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Research Article

Novel Soliton Solutions and Wave Interactions for the Nonlinear Fisher Equation Using Hirota's Method: Applications in Plasma, Optics, and Material Sciences

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Abstract: The nonlinear Fisher equation, which can be applied to crystallization, fluid dynamics, fiber optics, plasma, and biological population models, is of particular importance. We build bilinear equations using Hirota's derivatives, and then we compute several kinds of solitons. We use the Hirota Bilinear Method (HBM) and the ansatz approach to develop Lump Solution (LS), Multi-Waves (MWs), Ma-Breathers (MBs), Kuznetsov-Ma-Breathers (KMBs), and Rogue Waves solutions (RWs) for the proposed model. In the domains of science and engineering, the developed wave solutions are highly significant. We also investigated the stability analysis of the proposed model by using the linear stability approach. Solutions for breathers could be applied to increase the effectiveness of solitons in plasma waves and optical communication systems. Lump wave solutions can be used to manipulate and control laser beams for material manufacturing or laser surgery, whereas rogue wave solutions can help ensure the safety of ships and oil rigs. Under certain constraints, we additionally investigate one, two, and other soliton interactions for suggested model. To anticipate the wave dynamics, specific 2D, 3D, and contour portraits are also examined with the help of computing software Mathematica. To regulate fusion as a potential energy source in the future, these interactions may be applied to plasma stability and containment. This work presents a novel contribution to the field by exploring soliton solutions of the nonlinear Fisher equation. To the best of our knowledge, this research has not been previously addressed in the literature. The suggested method provides a more powerful computational framework for examining Non Linear Evolution Equations (NLEEs) in engineering and mathematical sciences and yields a wide variety of solutions.

Keywords: nonlinear Fisher equation, Hirota bilinear method, ansatz functions, nonlinear waves

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1. Introduction

It is widely acknowledged that almost all non-linear physical processes may be described by mathematical equations, commonly referred to as non-linear Differential Equations (DEs). Nonlinear Partial Differential Equations (NLPDEs), which are particularly important in this discipline, have been extensively used to study nonlinear phenomena. In many different fields, NLPDEs have shown themselves to be an excellent tool for defining a wide range of physical processes [1–5]. Recently, the Fisher Equation (FE) has received a lot of attention. A key role for soliton theory is played by soliton types, which include dark soliton, bright soliton, unique soliton, and others that have been described in literature. Numerous methods have been put forth to investigate various physical phenomena connected to nonlinear wave equations because of their important mathematical characteristics and broad range of applications. It is acknowledged that the efficacy of various strategies varies based on the particular situation; some methods may work well for some problems but not for others. Understanding the physical relevance and qualitative characteristics of many occurrences depends critically on the analytical solutions of NLPDEs. In the literature, a number of techniques have been effectively developed and applied to get analytical answers for NLPDEs, including neural networks approaches [6, 7], modified exp-function scheme [8–10], exponential rational function scheme [11, 12], inverse scattering approach [13, 14], generalized unified approach [15, 16], Hirota Bilinear Method (HBM) [17–19], generalized Kudryashov scheme [20–22], extended direct algebraic scheme [23, 24], extended simple equation approach [25], planner dynamical system scheme [26, 27], and many other. There are various kinds of solutions that are discussed. Significantly, solutions like [28–31] are very noteworthy. Solitons are nonlinear wave packets or pulses that can propagate continuously over extremely large distances without deteriorating or changing shape. Usually, the existence of these pulses in optical fibers depends on a delicate balancing act between the effects of self-phase oscillation and group velocity dispersion. To comprehend nonlinear processes, it is crucial to construct soliton solutions to the Non Linear Evolution Equations (NLEEs) that arise in nonlinear research. Solitons are used extensively in applied mathematics, physics, optical science, and challenges in engineering. In an optical system for communication, placing the solitons close to one another is crucial for increasing the fiber's ability to transport information. There are numerous recently developed nonlinear models that are completely integrated. These systems, which explain the self-interaction of single solitons, can be characterized as flows in multisoliton fields. As a result, there is now more interest in investigating "exact solutions" for NLPDEs.

The goal of this work is to find soliton solutions of the nonlinear Fisher equation, which is important in mathematical physics and engineering, using the HBM and ansatz approaches. These techniques are highly effective for N-solitons and solutions of nonlinear waves for NLEEs arising in optics, plasma physics, fluid dynamics and quantum mechanics. By providing analytical solutions for nonlinear waves, these techniques offer valuable analytical insight into phenomena such as optical pulse transmission, and reaction-diffusion processes in biology. In mathematical physics and engineering, this equation is used to investigate the various physical characteristics of nonlinear waves [33, 34]. The wave solution explains the transition front by moving at a steady speed from one homogeneous to another. A sine wave is therefore described in physics and mathematics as a self-sustaining, solitary wave that maintains its size and shape while traveling at a steady speed. The interplay between wave speed and frequency is referred to as dispersive effects. The physical system is described by solutions to a generic class of Partial Differential Equations (PDEs). These equations are dispersive and mildly nonlinear. The Nonlinear Fisher Equation (NLFE) is described as [35]:

$$w_t = w_{xx} + \alpha(1 - w^{\delta})(w - a). \tag{1}$$

Taking $\alpha = 1$, $\delta = 1$, and $a = 0$ yields Eq. (1) as

$$w_t - w_{xx} - w + w^2 = 0. \tag{2}$$

It explains how diffusion and reaction interact in this process. This equation was proposed by Fisher as a model for the spread of a mutant gene, where $w(x, t)$ represents the beneficial gene density. This equation is used in population dynamics and chemical kinetics, which covers challenges like the neutron population in a nuclear reaction and the nonlinear evolution of a population in a one-dimensional space [36, 37]. Over the past few years, numerous researchers have developed distinct methods for producing findings for the NLFE. Using the homotopy perturbation technique, Ağırseven and Öziş discovered analytical solutions for proposed model [38]. To obtain exact solutions for this equation, Yaun et al. [39] employed the complex technique. Feng et al. used the Cole-Hop transformation and the first integrable scheme to find solutions for the proposed equation [40]. Wu et al. used the sub-super solution technique to find solutions in traveling waves for the fisher type equation [41]. Wazwaz et al. established precise solutions for Fisher type equations using efficient method [42]. Yildirim et al. established explicit solutions for suggested equations using the differential transform technique [43].

In the present study, we use the HBM and ansatz approaches to derive several new, more general, and interesting soliton solutions such as Lump Solution (LS), Multi-Waves (MWs), Ma-Breathers (MBs), Kuznetsov-Ma-Breathers (KMBs), and Rogue Waves solutions (RWs) for the well-known nonlinear equation. The proposed techniques are highly effective for PDEs that admit bilinearization. However, they may have some limitations when apply to higher-dimensional models or the models that do not admit bilinear form. To the best of our knowledge, this study has never been reported before. By choosing the appropriate parameter choices, specific findings are displayed in 3D, contour and 2D graphs to illustrate the physical behavior of solutions. We also examined the stability analysis that validate soliton solutions. This study extend theocratical understanding proposed equation and its applications in diverse field of engineering.

The remaining work is adjusted as follows: Section 2 discusses the 1-soliton, 2-soliton, and N-soliton for Eq. (1). In Section 3, the ansatz approach is used to calculate the significant lump solutions. Section 4 covers MWs solutions. We use a hyperbolic function ansatz to calculate RWs solutions in Section 5. In Section 6, the exp function transformation is used to construct the useful interpretation of MBs. Section 7 provides the KMBs solutions. Section 8 is structured using the solution interpretations and graphical depiction. Section 9 discusses comparison analysis. Stability analysis is disclosed in Section 10. Lastly, Section 11 offers conclusions.

2. Soliton interactions

We apply the ansatz in Eq. (2) [44]:

$$w(x, t) = \frac{g(x, t)}{f(x, t)}. \tag{3}$$

Applying Eq. (3) to Eq. (2) yields

$$(D_t - D_x^2)g.f = 0, \tag{4}$$

$$D_x^2(f.f) = 0, \tag{5}$$

$$g - f = 0, \tag{6}$$

where Hirota bilinear operator D is defined as

$$D_t^k D_x^l (g.f) = \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^k \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^l g(x, t) f(x, t)|_{t'=t, x'=x}. \tag{7}$$

Here, we examine the series expansion of g and f for a small parameter τ.

$$g = \sum_{j=1}^{\infty} \tau^j g^j, \tag{8}$$

$$f = 1 + \sum_{j=1}^{\infty} \tau^j f^j. \tag{9}$$

2.1 *One soliton solution*

In order to determine the one soliton, we assume

$$g = \tau g_1, \; f = 1 + \tau f_1. \tag{10}$$

By replacing Eq. (10) into Eq. (4) and equating the coefficient of τ, we obtain

$$(g_1)_t - (g_1)_{xx} = 0. \tag{11}$$

Now, we assume

$$g_1 = e^{\lambda x + \mu t + c_1}. \tag{12}$$

Combining Eq. (12) with Eq. (11) yields

$$\mu = \lambda^2, \tag{13}$$

$$g_1 = e^{\lambda x + \lambda^2 t + c_1}. \tag{14}$$

By replacing Eq. (10) into Eq. (6) and equating the coefficient of τ, we obtain

$$g_1 - f_1 = 0 \Rightarrow g_1 = f_1. \tag{15}$$

Thus, the 1-soliton solution is

$$w_1(x, t) = \frac{g}{f} = \frac{\tau e^{\lambda x+\lambda^2 t+c_1}}{1+\tau e^{\lambda x+\lambda^2 t+c_1}}, \tag{16}$$

where

$$\tau = 1.$$

2.2 *Two soliton solution*

In order to determine the one soliton, we assume

$$g = \tau g_1 + \tau^2 g_2, \;\; f = 1+\tau f_1 + \tau^2 f_2. \tag{17}$$

By inserting Eq. (17) into Eq. (4) and equating the coefficient of τ^2, we obtain

$$(g_1)_t f_1 - (f_1)_t g_1 - (g_2)_{xx} - 2f_1(g_1)_{xx} - (g_2)_{xx} + 2(f_1)_x(g_1)_x - (f_1)_{xx} g_1 = 0. \tag{18}$$

Now, we assume

$$g_2 = e^{\lambda_1 x+\mu_1 t+c_2}. \tag{19}$$

From Eq. (19) and Eq. (22), we have

$$\mu_1 = \lambda_1^2, \tag{20}$$

$$g_2 = e^{\lambda_1 x+\lambda_1^2 t+c_2}. \tag{21}$$

From Eq. (17) into Eq. (6) and equating the coefficient of τ^2, we obtain

$$g_2 - f_2 = 0 \Rightarrow g_2 = f_2. \tag{22}$$

Thus, the 2-soliton solution is

$$w_2(x, t) = \frac{g}{f} = \frac{\tau e^{\lambda x+\lambda^2 t+c_1} + \tau^2 e^{\lambda_1 x+\lambda_1^2 t+c_2}}{1+\tau e^{\lambda x+\lambda^2 t+c_1} + \tau^2 e^{\lambda_1 x+\lambda_1^2 t+c_2}}, \tag{23}$$

where

$$\tau = 1.$$

In the same way, we can collect N-solitons by utilizing the ansatz as given in [45, 46].

$$f = \sum_{w=0,\,1} \exp\left(\sum_{i=1}^{n} w_i \omega_i + \sum_{i<j} \lambda_{ij} w_i w_j\right), \tag{24}$$

where w indicates that each w_i takes either 0 or 1, and $\omega_i = m_{1,\,i}x_1 + m_{2,\,i}x_2 + \ldots + m_{M,\,i}x_M + \omega_{i,\,0}$, $1 \leq i \leq n$ and $\omega_{i,\,0}$ arbitrary phase shifts. The Hirota bilinear form transforms the NLPDE into a symmetric bilinear equation through a suitable variable substitution, enabling systematic construction of N-soliton solutions via exponential-function expansion.

3. Lump soliton solution

The subsequent ansatz is used for lump solutions of Eq. (2) [47]:

$$w = 2[\ln f(x,\,t)]_{xx}, \tag{25}$$

and derive the subsequent form

$$\begin{aligned} &2f^2{f_x}^2 + 4ff_t{f_x}^2 + 16{f_x}^4 - 4f^2 f_x f_{xt} - 2f^3 f_{xx} - 2f^2 f_t f_{xx} \\ &- 32f{f_x}^2 f_{xx} + 10f^2 f_{xx}^2 + 2f^3 f_{xxt} + 8f^2 f_x f_{xxx} - 2f^3 f_{xxxx} = 0 \end{aligned} \tag{26}$$

We can now assume that the function f in Eq. (27) as [47]:

$$f = \tau_1^2 + \tau_2^2 + v_7, \tag{27}$$

where $\tau_1 = v_1x + v_2t + v_3$, $\tau_2 = v_4x + v_5t + v_6$, and v_i $(0 < i \leq 7)$ are constants. From Eq. (27), and Eq. (26) and the solution of equations derived from the coefficients of x and t.

When $v_1 = v_6 = 0$,

$$
\begin{cases}
v_2 = -\dfrac{\sqrt{\dfrac{44v_4^2+3_5^2+2\sqrt{484v_4^4+88v_4^2v_5^2+v_5^4}}{-5v_5^2+88v_4^2}}}{v_5}, \\
v_3 = -\dfrac{v_5}{\sqrt{\dfrac{44v_4^2+3_5^2+2\sqrt{484v_4^4+88v_4^2v_5^2+v_5^4}}{-5v_5^2+88v_4^2}}}, \\
v_7 = -\dfrac{484v_4^4\left(44v_4^2+3_5^2+2\sqrt{484v_4^4+88v_4^2v_5^2+v_5^4}\right)}{v_5^2\left(-5v_5^2+88v_4^2+88v_4^2\right)}.
\end{cases} \tag{28}
$$

By combining these with Eq. (25), we have

$$
w = \frac{2\left(-4v_4^2(v_5t+v_4x)^2+2_4^2\left((v_5t+v_4x)^2+\left(-\dfrac{v_5}{\sqrt{\lambda}}-\dfrac{v_4^2t\sqrt{\lambda}}{v_5}\right)^2-\dfrac{484v_4^4\lambda}{v_4^2}\right)\right)}{\left((v_5t+v_4x)^2+\left(-\dfrac{v_5}{\sqrt{\lambda}}-\dfrac{v_4^2t\sqrt{\lambda}}{v_5}\right)^2-\dfrac{484v_4^4\lambda}{v_5^2}\right)^2}, \tag{29}
$$

where

$$
\lambda = \frac{44v_4^2+3_5^2+2\sqrt{484v_4^4+88v_4^2v_5^2+v_5^4}}{-5v_5^2+88v_4^2}.
$$

4. Multi-waves soliton solutions

For MWs solution, consider f as [47]:

$$
\begin{cases}
f = f_0\cosh\tau_1 + f_1\cos\tau_2 + f_2\cosh\tau_3 + v_{10}, \\
\tau_1 = v_1x + v_2t + v_3, \\
\tau_2 = v_4x + v_5t + v_6, \\
\tau_3 = v_7x + v_8t + v_9,
\end{cases} \tag{30}
$$

where v_i $(0 < i \leq 10)$ are real constants. Using Eq. (30) into Eq. (26) and the solution of equations derived from the coefficients of x and t.

When $v_{10}= v_6 = 0$,

$$\begin{cases} v_1 = \dfrac{\sqrt{-140v_7^2+42+14\sqrt{128v_7^4-32v_7^2+9}}}{v_4}, \\ v_4 = 0, \\ v_5 = 0, \\ v_8 = 0. \end{cases} \tag{31}$$

By combining these with Eq. (25), we have

$$w = \frac{2\left(\Delta_1\Delta_2 - \Delta_3\right)}{\left(f_1 + f_2\cosh\left(v_9 + v_7x\right) + f_0\cosh\left(v_3 + v_2t + x\lambda\right)\right)^2}, \tag{32}$$

where

$$\lambda = \sqrt{\frac{-140v_7^2+42+14\sqrt{128v_7^4-32v_7^2+9}}{14}},\ \Delta_1 = \left(f_1 + f_2\cosh\left(v_9 + v_7x\right) + f_0\cosh\left(v_3 + v_2t + x\lambda\right)\right),$$

$$\Delta_2 = \left(v_7^2 f_2\cosh\left(v_9 + v_7x\right) + f_0\lambda^2\cosh\left(v_3 + v_2t + x\lambda\right)\right),\ \Delta_3 = \left(v_7 f_2\sinh\left(v_9 + v_7x\right) + f_0\lambda\sinh\left(av_3 + v_2t + x\lambda\right)\right)^2.$$

5. Rogue waves solutions

For RWs solution, we utilize the subsequent ansatz [48]:

$$\begin{cases} f(x,\, t) = \tau_1^2 + \tau_2^2 + m_1\cosh(\alpha(x,\, t)) + v_7, \\ \tau_1 = v_1x + v_2t + v_3, \\ \tau_2 = v_4x + v_5t + v_6, \\ \alpha(x,\, t) = b_1x + b_2t, \end{cases} \tag{33}$$

where v_i $(0 < i \leq 10)$, b_1, b_2 are real constants. We can obtain the precise equations that generate parameter values by inserting Eq. (33) into Eq. (26).

When $v_1 = b_2 = v_6 = 0$, the subsequent solutions are obtained:

$$
\begin{cases}
v_2 = -\dfrac{3v_7b_1^4 + 44v_4^2b_1^2 + 3v_7b_1^2 - 2v_4^2}{\sqrt{-\dfrac{3v_7b_1^4 + 44v_4^2b_1^2 + 3v_7b_1^2 - 2v_4^2}{6b_1^2 + 6}}\, b_1}, \\[2ex]
v_3 = \dfrac{\sqrt{-\dfrac{3v_7b_1^4 + 44v_4^2b_1^2 + 3v_7b_1^2 - 2v_4^2}{6b_1^2 + 6}}}{b_1}, \\[2ex]
v_5 = \dfrac{\sqrt{9(v_7b_1^6 + v_7b_1^2) + 132v_4^2b_1^4 + 18v_7b_1^4 + 126v_4^2b_1^2 - 6v_4^2}}{8b_1}.
\end{cases}
\tag{34}
$$

By combining these with Eq. (25), we have

$$
w = \frac{2\Delta_1\Delta_2}{\left(v_7 + \left(-\dfrac{t\Delta}{b_1\lambda} + \dfrac{\lambda}{b_1}\right)^2 + (v_4x + t\mu)^2 + m_1\cosh(b_1x)\right)^2},
\tag{35}
$$

where

$$
\lambda = \sqrt{-\frac{3v_7b_1^4 + 44v_4^2b_1^2 + 3v_7b_1^2 - 2v_4^2}{6b_1^2 + 6}},
$$

$$
\Delta = 3v_7b_1^4 + 44v_4^2b_1^2 + 3v_7b_1^2 - 2v_4^2,
$$

$$
\mu = \frac{\sqrt{9(v_7b_1^6 + v_7b_1^2) + 132v_4^2b_1^4 + 18v_7b_1^4 + 126v_4^2b_1^2 - 6v_4^2}}{8b_1},
$$

$$
\Delta_1 = \left(v_7 + \left(-\frac{t\Delta}{b_1\lambda} + \frac{\lambda}{b_1}\right)^2 + (v_4x + t\mu)^2 + m_1\cosh(b_1x)\right),
$$

$$
\Delta_2 = \left(2v_4^2 + b_1^2m_1\cosh(b_1x)\right) - \left(2v_4(v_4x + t\mu) + b_1m_1\sinh(b_1x)\right)^2,
$$

$$
(3v_7b_1^4 + 44v_4^2b_1^2 + 3v_7b_1^2 - 2v_4^2)(6b_1^2 + 6) < 0,
$$

$$
\left(9(v_7b_1^6 + v_7b_1^2) + 132v_4^2b_1^4 + 18v_7b_1^4 + 126v_4^2b_1^2 - 6v_4^2\right)(8b_1) > 0.
$$

6. Ma-breathers solution

For MBs solution, we use the subsequent ansatz [48]:

$$f(x,t)=\mathrm{e}^{-\mathrm{i}(q_1x)}\mathrm{e}^{l_1t+l_2}+m_1\mathrm{e}^{2(l_1t+l_2)}+\mathrm{e}^{\mathrm{i}(q_1x)}+a_1, \tag{36}$$

where a_1, q_1, l_1, l_2, and m_1 are real parameters. The Eq. (36) is inserted into Eq. (26) to yield a set of equations. We obtain the subsequent solutions by solving system:

$$a_1=0,\ \ l_1=\frac{1}{4}\left(-1+2q_1^2\right). \tag{37}$$

By combining these with Eq. (25), we have

$$w=\frac{2\left(\Delta_2-\Delta_1\right)}{\Delta_2^2} \tag{38}$$

where

$$\Delta_1=\left(\mathrm{i}\mathrm{e}^{\mathrm{i}q_1x}q_1-\mathrm{i}\mathrm{e}^{l_2+\frac{1}{4}\left(-1+2q_1^2\right)t-\mathrm{i}q_1x}q_1\right)^2,$$

$$\Delta_2=\left(\mathrm{e}^{\mathrm{i}q_1x}+\mathrm{e}^{l_2+\frac{1}{4}\left(-1+2q_1^2\right)t-\mathrm{i}q_1x}+\mathrm{e}^{2\left(l_2+\frac{1}{4}\left(-1+2q_1^2\right)t\right)}m_1\right),$$

$$\Delta_3=\left(-\mathrm{e}^{\mathrm{i}q_1x}q_1^2-\mathrm{e}^{l_2+\frac{1}{4}\left(-1+2q_1^2\right)t-\mathrm{i}q_1x}q_1^2\right).$$

7. Kuznetsov-Ma-breather

For KMBs solution, we utilize the subsequent ansatz [48]:

$$f(x,t)=\mathrm{e}^{-q_1(x-b_1t)}+k_1\cos\left(p\left(x+b_1t\right)\right)+k_2\cos\left(q\left(x-b_1t\right)\right), \tag{39}$$

where k_1, q_1, q, k_2, and b_1 are real parameters. The Eq. (39) is inserted into Eq. (26) to yield a set of equations that determine the values of the coefficients. We obtain the subsequent solutions:

$$q=-\frac{\sqrt{1-b_1q_1+6q_1^2-\sqrt{1-2b_1q_1+8q_1^2+b_1^2q_1^2-16b_1q_1^3+32q_1^4}}}{\sqrt{2}},\quad k_2=0. \tag{40}$$

By combining these with Eq. (25), we have

$$w=\frac{2\Delta_1\Delta_2\left(3\mathrm{e}^{-3(-b_1t+x)}+k_1\lambda\sin\left(\left(b_1t+x\right)\lambda\right)\right)^2}{\Delta_1^2}, \tag{41}$$

where

$$\Delta_1 = \left(\mathrm{e}^{-3(-b_1 t + x)} + k_1 \cos\left((b_1 t + x)\lambda\right)\right),$$

$$\Delta_2 = \left(9\mathrm{e}^{-3(-b_1 t + x)} - k_1 \lambda^2 \cos\left((b_1 t + x)\lambda\right)\right),$$

$$1 - 2b_1 q_1 + 8q_1^2 + b_1^2 q_1^2 - 16 b_1 q_1^3 + 32 q_1^4 > 0.$$

8. Results and discussion

Several investigations have been conducted on the Fisher equation. Chen and Zhang investigated multiple soliton solutions of Fisher equation by using tanh function approach [49]. Khan et al. investigated the same model with modified simple equation approach [50]. Iqbal et al. studied the proposed model to obtain mixed soliton solutions by using an explicit approach. Pinar and Kocak studied the same model and obtained multiple soliton solutions [51]. Eq. (16), Eq. (23), and Eq. (24), which describe one soliton, two solitons, and N solitons, respectively, provide the HBM that we used to study the nonlinear Fisher equation for multiple soliton for the first time in literature. Instead of using an analytical approach, the HBM is utilized to solve many soliton equations algebraically. A single soliton solution can be found using the traveling wave ansatz, however several soliton solutions for different NLEEs can be found with the aid of HBM. First, we analyzed one soliton, two soliton, and extended the concept to N-soliton for Eq. (2) via HBM, and the resulting profiles are presented. The one soliton patterns for the solution in Eq. (16) with $\lambda = -1.5$, $c_1 = -4$, are displayed in our Figure 1. In Figure 2, we presented two soliton pattern for the solution in Eq. (24) when $\lambda = -0.05$, $\lambda_1 = 0.5$, $c_1 = 4$, $c_2 = -0.05$. Using the Eq. (26) and the ansatz approach, we have obtained lump solutions for Eq. (2), and their graphs are presented. Figures 3 and 4 illustrates lump solution by considering parameter values $v_4 = -0.99$, $v_5 = 5$, and $v_4 = -6$, $v_5 = 5$, respectively. Figure 3 depicts peak bright soliton solution, while change in the wave profile can be seen by varying the parameter values in Figure 4. Figure 5 shows bright soliton solutions for Eq. (32) choosing $v_3 = 0.5$, $v_2 = -0.5$, $v_7 = -1.6$, $v_9 = -4$, $f_0 = 5$, $f_1 = 8$, $f_2 = 7$. In Figures 6 and 7, we obtain two peak bright multi-waves solutions by choosing $v_3 = 0.05$, $v_2 = 5$, $v_7 = -1.6$, $v_9 = -4$, $f_0 = 0.05$, $f_1 = 10$, $f_2 = 0.07$, and $v_3 = 0.05$, $v_2 = 5$, $v_7 = -0.6$, $v_9 = -4$, $f_0 = 0.05$, $f_1 = 10$, $f_2 = 0.007$, respectively. Figure 8 illustrates peak bright face of solutions for Eq. (35), when $v_4 = 0.05$, $v_3 = -0.5$, $v_5 = -5$, $v_7 = 0.5$, $m_1 = -9$, $b_1 = 0.5$. Figure 9 shows two peak bright face of solutions for Eq. (35), when $v_4 = 2.5$, $v_3 = -0.5$, $v_5 = -0.5$, $v_7 = 0.5$, $m_1 = -5$, $b_1 = 1.5$. Figure 10 depicts multiple peak bright faces of the solution for Eq. (38) choosing $q_1 = 0.99$, $l_2 = 0.2$, $m_1 = 0.5$. Figure 11 depicts periodic solution for Eq. (38) choosing $q_1 = 3$, $l_2 = 0.05$, $m_1 = 5$. Figure 12 shows MBs solution for Eq. (38), when $q_1 = 3$, $l_2 = 0.5$, $m_1 = 5$. Figure 13 shows peak bright soliton profile for Eq. (41), when $k_2 = 1$, $k_1 = -8$, $b_1 = -20$, $q_1 = -0.05$.

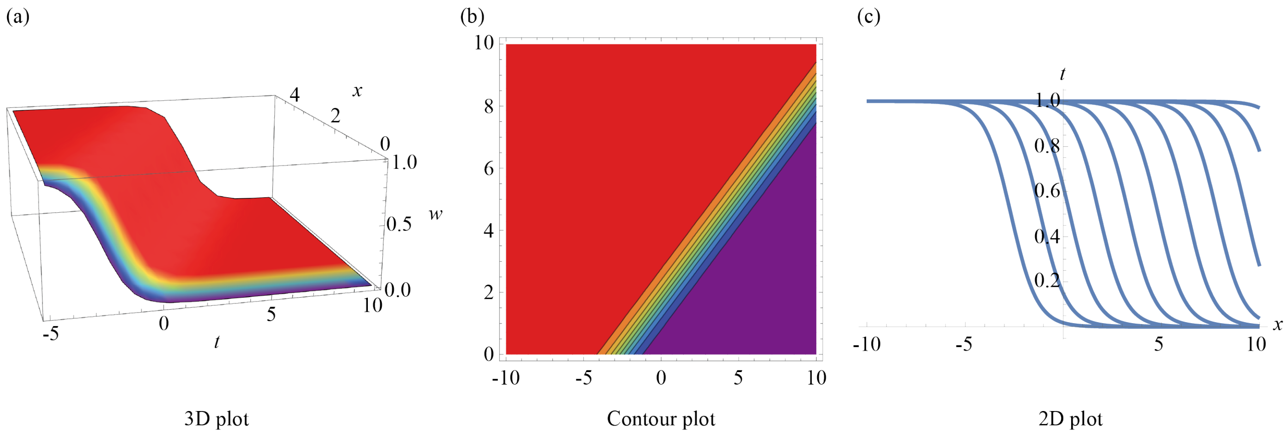


Figure 1. The 3D, contour and 2D profiles of 1-soliton for Eq. (16), representing anti-kink type soliton when $\lambda = -1.5$, $c_1 = -4$

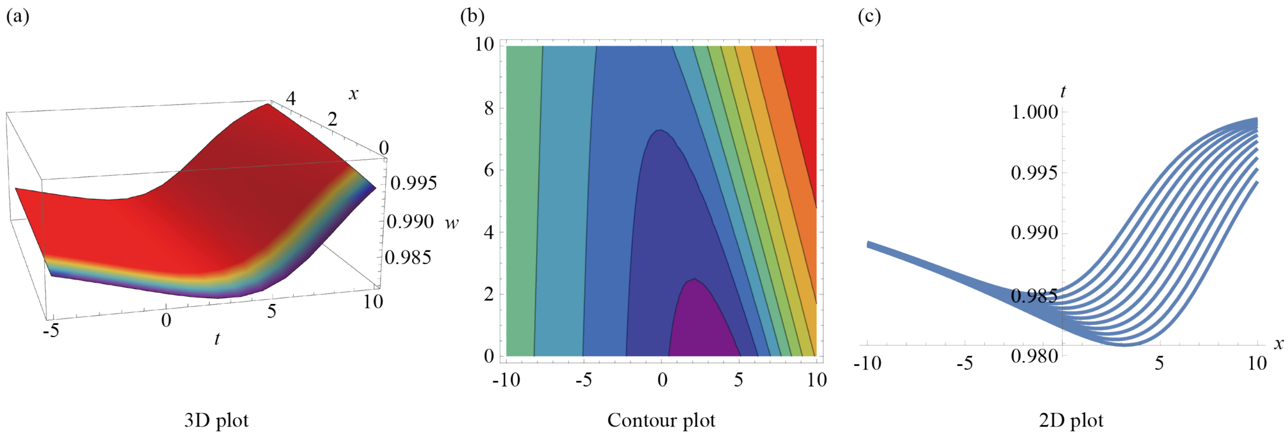


Figure 2. The 3D, contour and 2D profiles of 2-solution for Eq. (23), representing dark shaped soliton when $\lambda = -0.05$, $\lambda_1 = 0.5$, $c_1 = 4$, $c_2 = -0.05$

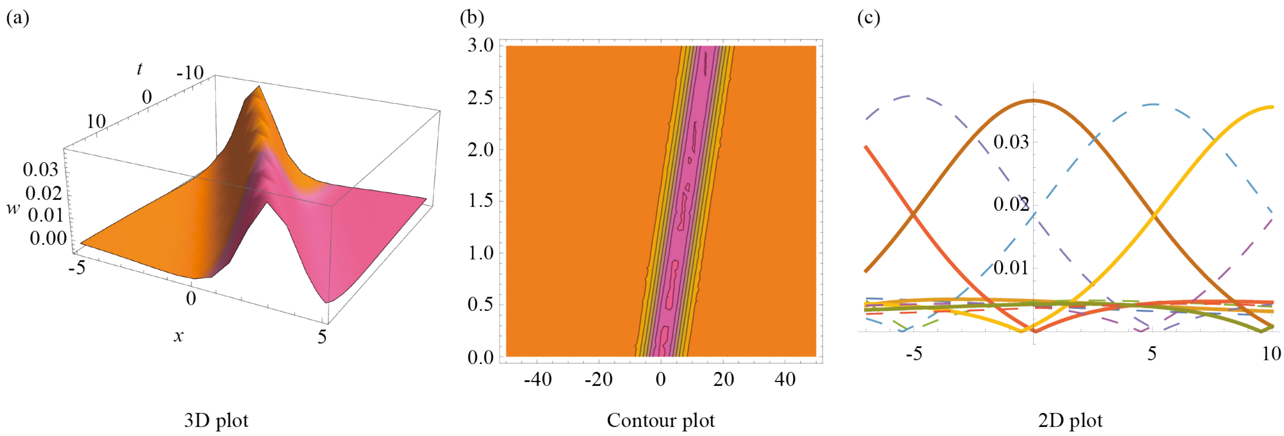


Figure 3. The 3D, contour and 2D profiles of lump solution for Eq. (29), depicting bright shaped soliton when $v_4 = -0.99$, $v_5 = 5$

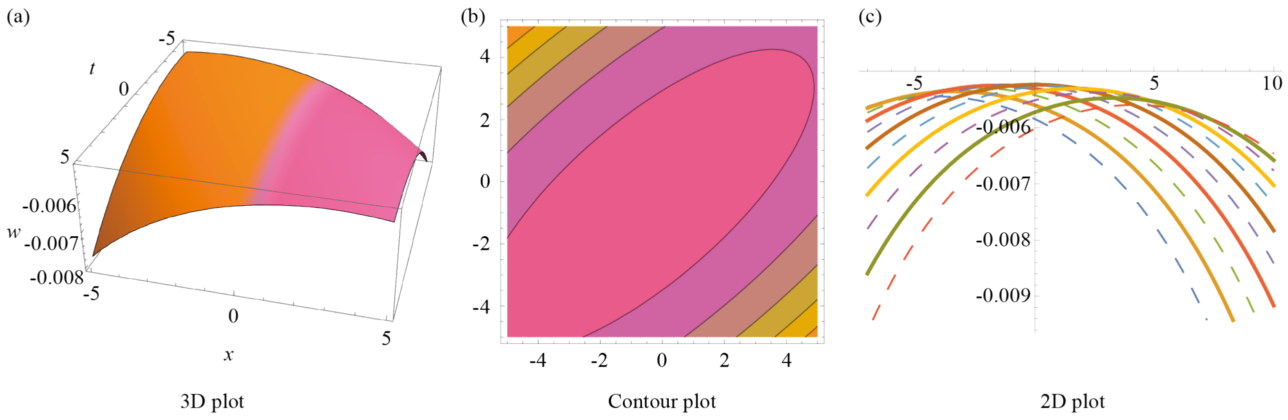


Figure 4. The 3D, contour and 2D profiles of lump solution for Eq. (29), depicting bright soliton when $v_4 = -6$, $v_5 = 5$

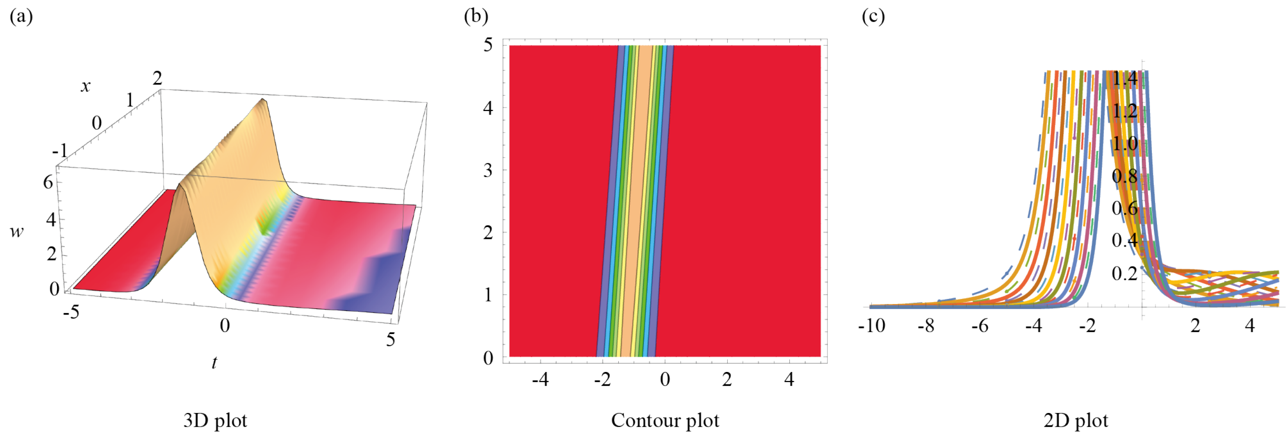


Figure 5. The 3D, contour and 2D profiles of multi-waves solution for Eq. (32), depicting peak amplitude when $v_3 = 0.5$, $v_2 = -0.5$, $v_7 = -1.6$, $v_9 = -4$, $f_0 = 5$, $f_1 = 8$, $f_2 = 7$

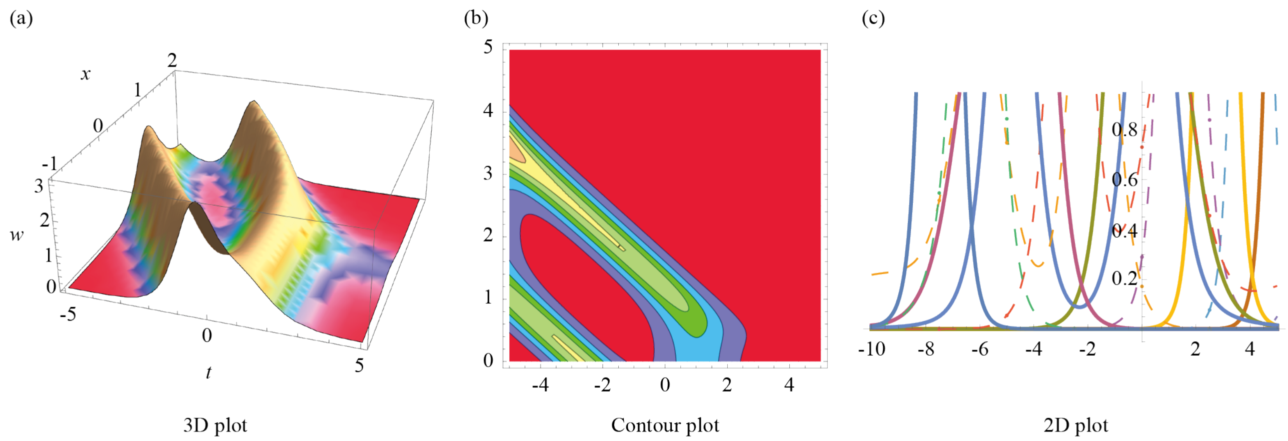


Figure 6. The 3D, contour and 2D profiles of multi-waves solution for Eq. (32), depicting peak amplitude with multiple waves when $v_3 = 0.05$, $v_2 = 5$, $v_7 = -1.6$, $v_9 = -4$, $f_0 = 0.05$, $f_1 = 10$, $f_2 = 0.07$

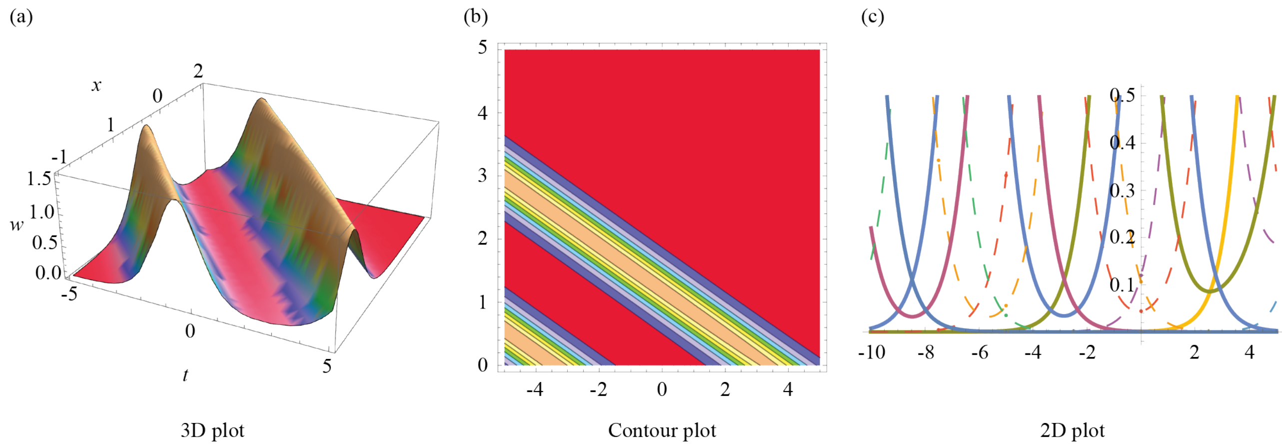


Figure 7. The 3D, contour and 2D profiles of multi-waves solution for Eq. (32), depicting peak amplitude with multiple waves when $v_3 = 0.05$, $v_2 = 5$, $v_7 = -0.6$, $v_9 = -4$, $f_0 = 0.05$, $f_1 = 10$, $f_2 = 0.007$

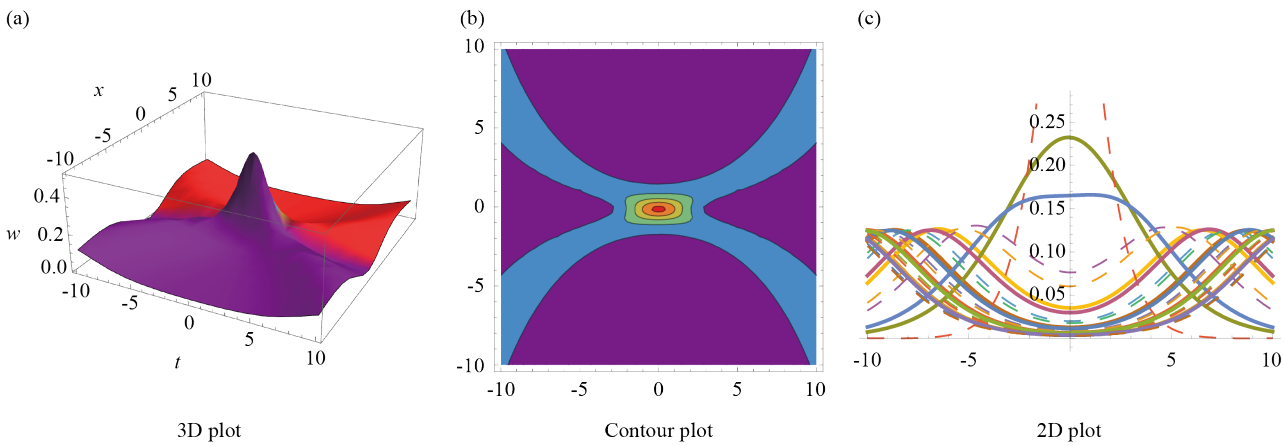


Figure 8. The 3D, contour and 2D profiles of rogue waves solution for Eq. (35), depicting sharp localized peak when $v_4 = 0.05$, $v_3 = -0.5$, $v_5 = -5$, $v_7 = 0.5$, $m_1 = -9$, $b_1 = 0.5$

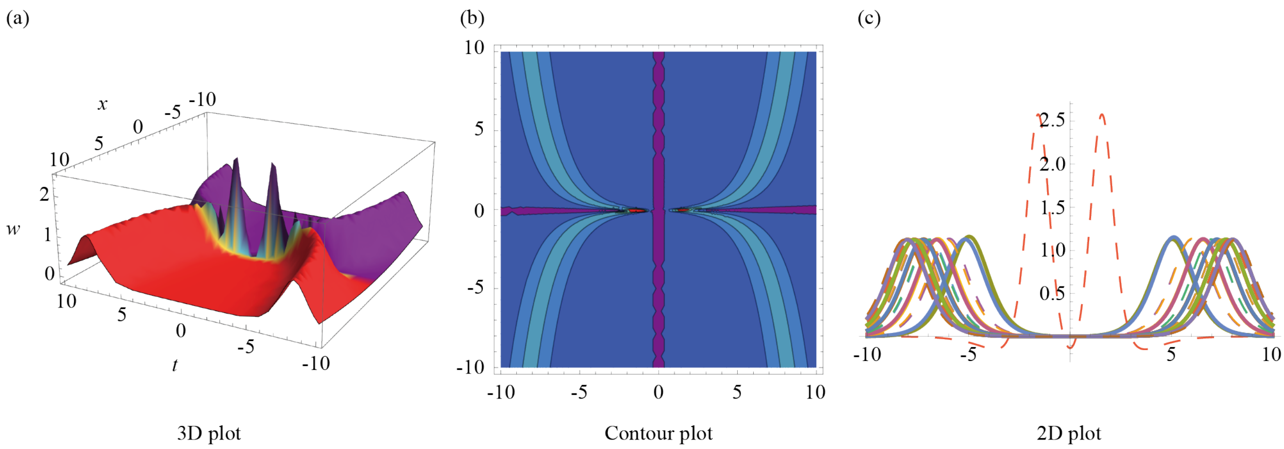
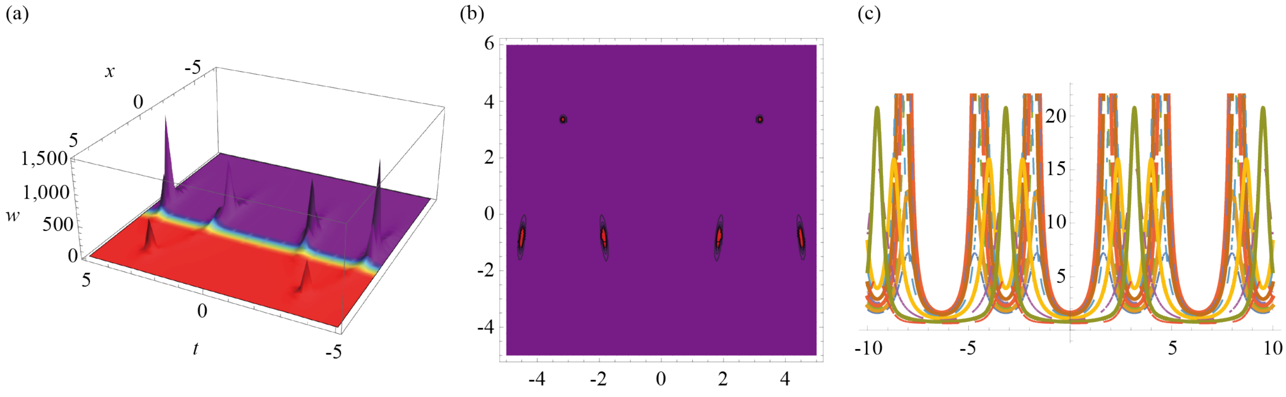
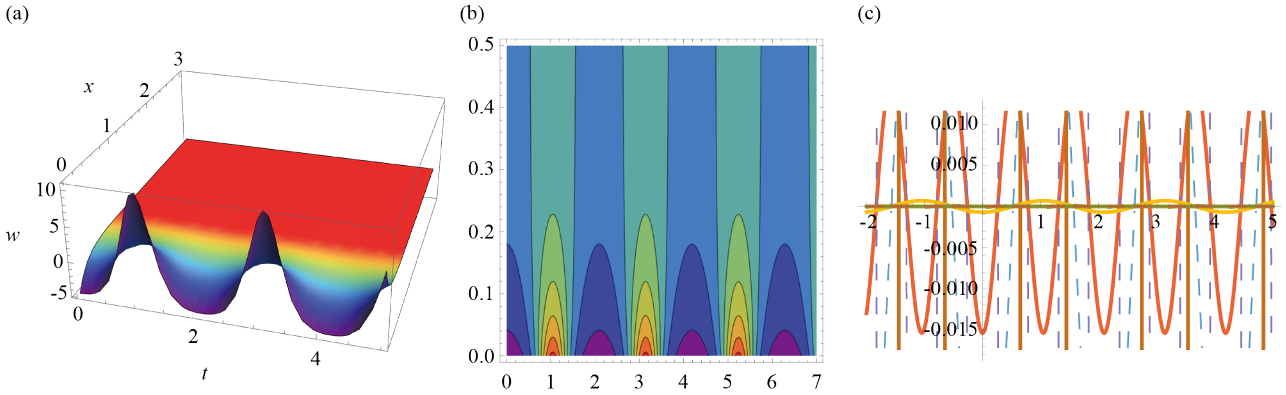


Figure 9. The 3D, contour and 2D profiles of rogue waves solution for Eq. (35), illustrating sharp localized peak when $v_4 = 2.5$, $v_3 = -0.5$, $v_5 = -0.5$, $v_7 = 0.5$, $m_1 = -5$, $b_1 = 1.5$



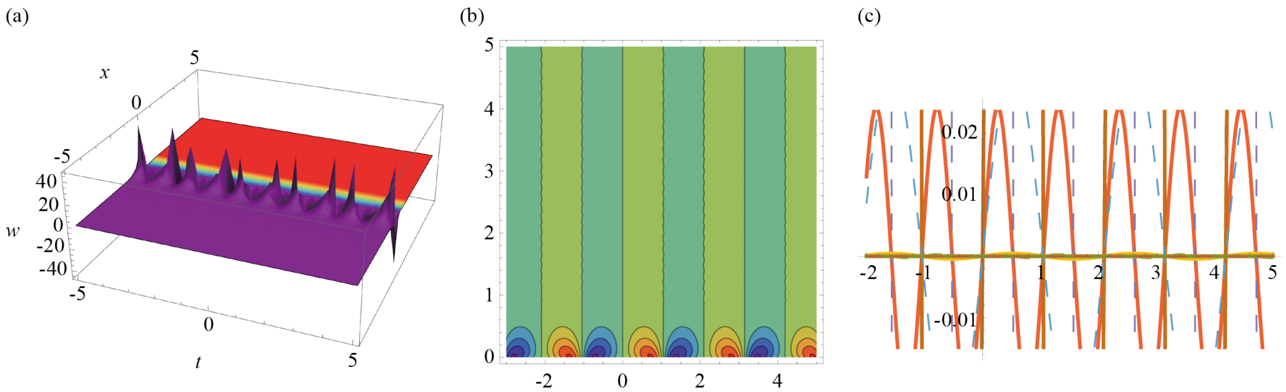
3D plot | Contour plot | 2D plot

Figure 10. The 3D, contour and 2D profiles of MBs solution for Eq. (38), exhibiting a localized oscillation in time when $q_1 = 0.99$, $l_2 = 0.2$, $m_1 = 0.5$



3D plot | Contour plot | 2D plot

Figure 11. The 3D, contour and 2D profiles of MBs solution for Eq. (38), depicting multiple positive and negative amplitudes when $q_1 = 3$, $l_2 = 0.05$, $m_1 = 5$



3D plot | Contour plot | 2D plot

Figure 12. The 3D, contour and 2D profiles of MBs solution for Eq. (38), depicting periodicity of amplitude when $q_1 = 3$, $l_2 = 0.5$, $m_1 = 5$

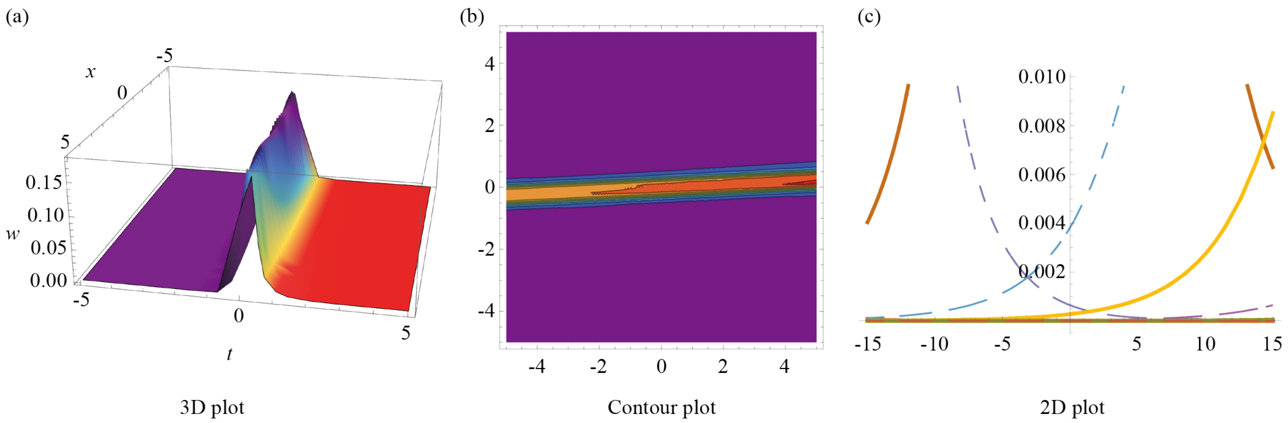


Figure 13. The 3D, contour and 2D profiles of KMBs solution for Eq. (41), depicting positive peak amplitude when $k_2 = 1$, $k_1 = -8$, $b_1 = -20$, $q_1 = -0.05$

9. Comparison analysis

Over the past few years, numerous researchers have developed distinct methods for the NLFE [38–43]. This section covers the comparison analysis of our solution with the solution derived in [35] (Table 1).

Table 1. Comparison analysis of our solutions with [35]

Solutions in [35]	Our solutions
(i) Employed the $\exp(-\Phi(\eta))$ function approach.	(i) Utilized Hirota bilinear approach and different ansatz transformations.
(ii) These solutions include trigonometric and hyperbolic solutions.	(ii) These solutions included exponential, rational, linear, hyperbolic trigonometric, and trigonometric functions.
(iii) By using suggested approach, the singular bright, combined bright-dark, singular dark, kink, and anti-kink solitons.	(iii) It yields the one soliton, two soliton, N-soliton, LS, MWs, MBs, KMBs, and RWs solutions. We also discussed the stability analysis of suggested model.

10. Stability analysis

This section of the study examines stability analysis, assuming that the perturbed solutions of Eq. (2) contain the subsequent form

$$w(x, t) = A_0 + \mu w(x, t). \tag{42}$$

The steady state solution of Eq. (2) can be easily observed for any constant A_0. By integrating Eq. (42) into Eq. (2), one obtains

$$\mu w_t - \mu w_{xx} + \mu^3 w^3 + A_0^3 + 3\mu^2 w^2 A_0 + 3A_0^2 \mu w - \mu w - A_0 = 0. \tag{43}$$

By linearizing the Eq. (43), we obtain

$$\mu w_t - \mu w_{xx} + 3A_0^2 \mu w - \mu w - A_0 = 0. \tag{44}$$

Assume the following solution to Eq. (44):

$$w(x, t) = e^{(i\sigma x + \eta t)}, \tag{45}$$

where σ is the normalized wave number. By inserting Eq. (45) into Eq. (44) and solving for η, we obtain

$$\eta(\sigma) = A_0(1 - 3A_0) - \sigma^2. \tag{46}$$

The real part is negative for all σ values, as shown by Eq. (46), hence any superposition of the solutions will seem to decay. Consequently, the dispersion is stable.

11. Conclusions

In this research, with the aid of HBM and ansatz function approaches, we investigated the nonlinear Fisher equation for one, two, and N-soliton solutions along with stability analysis. In contemporary telecommunications networks, a single soliton can be utilized to send data over great distances without experiencing appreciable signal deterioration, whereas two soliton solutions can be used to investigate the interaction of two water waves. We also studied LS, MWs, MBs, KMBs, and RWs solutions. These solutions could be utilized for localized disruptions in water waves and the regulation of light pulse propagation in fiber optics. Additionally, localized chemical or electrical disruptions in biological systems, including chemical reaction-diffusion systems and neural networks, can be modeled using these solutions. Above all, we discovered RWs solutions that have multiple applications in wave energy harvesting, marine and ocean engineering, insuring and risk evaluation, weather prediction, oceanography, and leisure and tourism. Furthermore, 2D and 3D visual representations are offered to illustrate the dynamical behavior of the identified solutions. We can better comprehend the dynamical properties and patterns of these solutions by using the contour profiles. As far as we are aware, these solutions are unique and have never been discovered previously. In addition to helping to construct more precise theoretical frameworks, the research deepens our understanding of soliton behavior in complex models. This approach can be used to handle a wide range of higher-dimensional nonlinear challenges that arise in mathematical physics and the applied sciences. The developed results have shown that the used approach is a potent, highly effective, and powerful. In the future, the these techniques will be used to solve a number of significant models in the domains of mathematical physics, engineering, and the natural sciences due to their significant performance.

Data availability statement

All data generated or analyzed during this study are included in this manuscript.

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Conflict of interest

The authors declare no conflict of interest.

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Appendix

We use the subsequent transformation as given in Eq. (25)

$$w(x,\,t)=2\frac{f(x,\,t)f_{xx}(x,\,t)-f_x(x,\,t)^2}{f(x,\,t)^2},$$

into

$$w_t-w_{xx}-w+w^2=0.$$

By calculating the required derivative as:

$$w_t=2.\frac{(f_tf_{xx}+ff_{xxt}-2f_xf_{xt})f^2-2(ff_{xx}-f_x^2)ff_t}{f^4},$$

and

$$w_{xx}=2.\frac{f^3f_{xxxx}-3f^2f_{xx}^2+2f_x^4}{f^4}-8.\frac{(f^3f_{xxx}-3f^2f_xf_{xx}+2ff_x^3)f_x}{f^5}.$$

By replacing these derivatives in considered equation and after simplification we have:

$$2f^2{f_x}^2+4ff_t{f_x}^2+16{f_x}^4-4f^2f_xf_{xt}-2f^3f_{xx}-2f^2f_tf_{xx}$$

$$-32ff_x{}^2f_{xx}+10f^2f_{xx}^2+2f^3f_{xxt}+8f^2f_xf_{xxx}-2f^3f_{xxxx}=0.$$