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Research Article

Fixed Point Theorems for Multipoint Chatterjea-Type Mappings

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Abstract: In this paper, we give a four point and a multipoint generalizations of the Chatterjea type mappings. Such mappings are repectively based on the mappings of four or n points of the space instead of two or three. We compare these mappings with some contractive type mappings in the literature. We prove several novel fixed point theorems supported by auxiliary lemmas and propositions in metric spaces for these mappings. Moreover, an illustrative example concerning the four-point Chatterjea type mappings is provided.

***Keywords*:** fixed point, multipoint, Chatterjea type mapping

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1. Introduction and preliminaries

Among the most well-known generalizations of Banach contraction theorem is Chatterjea's [1] theorem. Banach [2], Kannan [3], and Chatterjea's [1] fixed point theorems are independent, and the latter two define the metric space's completeness, as we observe. Chatterjea's results were highly intriguing, and they led to numerous more research avenues and generalizations such as [4–12]. Various fixed point theorems have been established for Chatterjea Type Mappings (CTM). Completeness in Metric Spaces (MS) can be effectively characterized via Chatterjea mappings [13].

In 1969, Nadler [14] introduced multi-valued contraction mappings, and later Ćirić [15] proposed generalized contraction mappings.

Petrov [16] recently studied a novel class of mappings, described as Mappings Contracting Perimeters of Triangles (MCPT). He derived a new result for this class and provided explicit examples of MCPT that do not satisfy the classical contraction condition. Also, the author demonstrated that such mappings are continuous. Later, generalized Kannan type and Generalized Chatterjea Type Mappings (GCTM) were introduced by Petrov and Bisht [17], Pacurar and Popescu [18] and Bisht and Petrov [19]. Bisht [20], Bisht and Petrov [21, 22], Bey et al. [23], and Petrov [24] also provide relevant insights into this topic.

In this work, a four-point generalization of the CTM is introduced. Unlike the classical approach, this generalization is based on mappings involving four distinct points in the space. In the last part of the section on the main result, we further generalize the concept to present $n-$point CTM. Furthermore, we prove some new lemmas and propositions and

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fixed point theorems in MS for these two types of mappings. Within this framework, several new fixed point theorems are established in the context of MS. Also, we give an example related to the four point CTM.

The first part serves as a lead-up to the sequel. The fundamental results of this manuscript require the following notation, definitions and theorems which we offer below.

Let $F(\Upsilon) = \{r \in V : \Upsilon r = r\}$, the set of all fixed point of Υ.

Chatterjea [1] formulated a fundamental fixed point theorem indicating that there is a fixed point for discontinuous mappings.

Theorem 1 [1] Let (V, ρ) be a Complete Metric Space (CMS) and $\Upsilon : V \to V$ be a mapping such that for all $w, r \in V$ the inequality

$$\rho(\Upsilon w, \Upsilon r) \leq \zeta\left(\rho(w, \Upsilon r) + \rho(r, \Upsilon w)\right) \tag{1}$$

is satisfied, where $\zeta \in \left[0, \frac{1}{2}\right)$. Then Υ has a unique fixed point.

Let $|V|$ represent the cardinality of a set V. Let (V, ρ) be a MS with $|V| \geq n$, and let $w_1, w_2, \ldots, w_n \in V$, where $n \geq 2$. The sum of distances taken over all distinct pairs of points in the set $\{w_1, w_2, \ldots, w_n\}$ is denoted by

$$S(w_1, w_2, \ldots, w_n) = \sum_{1 \leq i < j \leq n} \rho(w_i, w_j).$$

Petrov [16] offered the following definition in 2023, generalizing the Banach contraction mappings.

Definition 1 [16] Let (V, ρ) be a MS with $|V| \geq 3$. The mapping $\Upsilon : V \to V$ is called a MCPT on V if there exists $\zeta \in [0, 1)$ such that

$$\rho(\Upsilon w, \Upsilon r) + \rho(\Upsilon w, \Upsilon z) + \rho(\Upsilon r, \Upsilon z) \leq \zeta\left(\rho(w, r) + \rho(w, z) + \rho(r, z)\right)$$

is satisfied for all three pairwise distinct points $w, r, z \in V$.

Petrov [16] showed that MCPT are continuous.

After that, Petrov and Bisht [17] gave the concept of generalized Kannan type mapping in the manner defined below.

Definition 2 [17] Let (V, ρ) be a MS with $|V| \geq 3$. The mapping $\Upsilon : V \to V$ is called a generalized Kannan type mapping on V if there exists $\zeta \in \left[0, \frac{2}{3}\right)$ such that

$$\rho(\Upsilon w, \Upsilon r) + \rho(\Upsilon w, \Upsilon z) + \rho(\Upsilon r, \Upsilon z) \leq \zeta\left(\rho(w, \Upsilon w) + \rho(r, \Upsilon r) + \rho(z, \Upsilon z)\right)$$

is satisfied for all three pairwise distinct points $w, r, z \in V$.

Later, Pacurar [18] proposed the following extension of the Chatterjea mapping

Definition 3 [18] Let (V, ρ) be a MS with $|V| \geq 3$. The mapping $\Upsilon : V \to V$ is called a GCTM on V if there exists $\zeta \in \left[0, \frac{1}{2}\right)$ such that

$$\rho(\Upsilon w, \Upsilon r) + \rho(\Upsilon w, \Upsilon z) + \rho(\Upsilon r, \Upsilon z) \leq \zeta \begin{pmatrix} \rho(w, \Upsilon r) + \rho(w, \Upsilon z) + \rho(r, \Upsilon w) \\ +\rho(r, \Upsilon z) + \rho(z, \Upsilon w) + \rho(z, \Upsilon r) \end{pmatrix}$$

is true for all three pairwise distinct points $w, r, z \in V$.

Definition 4 [24] Let (V, ρ) be a MS with $|V| \geq n$, for $n \geq 2$. The mapping $\Upsilon : V \to V$ is said to be a Mapping Contracting the Total Pairwise Distances (MCTPD) between n points if there exists $\zeta \in [0, 1)$ such that

$$S(\Upsilon w_1, \Upsilon w_2, \ldots, \Upsilon w_n) \leq \zeta S(w_1, w_2, \ldots, w_n) \tag{2}$$

is satisfied for all n pairwise distinct points $w_1, w_2, \ldots, w_n \in V$.

If $n = 4$ is taken in (2), then

$$\begin{pmatrix} \rho(\Upsilon w, \Upsilon r) + \rho(\Upsilon w, \Upsilon z) + \rho(\Upsilon r, \Upsilon z) \\ + \rho(\Upsilon w, \Upsilon t) + \rho(\Upsilon r, \Upsilon t) + \rho(\Upsilon z, \Upsilon t) \end{pmatrix} \leq \zeta \begin{pmatrix} \rho(w, r) + \rho(w, z) + \rho(r, z) \\ + \rho(w, t) + \rho(r, t) + \rho(z, t) \end{pmatrix}$$

for all four pairwise distinct points $w, r, z, t \in V$. Also, Υ is called contracting the sum of diagonol and perimeters of rectangles on V.

Remark 1 Note that $w, r, z, t \in V$ must be pairwise distinct. Otherwise, it is obvious that this definition is the same as the definition of a MCPT contraction mapping.

2. Main results

Within this section, we propose our four-point generalization of the CTM. We call it four-point CTM. To clarify this new concept and underline its importance, an illustrative example is provided. Moreover, we prove some new fixed point results in MS by using these new mappings.

Definition 5 Let (V, ρ) be a MS with $|V| \geq 4$. A self-mapping $\Upsilon : V \to V$ is called four-point CTM if there exists a constant $\zeta \in \left[0, \frac{1}{3}\right)$ satisfying the inequality

$$\begin{pmatrix} \rho(\Upsilon w, \Upsilon r) + \rho(\Upsilon r, \Upsilon z) + \rho(\Upsilon z, \Upsilon w) \\ + \rho(\Upsilon w, \Upsilon t) + \rho(\Upsilon r, \Upsilon t) + \rho(\Upsilon z, \Upsilon t) \end{pmatrix} \leq \zeta \begin{pmatrix} \rho(w, \Upsilon r) + \rho(w, \Upsilon z) + \rho(w, \Upsilon t) \\ + \rho(r, \Upsilon w) + \rho(r, \Upsilon z) + \rho(r, \Upsilon t) \\ + \rho(z, \Upsilon w) + \rho(z, \Upsilon r) + \rho(z, \Upsilon t) \\ + \rho(t, \Upsilon w) + \rho(t, \Upsilon r) + \rho(t, \Upsilon z) \end{pmatrix} \tag{3}$$

for all four pairwise distinct points $w, r, z, t \in V$.

Definition 6 [1] Let (V, ρ) be a MS and $\Upsilon : V \to V$ be a mapping. A point $w \in V$ is said to be periodic point of period n if $\Upsilon^n(w) = w$. The smallest such positive integer n for which this condition holds is referred to as the prime period of the point w.

The following example exhibits that our four-point CTM is neither a Chatterjea mapping nor a GCTM [18].

Example 1 Let (V, ρ) be a MS such that $V = \{w, r, z, t\}$, $\rho(w, r) = \frac{7}{10}$, $\rho(w, z) = \rho(w, t) = \rho(r, t) = \rho(r, z) = \rho(z, t) = 1$ and $\Upsilon : V \to V$ be a mapping such that $\Upsilon w = w$, $\Upsilon r = r$, $\Upsilon z = x$ and $\Upsilon t = y$.

For $w, r \in V$ in (1), we have $\frac{7}{10} \leq \zeta \left(\frac{7}{10} + \frac{7}{10}\right)$ which is not true for $\zeta \in \left[0, \frac{1}{2}\right)$. So, that Υ is not Chatterjea mapping.

Next, for three points $w, r, z \in V$, we have

$$\rho(\Upsilon w, \Upsilon r)+\rho(\Upsilon w, \Upsilon z)+\rho(\Upsilon r, \Upsilon z) \leq \zeta\left(\begin{array}{c} \rho(w, \Upsilon r)+\rho(w, \Upsilon z)+\rho(r, \Upsilon w) \\ +\rho(r, \Upsilon z)+\rho(z, \Upsilon w)+\rho(z, \Upsilon r) \end{array}\right).$$

This gives

$$2\rho(w, r) \leq \zeta\left(3\rho(w, r)+\rho(z, w)+\rho(z, r)\right).$$

Plugging in the values, we obtain

$$\frac{14}{10} \leq \zeta\left(\frac{21}{10}+2\right)$$

so that $\zeta \geq \frac{14}{41} > \frac{1}{3}$. Consequently, Υ is not a GCTM.

However, for four points $w, r, z, t \in V$,

$$\left(\begin{array}{c} \rho(\Upsilon w, \Upsilon r)+\rho(\Upsilon r, \Upsilon z)+\rho(\Upsilon z, \Upsilon w) \\ +\rho(\Upsilon w, \Upsilon t)+\rho(\Upsilon r, \Upsilon t)+\rho(\Upsilon z, \Upsilon t) \end{array}\right) \leq \zeta\left(\begin{array}{c} \rho(w, \Upsilon r)+\rho(w, \Upsilon z)+\rho(w, \Upsilon t) \\ +\rho(r, \Upsilon w)+\rho(r, \Upsilon z)+\rho(r, \Upsilon t) \\ +\rho(z, \Upsilon w)+\rho(z, \Upsilon r)+\rho(z, \Upsilon t) \\ +\rho(t, \Upsilon w)+\rho(t, \Upsilon r)+\rho(t, \Upsilon z) \end{array}\right)$$

$$\left(\begin{array}{c} \rho(w, r)+\rho(r, w)+\rho(w, w) \\ +\rho(w, r)+\rho(r, r)+\rho(w, r) \end{array}\right) \leq \zeta\left(\begin{array}{c} \rho(w, r)+\rho(w, w)+\rho(w, r)+\rho(r, w) \\ +\rho(r, w)+\rho(r, r)+\rho(z, w)+\rho(z, r) \\ +\rho(z, r)+\rho(t, w)+\rho(t, r)+\rho(t, w) \end{array}\right)$$

or

$$4\rho(w, r) \leq \zeta\left(4\rho(w, r)+\rho(z, w)+2\rho(z, r)+2\rho(t, w)+\rho(t, r)\right)$$

and we get

$$\frac{28}{10} \leq \zeta\left(\frac{28}{10}+6\right).$$

Thus, $\zeta \geq \frac{7}{22} \in \left[0, \frac{1}{3}\right)$ and we conclude that Υ is a four-point CTM.

On the other hand, the next proposition establishes the relationship between CTM and four-point CTM.

Proposition 1 Let $\Upsilon : V \to V$ be a CTM with $\zeta \in \left[0, \frac{1}{3}\right)$. Then Υ is a four-point CTM.

Proof. Let (V, ρ) be a CMS such that $|V| \geq 4$ and let the mapping $\Upsilon : V \to V$ be four-point CTM and let $w_i \in V$, $i = 1, 2, 3, 4$ pairwise distinct points. Considering CTM, we get the following six cases:

$$\rho(\Upsilon w_1, \Upsilon w_2) \leq \zeta \left[\rho(w_1, \Upsilon w_2) + \rho(w_2, \Upsilon w_1)\right],$$

$$\rho(\Upsilon w_2, \Upsilon w_3) \leq \zeta \left[\rho(w_2, \Upsilon w_3) + \rho(w_3, \Upsilon w_2)\right],$$

$$\rho(\Upsilon w_1, \Upsilon w_3) \leq \zeta \left[\rho(w_1, \Upsilon w_3) + \rho(w_3, \Upsilon w_1)\right],$$

$$\vdots$$

$$\rho(\Upsilon w_3, \Upsilon w_4) \leq \zeta \left[\rho(w_3, \Upsilon w_4) + \rho(w_4, \Upsilon w_3)\right].$$

When the right and left parts of the aforementioned inequalities are summarized, we get

$$S(\Upsilon w_1, \Upsilon w_2, \Upsilon w_3, \Upsilon w_4) \leq \zeta \begin{pmatrix} \sum\limits_{i=2}^{4} \rho\left(w_1, \Upsilon w_i\right) + \sum\limits_{\substack{i=1 \\ i \neq 2}}^{4} \rho\left(w_2, \Upsilon w_i\right) \\ \\ + \sum\limits_{\substack{i=1 \\ i \neq 3}}^{4} \rho\left(w_3, \Upsilon w_i\right) + \sum\limits_{i=1}^{3} \rho\left(w_4, \Upsilon w_i\right) \end{pmatrix}.$$

The desired assertion is now obvious. □

Next, to prove that four-point CTM is a Chatterjea contraction under certain restrictions, we need to prove the following lemma.

Lemma 1 Let (V, ρ) be a CMS with $|V| \geq 4$ and $\Upsilon : V \to V$ be a continuous four-point CTM. If z is an accumulation point of V, then

$$(1 - 2\zeta)\,\rho(\Upsilon z, \Upsilon r) \leq \zeta\left(3\rho(z, \Upsilon r) + \rho(r, \Upsilon z)\right) \tag{4}$$

for all $r \in V$.

Proof. Given any accumulation point $z \in V$, and any $r \in V$. If $z = r$, then it is obvious that (4) is true. Suppose that $z \neq r$. Since z is an accumulation point of V, there exists a sequence $w_k \to z$ such that $w_k \neq z$, $w_k \neq r$ for all k and all w_k are pairwise distinct. Thus, by (2), we have

$$S(\Upsilon z, \Upsilon r, \Upsilon w_{k+1}, \Upsilon w_{k+2}) \leq \zeta \begin{pmatrix} \rho(z, \Upsilon r) + \rho(r, \Upsilon z) + \rho(r, \Upsilon w_{k+1}) \\ +\rho(r, \Upsilon w_{k+2}) + \rho(z, \Upsilon w_{k+1}) + \rho(z, \Upsilon w_{k+2}) \\ +\rho(w_{k+1}, \Upsilon r) + \rho(w_{k+2}, \Upsilon r) + \rho(w_{k+1}, \Upsilon z) \\ +\rho(w_{k+2}, \Upsilon z) + \rho(w_{k+1}, \Upsilon w_{k+2}) + \rho(w_{k+2}, \Upsilon w_{k+1}) \end{pmatrix}.$$

Also, we have

$$S(\Upsilon z, \Upsilon r, \Upsilon w_{k+1}, \Upsilon w_{k+2}) = \begin{pmatrix} \rho(\Upsilon r, \Upsilon z) + \rho(\Upsilon r, \Upsilon w_{k+1}) + \rho(\Upsilon r, \Upsilon w_{k+2}) \\ +\rho(\Upsilon z, \Upsilon w_{k+1}) + \rho(\Upsilon z, \Upsilon w_{k+1}) + \rho(\Upsilon w_{k+1}, \Upsilon w_{k+2}) \end{pmatrix}.$$

Thus, we get

$$\begin{pmatrix} \rho(\Upsilon r, \Upsilon z) + \rho(\Upsilon r, \Upsilon w_{k+1}) + \rho(\Upsilon r, \Upsilon w_{k+2}) \\ +\rho(\Upsilon z, \Upsilon w_{k+1}) + \rho(\Upsilon z, \Upsilon w_{k+1}) + \rho(\Upsilon w_{k+1}, \Upsilon w_{k+2}) \end{pmatrix}$$

$$\leq \zeta \begin{pmatrix} \rho(z, \Upsilon r) + \rho(r, \Upsilon z) + \rho(r, \Upsilon w_{k+1}) \\ +\rho(r, \Upsilon w_{k+2}) + \rho(z, \Upsilon w_{k+1}) + \rho(z, \Upsilon w_{k+2}) \\ +\rho(w_{k+1}, \Upsilon r) + \rho(w_{k+2}, \Upsilon r) + \rho(w_{k+1}, \Upsilon z) \\ +\rho(w_{k+2}, \Upsilon z) + \rho(w_{k+1}, \Upsilon w_{k+2}) + \rho(w_{k+2}, \Upsilon w_{k+1}) \end{pmatrix}.$$

Since $\rho(w_k, z) \to 0$, due to the continuity of Υ, we have $\Upsilon w_k \to \Upsilon z$, $\rho(\Upsilon r, \Upsilon w_k) \to \rho(\Upsilon r, \Upsilon z)$, $\rho(w_k, \Upsilon w_k) \to \rho(z, \Upsilon z)$. Letting $k \to \infty$, we have

$$3\rho(\Upsilon z, \Upsilon r) \leq \zeta(3(\rho(z, \Upsilon r) + \rho(r, \Upsilon z)) + 6\rho(z, \Upsilon z))$$

$$\rho(\Upsilon z, \Upsilon r) \leq \zeta((\rho(z, \Upsilon r) + \rho(r, \Upsilon z)) + 2\rho(z, \Upsilon z)).$$

Using triangular inequality, we have $\rho(z, \Upsilon z) \leq \rho(z, \Upsilon r) + \rho(\Upsilon r, \Upsilon z)$. Applying the triangle inequality to the inequality above, we obtain

$$(1 - 2\zeta)\rho(\Upsilon z, \Upsilon r) \leq \zeta(3\rho(z, \Upsilon r) + \rho(r, \Upsilon z)).$$

$\square$

Proposition 2 Let (V, ρ) be a MS with $|V| \geq 4$ and $\Upsilon : V \to V$ be a continuous four-point CTM with $\zeta \in \left[0, \frac{1}{6}\right)$. If every point of V is accumulation point, then, Υ is a Chatterjea contraction.

Proof. Interchanging the roles of r and z in (4), we obtain the following inequality:

$$(1-2\zeta)\,\rho\,(\Upsilon r, \Upsilon z) \leq \zeta\,(3\rho\,(r, \Upsilon z) + \rho\,(z, \Upsilon r)). \tag{5}$$

Adding (4) and (5) yields

$$\rho\,(\Upsilon r, \Upsilon z) \leq \frac{2\zeta}{(1-2\zeta)}\,(\rho\,(r, \Upsilon z) + \rho\,(z, \Upsilon r)).$$

Since $\zeta \in \left[0, \frac{1}{6}\right)$, we have $\frac{2\zeta}{(1-2\zeta)} \in \left[0, \frac{1}{2}\right)$ and hence the proof. $\square$

Proposition 3 Let (V, ρ) be a MS with $|V| \geq 4$ and $\Upsilon : V \to V$ be a mapping contracting the sum of diagonals and perimeters of rectangles on V with $\alpha \in \left[0, \frac{1}{4}\right)$. Then Υ is a four-point CTM.

Proof. By the supposition, (2) holds for all four pairwise distinct points $w_1, w_2, w_3, w_4 \in V$, using the triangle inequalities

$$\rho\,(w_i, w_j) \leq \rho\,(w_i, \Upsilon w_j) + \rho\,(\Upsilon w_i, \Upsilon w_j) + \rho\,(w_j, \Upsilon w_i)$$

for each pair, we obtain

$$\begin{aligned} S(w_1, w_2, w_3, w_4) &\leq \sum_{i=1}^{4}\sum_{\substack{j=1\\ i\neq j}}^{4} \rho\,(w_i, \Upsilon w_j) + \sum_{1\leq i\leq j\leq 4} \rho\,(\Upsilon w_i, \Upsilon w_j) \\ &\leq \sum_{i=1}^{4}\sum_{\substack{j=1\\ i\neq j}}^{4} \rho\,(w_i, \Upsilon w_j) + S(\Upsilon w_1, \Upsilon w_2, \Upsilon w_3, \Upsilon w_4). \end{aligned} \tag{6}$$

Substituting (6) into (2), we get that

$$S(\Upsilon w_1, \Upsilon w_2, \Upsilon w_3, \Upsilon w_4) \leq \alpha \left(\sum_{i=1}^{4}\sum_{\substack{j=1\\ i\neq j}}^{4} \rho\,(w_i, \Upsilon w_j) + S(\Upsilon w_1, \Upsilon w_2, \ldots, \Upsilon w_n) \right).$$

So, we have

$$S(\Upsilon w_1, \Upsilon w_2, \Upsilon w_3, \Upsilon w_4) \leq \frac{\alpha}{1-\alpha} \left(\sum_{i=1}^{n}\sum_{\substack{j=1\\ i\neq j}}^{n} \rho\,(w_i, \Upsilon w_j) \right).$$

Since $\alpha \in \left[0, \frac{1}{4}\right)$, we obtain that $\zeta = \frac{\alpha}{1-\alpha} \in \left[0, \frac{1}{3}\right)$. Thus, Υ is an four-point CTM. □

Now, we present our main theorems on the existence of at most three fixed points.

Theorem 2 Let (V, ρ) be a CMS with $|V| \geq 4$ and let $\Upsilon : V \to V$ be a self mapping satisfying:

(i) Υ has no periodic points of prime periods 2 and 3.

(ii) Υ is a four point CTM.

Then Υ has a fixed point in V and the number of fixed points is at most 3.

Proof. Let $w_0 \in V$, $w_1 = \Upsilon w_0$, $w_2 = \Upsilon w_1$, $\ldots$, $w_{n+1} = \Upsilon w_n$. Suppose that w_n is not a fixed point of the mapping Υ for every $n = 0, 1, \ldots$. Then, in particular, we have $w_n = \Upsilon w_{n-1} \neq w_{n-1}$, $w_{n+1} = \Upsilon(\Upsilon w_{n-1}) \neq w_{n-1}$, and $w_{n+2} = \Upsilon^2(\Upsilon w_n) \neq w_{n+1}$ for every $n = 1, 2, \ldots$. Hence, by condition (i), w_{n-1}, w_n, w_{n+1} and w_{n+2} are mutually distinct for every $n = 1, 2, \ldots$. We begin proof by taking $w = w_0$, $r = w_1$, $z = w_2$, and $t = w_3$ in (3), which characterizes the four point Chatterjea type condition, we get

$$\begin{pmatrix} \rho(\Upsilon w_0, \Upsilon w_1) + \rho(\Upsilon w_1, \Upsilon w_2) + \rho(\Upsilon w_2, \Upsilon w_0) \\ +\rho(\Upsilon w_0, \Upsilon w_3) + \rho(\Upsilon w_1, \Upsilon w_3) + \rho(\Upsilon w_2, \Upsilon w_3) \end{pmatrix}$$

$$\leq \zeta \begin{pmatrix} \rho(w_0, \Upsilon w_1) + \rho(w_0, \Upsilon w_2) + \rho(w_0, \Upsilon w_3) + \rho(w_1, \Upsilon w_0) \\ +\rho(w_1, \Upsilon w_2) + \rho(w_1, \Upsilon w_3) + \rho(w_2, \Upsilon w_0) + \rho(w_2, \Upsilon w_1) \\ +\rho(w_2, \Upsilon w_3) + \rho(w_3, \Upsilon w_0) + \rho(w_3, \Upsilon w_1) + \rho(w_3, \Upsilon w_1) \end{pmatrix}$$

and

$$\begin{pmatrix} \rho(w_1, w_2) + \rho(w_2, w_3) + \rho(w_3, w_1) \\ +\rho(w_1, w_4) + \rho(w_2, w_4) + \rho(w_3, w_4) \end{pmatrix}$$

$$\leq \zeta \begin{pmatrix} \rho(w_0, w_2) + \rho(w_0, w_3) + \rho(w_0, w_4) + \rho(w_1, w_1) \\ +\rho(w_1, w_3) + \rho(w_1, w_4) + \rho(w_2, w_1) + \rho(w_2, w_2) \\ +\rho(w_2, w_4) + \rho(w_3, w_1) + \rho(w_3, w_2) + \rho(w_3, w_2) \end{pmatrix}$$

$$= \zeta \begin{pmatrix} \rho(w_0, w_2) + \rho(w_0, w_3) + \rho(w_0, w_4) + 2\rho(w_1, w_3) + \rho(w_1, w_4) \\ +\rho(w_2, w_1) + \rho(w_2, w_4) + 2\rho(w_3, w_2) \end{pmatrix}.$$

So, we can write

$$(1-\zeta) \begin{pmatrix} \rho(w_1, w_2) + \rho(w_2, w_3) + \rho(w_1, w_3) \\ +\rho(w_1, w_4) + \rho(w_2, w_4) \end{pmatrix} + \rho(w_3, w_4)$$

$$\leq \zeta\left(\rho(w_0, w_2)+\rho(w_0, w_3)+\rho(w_0, w_4)+\rho(w_1, w_3)+\rho(w_3, w_2)\right).$$

Exploiting the triangle inequality $\rho(w_0, w_4) \leq \rho(w_0, w_1)+\rho(w_1, w_2)+\rho(w_2, w_4)$ coupled with $\zeta\rho(w_3, w_4) < \rho(w_3, w_4)$, we get

$$(1-\zeta)\begin{pmatrix} \rho(w_1, w_2)+\rho(w_2, w_3)+\rho(w_1, w_3) \\ +\rho(w_1, w_4)+\rho(w_2, w_4)+\rho(w_3, w_4) \end{pmatrix}$$

$$< (1-\zeta)\begin{pmatrix} \rho(w_1, w_2)+\rho(w_2, w_3)+\rho(w_1, w_3) \\ +\rho(w_1, w_4)+\rho(w_2, w_4) \end{pmatrix}+\rho(w_3, w_4)$$

$$\leq \zeta\begin{bmatrix} \rho(w_0, w_2)+\rho(w_0, w_3)+\rho(w_0, w_1)+\rho(w_1, w_2) \\ +\rho(w_2, w_4)+\rho(w_1, w_3)+\rho(w_3, w_2) \end{bmatrix}.$$

Now, we have

$$(1-\zeta)\begin{pmatrix} \rho(w_1, w_2)+\rho(w_2, w_3)+\rho(w_1, w_3)+\rho(w_1, w_4) \\ +\rho(w_2, w_4)+\rho(w_3, w_4) \end{pmatrix}-\zeta\rho(w_2, w_4)$$

$$< \zeta\left(\rho(w_0, w_2)+\rho(w_0, w_3)+\rho(w_0, w_1)+\rho(w_1, w_2)+\rho(w_1, w_3)+\rho(w_2, w_3) \right)$$

and

$$(1-2\zeta)\left(\rho(w_1, w_2)+\rho(w_2, w_3)+\rho(w_1, w_3)+\rho(w_1, w_4)+\rho(w_2, w_4)+\rho(w_3, w_4) \right)$$

$$< \zeta\left(\rho(w_0, w_1)+\rho(w_1, w_2)+\rho(w_0, w_2)+\rho(w_0, w_3)+\rho(w_1, w_3)+\rho(w_2, w_3) \right).$$

Let $A_n = \rho(w_n, w_{n+1})+\rho(w_{n+1}, w_{n+2})+\rho(w_n, w_{n+2})+\rho(w_n, w_{n+3})+\rho(w_{n+1}, w_{n+3})+\rho(w_{n+2}, w_{n+3})$. Then we get $A_0 = \rho(w_0, w_1)+\rho(w_1, w_2)+\rho(w_0, w_2)+\rho(w_0, w_3)+\rho(w_1, w_3)+\rho(w_2, w_3)$ and $A_1 = \rho(w_0, w_1)+\rho(w_1, w_2)+\rho(w_0, w_2)+\rho(w_0, w_3)+\rho(w_1, w_3)+\rho(w_2, w_3)$. So we have

$$(1-2\zeta)A_1 < \zeta A_0 \Rightarrow A_1 < \frac{\zeta}{1-2\zeta}A_0.$$

Since $\zeta \in \left[0, \frac{1}{3}\right)$, we have $\gamma = \frac{\zeta}{1-2\zeta} \in [0, 1)$. Thus, we get

$$A_1 < \gamma A_0.$$

By taking $w = w_1,\ r = w_2,\ z = w_3,$ and $t = w_4$ in (3), we get

$$\begin{pmatrix} \rho(\Upsilon w_0, \Upsilon w_1) + \rho(\Upsilon w_1, \Upsilon w_2) + \rho(\Upsilon w_2, \Upsilon w_0) \\ +\rho(\Upsilon w_0, \Upsilon w_3) + \rho(\Upsilon w_1, \Upsilon w_3) + \rho(\Upsilon w_2, \Upsilon w_3) \end{pmatrix}$$

$$\leq \zeta \begin{pmatrix} \rho(w_0, \Upsilon w_1) + \rho(w_0, \Upsilon w_2) + \rho(w_0, \Upsilon w_3) + \rho(w_1, \Upsilon w_0) \\ +\rho(w_1, \Upsilon w_2) + \rho(w_1, \Upsilon w_3) + \rho(w_2, \Upsilon w_0) + \rho(w_2, \Upsilon w_1) \\ +\rho(w_2, \Upsilon w_3) + \rho(w_3, \Upsilon w_0) + \rho(w_3, \Upsilon w_1) + \rho(w_3, \Upsilon w_1) \end{pmatrix}$$

and using a similar argument, we get

$$(1-2\zeta)\left(\rho(w_2, w_3) + \rho(w_3, w_4) + \rho(w_2, w_4) + \rho(w_2, w_5) + \rho(w_3, w_5) + \rho(w_4, w_5) \right)$$

$$< \zeta \left(\rho(w_1, w_2) + \rho(w_2, w_3) + \rho(w_1, w_3) + \rho(w_1, w_4) + \rho(w_2, w_4) + \rho(w_3, w_4) \right).$$

Thus, we have $A_2 < \gamma A_1$. Inductively,

$$A_n < \gamma A_{n-1} < \gamma^2 A_{n-2} < \cdots < \gamma^{n-1} A_1 < \gamma^n A_0.$$

As in the proof of main theorem in [16], we can conclude that $\{w_n\}$ is a Cauchy sequence. Since (V, ρ) is complete, we can conclude that $\{w_n\}$ has a limit, say $\varkappa \in V$.

Let us show that $\varkappa$ is a fixed point of Υ. We will show that there exists a subsequence $\left\{w_{n_k}\right\}_{k\geq 0}$ such that w_{n_k}, w_{n_k+1}, w_{n_k+2} and $\varkappa$ are pairwise distinct for every $k = 0, 1, \ldots$. Actually $\varkappa \notin \{w_n\}$. By taking $w = w_{n_k},\ r = w_{n_k+1},$ $z = w_{n_k+2},$ and $t = \varkappa$ in (3) we get

$$\begin{pmatrix} \rho(\Upsilon w_{n_k}, \Upsilon w_{n_k+1}) + \rho(\Upsilon w_{n_k+1}, \Upsilon w_{n_k+2}) + \rho(\Upsilon w_{n_k+2}, \Upsilon w_{n_k}) \\ +\rho(\Upsilon w_{n_k}, \Upsilon\varkappa) + \rho(\Upsilon w_{n_k+1}, \Upsilon\varkappa) + \rho(\Upsilon w_{n_k+2}, \Upsilon\varkappa) \end{pmatrix}$$

$$\leq \zeta \begin{pmatrix} \rho(w_{n_k}, \Upsilon w_{n_k+1}) + \rho(w_{n_k}, \Upsilon w_{n_k+2}) + \rho(w_{n_k}, \Upsilon\varkappa) + \rho(w_{n_k+1}, \Upsilon w_{n_k}) \\ +\rho(w_{n_k+1}, \Upsilon w_{n_k+2}) + \rho(w_{n_k+1}, \Upsilon\varkappa) + \rho(w_{n_k+2}, \Upsilon w_{n_k}) + \rho(w_{n_k+2}, \Upsilon w_{n_k+1}) \\ +\rho(w_{n_k+2}, \Upsilon\varkappa) + \rho(\varkappa, \Upsilon w_{n_k}) + \rho(\varkappa, \Upsilon w_{n_k+1}) + \rho(\varkappa, \Upsilon w_{n_k+2}) \end{pmatrix}$$

$$\begin{pmatrix} \rho(w_{n_k+1}, w_{n_k+2}) + \rho(w_{n_k+2}, w_{n_k+3}) + \rho(w_{n_k+3}, w_{n_k+1}) \\ +\rho(w_{n_k+1}, \Upsilon\varkappa) + \rho(w_{n_k+1}, \Upsilon\varkappa) + \rho(w_{n_k+3}, \Upsilon\varkappa) \end{pmatrix}$$

$$\leq \zeta \begin{pmatrix} \rho(w_{n_k}, w_{n_k+2}) + \rho(w_{n_k}, w_{n_k+3}) + \rho(w_{n_k}, \Upsilon\varkappa) + \rho(w_{n_k+1}, w_{n_k+1}) \\ +\rho(w_{n_k+1}, w_{n_k+3}) + \rho(w_{n_k+1}, \Upsilon\varkappa) + \rho(w_{n_k+2}, w_{n_k+1}) + \rho(w_{n_k+2}, w_{n_k+2}) \\ +\rho(w_{n_k+2}, \Upsilon\varkappa) + \rho(\varkappa, w_{n_k+1}) + \rho(\varkappa, w_{n_k+2}) + \rho(\varkappa, w_{n_k+3}) \end{pmatrix}.$$

By taking the limit as $k \to \infty$ and using the fact that $\{w_n\}$ is a Cauchy sequence with limit $\varkappa \in V$, we obtain

$$3\rho(\Upsilon\varkappa, \varkappa) \leq 3\zeta\rho(\Upsilon\varkappa, \varkappa).$$

Since $\zeta \in \left[0, \frac{1}{3}\right)$, we get $\rho(\Upsilon\varkappa, \varkappa) = 0,$ that is, $\varkappa$ is a fixed point of Υ.

Next, assume that there exist at least four distinct fixed points $w,\ r,\ z$ and t. Afterward, $\Upsilon w = w,\ \Upsilon r = r,\ \Upsilon z = z,$ and $\Upsilon t = t$. Using (3),

$$\begin{pmatrix} \rho(w, r) + \rho(r, z) + \rho(z, w) \\ +\rho(w, t) + \rho(r, t) + \rho(z, t) \end{pmatrix} \leq \zeta \begin{pmatrix} \rho(w, r) + \rho(w, z) + \rho(w, t) + \rho(r, w) \\ +\rho(r, z) + \rho(r, t) + \rho(z, w) + \rho(z, r) \\ +\rho(z, t) + \rho(t, w) + \rho(t, r) + \rho(t, z) \end{pmatrix}$$

$$\leq 2\zeta \begin{pmatrix} \rho(w, r) + \rho(r, z) + \rho(z, w) \\ +\rho(w, t) + \rho(r, t) + \rho(z, t) \end{pmatrix}.$$

which is a contradiction, because of $\zeta \in \left[0, \frac{1}{3}\right)$. So, the mapping admits at most three fixed points. □

We now extend the idea of four point CTM to n-point CTM as follows.

Definition 7 Let (V, ρ) be a MS with $|V| \geq n$ where $n \geq 3.$ The mapping $\Upsilon : V \to V$ is said to be n-point CTM if there exists $\zeta \in \left[0, \frac{1}{3}\right)$ such that the inequality

$$S(\Upsilon w_1, \Upsilon w_2, \ldots, \Upsilon w_n) \leq \lambda \left(\sum_{i=1}^{n} \sum_{\substack{j=1 \\ i \neq j}}^{n} \rho(w_i, \Upsilon w_j) \right) \tag{7}$$

holds for all pairwise distinct points $w_i \in V$, $i = 1, 2, ..., n,\ n \geq 3.$

Remark 2 We know that

$$\sum_{i=1}^{n}\sum_{\substack{j=1\\ i\neq j}}^{n}\rho\left(w_i,\,\Upsilon w_j\right)=\left(\begin{array}{c}\sum\limits_{j=2}^{n}\rho\left(w_1,\,\Upsilon w_j\right)+\sum\limits_{\substack{j=1\\ j\neq 2}}^{n}\rho\left(w_2,\,\Upsilon w_j\right)+\sum\limits_{\substack{j=1\\ j\neq 3}}^{n}\rho\left(w_3,\,\Upsilon w_j\right)\\ \\ +\cdots+\sum\limits_{\substack{j=1\\ j\neq n-1}}^{n}\rho\left(w_{n-1},\,\Upsilon w_j\right)+\sum\limits_{j=1}^{n-1}\rho\left(w_n,\,\Upsilon w_j\right)\end{array}\right).$$

Theorem 3 Let $(V,\,\rho)$ be a CMS with $|V|\geq n,$ where $n\geq 3$ and let the mapping $\Upsilon:V\to V$ be such that:
(i) Υ has no periodic points of prime periods 2, ..., $n-1$;
(ii) Υ is n-point CTM.
Then, Υ has a fixed point in V and the number of fixed points is at most $n-1$.

Proof. Let us consider a sequence defined by $w_1\in V$, and $w_{n+1}=\Upsilon w_n$ for all $n\in\mathbb{N}$. Assume that none of the elements in this sequence is a fixed point of Υ, i.e., $\Upsilon w_n\neq w_n$ for every $n=1,\,2,\,\ldots$. Since Υ has no periodic points of prime periods 2, ..., $n-1$, all n consecutive terms of $\{w_m\}$ are distinct. Now, we begin proof by taking $w_1,\,w_2,\,\ldots,\,w_n\in V$ in (7), which characterizes n-point Chatterjea-type condition, we get

$$S\left(\Upsilon w_1,\,\Upsilon w_2,\,\ldots,\,\Upsilon w_n\right)\leq\zeta\left(\begin{array}{c}\sum\limits_{i=2}^{n}\rho\left(w_1,\,\Upsilon w_i\right)+\sum\limits_{\substack{i=1\\ i\neq 2}}^{n}\rho\left(w_2,\,\Upsilon w_i\right)+\sum\limits_{\substack{i=1\\ i\neq 3}}^{n}\rho\left(w_3,\,\Upsilon w_i\right)\\ \\ +\cdots+\sum\limits_{\substack{i=1\\ i\neq n-1}}^{n}\rho\left(w_{n-1},\,\Upsilon w_i\right)+\sum\limits_{i=1}^{n-1}\rho\left(w_n,\,\Upsilon w_i\right)\end{array}\right).$$

Using triangle inequality together with $\zeta\rho\left(w_n,\,\Upsilon w_n\right)<\rho\left(w_n,\,\Upsilon w_n\right),$ we have

$$(1-\zeta)\,S\left(\Upsilon w_1,\,\Upsilon w_2,\,\ldots,\,\Upsilon w_n\right)\leq\zeta\left(S\left(w_1,\,w_2,\,\ldots,\,w_n\right)+\rho\left(w_n,\,\Upsilon w_n\right)\right)$$

$$(1-2\zeta)\,S\left(\Upsilon w_1,\,\Upsilon w_2,\,\ldots,\,\Upsilon w_n\right)\leq\zeta\left(S\left(w_1,\,w_2,\,\ldots,\,w_n\right)\right).$$

Let $S_m=S\left(w_m,\,w_{m+1},\,\ldots,\,w_{m+n}\right)$. Then we get $S_1=S\left(w_1,\,w_2,\,\ldots,\,w_n\right)$ and $S_2=S\left(\Upsilon w_1,\,\Upsilon w_2,\,\ldots,\,\Upsilon w_n\right)=S\left(w_2,\,w_3,\,\ldots,\,w_{n+1}\right)$. Thus we can write

$$(1-2\zeta)\,S_2\leq\zeta S_1\Rightarrow S_2<\frac{\zeta}{1-2\zeta}S_1.$$

Let $\gamma=\frac{\zeta}{1-2\zeta}$. Since $\zeta\in\left[0,\,\frac{1}{3}\right),$ we have $\gamma\in[0,\,1)$ and

$$S_2<\gamma S_1.$$

If we take $w_2,\,w_3,\,\ldots,\,w_{n+1}\in V$ in (7), we get

$$(1-\zeta)S(\Upsilon w_2, \Upsilon w_3, \ldots, \Upsilon w_{n+1}) \leq \zeta \begin{pmatrix} \sum\limits_{i=3}^{n+1} \rho(w_2, \Upsilon w_i) + \sum\limits_{\substack{i=2 \\ i\neq 3}}^{n+1} \rho(w_3, \Upsilon w_i) \\ \\ +\cdots+ \sum\limits_{\substack{i= \\ i\neq n}}^{n-1} \rho(w_n, \Upsilon w_i) + \sum\limits_{i=2}^{n} \rho(w_{n+1}, \Upsilon w_i) \end{pmatrix}.$$

Using triangle inequality together wih $\zeta\rho(w_{n+1}, \Upsilon w_{n+1}) < \rho(w_{n+1}, \Upsilon w_{n+1})$, we have

$$(1-\zeta)S(\Upsilon w_2, \Upsilon w_3, \ldots, \Upsilon w_{n+1}) \leq \zeta(S(w_2, w_3, \ldots, w_{n+1}) + \rho(w_{n+1}, \Upsilon w_{n+1}))$$

$$(1-2\zeta)S(\Upsilon w_2, \Upsilon w_3, \ldots, \Upsilon w_{n+1}) < \zeta(S(w_2, w_3, \ldots, w_{n+1})).$$

So, we have

$$(1-2\zeta)S_3 < \zeta S_2 \Rightarrow S_3 < \frac{\zeta}{1-2\zeta}S_2.$$

Since $\gamma = \frac{\zeta}{1-2\zeta}$ and $\zeta \in \left[0, \frac{1}{3}\right)$, we have $\gamma \in [0, 1)$ and

$$S_3 < \gamma S_2 < \gamma^2 S_1.$$

By taking $w_m, w_{m+1}, \ldots, w_{m+n} \in V$ in (7), we get that

$$(1-\zeta)S(\Upsilon w_m, \Upsilon w_{m+1}, \ldots, \Upsilon w_{m+n})$$

$$\leq \zeta \begin{pmatrix} \sum\limits_{i=m+1}^{m+n} \rho(w_m, \Upsilon w_i) + \sum\limits_{\substack{i=m \\ i\neq m+1}}^{m+n} \rho(w_{m+1}, \Upsilon w_i) + \sum\limits_{\substack{i=m \\ i\neq m+2}}^{m+n} \rho(w_3, \Upsilon w_i) \\ \\ +\cdots+ \sum\limits_{\substack{i=m \\ i\neq m+n-1}}^{m+n} \rho(w_{m+n-1}, \Upsilon w_i) + \sum\limits_{i=1}^{m+n-1} \rho(w_{m+n}, \Upsilon w_i) \end{pmatrix}.$$

If we continue similar process, we have

$$(1-\zeta)S(\Upsilon w_m, \Upsilon w_{m+1}, \ldots, \Upsilon w_{m+n}) \leq \zeta(S(w_m, w_{m+1}, \ldots, w_{m+n}) + \rho(w_{m+n}, \Upsilon w_{m+n}))$$

$$(1-2\zeta)S(\Upsilon w_m, \Upsilon w_{m+1}, \ldots, \Upsilon w_{m+n}) \leq \zeta(S(w_m, w_{m+1}, \ldots, w_{m+n})).$$

So, we get

$$S_m < \frac{\zeta}{1-2\zeta} S_{m-1}.$$

Taking $\gamma = \frac{\zeta}{1-2\zeta}$, we have

$$S_m < \gamma S_{m-1}.$$

By induction, we obtain

$$S_m < \gamma S_{m-1} < \gamma^2 S_{m-2} < \cdots < \gamma^{n-2} S_2 < \gamma^{n-1} S_1.$$

As in the proof of the main theorem in [16], we can conclude that $\{w_m\}$ is a Cauchy sequence. Since (V, ρ) is complete, we can conclude that $\{w_m\}$ has a limit, say $\varkappa$ in V.

Let us show that $\varkappa$ is a fixed point of Υ. We will show that there exists a subsequence $\left\{w_{m_k}\right\}_{k\geq 0}$ such that w_{m_k}, w_{m_k+1}, $\ldots$, w_{m_k+n-1}, and $\varkappa$ are pairwise distinct for every $k = 0, 1, \ldots$. Actually $\varkappa \notin \{w_m\}$. By taking w_{m_k}, $w_{m_k+1}, \ldots, w_{m_k+n-2}$, and $\varkappa$ in (7) we get

$$S\left(\Upsilon w_{m_k}, \Upsilon w_{m_k+1}, \ldots, \Upsilon w_{m_k+n-2}, \varkappa\right)$$

$$\leq \zeta \begin{pmatrix} \sum\limits_{i=1}^{n-2} \rho\left(w_{m_k}, \Upsilon w_{m_k+i}\right) + \sum\limits_{\substack{i=0 \\ i\neq 1}}^{n-2} \rho\left(w_{m_k+1}, \Upsilon w_{m_k+i}\right) \\ + \sum\limits_{\substack{i=0 \\ i\neq 2}}^{n-2} \rho\left(w_{m_k+2}, \Upsilon w_{m_k+i}\right) + \cdots + \sum\limits_{i=0}^{n-3} \rho\left(w_{m_k+n-2}, \Upsilon w_{m_k+i}\right) \\ + \sum\limits_{i=0}^{n-2} \rho\left(\varkappa, \Upsilon w_{m_k+i}\right) + \sum\limits_{i=0}^{n-2} \rho\left(\Upsilon w_{m_k+i}, \Upsilon \varkappa\right) \end{pmatrix}.$$

Also we have

$$S\left(\Upsilon w_{m_k}, \Upsilon w_{m_k+1}, \ldots, \Upsilon w_{m_k+n-2}, \varkappa\right)$$

$$= \sum_{0\leq i<j\leq n-2} \rho\left(\Upsilon w_{m_k+i}, \Upsilon w_{m_k+j}\right) + \sum_{0\leq i\leq n-2} \rho\left(\Upsilon w_{m_k+i}, \Upsilon \varkappa\right).$$

Rearranging and by letting $k \to \infty$, we get

$$(n-1)\,\rho(\Upsilon\varkappa, \varkappa) \leq \zeta\,(n-1)\,\rho(\Upsilon\varkappa, \varkappa).$$

Since $\zeta \in \left[0, \frac{1}{3}\right)$, we get $\rho(\Upsilon\varkappa, \varkappa) = 0$, that is, $\varkappa$ is a fixed point of Υ.

Next, assume that there exist at least n distinct fixed points $w_1, w_2, \ldots, w_n$. Then $\Upsilon w_i = w_i$, $i = 1, 2, ..., n$, $n \geq 3$. Using (7), we have

$$S(\Upsilon w_1, \Upsilon w_2, \ldots, \Upsilon w_n) \leq \zeta \left(\begin{array}{c} \sum\limits_{i=2}^{n} \rho(w_1, \Upsilon w_i) + \sum\limits_{\substack{i=1 \\ i\neq 2}}^{n} \rho(w_2, \Upsilon w_i) + \sum\limits_{\substack{i=1 \\ i\neq 3}}^{n} \rho(w_3, \Upsilon w_i) \\ + \cdots + \sum\limits_{\substack{i=1 \\ i\neq n-1}}^{n} \rho(w_{n-1}, \Upsilon w_i) + \sum\limits_{i=1}^{n-1} \rho(w_n, \Upsilon w_i) \end{array} \right).$$

Since w_i, $i = 1, 2, ..., n$ with $n \geq 3$ are distinct fixed points, we get

$$S(w_1, w_2, \ldots, w_n) \leq 2\zeta S(w_1, w_2, \ldots, w_n).$$

which is a contradiction, because of $\zeta \in \left[0, \frac{1}{3}\right)$. So, the mapping admits at most $n-1$ fixed points. □

Proposition 4 Let $\Upsilon : V \to V$ be a CTM with $\zeta \in \left[0, \frac{1}{3}\right)$. Then, Υ is n-point CTM.

Proof. Let (V, ρ) be a CMS with $|V| \geq n \geq 3$ and the mapping $\Upsilon : V \to V$ be n-point CTM and let $w_i \in V$, $i = 1, 2, ..., n$, pairwise distinct points. Considering Chatterjea mapping, we get the following $\binom{n}{2}$ cases.

$$\rho(\Upsilon w_1, \Upsilon w_2) \leq \zeta\left(\rho(w_1, \Upsilon w_2) + \rho(w_2, \Upsilon w_1)\right),$$

$$\rho(\Upsilon w_2, \Upsilon w_3) \leq \zeta\left(\rho(w_2, \Upsilon w_3) + \rho(w_3, \Upsilon w_2)\right),$$

$$\rho(\Upsilon w_1, \Upsilon w_3) \leq \zeta\left(\rho(w_1, \Upsilon w_3) + \rho(w_3, \Upsilon w_1)\right),$$

$$\vdots$$

$$\rho(\Upsilon w_{n-1}, \Upsilon w_n) \leq \zeta\left(\rho(w_{n-1}, \Upsilon w_n) + \rho(w_n, \Upsilon w_{n-1})\right).$$

When the right and left parts of the aforementioned inequalities are summarized, we get

$$S(\Upsilon w_1, \Upsilon w_2, \ldots, \Upsilon w_n) \leq \zeta \left(\begin{array}{c} \sum\limits_{i=2}^{n} \rho(w_1, \Upsilon w_i) + \sum\limits_{\substack{i=1 \\ i\neq 2}}^{n} \rho(w_2, \Upsilon w_i) + \sum\limits_{\substack{i=1 \\ i\neq 3}}^{n} \rho(w_3, \Upsilon w_i) \\ + \cdots + \sum\limits_{\substack{i=1 \\ i\neq n-1}}^{n} \rho(w_{n-1}, \Upsilon w_i) + \sum\limits_{i=1}^{n-1} \rho(w_n, \Upsilon w_i) \end{array} \right).$$

So, the desired assertion is obtained. □

Lemma 2 Let (V, ρ) be a CMS with $|V| \geq n,$ where $n \geq 3$ and let $\Upsilon : V \to V$ be an n-point CTM $(3 \leq n \leq |V|,$ $n \in \mathbb{N})$. If z is an accumulation point of V and Υ is continuous, then the inequality

$$(1-\zeta(n-2))\,\rho(\Upsilon z, \Upsilon r) \leq \zeta\left((n-1)\,\rho(z, \Upsilon r)+\rho(r, \Upsilon z)\right) \tag{8}$$

holds for all $r \in V$.

Proof. Given any accumulation point $z \in V$, and any $r \in V$. If $z = r$, then it is obvious that (8) is provided. Suppose that $z \neq r$, since z is an accumulation point of V, there exists a sequence $w_k \to z$ such that $w_k \neq z$, $w_k \neq r$ for all k and all w_k are pairwise distinct. Thus, by (7), we have

$$S(\Upsilon z, \Upsilon r, \Upsilon w_{k+1}, \Upsilon w_{k+2}, \ldots, \Upsilon w_{k+n-2})$$

$$\leq \zeta \begin{pmatrix} \rho(z, \Upsilon r)+\rho(r, \Upsilon z)+\sum_{i=1}^{n-2} \rho(r, \Upsilon w_{k+i}) \\ +\sum_{i=1}^{n-2} \rho(z, \Upsilon w_{k+i})+\sum_{i=1}^{n-2} \rho(w_{k+i}, \Upsilon r) \\ +\sum_{i=1}^{n-2} \rho(w_{k+i}, \Upsilon z)+\sum_{i=1}^{n-2} \sum_{\substack{j=1 \\ i \neq j}}^{n-2} \rho\left(w_{k+i}, \Upsilon w_{k+j}\right) \end{pmatrix}$$

$$\leq \zeta \begin{pmatrix} \rho(z, \Upsilon r)+\rho(r, \Upsilon z)+\sum_{i=1}^{n-2} \rho(r, \Upsilon w_{k+i}) \\ +\sum_{i=1}^{n-2} \rho(z, \Upsilon w_{k+i})+\sum_{i=1}^{n-2} \rho(w_{k+i}, \Upsilon r)+\sum_{i=1}^{n-2} \rho(w_{k+i}, \Upsilon z) \\ \sum_{i=2}^{n-2} \rho(w_1, \Upsilon w_j)+\sum_{\substack{j=1 \\ j \neq 2}}^{n-2} \rho(w_2, \Upsilon w_j)+\sum_{\substack{j=1 \\ j \neq 3}}^{n-2} \rho(w_3, \Upsilon w_j) \\ +\cdots+\sum_{\substack{j=1 \\ j \neq n-1}}^{n-2} \rho(w_{n-1}, \Upsilon w_j)+\sum_{i=1}^{n-3} \rho(w_n, \Upsilon w_j) \end{pmatrix}.$$

Also, we have

$$S(\Upsilon z, \Upsilon r, \Upsilon w_{k+1}, \Upsilon w_{k+2}, \ldots, \Upsilon w_{k+n-2}) = \rho(\Upsilon r, \Upsilon z)+\sum_{i=1}^{n-2} \rho(\Upsilon r, \Upsilon w_{k+i})+\sum_{i=1}^{n-2} \rho(\Upsilon z, \Upsilon w_{k+i}).$$

Thus, we get

$$\rho(\Upsilon z, \Upsilon r)+\sum_{i=1}^{n-2} \rho(\Upsilon r, \Upsilon w_{k+i})+\sum_{i=1}^{n-2} \rho(\Upsilon z, \Upsilon w_{k+i})$$

$$\leq \zeta \left(\begin{array}{c} \rho(z, \Upsilon r) + \rho(r, \Upsilon z) + \sum\limits_{i=1}^{n-2} \rho(r, \Upsilon w_{k+i}) \\ + \sum\limits_{i=1}^{n-2} \rho(z, \Upsilon w_{k+i}) + \sum\limits_{i=1}^{n-2} \rho(w_{k+i}, \Upsilon r) + \sum\limits_{i=1}^{n-2} \rho(w_{k+i}, \Upsilon z) \\ \sum\limits_{i=2}^{n-2} \rho(w_1, \Upsilon w_j) + \sum\limits_{\substack{j=1 \\ j \neq 2}}^{n-2} \rho(w_2, \Upsilon w_j) + \sum\limits_{\substack{j=1 \\ j \neq 3}}^{n-2} \rho(w_3, \Upsilon w_j) \\ + \cdots + \sum\limits_{\substack{j=1 \\ j \neq n-1}}^{n-2} \rho(w_{n-1}, \Upsilon w_j) + \sum\limits_{i=1}^{n-3} \rho(w_n, \Upsilon w_j) \end{array} \right).$$

Since $\rho(w_k, z) \to 0$ as $k \to \infty$, due to the continuity of Υ, we have $\Upsilon w_k \to \Upsilon z$, $\rho(\Upsilon r, \Upsilon w_k) \to \rho(\Upsilon r, \Upsilon z)$, $\rho(w_k, \Upsilon w_k) \to \rho(z, \Upsilon z)$. Letting $k \to \infty$, we have

$$(n-1)\rho(\Upsilon z, \Upsilon r) \leq \zeta((n-1)(\rho(z, \Upsilon r) + \rho(r, \Upsilon z)) + (n-2)(n-1)\rho(z, \Upsilon z)).$$

Dividing both sides of above inequality by $n-1$, we get

$$\rho(\Upsilon z, \Upsilon r) \leq \zeta((\rho(z, \Upsilon r) + \rho(r, \Upsilon z)) + (n-2)\rho(z, \Upsilon z)).$$

Using the triangular inequality, we have $\rho(z, \Upsilon z) \leq \rho(z, \Upsilon r) + \rho(\Upsilon r, \Upsilon z)$. Applying this inequality to the inequality above, we obtain

$$1-(n-2)\zeta\rho(\Upsilon z, \Upsilon r) \leq \zeta((n-1)\rho(z, \Upsilon r) + \rho(r, \Upsilon z)).$$

□

Proposition 5 Let (V, ρ) be a MS and $\Upsilon : V \to V$ be a continuous n-point CTM $\zeta \in \left[0, \frac{1}{2n-2}\right)$ with ($3 \leq n \leq |V|$, $n \in \mathbb{N}$). If every point of V is accumulation point, then, Υ is a Chatterjea contraction.

Proof. By Lemma 1, along with (8) we also obtain the following inequality

$$1-(n-2)\zeta\rho(\Upsilon r, \Upsilon z) \leq \zeta((n-1)\rho(r, \Upsilon z) + \rho(z, \Upsilon r)). \tag{9}$$

The addition of the left- and right-hand sides of (8) and (9), followed by division by 2, gives

$$\rho(\Upsilon r, \Upsilon z) \leq \frac{\zeta n}{2(1-(n-2)\zeta)}(\rho(r, \Upsilon z) + \rho(z, \Upsilon r)).$$

Since $\zeta \in \left[0, \frac{1}{2n-2}\right)$, we have $\frac{\zeta n}{2(1-(n-2)\zeta)} \in \left[0, \frac{1}{2}\right)$. Thus the proof is completed. □

Proposition 6 Let (V, ρ) be a MS and $\Upsilon : V \to V$ be a MCTPD between n points ($3 \le n \le |V|, n \in \mathbb{N}$) with $\alpha \in \left[0, \frac{1}{4}\right)$. Then, Υ is an n-point CTM.

Proof. By the supposition (7) holds for all pairwise distinct points $w_1, w_2, \ldots, w_n \in V$. Using the triangle inequalities for each pair

$$\rho(w_i, w_j) \le \rho(w_i, \Upsilon w_j) + \rho(\Upsilon w_i, \Upsilon w_j) + \rho(w_j, \Upsilon w_i)$$

we obtain

$$\begin{aligned} S(w_1, w_2, \ldots, w_n) &\le \sum_{i=1}^{n} \sum_{\substack{j=1 \\ i \ne j}}^{n} \rho(w_i, \Upsilon w_j) + \sum_{1 \le i \le j \le n} \rho(\Upsilon w_i, \Upsilon w_j) \\ &\le \sum_{i=1}^{n} \sum_{\substack{j=1 \\ i \ne j}}^{n} \rho(w_i, \Upsilon w_j) + S(\Upsilon w_1, \Upsilon w_2, \ldots, \Upsilon w_n). \end{aligned} \tag{10}$$

Substituting (10) into (7), we get that

$$S(\Upsilon w_1, \Upsilon w_2, \ldots, \Upsilon w_n) \le \alpha \left(\sum_{i=1}^{n} \sum_{\substack{j=1 \\ i \ne j}}^{n} \rho(w_i, \Upsilon w_j) + S(\Upsilon w_1, \Upsilon w_2, \ldots, \Upsilon w_n) \right).$$

So, we have

$$S(\Upsilon w_1, \Upsilon w_2, \ldots, \Upsilon w_n) \le \frac{\alpha}{1-\alpha} \left(\sum_{i=1}^{n} \sum_{\substack{j=1 \\ i \ne j}}^{n} \rho(w_i, \Upsilon w_j) \right).$$

satisfies for all n pairwise distinct points. Since $\alpha \in \left[0, \frac{1}{4}\right)$, we obtain that $\zeta = \frac{\alpha}{1-\alpha} \in \left[0, \frac{1}{3}\right)$. Thus, Υ is an n-point CTM. □

Corollary 1 Let (V, ρ) be a MS with $|V| \ge 4$ and $\Upsilon : V \to V$ be a continuous four-point CTM with $\zeta \in \left[0, \frac{1}{6}\right)$. Whenever every point of V is an accumulation point and Υ is continuous, Υ is a Chatterjea contraction.

Corollary 2 Theorem 3 generalizes some consequences of [1, 16, 17, 19].

3. Conclusion

In this study, after investigating GCTM, we propose the n-point generalization of the CTM. Unlike the classical approach, this generalization is based on mappings involving n distinct points in the space. An example is given where it is not Chatterjea mapping but the four-point CTM. Moreover, we establish several new fixed point results, supported by auxiliary lemmas and propositions for such mappings in the context of complete metric spaces. In addition, we compare *n*-point CTM with some contractive type mappings in the literature and we can say that our results generalize some consequences of [1, 16, 17, 19].

Author contributions

Both the authors wrote, read, and approved the final manuscript.

Data availability

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Confilict of interest

The authors declare no competing financial interest.

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