N} \bar{\phi}_j Q_{kj} = \gamma(\tau_k) f\left(\delta(\tau_k),\ \bar{\phi}_k,\ \bar{\theta}_k,\ \bar{u}_k\right),\ \ k = 0,\ 1,\ 2,\ \ldots,\ N, \\ \sum_{j=0}^{N} \bar{\theta}_j Q_{kj} = \gamma(\tau_k) g\left(\delta(\tau_k),\ \bar{\theta}_k,\ \bar{v}_k\right),\ \ k = 0,\ 1,\ 2,\ \ldots,\ N, \\ \bar{\phi}_0 = \alpha,\ \bar{\theta}_0 = \beta. \end{cases} \tag{14}$$

The variables of problem (13)-(14) are $(\bar{\phi}_k,\ \bar{\theta}_k,\ \bar{u}_k,\ \bar{v}_k)$, $k = 0,\ 1,\ 2,\ \ldots,\ N$. Now by solving the NLP problem (13)-(14), the optimal solutions $(\bar{\phi}_k^*,\ \bar{\theta}_k^*,\ \bar{u}_k^*,\ \bar{v}_k^*)$ for $k = 0,\ 1,\ 2,\ \ldots,\ N$ are obtained and by substituting in equations (8), the approximate optimal solution of problem (6)-(7) is gained as

$$\phi(\omega) \simeq \sum_{j=0}^{N} \bar{\phi}_j^* Z_j\left(\frac{\omega-1}{\omega+1}\right),\ \ \theta(\omega) \simeq \sum_{j=0}^{N} \bar{\theta}_j^* Z_j\left(\frac{\omega-1}{\omega+1}\right),\ \omega \in [0,\ \infty), \tag{15}$$

$$u(\omega) \simeq \sum_{j=0}^{N} \bar{u}_j^* Z_j\left(\frac{\omega-1}{\omega+1}\right),\ \ v(\omega) \simeq \sum_{j=0}^{N} \bar{v}_j^* Z_j\left(\frac{\omega-1}{\omega+1}\right),\ \omega \in [0,\ \infty). \tag{16}$$

In next subsection, we use the KKT optimality conditions for solving the NLP problem (13)-(14).

3.3 *KKT optimality conditions for ITS problem*

Consider the NLP problem (13)-(14) and define $F_k(\bar{\phi},\ \bar{\theta},\ \bar{u})$ and $G_k(\bar{\phi},\ \bar{v})$ as

$$\begin{cases} F_k(\bar{\phi},\ \bar{\theta},\ \bar{u}) = \sum_{j=0}^{N} \bar{\phi}_j Q_{kj} - \gamma(\tau_k) f(\delta(\tau_k),\ \bar{\phi}_k,\ \bar{\theta}_k,\ \bar{u}_k),\ \ k = 0,\ 1,\ 2,\ \ldots,\ N, \\ \\ G_k(\bar{\theta},\ \bar{v}) = \sum_{j=0}^{N} \bar{\theta}_j Q_{kj} - \gamma(\tau_k) g(\delta(\tau_k),\ \bar{\theta}_k,\ \bar{v}_k),\ \ k = 0,\ 1,\ 2,\ \ldots,\ N, \end{cases} \tag{17}$$

where

$$\bar{\phi} = (\bar{\phi}_0,\ \bar{\phi}_1 \cdots,\ \bar{\phi}_N),\ \bar{\theta} = (\bar{\theta}_0,\ \bar{\theta}_1 \cdots,\ \bar{\theta}_N),\ \bar{u} = (\bar{u}_0,\ \bar{u}_1 \cdots,\ \bar{u}_N),\ \bar{v} = (\bar{v}_0,\ \bar{v}_1 \cdots,\ \bar{v}_N). \tag{18}$$

Now by relation (17), the problem (13)-(14) is written as

$$\text{Minimize} \quad J^N(\bar{\phi},\ \bar{\theta}) = \sum_{k=0}^{N} w_k \gamma(\tau_k)(\bar{\phi}_k - \bar{\theta}_k)^2 \tag{19}$$

$$\text{subject to} \begin{cases} F_k(\bar{\phi},\ \bar{\theta},\ \bar{u}) = 0,\ \ k = 0,\ 1,\ 2,\ \ldots,\ N, \\ \\ G_k(\bar{\theta},\ \bar{v}) = 0,\ \ k = 0,\ 1,\ 2,\ \ldots,\ N. \\ \\ \bar{\phi}_0 = \alpha,\ \bar{\theta}_0 = \beta. \end{cases} \tag{20}$$

In order to solve this NLP problem, we find the necessary conditions of optimality and define the Lagrange auxiliary function as

$$L(\bar{\phi},\ \bar{\theta},\ \bar{u},\ \bar{v},\ \lambda,\ \mu) = J^N(\bar{\phi},\ \bar{\theta}) + \lambda F(\bar{\phi},\ \bar{\theta},\ \bar{u}) + \mu G(\bar{\theta},\ \bar{v}),$$

where $F = (F_0,\ F_1,\ ...,\ F_N)$ and $G = (G_0,\ G_1,\ ...,\ G_N)$, and further $\lambda = (\lambda_0,\ \lambda_1,\ ...,\ \lambda_N)$ and $\mu = (\mu_0,\ \mu_1,\ ...,\ \mu_N)$ are the Lagrange multipliers. The KKT necessary optimality conditions for NLP problem (37)-(38) can be gained as

$$\begin{cases} \dfrac{\partial L}{\partial \bar{\phi}_i} = 0,\ \ i = 0,\ 1,\ 2,\ \ldots,\ N \\ \dfrac{\partial L}{\partial \bar{\theta}_i} = 0,\ \ i = 0,\ 1,\ 2,\ \ldots,\ N \\ \dfrac{\partial L}{\partial \lambda_i} = 0,\ \ i = 0,\ 1,\ 2,\ \ldots,\ N \\ \dfrac{\partial L}{\partial \mu_i} = 0,\ \ i = 0,\ 1,\ 2,\ \ldots,\ N \\ \dfrac{\partial L}{\partial \bar{u}_i} = 0,\ \ i = 0,\ 1,\ 2,\ \ldots,\ N \\ \dfrac{\partial L}{\partial \bar{v}_i} = 0,\ \ i = 0,\ 1,\ 2,\ \ldots,\ N \\ \bar{\phi}_0 = \alpha,\ \bar{\theta}_0 = \beta. \end{cases} \tag{21}$$

Hence we have

$$
\begin{cases}
2w_i\gamma(\tau_i)(\bar{\phi}_i-\bar{\theta}_i)+\sum_{k=0}^{N}\lambda_k\hat{F}_{ki}(\bar{\phi},\ \bar{\theta},\ \bar{u})=0,\ \ i=0,\ 1,\ 2,\ \dots,\ N,\\
-2w_i\gamma(\tau_i)(\bar{\phi}_i-\bar{\theta}_i)+\sum_{k=0}^{N}\lambda_k\tilde{F}_{ki}(\bar{\phi},\ \bar{\theta},\ \bar{u})+\sum_{k=0}^{N}\mu_k\hat{G}_{ki}(\bar{\theta},\ \bar{v})=0,\ \ i=0,\ 1,\ 2,\ \dots,\ N,\\
\sum_{j=0}^{N}\bar{\phi}_jQ_{ij}-\gamma(\tau_i)f(\delta(\tau_i),\ \bar{\phi}_i,\ \bar{\theta}_i,\ \bar{u}_i)=0,\ \ i=0,\ 1,\ 2,\ \dots,\ N,\\
\sum_{j=0}^{N}\bar{\theta}_jQ_{ij}-\gamma(\tau_i)g(\delta(\tau_i),\ \bar{\theta}_i,\ \bar{v}_i)=0,\ \ i=0,\ 1,\ 2,\ \dots,\ N,\\
\sum_{k=0}^{N}\lambda_k\bar{F}_{ki}(\bar{\phi},\ \bar{\theta},\ \bar{u})=0,\ \ i=0,\ 1,\ 2,\ \dots,\ N,\\
\sum_{k=0}^{N}\mu_k\bar{G}_{ki}(\bar{\theta},\ \bar{v})=0,\ \ i=0,\ 1,\ 2,\ \dots,\ N,
\end{cases}
\tag{22}
$$

where

$$
\frac{\partial J^N(\bar{\phi},\ \bar{\theta})}{\partial\bar{\phi}_i}=2w_i\gamma(\tau_i)(\bar{\phi}_i-\bar{\theta}_i),\quad \frac{\partial J^N(\bar{\phi},\ \bar{\theta})}{\partial\bar{\theta}_i}=-2w_i\gamma(\tau_i)(\bar{\phi}_i-\bar{\theta}_i),
\tag{23}
$$

$$
\hat{F}_{ki}(\bar{\phi},\ \bar{\theta},\ \bar{u})=\frac{\partial F_k}{\partial\bar{\phi}_i}=\begin{cases}Q_{ii}-\gamma(\tau_i)\dfrac{\partial f}{\partial\bar{\phi}_i}, & i=k,\\ Q_{ki}, & i\neq k,\end{cases}\quad \tilde{F}_{ki}(\bar{\phi},\ \bar{\theta},\ \bar{u})=\frac{\partial F_k}{\partial\bar{\theta}_i}=\begin{cases}-\gamma(\tau_i)\dfrac{\partial f}{\partial\bar{\theta}_i}, & i=k,\\ 0, & i\neq k,\end{cases}
\tag{24}
$$

$$
\bar{F}_{ki}(\bar{\phi},\ \bar{\theta},\ \bar{u})=\frac{\partial F_k}{\partial\bar{u}_i}=\begin{cases}-\gamma(\tau_i)\dfrac{\partial f}{\partial\bar{u}_i}, & i=k,\\ 0, & i\neq k,\end{cases}\quad \hat{G}_{ki}(\bar{\theta},\ \bar{v})=\frac{\partial G_k}{\partial\bar{\theta}_i}=\begin{cases}Q_{ii}-\gamma(\tau_i)\dfrac{\partial g}{\partial\bar{\theta}_i}, & i=k,\\ Q_{ki}, & i\neq k,\end{cases},
\tag{25}
$$

$$
\bar{G}_{ki}(\bar{\theta},\ \bar{v})=\frac{\partial G_k}{\partial\bar{v}_i}=\begin{cases}-\gamma(\tau_i)\dfrac{\partial g}{\partial\bar{v}_i}, & i=k,\\ 0, & i\neq k.\end{cases}
\tag{26}
$$

Now by solving algebraic system (22), we can obtain unknown coefficients $\bar{\phi}$, $\bar{\theta}$, $\bar{u}$ and $\bar{v}$. Also, by (15)-(16) we can approximate the optimal solution of problem (6)-(7). Note that according to the Theorem 3.1, $\big(\phi(\cdot),\ \theta(\cdot),\ u(\cdot),\ v(\cdot)\big)$ defined by (15) and (16) is an approximate solution for the original ITS problem.

4. Implementing method for FTS problem

The same as previous section, we implement the method in three subsection. In the following, we first propose a Finite-Time Optimal Control (FTOC) problem to solve the FTS problem.

4.1 *Proposing a FTOC problem*

In this subsection, a FTOC problem is proposed to solve the FTS problem as follows

$$\text{Minimize} \quad J(\phi,\ \theta) = \int_0^T \|\phi(\omega) - \theta(\omega)\|_2^2\, d\omega \tag{27}$$

$$\text{subject to} \begin{cases} \dot{\phi}(\omega) = f\big(\omega,\ \phi(\omega),\ \theta(\omega),\ u(\omega)\big),\ 0 \le \omega \le T, \\ \dot{\theta}(\omega) = g\big(\omega,\ \theta(\omega),\ v(\omega)\big),\ 0 \le \omega \le T, \\ \phi(0) = \alpha,\ \theta(0) = \beta,\ \phi(T) = \theta(T). \end{cases} \tag{28}$$

Theorem 4.1 If $\big(\phi^*(\cdot),\ \theta^*(\cdot),\ u^*(\cdot), v^*(\cdot)\big)$ is an optimal solution of FTOC problem (27)-(28) then there exists time $\bar{\omega}_1 \in [0,\ T]$ such that $\phi^*(\omega) = \theta^*(\omega)$ for all $\omega \in [\bar{\omega}_1,\ T]$.

Proof. Assume $\big(\bar{\phi}(\cdot),\ \bar{\theta}(\cdot),\ \bar{u}(\cdot),\ \bar{v}(\cdot)\big)$ is a solution for FTS problem. So there is $\omega_1 \in [0,\ T)$ such that $\phi(\omega) = \theta(\omega)$ for all $\omega \in [\omega_1,\ T]$. Now by contradiction, and continuity of functions $\phi(\cdot)$ and $\theta(\cdot)$, there exists $\omega_2 \in [0,\ T)$ such that $\omega_2 > \omega_1$ and $\int_{\omega_2}^T \|\phi^*(\omega) - \theta^*(\omega)\|_2^2\, d\omega > 0$. Define

$$\tilde{\phi}(\omega) = \begin{cases} \phi^*(\omega), & 0 \le \omega \le \omega_2, \\ \bar{\phi}(\omega), & \omega_2 < \omega \le T, \end{cases} \quad \tilde{\theta}(\omega) = \begin{cases} \theta^*(\omega), & 0 \le \omega \le \omega_2, \\ \bar{\theta}(\omega), & \omega_2 < \omega \le T, \end{cases} \tag{29}$$

$$\tilde{u}(\omega) = \begin{cases} u^*(\omega), & 0 \le \omega \le \omega_2, \\ \bar{u}(\omega), & \omega_2 < \omega \le T, \end{cases} \quad \tilde{v}(\omega) = \begin{cases} v^*(\omega), & 0 \le \omega \le \omega_2, \\ \bar{v}(\omega), & \omega_2 < \omega \le T. \end{cases} \tag{30}$$

It is trivial that $\big(\tilde{\phi}(\cdot),\ \tilde{\theta}(\cdot),\ \tilde{u}(\cdot),\ \tilde{v}(\cdot)\big)$ satisfies system (28) almost everywhere. Moreover, we get

$$J(\tilde{\phi}, \tilde{\theta}) = \int_0^{\omega_2} \|\phi^*(\omega) - \theta^*(\omega)\|_2^2\, d\omega + \int_{\omega_2}^T \|\bar{\phi}(\omega) - \bar{\theta}(\omega)\|_2^2\, d\omega \tag{31}$$

$$< \int_0^{\omega_2} \|\phi^*(\omega) - \theta^*(\omega)\|_2^2\, d\omega + \int_{\omega_2}^T \|\phi^*(\omega) - \theta^*(\omega)\|_2^2\, d\omega = J(\phi^*, \theta^*), \tag{32}$$

which this contradicts with the optimality of $\big(\phi^*(\cdot),\ \theta^*(\cdot),\ u^*(\cdot),\ v^*(\cdot)\big)$.

4.2 *Proposing a PS method for FTS problem*

In this section, we will solve FTS problem by a PS method. Since problem FTOC (27)-(28) is on the interval $[0,\ T]$, we first change it to the interval $[-1,\ 1]$ by defining variable $\omega = \delta(\tau) = \dfrac{T}{2}(\tau + 1)$ where $\tau \in [-1,\ 1]$. Define

$$\Phi(\tau) = \phi(\delta(\tau)),\ \Theta(\tau) = \theta(\delta(\tau)),\ U(\tau) = u(\delta(\tau)),\ V(\tau) = v(\delta(\tau)).$$

Now, the problem (27) becomes as follows

$$\text{Minimize} \quad J(\Phi,\, \Theta) = \int_{-1}^{1} \frac{T}{2} \|\Phi(\tau) - \Theta(\tau)\|_2^2 \, d\tau \tag{33}$$

$$\text{subject to} \begin{cases} \dot{\Phi}(\tau) = \dfrac{T}{2} f\big(\delta(\tau),\, \Phi(\tau),\, \Theta(\tau),\, U(\tau)\big),\; -1 \le \tau \le 1, \\ \\ \dot{\Theta}(\tau) = \dfrac{T}{2} g\big(\delta(\tau),\, \Theta(\tau),\, V(\tau)\big),\; -1 \le \tau \le 1, \\ \\ \Phi(-1) = \alpha,\, \Theta(-1) = \beta,\, \Phi(1) = \Theta(1). \end{cases} \tag{34}$$

The problem (33)-(34) is also solved like solving problem (6)-(7) with the help of LGL colocalization points and a PS method. Notice that the LGL points $\{\tau_k\}_{k=0}^{N}$ are the roots of polynomial $h(\tau) = (1-\tau^2)\dot{P}_N(\tau)$ where $P_N(\cdot)$ is the Legendre polynomial of degree N. Define $F_k(\bar{\phi},\, \bar{\theta},\, \bar{u})$ and $G_k(\bar{\phi},\, \bar{v})$ as follows

$$\begin{cases} F_k(\bar{\phi},\, \bar{\theta},\, \bar{u}) = \sum_{j=0}^{N} \bar{\phi}_j Q_{kj} - \dfrac{T}{2} f(\delta(\tau_k),\, \bar{\phi}_k,\, \bar{\theta}_k,\, \bar{u}_k),\; k = 0,\, 1,\, 2,\, \ldots,\, N, \\ \\ G_k(\bar{\theta},\, \bar{v}) = \sum_{j=0}^{N} \bar{\theta}_j Q_{kj} - \dfrac{T}{2} g(\delta(\tau_k),\, \bar{\theta}_k,\, \bar{v}_k),\; k = 0,\, 1,\, 2,\, \ldots,\, N, \end{cases} \tag{35}$$

where $(\bar{\phi},\, \bar{\theta},\, \bar{u},\, \bar{v})$ is defined by (18) and $R = \big(R_{kj}\big)_{(N+1)\times(N+1)}$ is the derivative matrix in this PS method given by

$$R_{kj} = \dot{Z}_j(\tau_k) = \begin{cases} -\dfrac{N(N+1)}{4}, & j = k = 0 \\ \\ \dfrac{P_N(\tau_k)}{P_N(\tau_j)(\tau_k - \tau_j)}, & j \ne k,\, 0 \le j,\, k \le N \\ \\ 0, & 1 \le j = k \le N-1 \\ \\ \dfrac{N(N+1)}{4}, & k = j = N, \end{cases} \tag{36}$$

where $Z_j(\cdot)$ is the Lagrange polynomial constructed based on the LGL points.

Lemma 4.1 Assume $\tau_0 = -1 < \tau_1 < \cdots < \tau_N = 1$ are LGL points in interval $[-1,\, 1]$. There exists a unique set of weights $\{w_k\}_{k=0}^{N}$ such that for every polynomial $q(\cdot)$ of degree $2N-1$ or less, we have

$$\int_{-1}^{1} q(t)dt = \sum_{k=0}^{N} w_k q(\tau_k),$$

where the weights come from the following relation

$$w_k = \frac{2}{N(N+1)} \cdot \frac{1}{(p_N(\tau_i))^2}, \quad k = 0,\ 1,\ 2,\ 3,\ \ldots,\ N.$$

Now by Lemma 4.1 and Equation (35), and the same as previous section, the problem (33)-(34) converted into the following NLP problem

$$\text{Minimize} \quad J^N(\bar{\phi},\ \bar{\theta}) = \sum_{k=0}^{N} \frac{T}{2} w_k (\bar{\phi}_k - \bar{\theta}_k)^2 \tag{37}$$

$$\text{subject to} \begin{cases} F_k(\bar{\phi},\ \bar{\theta},\ \bar{u}) = 0,\ k = 0,\ 1,\ 2,\ \ldots,\ N, \\ G_k(\bar{\theta},\ \bar{v}) = 0,\ k = 0,\ 1,\ 2,\ \ldots,\ N, \\ \bar{\phi}_0 = \alpha,\ \bar{\theta}_0 = \beta,\ \bar{\phi}_N = \bar{\theta}_N. \end{cases} \tag{38}$$

In next subsection, we obtain the KKT conditions for solving the NLP problem (37)-(38).

4.3 *KKT optimality conditions for FTS problem*

As in the previous section, in order to find the necessary conditions of optimality, we can define the Lagrange auxiliary function and obtain the KKT optimality conditions for the NLP problem (37)-(38), as the following algebraic system

$$\begin{cases} w_i T(\bar{\phi}_i - \bar{\theta}_i) + \sum_{k=0}^{N} \lambda_k \hat{F}_{ki}(\bar{\phi},\ \bar{\theta},\ \bar{u}) = 0,\ i = 0,\ 1,\ 2,\ \ldots,\ N, \\ -w_i T(\bar{\phi}_i - \bar{\theta}_i) + \sum_{k=0}^{N} \lambda_k \tilde{F}_{ki}(\bar{\phi},\ \bar{\theta},\ \bar{u}) + \sum_{k=0}^{N} \mu_k \hat{G}_{ki}(\bar{\theta},\ \bar{v}) = 0,\ i = 0,\ 1,\ 2,\ \ldots,\ N, \\ \sum_{j=0}^{N} \bar{\phi}_j R_{ij} - \frac{T}{2} f(\delta(\tau_i),\ \bar{\phi}_i,\ \bar{\theta}_i,\ \bar{u}_i) = 0,\ i = 0,\ 1,\ 2,\ \ldots,\ N, \\ \sum_{j=0}^{N} \bar{\theta}_j R_{ij} - \frac{T}{2} g(\delta(\tau_i),\ \bar{\theta}_i,\ \bar{v}_i) = 0,\ i = 0,\ 1,\ 2,\ \ldots,\ N, \\ \sum_{k=0}^{N} \lambda_k \bar{F}_{ki}(\bar{\phi},\ \bar{\theta},\ \bar{u}) = 0,\ i = 0,\ 1,\ 2,\ \ldots,\ N, \\ \sum_{k=0}^{N} \mu_k \bar{G}_{ki}(\bar{\theta},\ \bar{v}) = 0,\ i = 0,\ 1,\ 2,\ \ldots,\ N, \\ \bar{\phi}_0 = \alpha,\ \bar{\theta}_0 = \beta,\ \bar{\phi}_N = \bar{\theta}_N, \end{cases} \tag{39}$$

where $F = (F_0,\ F_1,\ \ldots,\ F_N)$ and $G = (G_0,\ G_1,\ \ldots,\ G_N)$, and further $\lambda = (\lambda_0,\ \lambda_1,\ \ldots,\ \lambda_N)$ and $\mu = (\mu_0,\ \mu_1,\ \ldots,\ \mu_N)$ are the Lagrange multipliers. Also,

$$\hat{F}_{ki}(\bar{\phi},\ \bar{\theta},\ \bar{u}) = \frac{\partial F_k}{\partial \bar{\phi}_i} = \begin{cases} R_{ii} - \frac{T}{2}\frac{\partial f}{\partial \bar{\phi}_i}, & i = k \\ R_{ki}, & i \neq k \end{cases} \quad \hat{F}_{ki}(\bar{\phi},\ \bar{\theta},\ \bar{u}) = \frac{\partial F_k}{\partial \bar{\theta}_i} = \begin{cases} -\frac{T}{2}\frac{\partial f}{\partial \bar{\theta}_i}, & i = k \\ 0, & i \neq k \end{cases} \tag{40}$$

$$\bar{F}_{ki}(\bar{\phi},\ \bar{\theta},\ \bar{u}) = \frac{\partial F_k}{\partial \bar{u}_i} = \begin{cases} -\frac{T}{2}\frac{\partial f}{\partial \bar{u}_i}, & i = k \\ 0, & i \neq k \end{cases} \quad \hat{G}_{ki}(\bar{\theta},\ \bar{v}) = \frac{\partial G_k}{\partial \bar{\theta}_i} = \begin{cases} R_{ii} - \frac{T}{2}\frac{\partial g}{\partial \bar{\theta}_i}, & i = k \\ R_{ki}, & i \neq k \end{cases} \tag{41}$$

$$\bar{G}_{ki}(\bar{\theta},\ \bar{v}) = \frac{\partial G_k}{\partial \bar{v}_i} = \begin{cases} -\frac{T}{2}\frac{\partial g}{\partial \bar{v}_i}, & i = k \\ 0, & i \neq k \end{cases} \tag{42}$$

Now, by solving system (39), we can obtain unknown coefficients $(\bar{\phi},\ \bar{\theta},\ \bar{u},\ \bar{v})$ which are the approximate solutions for the NLP problem (37)-(38). Also, these solutions can be used to approximate the solution of FTOC problem (27)-(28).

Remark 4.1 Notice that the existence of a solution of algebraic systems and their convergence to the optimal solutions of (ITOC or FTOC) problems are discussed in some works (see [23–25, 27]). Also some generalization of convergence analysis of PS methods is proposed by Noori Skandari et al. [28, 30, 32, 33].

5. Levenberg-Marquardt algorithm for the gained algebraic systems

The Levenberg-Marquardt algorithm is a highly effective method for solving square and non-square systems of algebraic nonlinear equations. Essentially, the algorithm modifies the Gauss-Newton method to handle cases where the problem is ill-conditioned or the initial guess is far from the solution. It introduces a damping parameter to switch between the Gauss-Newton method and gradient descent. We present the Levenberg-Marquardt algorithm (implemented in MATLAB software) to solve square algebraic systems (22) and (39). These systems has $6(N+1)$ equations and $6(N+1)$ unknown variables $(\bar{\phi}_i,\ \bar{\theta}_i,\ \bar{u}_i,\ \bar{v}_i,\ \lambda_i,\ \mu_i)$, $i = 0,\ 1,\ 2,\ \ldots,\ N$. For more clarification, let us define these systems as $H(X) = 0$ where

$$X = (x_1,\ x_2,\ \ldots,\ x_{6N+6}) = \left((\bar{\phi}_i,\ \bar{\theta}_i,\ \bar{u}_i,\ \bar{v}_i,\ \lambda_i,\ \mu_i) : i = 0,\ 1,\ 2,\ \ldots,\ N\right), \tag{43}$$

and $H(X) = (H_1(X),\ H_2(X),\ \ldots,\ H_{6N+6}(X))$. Start with an initial guess $X^{(0)}$ for the unknown vector X, and compute the Jacobian matrix $J^H(X)$, which contains the first derivatives of each equation with respect to each unknown, given by $J^H_{ij} = \frac{\partial H_i}{\partial X_j}$ and update the solution vector X using the following iterative formula

$$X^{(r+1)} = X^{(r)} - \left[J^H\left(X^{(r)}\right)^T J^H\left(X^{(r)}\right) + \mu I\right]^{-1} J^H\left(X^{(r)}\right)^T H\left(X^{(r)}\right), \quad r = 0,\ 1,\ 2,\ \ldots. \tag{44}$$

Here, μ is the damping parameter, and I is the identity matrix. The parameter μ is adjusted dynamically at each iteration. This algorithm is robust and can handle scenarios where the initial guess is far from the final solution. By combining the fast convergence of the Gauss-Newton method with the stability of gradient descent, it achieves reliable results. Convergence can be monitored by evaluating the norm of the residual vector $H\left(X^{(r+1)}\right)$. If this norm falls below a predefined threshold, the algorithm is considered to have successfully converged to a solution.

6. Numerical examples

In this section, five comparative examples are given to illustrate the effectiveness of the presented approaches for ITS and FTS problems. Notice that in Examples 1 and 2, we utilize relations $\Phi(\tau) = \phi(\delta(\tau))$, $\Theta(\tau) = \theta(\delta(\tau))$, $U(\tau) = u(\delta(\tau))$ and $V(\tau) = v(\delta(\tau))$ where $\omega = \delta(\tau) = \dfrac{1+\tau}{1-\tau}$, for $\tau \in [-1,\ 1)$.

Example 1 Consider the following infinite-time nonlinear control system

$$\begin{cases} \dot{\phi}(\omega) = \dfrac{1}{1+\theta^2(\omega)} + \phi(\omega)\theta(\omega) + u(\omega)(\phi(\omega)-1),\ \omega \geq 0 \\ \dot{\theta}(\omega) = \theta(\omega)\tanh\left(\theta(\omega)\right) + v(\omega),\ \omega \geq 0 \\ \phi(0) = \phi_0,\ \theta(0) = \theta_0. \end{cases} \tag{45}$$

To synchronize trajectories $\phi(\cdot)$ and $\theta(\cdot)$, we solve the corresponding algebraic system (22) and show the results for different initial conditions ϕ_0 and θ_0 in Figures 1 and 2. It can be seen that the ITS problem is solved with a relatively high convergence rate and the suggested method has a good capability to solve such a problem.

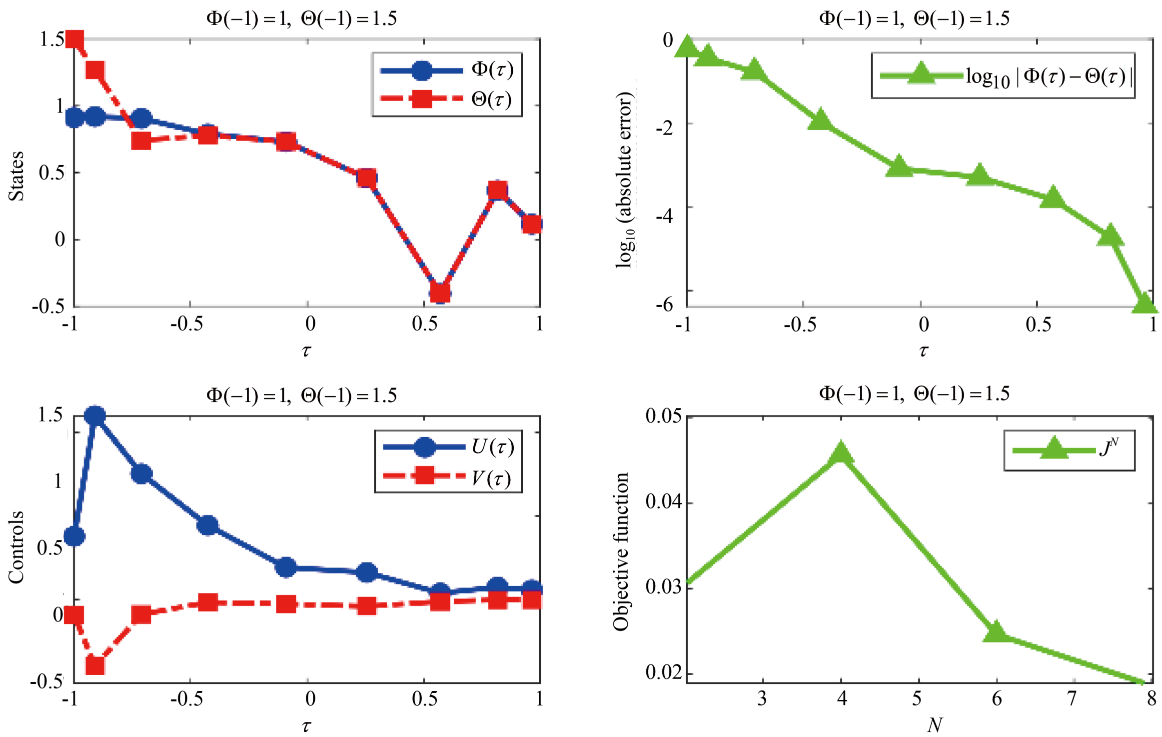

Figure 1. The solution of ITS problem by suggested method (with $N = 8$, $\phi_0 = 1$, $\theta_0 = 1.5$) for Example 1

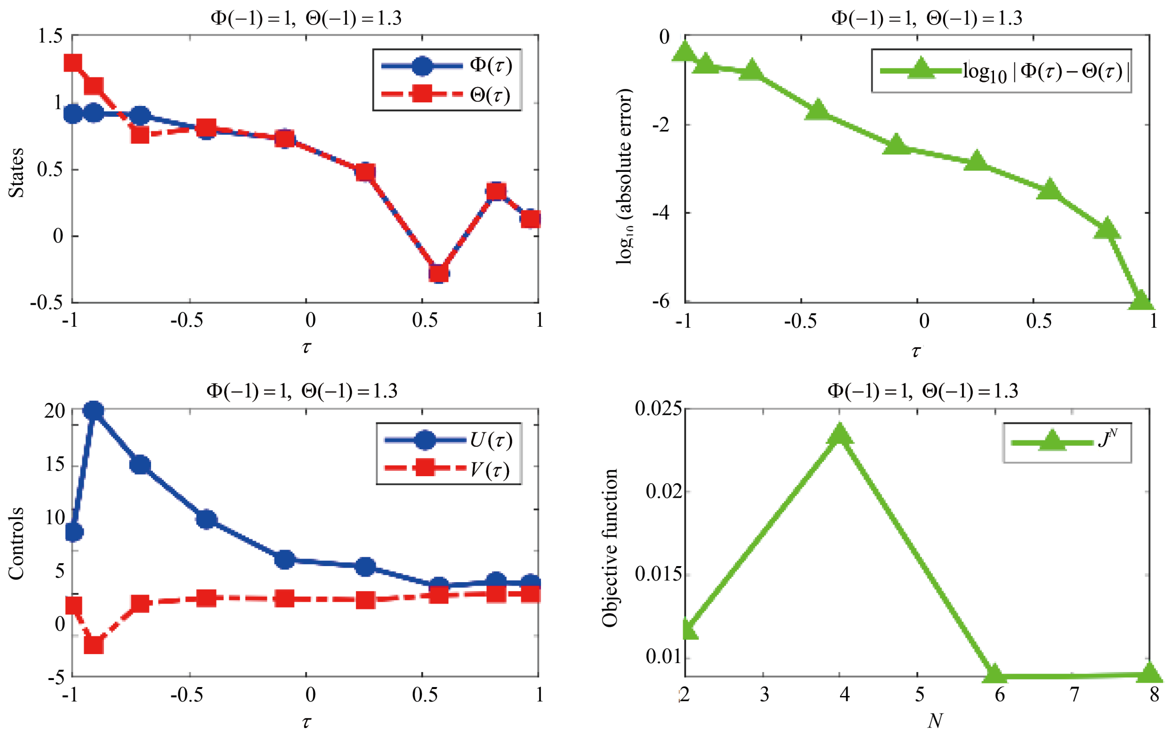

Figure 2. The solution of ITS problem by suggested method (with $N = 8$, $\phi_0 = 1$, $\theta_0 = 1.3$) for Example 1

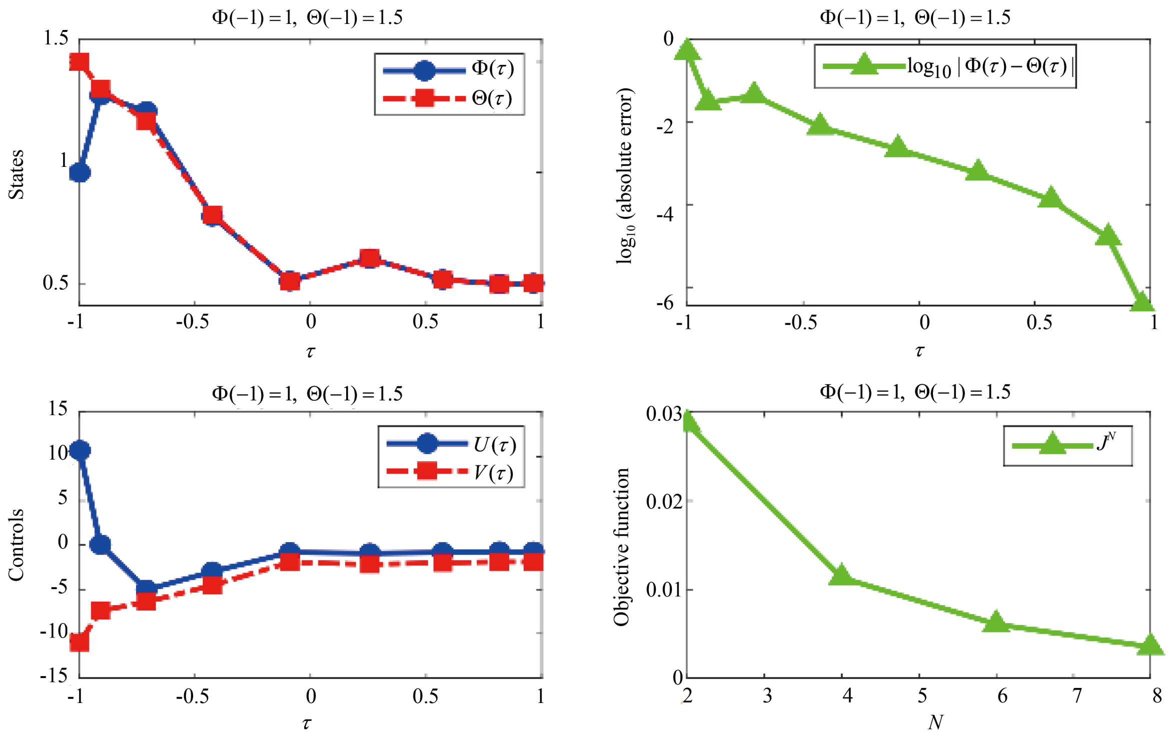

Figure 3. The solution of ITS problem by suggested method (with $N = 8$, $\phi_0 = 1$, $\theta_0 = 1.5$) for Example 2

Example 2 Consider the following infinite-time nonlinear control system

$$\begin{cases} \dot{\phi}(\omega) = \phi^2(\omega) + \theta(\omega) + u(\omega), & \omega \geq 0 \\ \dot{\theta}(\omega) = e^{\theta(\omega)} + \theta^2(\omega) + v(\omega), & \omega \geq 0 \\ \phi(0) = \phi_0, \ \theta(0) = \theta_0. \end{cases} \tag{46}$$

To achieve synchronization between the trajectories $\phi(\cdot)$ and $\theta(\cdot)$, we solve the associated algebraic system (22) and illustrate the outcomes for various initial conditions ϕ_0 and θ_0 in Figures 3 and 4. The results indicate that the ITS problem is resolved with a relatively rapid convergence rate. These findings underscore the effectiveness and reliability of the proposed method in handling such synchronization tasks.

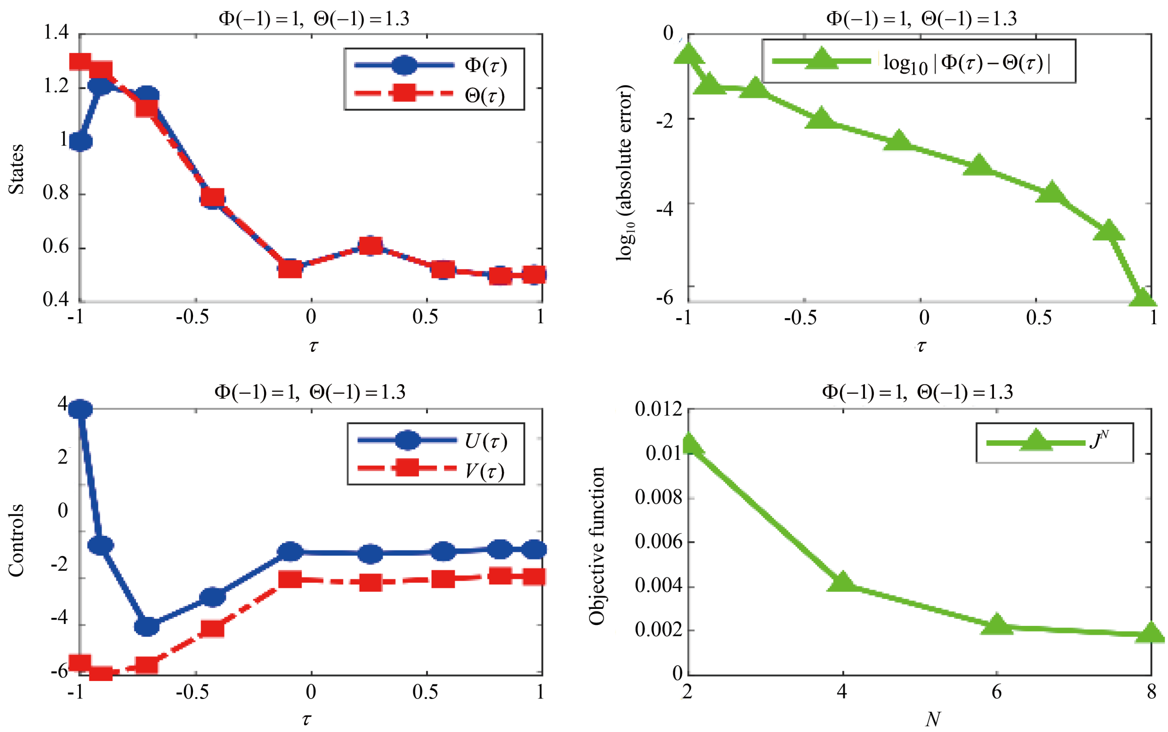

Figure 4. The solution of ITS problem by suggested method (with $N = 8$, $\phi_0 = 1$, $\theta_0 = 1.3$) for Example 2

Example 3 Consider the following finite-time nonlinear control system

$$
\begin{cases}
\dot{\phi}(\omega) = \sin\big(\pi\theta(\omega)\big) + \cos\big(\pi\phi(\omega)\big) + \omega u(\omega),\ 0 \le \omega \le 1, \\
\dot{\theta}(\omega) = \tan\big(\theta(\omega)\big) + \dfrac{\theta(\omega)}{1+(\theta)^2} + v(\omega),\ 0 \le \omega \le 1, \\
\phi(0) = \phi_0,\ \theta(0) = \theta_0.
\end{cases}
\tag{47}
$$

To evaluate the synchronization of the trajectories $\phi(\cdot)$ and $\theta(\cdot)$, we solve the associated algebraic system (39). Figures 5 and 6 present the results for various initial conditions $(\phi_0,\ \theta_0)$, demonstrating that the FTS problem is resolved with a rapid convergence rate. This highlights the robustness and efficiency of our proposed method.

Now we want to compare our method with other well-known methods for solving FTS problems such as OFCBPLS [8], AFC [9], PFTFS [10], and IFC [11]. We apply these methods to solve the FTS problem discussed in Example 3. It can be seen in Figure 7, AFC and PFTFS methods fail to achieve synchronization. In contrast, as shown in Figure 8, the OFCBPLS method and IFC are a little successful. However, comparing the absolute errors shows that our method is more effective and achieves a lower synchronization error and superior performance.

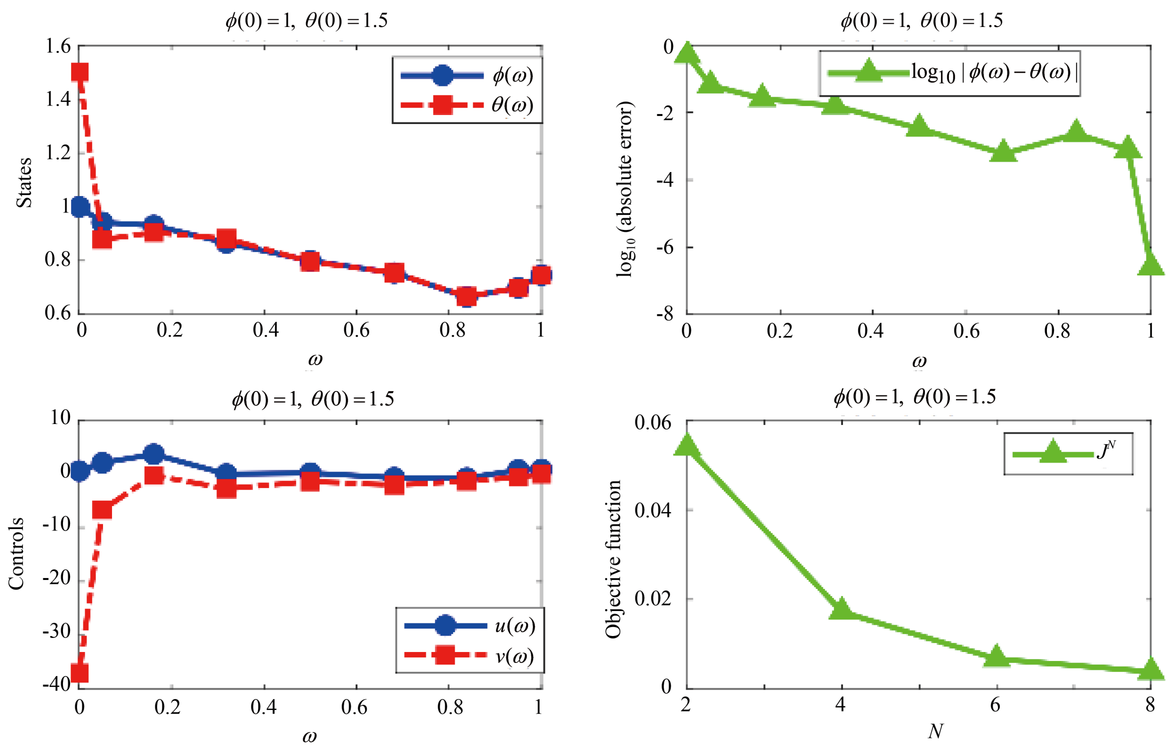

Figure 5. The solution of FTS problem by suggested method (with $N = 8$, $\phi_0 = 1$, $\theta_0 = 1.5$) for Example 3

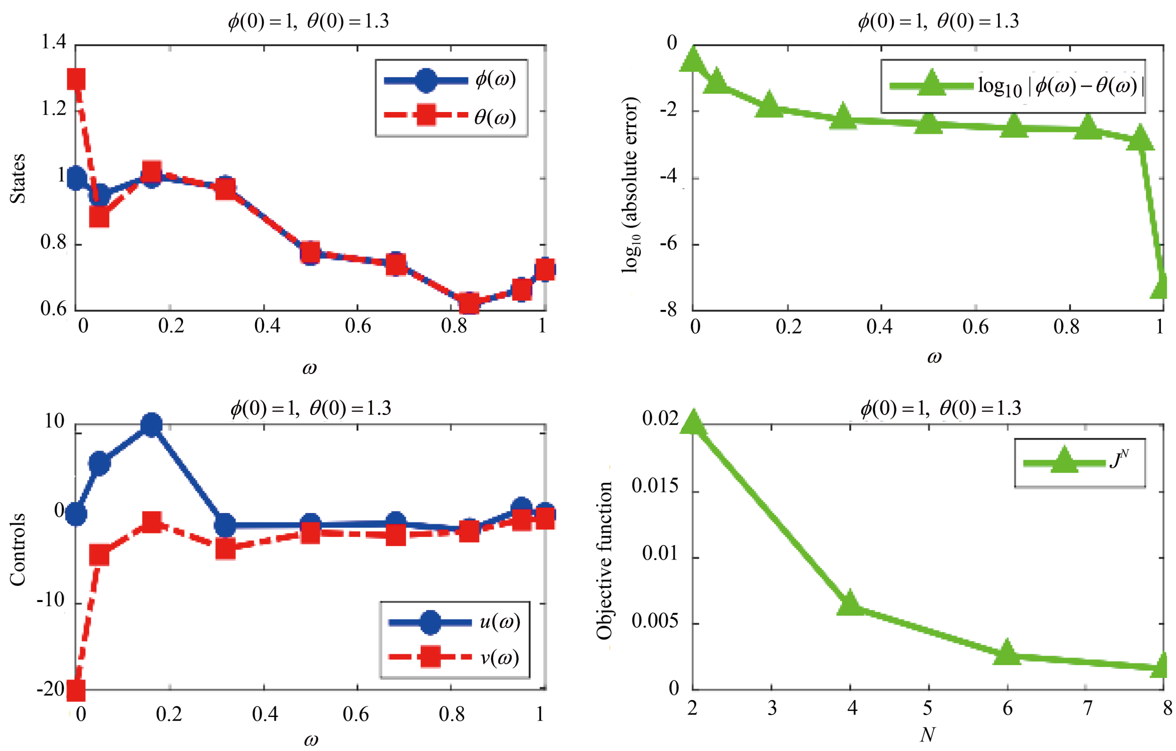

Figure 6. The solution of FTS problem by suggested method (with $N = 8$, $\phi_0 = 1$, $\theta_0 = 1.3$) for Example 3

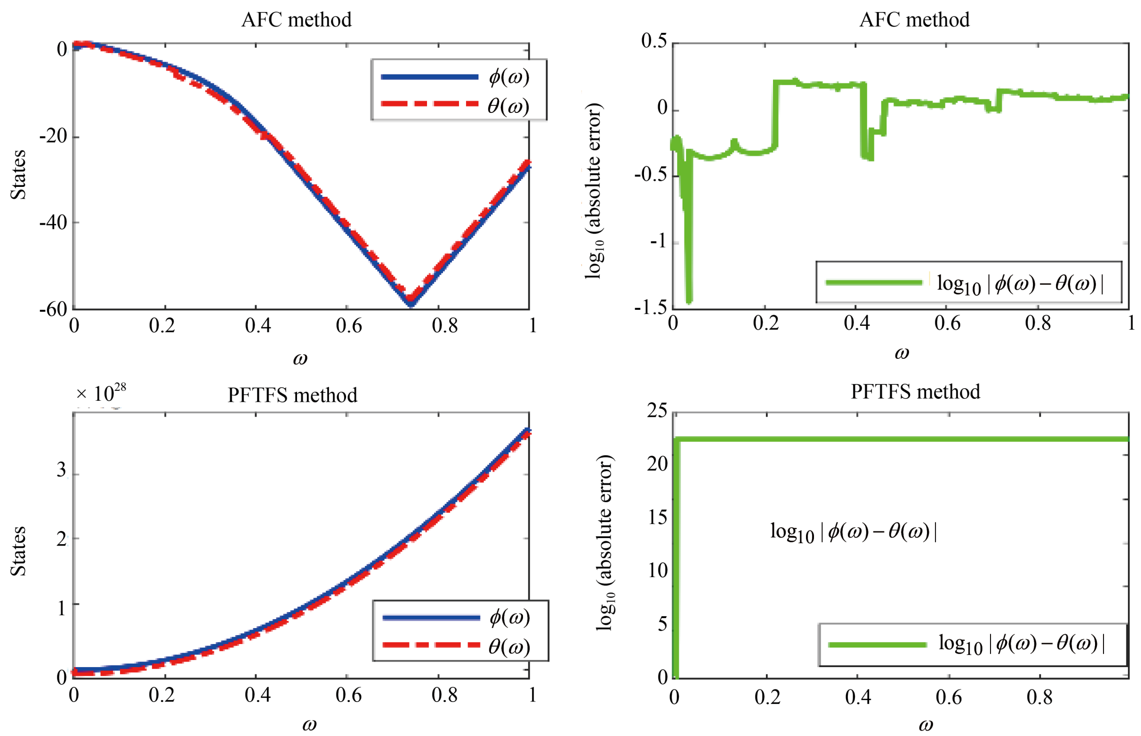

Figure 7. The failed results of solving the FTS problem (with $\phi_0 = 1$, $\theta_0 = 1.5$) by AFC and PFTFS methods for Example 3

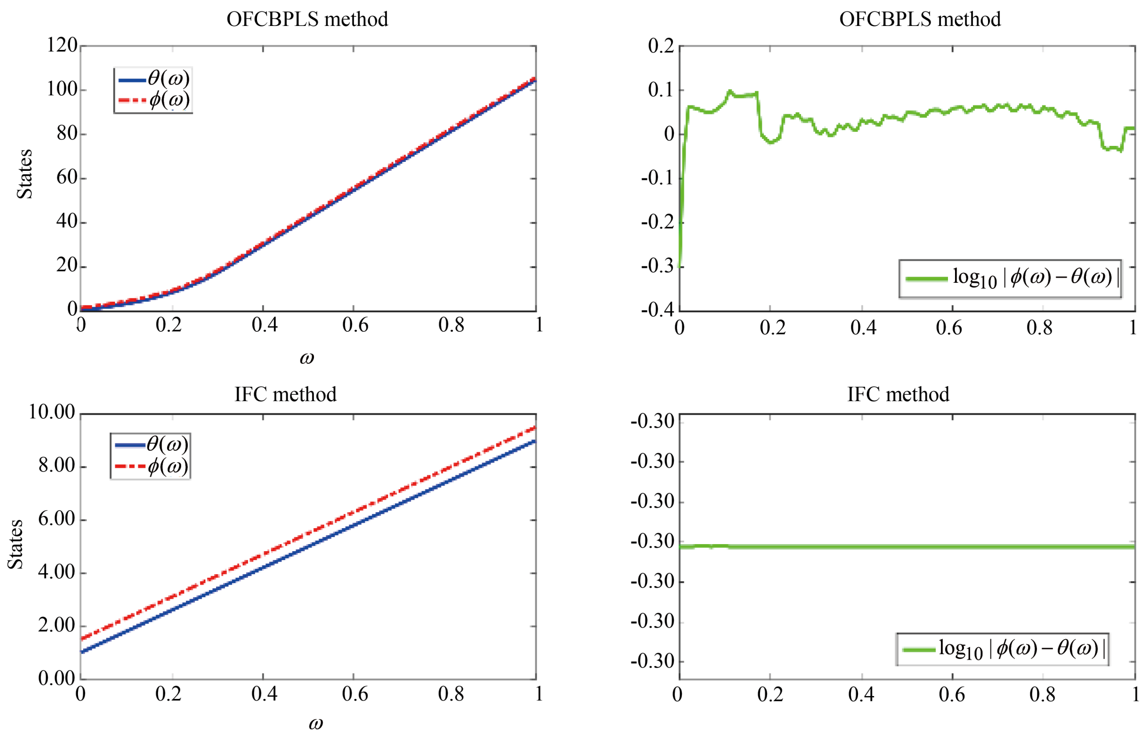

Figure 8. The solution of FTS problem (with $\phi_0 = 1$, $\theta_0 = 1.5$) by OFCBPLS and IFC methods for Example 3

Example 4 Consider the following finite-time nonlinear control system

$$\begin{cases} \dot{\phi}(\omega) = \phi^2(\omega) + \theta(\omega)\sin(\omega) + u(\omega), & 0 \le \omega \le 1 \\ \dot{\theta}(\omega) = \sin\big(\pi\theta(\omega)\big) + \cos\big(\omega\theta(\omega)\big) + v(\omega), & 0 \le \omega \le 1 \\ \phi(0) = \phi_0, \ \theta(0) = \theta_0. \end{cases} \tag{48}$$

To synchronize the trajectories $\phi(\cdot)$ and $\theta(\cdot)$, we solve the associated algebraic system (39). The results for various initial conditions $(\phi_0, \ \theta_0)$, presented in Figures 9 and 10, demonstrate that the FTS problem is successfully resolved with a rapid convergence rate, affirming the robustness and efficacy of the proposed method. A comparative analysis further highlights its performance. Established PFTFS and IFC methods, fail to achieve synchronization for Problem Example 4 as shown in Figure 11. In contrast, as shown in Figure 12, while the OFCBPLS and AFC methods are successful, our method achieves a lower synchronization error and superior performance, reveals that our approach yields superior results with consistently lower synchronization error.

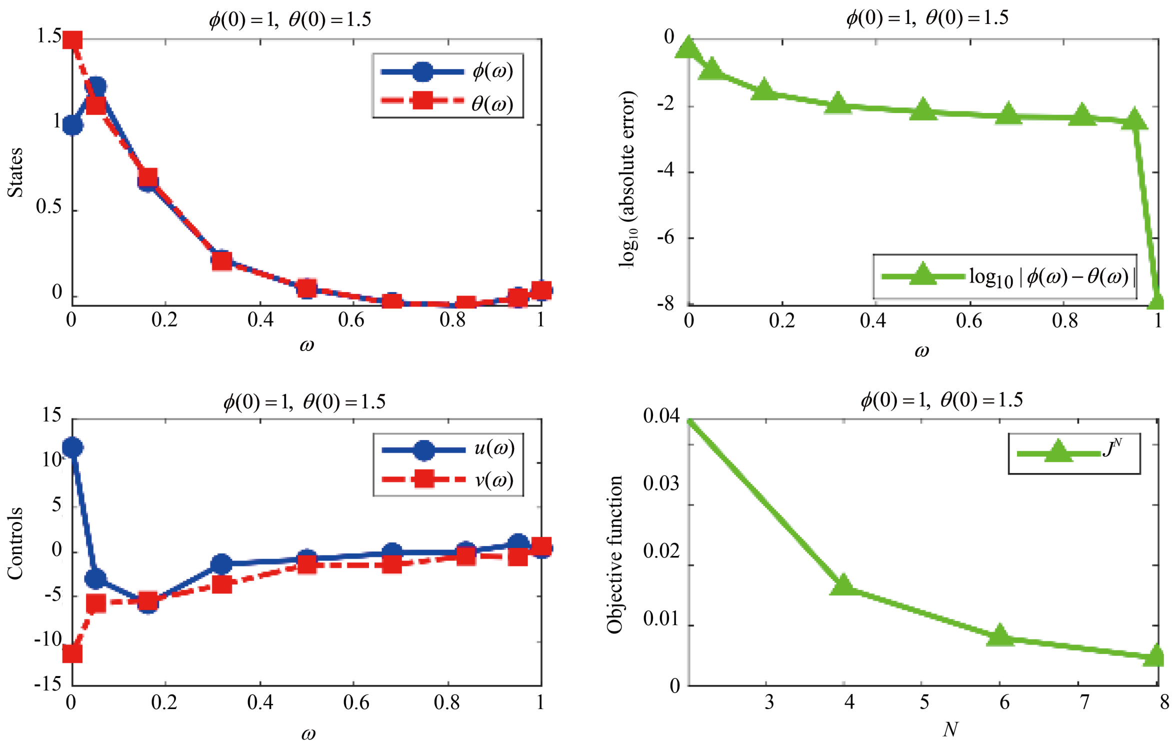

Figure 9. The solution of FTS problem by suggested method (with $N = 8$, $\phi_0 = 1$, $\theta_0 = 1.5$) for Example 4

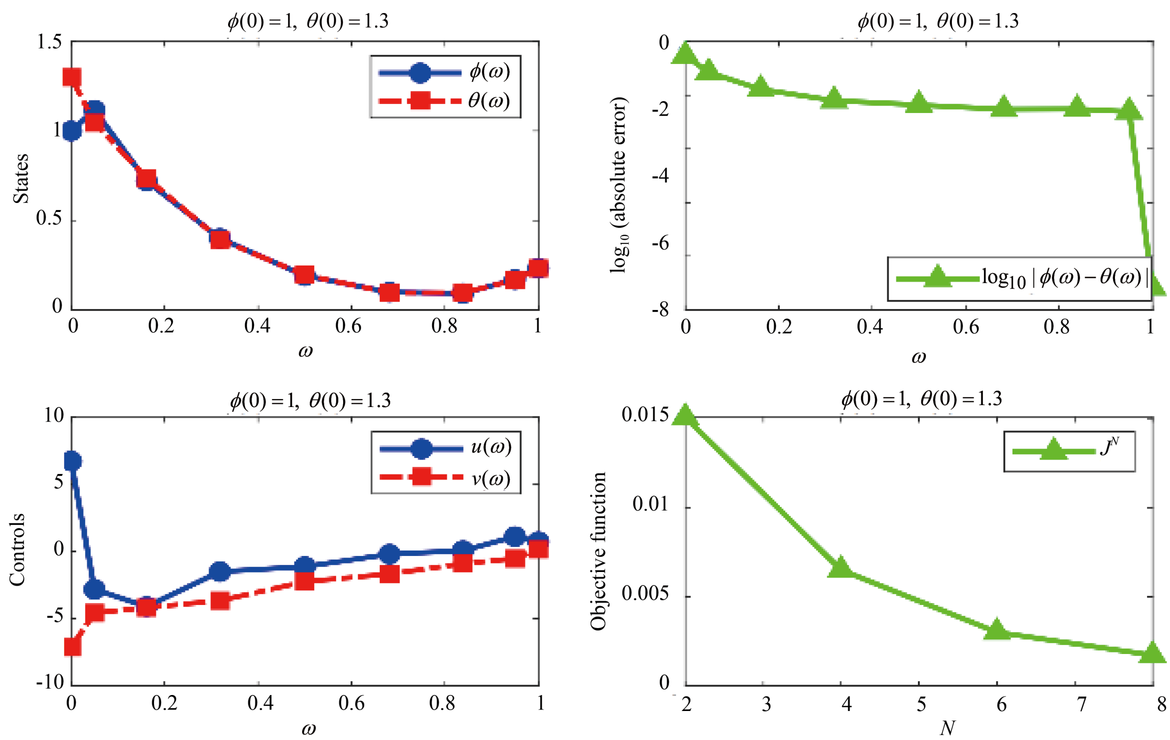

Figure 10. The solution of FTS problem by suggested method (with $N = 8$, $\phi_0 = 1$, $\theta_0 = 1.3$) for Example 4

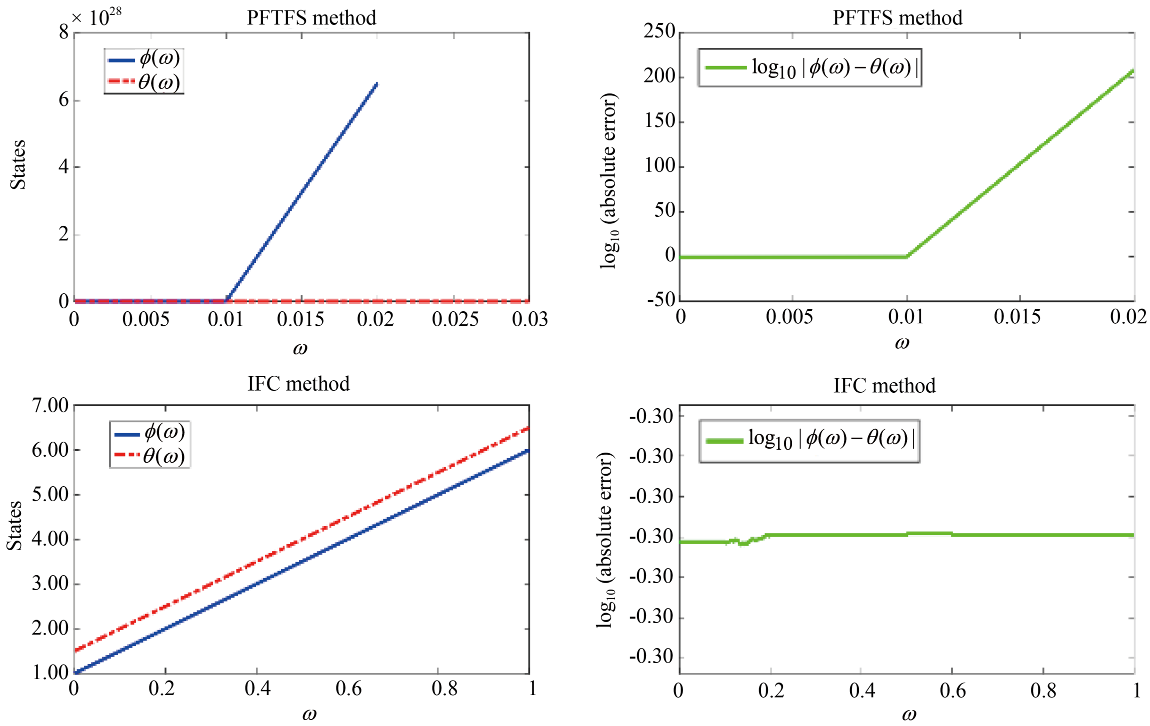

Figure 11. The failed results of solving the FTS problem (with $\phi_0 = 1$, $\theta_0 = 1.5$) by PFTFS and IFC methods for Example 4

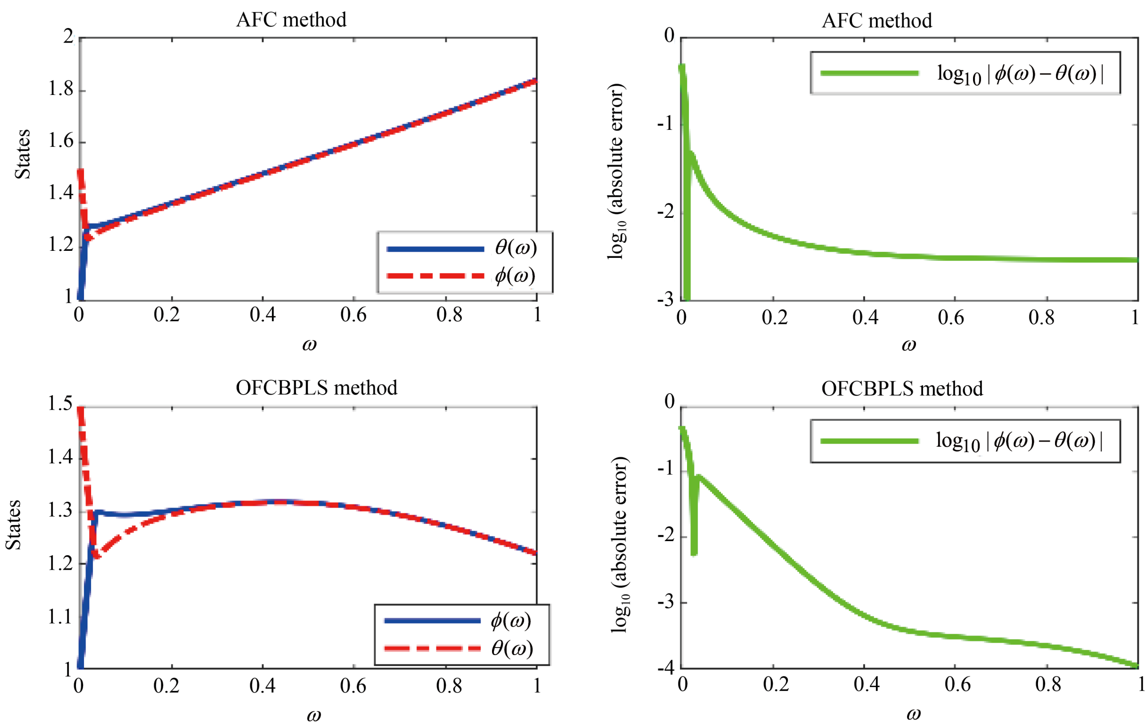

Figure 12. The solution of FTS problem (with $\phi_0 = 1$, $\theta_0 = 1.5$) by AFC and OFCBPLS methods for Example 4

Example 5 Consider the following finite-time nonlinear control system

$$\begin{cases} \dot{\phi}(\omega) = \phi(\omega) - \phi^2(\omega), & 0 \le \omega \le 2 \\ \dot{\theta}(\omega) = \theta(\omega) + \sin(\theta(\omega)) + v(\omega), & 0 \le \omega \le 2 \\ \phi(0) = 2,\ \theta(0) = 1. \end{cases} \tag{49}$$

To synchronize the trajectories $\phi(\cdot)$ and $\theta(\cdot)$, we solve the associated algebraic system (39). The results presented in Figure 13, demonstrates that the FTS problem is successfully resolved with a rapid convergence rate, confirming the method's robustness and competence. The superiority of our approach is further established through a comparative analysis. The established methods, including LB [7], IFC, PFTFS and OFCBPLS methods fail to achieve synchronization for this problem, as evidenced by the results in Figures 14 and 15. Consequently, the comparative results confirm the distinct advantage of our proposed method in effectively solving the presented FTS problem in Example 5.

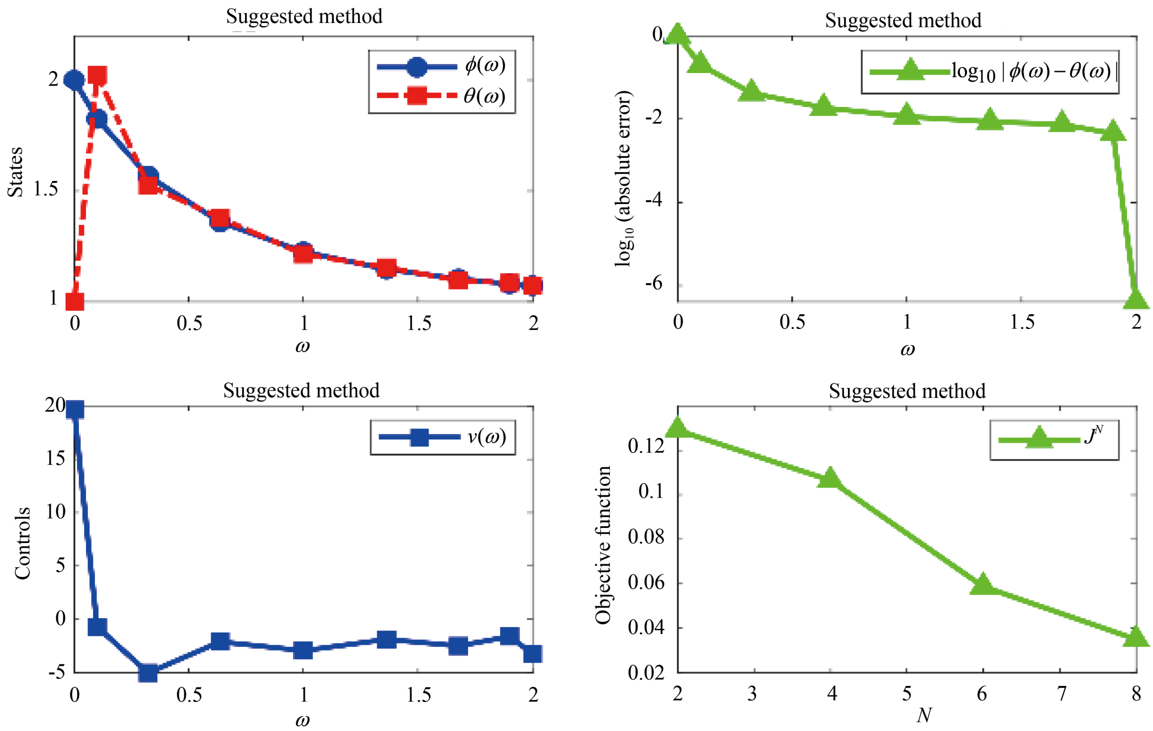

Figure 13. The solution of FTS problem by suggested method (with N = 8) for Example 5

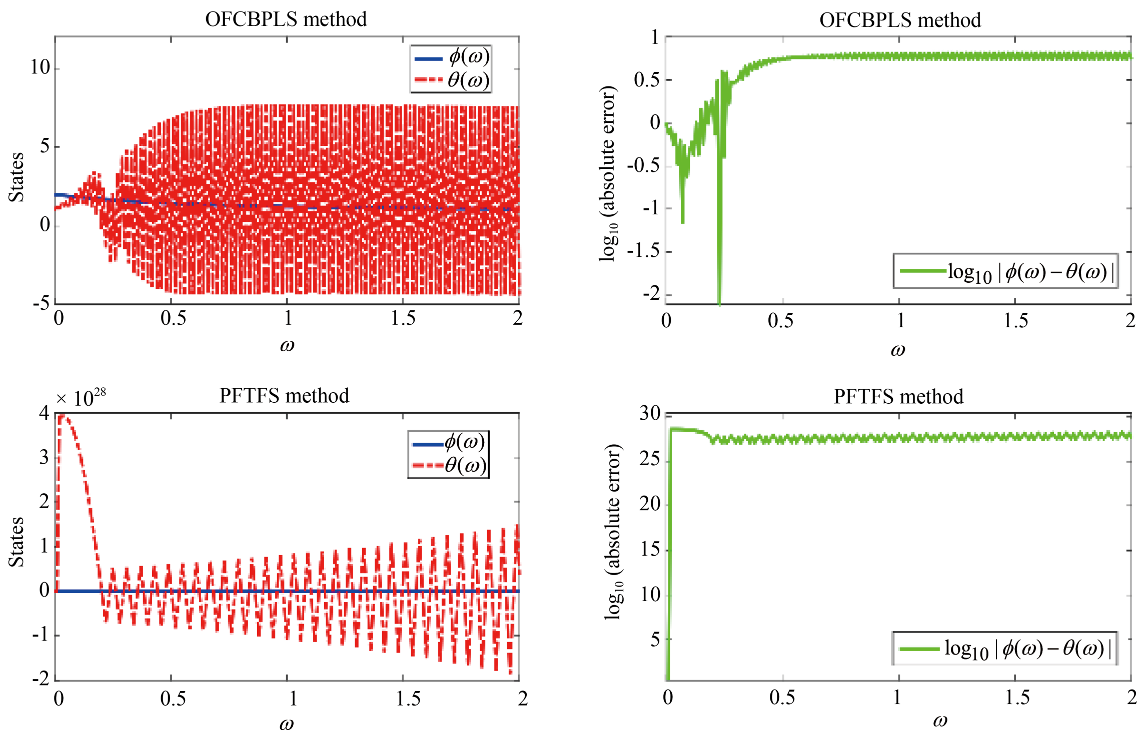

Figure 14. The failed results of solving the FTS problem by OFCBPLS and PFTFS methods for Example 5

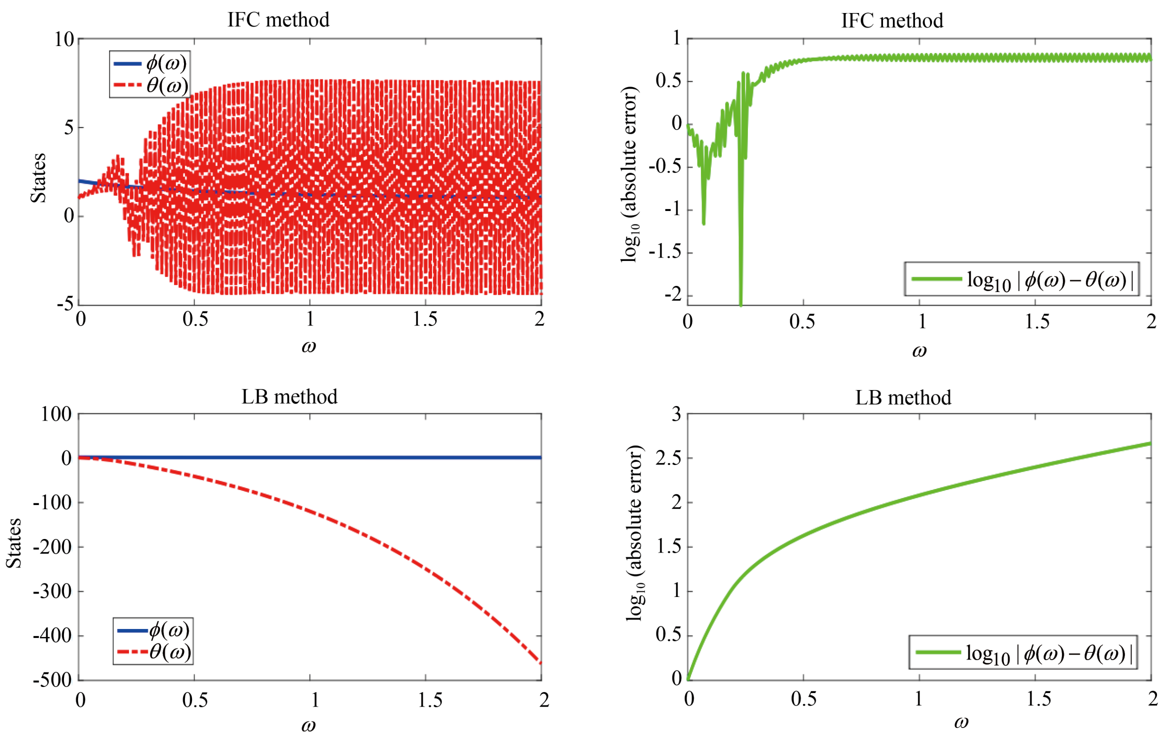

Figure 15. The failed results of solving the FTS problem by IFC and LB methods for Example 5

7. Conclusions and suggestions

This paper has introduced a unified PS framework for solving both FTS and ITS problems in nonlinear dynamical systems. The core of our methodology was the novel reformulation of these synchronization problems as equivalent optimal control problems, for which we provided rigorous proofs that their optimal solutions guarantee synchronization. By leveraging the high accuracy of Legendre pseudo-spectral methods, utilizing LGR points for ITS and LGL points for FTS, we transcribed these continuous-time problems into tractable NLP problems. The resulting systems of algebraic equations, derived from the KKT optimality conditions, were solved efficiently using the Levenberg-Marquardt algorithm. The numerical simulations unequivocally demonstrate the effectiveness and superiority of the proposed method. In numerical simulation, our approach consistently outperformed five established synchronization techniques, methods such as LB [7], OFCBPLS [8], AFC [9] for practical fixed-time synchronization, PFTFS [10], and IFC [11] for chaos synchronization, in terms of convergence speed, precision, and reliability. A key advantage of our method is its independence from the complex and often challenging task of constructing Lyapunov functions, relying instead on a robust computational framework with exponential convergence properties.

The versatility of the proposed framework opens numerous avenues for future research. Building on this work, we suggest the following promising directions:

1. Extension to complex system classes: The method can be generalized to synchronize systems governed by more complex dynamics, such as:

• Fractional fixed-order and variable-order dynamical systems: This would significantly broaden applicability in modeling viscoelastic materials, biological tissues, and other systems with memory effects.

• Systems with state and input delays: Investigating synchronization in delayed systems is crucial for applications in networked control, biology, and chemistry.

• Stochastic and hybrid dynamical systems: Extending the framework to handle random perturbations and switching dynamics would enhance its robustness for real-world applications.

2. Application to specific engineering challenges: The framework's potential for real-world impact can be explored by applying it to high-value engineering problems, including:

• Precision synchronization of multi-robot systems: For coordinated motion planning and collaborative task execution in swarms of drones or autonomous vehicles.

• Secure communication schemes: Designing new chaos-based cryptographic systems where the high accuracy of the PS method ensures reliable synchronization between transmitter and receiver.

• Control of pathological neural synchronization: Exploring its use in designing Deep Brain Stimulation (DBS) protocols to disrupt synchrony in neurological disorders like Parkinson's disease.

3. Algorithmic and computational enhancements:

• Developing an adaptive PS method that dynamically adjusts the number and location of collocation points to improve efficiency for problems with sharp transients or localized phenomena.

• Implementing a real-time iteration scheme based on the proposed NLP formulation to enable Model Predictive Control (MPC) for online synchronization tasks.

In conclusion, the PS framework developed in this paper provides a powerful, accurate, and versatile tool for synchronization. We are confident that its future extensions and applications will make substantial contributions to the fields of optimal control and complex systems theory.

Availability of data and material

There is no data and material outside the article.

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Authors'contributions

AH carried out the research, study, methodology and writing. MHNS carried out the supervisor role, MATLAB program and methodology. DB carried out the advisor role.

Conflict of interest

The authors declare that they have no competing interests.

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