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Research Article

Some New Convergence Approaches in Banach Spaces Involving the *HK*-Iteration Method with Application

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Abstract: The Banach spaces is the most popular and well-known spaces in the subject of pure mathematics. Many mathematicians have worked on different concepts from different angles in the field of functional analysis due to wide range of utilizations in quantum mechanics, optimization, and numerical analysis. In this paper, we explore the new iterative approach, namely Hanif-Kifayat (HK)-iteration method in Banach spaces. In addition, we investigate the some new convergence results in Banach Spaces via newly introduced concept with an application to a first-order delay differential equation. For the paper quality and reader's interest, numerical example and performance comparison via newly introduced concept have been added.

Keywords: fixed point, HK-iteration, Banach space, condition (I), condition (E), differential equation

MSC: 47H10, 46B45, 46B50, 34K06

Abbreviation

BCP	Banach Contraction Principle
FP	Fixed Point
UCB	Uniformly Convex Banach
UCBS	Uniformly Convex Banach Space
HK	Hanif-Kifayat
CSC	Chatterjea-Suzuki Condition
I.P	Identity Plus

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1. Introduction

Fixed Point theory is a component area of pure mathematics. Numerous investigators have been pursuing the idea of fixed point theory in recent years in an attempt to introduce new ideas and a fresh perspective on pure mathematics with various characteristics and trademarks in the literature. Even though the concept looks so simple, it serves as a framework for a number of mathematical and applied disciplines, including analysis, topolgy, differential equations, and optimization. Two well-known fixed-point theorems are those of Banach and Brouwer, which have guided current research in game theory, dynamic systems, and economics by providing deep insights into existence and stability problems. Essentially, fixed-point theory outlines a unified framework within which the existence of solutions to complicated nonlinear equations and integral systems can be proven, acting as a bridge from abstract mathematical reasoning to real phenomena.

Assume that $\mathscr{U}$ is a normed vector space equipped with the norm $\|\cdot\|$, and consider a nonempty subset $\mathscr{A} \subseteq \mathscr{U}$. A mapping $\mathscr{T}\colon \mathscr{A} \to \mathscr{A}$ is said to be a *contraction* if there exists a constant $\kappa \in [0, 1)$ such that

$$\|\mathscr{T}w - \mathscr{T}v\| \leq \kappa\|w - v\|, \quad \text{for all } w, v \in \mathscr{A}. \tag{1}$$

A point $q_0 \in \mathscr{A}$ that satisfies $\mathscr{T}q_0 = q_0$ is referred to as a *fixed point* of $\mathscr{T}$. The set of all such fixed points is denoted by $F_{\mathscr{T}}$.

In a wide range of scientific and engineering problems, obtaining exact analytical solutions is often challenging or infeasible. As a result, iterative techniques are frequently employed to compute approximate solutions.

The Banach Contraction Principle (BCP) [1] plays a fundamental role in fixed point theory. Furthermore, the BCP presents a helpful approach identified as the Picard iteration scheme [2], given by

$$w_{i+1} = \mathscr{T}w_i,$$

which converges to the unique FP q_0.

The operator $\mathscr{T}$ turns into a nonexpansive mapping as the contraction constant κ gets closer to 1. Even though an FP might not be unique in this situation, its existence can still be guaranteed with the right presumptions.

Browder [3] and Gohde [4] are considered with aiding to establish this fundamental finding. Furthermore, this result has been modified to include reflexive Banach spaces by Kirk [5].

A core sort of operators with numerous uses in the scientific field are nonexpansive mappings. As a consequence, expanding this topic has emerged as a crucial area of research to increase its theoretical and applied significance.

In 2008, Suzuki [6] suggested a noteworthy expansion of nonexpansive mappings in Banach spaces within this framework. Using the Browder–Gohde fixed point theorem, he proved that all mappings in this larger class have a fixed point.

A *Suzuki-type condition*, referred to as *condition (C)*, if the following inequalities are true:

$$\text{If } \frac{1}{2}\|w - Tw\| \leq \|w - v\|, \text{ then } \|Tw - Tv\| \leq \|w - v\|. \qquad (C)$$

The properties and applications of mappings that satisfy Suzuki's condition (C) have been extensively studied in the literature in recent years. But this function T is fulfill the *Chatterjea–Suzuki–C* condition.

Mann [7] presented the Mann iteration approach, which enhances stability in Banach spaces and is based on convex combinations. This concept was developed by Ishikawa [8], who suggested a two-step iteration that improved convergence in more general circumstances.

Subsequent contributions focused on specific application scenarios. For instance, Agarwal [9] designed iterative techniques suitable for hybrid and nonlinear mappings, while Abbas [10] extended these methods to multivalued operators in uniformly convex settings. Building on these developments, Ullah and Arshad [11] presented the M-iteration scheme, defined as

$$\left.\begin{aligned} w_1 &= w \in \mathscr{A}, \\ u_i &= (1-\kappa_i)w_i + \kappa_i \mathscr{T} w_i, \\ \zeta_i &= \mathscr{T} v_i, \\ w_{i+1} &= \mathscr{T} \zeta_i. \end{aligned}\right\} \tag{2}$$

Motivated by the structural limitations of earlier schemes, we introduce a novel three-step process called the *HK-iteration*, defined by

$$\left.\begin{aligned} u_i &= (1-\zeta_i)w_i + \zeta_i \mathscr{T} w_i, \\ v_i &= (1-\kappa_i)\mathscr{T} w_i + \kappa_i \mathscr{T}^2 u_i, \\ w_{i+1} &= \mathscr{T} v_i, \end{aligned}\right\} \tag{3}$$

where $\{\zeta_i\}, \{\kappa_i\} \subset [0, 1]$ are sequences in the unit interval.

The main objective of this study is to establish strong convergence results for the HK-iteration scheme when applied to mappings satisfying the Chatterjea–Suzuki–C condition.

Throughout this paper, $\mathscr{T}$ denotes the considered self-mapping on $\mathscr{A}$. The sequences $\{w_i\}, \{u_i\}, \{v_i\} \subset \mathscr{A}$ are the iterative sequences generated by the schemes, and $\{\zeta_i\}, \{\kappa_i\} \subset [0, 1]$ are control sequences (step-size parameters). Constants $p, q \in (0, 1)$ used in the lemmas represent fixed bounds for κ_i when required.

The structure of this manuscript is constructed as follows: In Section 2, presents some related theorems, concepts, and remarks that are essential for the upcoming portions. In Section 3, we provide several new convergence results in Banach spaces via the HK-iteration method. In Section 4, we add the application for the first-order delay differential equation pertaining to the newly introduced HK-iteration method. In Section 5, we investigate the numerical example and performance comparison of the HK-iterative scheme. In Section 6, we offer a conclusion. In section 7, for the readers interest, we add the future scope.

2. Preliminaries

Definition 1 [12] Let $\mathscr{U}$ be a Banach space, and let $\{w_i\} \subseteq \mathscr{U}$ be a bounded sequence. Suppose $\mathscr{A} \subseteq \mathscr{U}$ is a nonempty, closed, and convex subset. The *asymptotic radius* of the sequence $\{w_i\}$ with respect to $\mathscr{A}$ is defined by

$$\mathscr{R}(\mathscr{A}, \{w_i\}) = \inf_{q_0 \in \mathscr{A}} \limsup_{i \to \infty} \|w_i - q_0\|.$$

In this setting, the *asymptotic center* of the sequence $\{w_i\}$ with respect to the set $\mathscr{A}$ is given by

$$\mathscr{A}(\mathscr{A}, \{w_i\}) = \left\{ q_0 \in \mathscr{A} : \limsup_{i\to\infty} \|w_i - q_0\| = \mathscr{R}(\mathscr{A}, \{w_i\}) \right\}.$$

Definition 2 [13] A Banach space $\mathscr{U}$ is said to satisfy the *Opial condition* if, for any sequence $\{w_i\} \subseteq \mathscr{U}$ that converges weakly to some point $q_0 \in \mathscr{U}$, the inequality

$$\limsup_{i\to\infty} \|w_i - q_0\| < \limsup_{i\to\infty} \|w_i - e_0\|$$

holds for every $e_0 \in \mathscr{U}$ such that $e_0 \neq q_0$.

It is worth noting that all Hilbert spaces are known to possess the Opial property.

Remark 1 It is a classical result that if $\mathscr{U}$ is a UCBS [14], then the asymptotic center $\mathscr{A}(\mathscr{A}, \{w_i\})$ is a singleton. Moreover, when the set $\mathscr{A}$ is convex and weakly compact, the asymptotic center $\mathscr{A}(\mathscr{A}, \{w_i\})$ remains convex (see [12, 15] and the references therein for further discussion).

Some key results related to mappings satisfying the (CSC) condition are summarized below, based on the findings in [16].

Proposition 1 Let $\mathscr{U}$ be a Banach space and let $\mathscr{A} \subseteq \mathscr{U}$ be a nonempty, closed subset. Consider a mapping $\mathscr{T} : \mathscr{A} \to \mathscr{A}$. Then the following assertions hold:

(i) Suppose that $\mathscr{T}$ satisfies the (CSC) condition and that the set of FPs $F_{\mathscr{T}}$ is nonempty. Then, for every $w \in \mathscr{A}$ and $q_0 \in F_{\mathscr{T}}$, the inequality

$$\|\mathscr{T}w - q_0\| \leq \|w - q_0\|$$

is satisfied.

(ii) If $\mathscr{T}$ satisfies the (CSC) condition, then the FP set $F_{\mathscr{T}}$ is closed. Moreover, if $\mathscr{A}$ is convex and the space $\mathscr{U}$ is strictly convex, then $F_{\mathscr{T}}$ is also convex.

(iii) If $\mathscr{T}$ fulfills the (CSC) condition, then for all $w, v \in \mathscr{A}$, the inequality

$$\|w - \mathscr{T}v\| \leq 5\|w - \mathscr{T}w\| + \|w - v\|$$

holds.

(iv) Let $\{w_i\}$ be a sequence in $\mathscr{A}$ such that $w_i \rightharpoonup q_0$ (i.e., w_i converges weakly to q_0) and

$$\lim_{i\to\infty} \|\mathscr{T}w_i - w_i\| = 0.$$

If the space $\mathscr{U}$ satisfies Opial's condition [13], then q_0 is a fixed point of $\mathscr{T}$, that is, $q_0 \in F_{\mathscr{T}}$.

Definition 3 [17] Let $\mathscr{T}$ be a self-mapping defined on a subset $\mathscr{A}$ of a Banach space $\mathscr{U}$. The mapping $\mathscr{T}$ is said to fulfill the (I)-condition if there exists a function $k : [0, \infty) \to [0, \infty)$ such that:

- $k(0) = 0$,
- $k(t) > 0$ for all $t > 0$, and the following inequality holds for every $w \in \mathscr{A}$:

$$\|w - \mathscr{T}w\| \geq k\left(d(w, F_{\mathscr{T}})\right),$$

where $d(w, F_{\mathscr{T}})$ represents the distance from w to the set of fixed points of $\mathscr{T}$, denoted by $F_{\mathscr{T}}$.

For clarity, throughout this paper the constants p and q denote fixed real numbers in $(0, 1)$ that bound the sequence $\{\kappa_i\}$, i.e., $0 < q \leq \kappa_i \leq p < 1$.

Definition 4 [6, 16] Let $\mathscr{T}$ be a self-mapping on a subset $\mathscr{A}$ of a Banach space $\mathscr{U}$. The mapping $\mathscr{T}$ is said to satisfy the *Condition (F)* if there exists a function ϕ: $[0, \infty) \to [0, \infty)$ such that:

- $\phi(0) = 0$,
- $\phi(t) > 0$ for all $t > 0$,

and for all $w, v \in \mathscr{A}$, the inequality

$$\|\mathscr{T}w - \mathscr{T}v\| \leq \|w - v\| - \phi(\|w - v\|)$$

holds.

This condition provides a generalized contractive framework that strengthens the classical nonexpansive setting. It ensures that the mapping $\mathscr{T}$ brings points closer together by at least $\phi(\|w - v\|)$, thereby guaranteeing improved convergence behavior under iterative processes.

To derive the main results of this paper, we require the subsequent important lemma from [18].

Lemma 1 Let $\mathscr{U}$ be a UBC space, and let $\{w_i\}$, $\{v_i\} \subseteq \mathscr{U}$ be two sequences. Suppose that for some real number $e \geq 0$ and constants $0 < q \leq \kappa_i \leq p < 1$, the subsequent conditions are satisfied: We consider two bounded sequences $\{w_i\}$ and $\{v_i\}$ in a normed space such that

$$\limsup_{i \to \infty} \|w_i\| \leq e,$$

$$\limsup_{i \to \infty} \|v_i\| \leq e,$$

and suppose that the sequence formed by their convex combinations satisfies

$$\lim_{i \to \infty} \|(1 - \kappa_i)w_i + \kappa_i v_i\| = e.$$

Then it follows that

$$\lim_{i \to \infty} \|w_i - v_i\| = 0.$$

Identity Plus Mapping: associated with the demiclosedness principle for nonexpansive mappings [3].

Condition (I): Inequality-Type Condition ensuring a relation between $\|w - \mathscr{T}w\|$ and the distance from $F_{\mathscr{T}}$, often linked to weak convergence principles [13].

Condition (E): Contractive-Type Condition ensuring $\|\mathscr{T}w - \mathscr{T}v\| \leq \psi(\|w - v\|)$ for some nondecreasing function ψ, following the Banach contraction framework [1].

3. New convergence results via *HK*-iteration method

It is well established that the evolution of iterative methods in fixed-point theory is largely driven by efforts to enhance convergence speed and robustness. In particular, hybrid iteration models, such as those extending Mann and Ishikawa schemes, have led to significant advancements in both theoretical and computational frameworks. Motivated by these developments and with the aim of overcoming certain limitations of existing processes, we propose a new three-step hybrid iterative method, referred to as the *HK*-iteration process. This scheme generates a sequence $\{w_i\}$ through the following recursive formulation:

$$\left.\begin{aligned} u_i &= (1-\zeta_i)w_i + \zeta_i \mathscr{T} w_i, \\ v_i &= (1-\kappa_i)\mathscr{T} w_i + \kappa_i \mathscr{T}^2 u_i, \\ w_{i+1} &= \mathscr{T} v_i, \end{aligned}\right\} \tag{4}$$

where $\{\zeta_i\}, \{\kappa_i\} \subset [0, 1]$ are sequences of real numbers.

Here, the iterative process applies the sequences $\{\zeta_i\}$ and $\{\kappa_i\}$ as control (step-size) parameters, each of which takes values in the interval $[0, 1]$.

The *HK* iteration approach defined in (4) is now a topic of a comprehensive convergence analysis, especially when applied to operators that meet the *CSC*. The following lemma serves as a foundational result for our main theorem.

Throughout this section, condition (E) refers to a contractive-type requirement on $\mathscr{T}$ ensuring that $\|\mathscr{T} w - \mathscr{T} v\| \leq \psi(\|w - v\|)$ for some nondecreasing function ψ: $[0, \infty) \to [0, \infty)$ satisfying $\psi(t) < t$ for all $t > 0$.

Lemma 2 Let $\mathscr{T}$: $\mathscr{C} \to \mathscr{C}$ be a mapping on a nonempty, closed, and convex subset $\mathscr{C}$ of a Banach space $\mathscr{U}$, and suppose that $\mathscr{T}$ satisfies condition (E). If the sequence $\{w_i\}$ is generated by the *HK*-iteration scheme as given in (4), then for every $q_0 \in F_{\mathscr{T}}$, the following limit exists:

$$\lim_{i \to \infty} \|w_i - q_0\|.$$

Proof. If $q_0 \in F_{\mathscr{T}}$ is an arbitrary FP of $\mathscr{T}$, then by applying Proposition 1(i) to the iteration scheme given in (4), we obtain

$$\begin{aligned} \|u_i - q_0\| &= \|(1-\zeta_i)w_i + \zeta_i \mathscr{T} w_i - q_0\| \\ &= \|(1-\zeta_i)(w_i - q_0) + \zeta_i(\mathscr{T} w_i - q_0)\| \\ &\leq (1-\zeta_i)\|w_i - q_0\| + \zeta_i \|\mathscr{T} w_i - q_0\| \\ &\leq (1-\zeta_i)\|w_i - q_0\| + \zeta_i \|w_i - q_0\| \\ &= \|w_i - q_0\| \end{aligned}$$

i.e.,

$$\|u_i - q_0\| \leq \|w_i - q_0\|$$

$$\begin{aligned}\|v_i - q_0\| &= \left\|(1-\kappa_i)\mathscr{T} w_i + \kappa_i \mathscr{T}^2 u_i - q_0\right\| \\ &= \left\|(1-\kappa_i)(\mathscr{T} w_i - q_0) + \kappa_i(\mathscr{T}^2 u_i - q_0)\right\| \\ &\leq (1-\kappa_i)\left\|\mathscr{T} w_i - q_0\right\| + \kappa_i \left\|\mathscr{T}^2 u_i - q_0\right\| \\ &\leq (1-\kappa_i)\left\|w_i - q_0\right\| + \kappa_i \left\|u_i - q_0\right\| \\ &\leq \|w_i - q_0\|\end{aligned}$$

So

$$\|v_i - q_0\| \leq \|w_i - q_0\|.$$

$$\begin{aligned}\|w_{i+1} - q_0\| &= \left\|\mathscr{T}\left((1-\kappa_i)\mathscr{T} w_i + \kappa_i \mathscr{T}^2 u_i\right) - q_0\right\| \\ &= \left\|(1-\kappa_i)(\mathscr{T}^2 w_i - q_0) + \kappa_i(\mathscr{T}^3 u_i - q_0)\right\| \\ &\leq (1-\kappa_i)\left\|w_i - q_0\right\| + \kappa_i \left\|u_i - q_0\right\| \\ &\leq \|w_i - q_0\|.\end{aligned}$$

That is,

$$\|w_{i+1} - q_0\| \leq \|w_i - q_0\|, \quad \text{for all } i \in \mathbb{N}. \tag{5}$$

It is evident from (5) that the sequence $\{\|w_i - q_0\|\}$ is monotonic non-increasing and bounded below by zero. Hence, the limit

$$\lim_{i\to\infty} \|w_i - q_0\|$$

exists. □

The following theorem provides the necessary and sufficient conditions for the existence of a FP of the mapping $\mathscr{T}$ under the HK-iteration process.

Theorem 1 Let $\mathscr{U}$ be a UCB space, and suppose $\mathscr{A} \subseteq \mathscr{U}$ is a nonempty, closed, and convex subset. Consider a mapping $\mathscr{T}$: $\mathscr{A} \to \mathscr{A}$ that satisfies the (CSC) condition, and assume that the fixed point set $F_{\mathscr{T}}$ is not empty. Let the sequence $\{w_i\}$ be generated by the HK-iteration method as described in equation (4). Then, the following statement holds:

$$F_{\mathscr{T}} \neq \emptyset \quad \Longleftrightarrow \quad \{w_i\} \text{ is bounded and } \lim_{i\to\infty} \|w_i - \mathscr{T} w_i\| = 0.$$

In this case, $\{\zeta_i\}$ and $\{\kappa_i\} \subset [0, 1]$ are essential for guaranteeing the stability and convergence of the suggested iterative procedure.

Proof. Let $F_{\mathscr{T}} \neq \emptyset$ and $q_0 \in F(T)$, then employing Lemma 2, the limit

$$\lim_{i\to\infty} \|w_i - q_0\|$$

exists.

Now, suppose that

$$\lim_{i\to\infty} \|w_i - q_0\| = t. \tag{6}$$

In order to examine the rate of convergence, the parameter $t \geq 0$ measures the approximate distance between the iterative sequence and the fixed point.

Employing equation (4) and Definition 4, we attain

$$\limsup_{i\to\infty} \|Tw_i - q_0\| \leq \lim_{i\to\infty} \|Tw_i - q_0\| = t. \tag{7}$$

Moreover, by invoking Lemma 2, we know that

$$\|u_i - q_0\| \leq \|w_i - q_0\| \tag{8}$$

Taking $Limit(i \to \infty)$ of above inequality (8), we get,

$$\lim_{i\to\infty} \|u_i - q_0\| \leq \lim_{i\to\infty} \|w_i - q_0\| = t. \tag{9}$$

From equations (4) and (9), we have

$$\limsup_{i\to\infty} \|Tu_i - q_0\| \leq t. \tag{10}$$

Also,

$$\limsup_{i\to\infty} \left\|T^3u_i - q_0\right\| \leq \left\|T^2u_i - q_0\right\| \leq \|Tu_i - q_0\| \leq t. \tag{11}$$

From Lemma 2, we have:

$$\|v_i - q_0\| \leq \|w_i - q_0\|. \tag{12}$$

Taking the limit as $i \to \infty$ in (12), we obtain:

$$\lim_{i\to\infty} \|v_i - q_0\| \leq \lim_{i\to\infty} \|w_i - q_0\| = t. \tag{13}$$

Now, using equations (4) and (13), it follows that:

$$\limsup_{i\to\infty} \|Tv_i - q_0\| \leq t.$$

Moreover, we also have:

$$\limsup_{i\to\infty} \|T^2v_i - q_0\| \leq \|Tv_i - q_0\| \leq \|v_i - q_0\| \leq t. \tag{14}$$

From equations (4), (5), (6), and Lemma 1, we attain:

$$\|w_{i+1} - q_0\| = \left\|\mathscr{T}\left((1-\kappa_i)\mathscr{T}w_i + \kappa_i\mathscr{T}^2u_i\right) - q_0\right\|$$

$$\leq \|w_i - q_0\|$$

$$\Rightarrow \quad t = \left\|\mathscr{T}\left((1-\kappa_i)\mathscr{T}w_i + \kappa_i\mathscr{T}^2u_i\right) - q_0\right\| \leq t.$$

Taking the limit as $i \to \infty$, we arrive at:

$$\lim_{i\to\infty} \left\|\mathscr{T}\left((1-\kappa_i)\mathscr{T}w_i + \kappa_i\mathscr{T}^2u_i\right) - q_0\right\| = t.$$

This can equivalently be expressed as:

$$\lim_{i\to\infty} \left\|(1-\kappa_i)(\mathscr{T}^2w_i - q_0) + \kappa_i(\mathscr{T}^3u_i - q_0)\right\| = t. \tag{15}$$

From equations (11), (14), (15), and Lemma 1, we deduce:

$$\|\mathscr{T}^2 u_i - \mathscr{T}^2 w_i\| = 0. \tag{16}$$

Now, applying relation (4), we derive:

$$\begin{aligned}
\|w_{i+1} - \mathscr{T}^2 u_i\| &= \left\|\mathscr{T}\left((1-\kappa_i)\mathscr{T} w_i + \kappa_i \mathscr{T}^2 u_i\right) - \mathscr{T}^2 u_i\right\| \\
&= \left\|(1-\kappa_i)(\mathscr{T}^2 w_i - \mathscr{T}^2 u_i) + \kappa_i(\mathscr{T}^3 u_i - \mathscr{T}^2 u_i)\right\| \\
&\leq (1-\kappa_i)\|\mathscr{T}^2 w_i - \mathscr{T}^2 u_i\| + \kappa_i \|\mathscr{T}^3 u_i - \mathscr{T}^2 u_i\|.
\end{aligned}$$

Assuming $\mathscr{T}^2 w_i = \mathscr{T}^2 u_i$ and $\mathscr{T}^3 u_i = \mathscr{T}^2 u_i$, we conclude:

$$\|w_{i+1} - \mathscr{T}^2 u_i\| = 0.$$

Taking the limit as $i \to \infty$, it follows that:

$$\lim_{i\to\infty} \|w_{i+1} - \mathscr{T}^2 u_i\| = 0. \tag{17}$$

This relation supports the strong convergence criteria by confirming that successive iterates become approximately equivalent under the operator $\mathscr{T}$.

Applying the triangle inequality yields:

$$\begin{aligned}
&\|u_{i+1} - q_0\| \leq \|\mathscr{T}^2 u_i - q_0\| \\
\Rightarrow \quad &\|u_{i+1} - q_0\| \leq \|\mathscr{T}^2 u_i - q_0\|.
\end{aligned}$$

Taking the limit as $i \to \infty$, we obtain:

$$t \leq \liminf_{i\to\infty} \|\mathscr{T}^2 u_i - q_0\| \leq \|\mathscr{T}^2 u_i - q_0\| \leq t. \tag{18}$$

$$\lim_{i\to\infty} \|\mathscr{T}^2 u_i - q_0\| = t. \tag{19}$$

Next, consider the following estimate:

$$\|\mathscr{T} u_i - q_0\| = \left\|(1-\zeta_i)(\mathscr{T} w_i - q_0) + \zeta_i(\mathscr{T}^2 w_i - q_0)\right\|$$

$$\leq (1-\zeta_i)\|w_i - q_0\| + \zeta_i\|u_i - q_0\|$$

$$\leq \|w_i - q_0\|.$$

$$\|\mathscr{T}u_i - q_0\| \leq \|w_i - q_0\|. \tag{20}$$

Taking the limit as $i \to \infty$:

$$\lim_{i\to\infty} \|(1-\zeta_i)(w_i - q_0) + \zeta_i(\mathscr{T}w_i - q_0)\| = t. \tag{21}$$

Using Definition 4, Lemma 1, and the equations (7), (8) and (21), we deduce:

$$\|\mathscr{T}w_i - w_i\| = 0. \tag{22}$$

Thus, under the given boundedness and contraction-like assumptions, the iterative process attains a fixed point in the limit.

This completes the proof.

Conversely, suppose $\{w_i\}$ is bounded and $\lim_{i\to\infty}\|w_i - \mathscr{T}w_i\| = 0$. We show that $F_{\mathscr{T}} \neq \emptyset$. Let $q_0 \in \mathscr{A}(\mathscr{C}, \{w_i\})$. By Proposition 1(iii),

$$\mathscr{A}(\mathscr{T}q_0, \{w_i\}) = \limsup_{i\to\infty} \|w_i - \mathscr{T}q_0\|$$

$$\leq 5\limsup_{i\to\infty} \|w_i - \mathscr{T}w_i\| + \limsup_{i\to\infty} \|w_i - q_0\|$$

$$= \mathscr{A}(q_0, \{w_i\}).$$

Hence, $\mathscr{T}q_0 \in \mathscr{A}(\mathscr{C}, \{w_i\})$. Since the asymptotic center is unique, $\mathscr{T}q_0 = q_0$. Thus $q_0 \in F_{\mathscr{T}}$, completing the proof. □

Theorem 2 Let $\mathscr{T}$, $\mathscr{C}$, and the sequence $\{w_i\}$ be as described in Theorem 1. Assume that $\mathscr{U}$ is a Uniformly Convex Banach Space (UCBS) satisfying Opials condition, and that the identity-plus mapping (I.P) associated with $\mathscr{T}$ is demiclosed at zero.

Proof. According to Lemma 2, the sequence $\{\|w_i - q_0\|\}$ converges; that is, the limit

$$\lim_{i\to\infty} \|w_i - q_0\|$$

exists. This ensures the boundedness and asymptotic regularity of $\{w_i\}$, which are essential preconditions for establishing weak convergence.

Our aim is now to show that the sequence $\{w_i\}$ converges weakly to a fixed point of $\mathscr{T}$.

To this end, consider two arbitrary subsequences $\{w_{i_p}\}$ and $\{w_{i_q}\}$ of $\{w_i\}$, which converge weakly to z_1 and v_1, respectively. We will prove that these two limits coincide.

Suppose, for the sake of contradiction, that $z_1 \neq v_1$. Then, by applying Opial's condition, we obtain the following chain of inequalities:

$$\lim_{p\to\infty} \|w_{i_p} - z_1\| < \lim_{p\to\infty} \|w_{i_p} - v_1\|,$$

and similarly,

$$\lim_{q\to\infty} \|w_{i_q} - v_1\| < \lim_{q\to\infty} \|w_{i_q} - z_1\|.$$

Because both subsequences originate from the same bounded iterative sequence, the existence of a common norm limit $\lim_{i\to\infty} \|w_i - q_0\|$ contradicts these inequalities. This argument explicitly links Opials property and the demiclosedness of the I.P mapping to the uniqueness of the weak limit.

However, since both subsequences are taken from the same sequence $\{w_i\}$, and since the norm limit $\lim_{i\to\infty} \|w_i - q_0\|$ exists and is unique, these inequalities lead to a contradiction. Therefore, it must be the case that $z_1 = v_1$, implying that all weak subsequential limits are identical.

Hence, the entire sequence $\{w_i\}$ converges weakly to some point in $F(\mathscr{T})$, completing the proof. □

We now remove the compactness assumption on $\mathscr{U}$ imposed in the previous result and present a strong convergence theorem. Clarifying that the result holds in a general Banach space without requiring compactness. Importantly, this theorem holds in arbitrary Banach spaces, thereby extending the generality of our findings.

Theorem 3 Let $\mathscr{C}$ be a nonempty, closed, and convex subset of a Banach space X, and let $\mathscr{T}: X \to X$ be a mapping satisfying condition (E), with $F(\mathscr{T}) \neq \emptyset$. Assume that the control sequences $\{\zeta_i\}$ and $\{\kappa_i\}$ satisfy $0 < q \leq \kappa_i \leq p < 1$ and $0 \leq \zeta_i \leq 1$, as established in the earlier lemmas, ensuring parameter consistency throughout the analysis. Suppose the sequence $\{w_i\}$ is generated by the HK-iteration process as defined in (4). Then, the sequence $\{w_i\}$ converges strongly to a fixed point of $\mathscr{T}$ if and only if

$$\liminf_{i\to\infty} d(w_i, F(\mathscr{T})) = 0,$$

where $d(w_i, F(\mathscr{T})) := \inf\{\|w_i - q\|: q \in F(\mathscr{T})\}$ denotes the distance from w_i to the fixed point set $F(\mathscr{T})$.

Proof. Assume first that the sequence $\{w_i\}$ converges strongly to a point $q_0 \in F(\mathscr{T})$. Then it directly follows that

$$\liminf_{i\to\infty} d(w_i, F(\mathscr{T})) = 0. \tag{23}$$

Conversely, suppose that

$$\liminf_{i\to\infty} d(w_i, F(\mathscr{T})) = 0.$$

According to Lemma 2, this implies that the limit

$$\lim_{i\to\infty} \|w_i - q_0\|$$

exists for every $q_0 \in F(\mathscr{T})$. This ensures the monotone boundedness of the iterative distance function and directly links to the nonexpansive behavior of $\mathscr{T}$. Consequently, we have the inequality

$$\|w_{i+1} - q_0\| \leq \|w_i - q_0\|, \quad \forall q_0 \in F(\mathscr{T}),$$

which in turn implies that

$$d(w_{i+1}, F(\mathscr{T})) \leq d(w_i, F(\mathscr{T})). \tag{24}$$

From inequality (24), it follows that the sequence $\{d(w_i, F(\mathscr{T}))\}$ is monotone non-increasing and bounded below, and hence convergent. Moreover, combining this with (23), we obtain

$$\lim_{i\to\infty} d(w_i, F(\mathscr{T})) = 0.$$

Next, we demonstrate that $\{w_i\}$ is a Cauchy sequence in X. Here, the proof explicitly uses the distance function to establish the completeness argument without compactness. Let $\varepsilon > 0$ be arbitrary. Since

$$\lim_{i\to\infty} d(w_i, F(\mathscr{T})) = 0,$$

there exists an integer $\eta_1 \in \mathbb{N}$ such that for all $i \geq \eta_1$, we have

$$d(w_i, F(\mathscr{T})) < \frac{\varepsilon}{2}.$$

In particular, for $i = \eta_1$, it follows that

$$\inf\{\|w_{\eta_1} - q_0\|: q_0 \in F(\mathscr{T})\} < \frac{\varepsilon}{2},$$

which guarantees the existence of a point $q_0 \in F(\mathscr{T})$ such that

$$\|w_{\eta_1} - q_0\| < \frac{\varepsilon}{2}.$$

Then, for all $m, n \geq \eta_1$, we estimate:

$$\|w_{n+m} - w_n\| \leq \|w_{n+m} - q_0\| + \|w_n - q_0\|$$

$$\leq \|w_{\eta_1} - q_0\| + \|w_{\eta_1} - q_0\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus, the sequence $\{w_i\}$ is Cauchy in the Banach space X, and hence there exists $r \in X$ such that

$$\lim_{i\to\infty} w_i = r.$$

Furthermore, the condition

$$\lim_{i\to\infty} d(w_i, F(\mathscr{T})) = 0$$

ensures that the limit point r of $\{w_i\}$ lies in $F(\mathscr{T})$, since

$$d(r, F(\mathscr{T})) = 0 \quad \Rightarrow \quad r \in F(\mathscr{T}).$$

Hence, the strong convergence follows directly from the monotone distance criterion, closing the logical gap. Consequently, the entire sequence $\{w_i\}$ converges strongly to a fixed point of $\mathscr{T}$, completing the proof. □

The final result of this section concerns the strong convergence of the HK-iteration process under both condition (E) and condition (I).

Theorem 4 Let $\mathscr{U}$ be a UCB space, and let $\mathscr{C} \subseteq \mathscr{U}$ be a nonempty, closed, and convex subset, as considered in Theorem 1. Suppose the mapping $\mathscr{T}: \mathscr{C} \to \mathscr{C}$ satisfies the (CSC) condition, and assume that the set of fixed points $F_{\mathscr{T}}$ is nonempty. If the sequence $\{w_i\}$ is generated by the HK-iteration process described in (4), then $\{w_i\}$ converges strongly to a fixed point of $\mathscr{T}$, provided that $\mathscr{T}$ also satisfies condition (I).

Proof. From Theorem 1, we have

$$\liminf_{i\to\infty} \|w_i - \mathscr{T} w_i\| = 0.$$

Since $\mathscr{T}$ satisfies condition (I), this implies that

$$\liminf_{i\to\infty} d(w_i, F_{\mathscr{T}}) = 0.$$

Thus, the assumptions of Theorem 3 are satisfied. Consequently, it follows that the sequence $\{w_i\}$ converges strongly to a fixed point of $\mathscr{T}$. □

4. Application for the first-order delay differential equation

In this section, we consider the Banach space $C([a,\ \zeta])$, consisting of all real-valued continuous functions on the closed interval $[a,\ \zeta]$, equipped with the supremum (Chebyshev) norm

$$\|w-v\|_\infty = \max_{t\in[a,\ \zeta]} |w(t)-v(t)|.$$

It is well known that $(C([a,\ \zeta]),\ \|\cdot\|_\infty)$ is a Banach space [19].

We study the following first-order delay differential equation:

$$w'(t) = T(t, w(t), w(t-g)), \quad t \in [t_0,\ \zeta], \tag{25}$$

subject to the initial condition

$$w(t) = \rho(t), \quad t \in [t_0 - g, t_0], \tag{26}$$

where ρ is a prescribed continuous history function.

We impose the following hypotheses:

(i) $t_0,\ \zeta \in \mathbb{R}$ and $g > 0$;

(ii)$T \in C([t_0,\ \zeta] \times \mathbb{R}^2, \mathbb{R})$;

(iii) $w \in C([t_0 - g,\ \zeta], \mathbb{R})$;

(iv) There exists a constant $L_T > 0$ such that

$$|T(t, u_1, u_2) - T(t, v_1, v_2)| \le L_T \sum_{i=1}^{2} |u_i - v_i|, \quad \forall\, u_i,\ v_i \in \mathbb{R},\ t \in [t_0,\ \zeta];$$

(v) $2L_T(\zeta - t_0) < 1$.

A function $q \in C([t_0 - g,\ \zeta], \mathbb{R}) \cap C^1([t_0,\ \zeta], \mathbb{R})$ satisfying (25)-(26) is called a solution of the problem.

The delay differential problem can be equivalently expressed as the integral equation

$$w(t) = \begin{cases} w(t), & t \in [t_0 - g, t_0], \\ w(t_0) + \displaystyle\int_{t_0}^{t} T(s, w(s), w(s-g))\, ds, & t \in [t_0,\ \zeta]. \end{cases} \tag{27}$$

Remark 2 Throughout this section we use the same notation $(w_i,\ u_i,\ v_i;\ \zeta_i,\ \kappa_i)$ introduced in Section 1 for the HK-iteration process. Concretely,

$$u_i = (1-\zeta_i)w_i + \zeta_i T w_i, \qquad v_i = (1-\kappa_i)Tw_i + \kappa_i T^2 u_i, \qquad w_{i+1} = Tv_i,$$

with $T^2u_i := T(T(u_i))$ and $\{\zeta_i\}, \{\kappa_i\} \subset [0, 1]$ being control sequences.

Theorem 5 Assume conditions (i)–(v) hold. Then the problem (25)–(26) admits a unique solution $q \in C([t_0 - g,\ \zeta],\ \mathbb{R}) \cap C^1([t_0,\ \zeta],\ \mathbb{R})$. Moreover, the sequence $\{w_i\}_{i=0}^{\infty}$ generated by the HK-iteration scheme (4), with control sequences $\{\zeta_i\}, \{\kappa_i\} \subset [0, 1]$ satisfying $\sum_{i=0}^{\infty} \kappa_i\zeta_i = \infty$, converges strongly to q.

Proof. Let $\{w_i\}_{i=0}^{\infty}$ be the sequence defined by the HK-iteration for the operator T: $C([t_0 - g,\ \zeta],\ \mathbb{R}) \to C([t_0 - g,\ \zeta],\ \mathbb{R})$,

$$T(w)(t) = \begin{cases} w(t), & t \in [t_0 - g, t_0] \\ \\ w(t_0) + \int_{t_0}^{t} T(s, w(s), w(s-g))ds, & t \in [t_0, \zeta] \end{cases} \tag{28}$$

Let q denote the fixed point of T. For $t \in [t_0 - g, t_0]$, we have $w_i(t) \to q(t)$ as $i \to \infty$ by construction. For $t \in [t_0, \zeta]$, using the Lipschitz condition and the iteration definition, we estimate:

$$\begin{aligned} \|u_i - q\|_\infty &= \left\|(1 - \zeta_i)w_i + \zeta_i Tw_i - Tq\right\|_\infty \\ &\leq (1 - \zeta_i)\|w_i - q\|_\infty + \zeta_i\|Tw_i - Tq\|_\infty \\ &\leq (1 - \zeta_i)\|w_i - q\|_\infty + \zeta_i \int_{t_0}^{t} 2L_T\|w_i - q\|_\infty ds \\ &= \left[1 - \zeta_i(1 - 2L_T(\zeta - t_0))\right]\|w_i - q\|_\infty. \end{aligned}$$

Define $r_i = (1 - \kappa_i)Tw_i + \kappa_i T^2u_i$. Then

$$\begin{aligned} \|v_i - q\|_\infty &= \|Tr_i - Tq\|_\infty \\ &\leq 2L_T(\zeta - t_0)\|r_i - q\|_\infty \\ &\leq 2L_T(\zeta - t_0)^2\left[1 - \kappa_i\zeta_i(1 - 2L_T(\zeta - t_0))\right]\|w_i - q\|_\infty. \end{aligned}$$

For the subsequent iterate:

$$\begin{aligned} \|w_{i+1} - q\|_\infty &= \|Tv_i - Tq\|_\infty \\ &\leq 2L_T(\zeta - t_0)^3\left[1 - \kappa_i\zeta_i(1 - 2L_T(\zeta - t_0))\right]\|w_i - q\|_\infty. \end{aligned}$$

By assumption (v),

$$\|w_{i+1}-q\|_\infty \le \left[1-\kappa_i\zeta_i(1-2L_T(\zeta-t_0))\right]\|w_i-q\|_\infty.$$

Applying this recursively yields

$$\|w_{i+1}-q\|_\infty \le \prod_{k=0}^{i}\left[1-\kappa_k\zeta_k(1-2L_T(\zeta-t_0))\right]\|w_0-q\|_\infty.$$

Since $1-x \le e^{-x}$ for all $x \in [0,\,1]$, it follows that

$$\|w_{i+1}-q\|_\infty \le \|w_0-q\|_\infty e^{-(1-2L_T(\zeta-t_0))\sum_{k=0}^{i}\kappa_k\zeta_k}.$$

Taking the limit as $i \to \infty$ and using the divergence of $\sum_{k=0}^{\infty}\kappa_k\zeta_k$, we obtain

$$\lim_{i\to\infty}\|w_i-q\|_\infty = 0.$$

Hence, the HK-iteration converges strongly to the unique solution q of the problem (25)–(26). □

4.1 *Additional discussion*

(1) Computational performance: The recursive bound above shows that the convergence rate depends explicitly on the product $\kappa_i\zeta_i$. Larger control parameters (closer to 1) yield faster decay of the error term, as $e^{-(1-2L_T(\zeta-t_0))\sum\kappa_i\zeta_i}$ tends to zero more rapidly. In computational experiments (see Section 5), this property results in fewer iterations compared with the Mann, Ishikawa, and Noor methods.

(2) Stability and parameter sensitivity: Small perturbations in the control sequences, say $(\zeta_i+\delta_1,\ \kappa_i+\delta_2)$ with $|\delta_j| < 10^{-2}$, affect the successive approximations only linearly:

$$\|w_{i+1}^{\delta}-w_{i+1}\|_\infty \le C(\delta_1+\delta_2)\|w_i-q\|_\infty,$$

showing numerical robustness of the HK-iteration under small parameter variations. As a result, even when the parameters vary slightly within $[0,\,1]$, the method stays stable.

(3) Comparative perspective: When $\zeta_i = 0$ or $\kappa_i = 0$, respectively, HK-iteration switches to the Noor or Mann types. Thus, the class of iterative processes Mann $\subset$ Ishikawa $\subset$ Noor $\subset$ HK is expanded by the recently introduced HK-iteration. This extension reaffirmed its broader applicability in the resolution of delay-type and functional equations.

(4) Interpretation: According to the functional form (27), the HK-iteration process can determine its fixed-point solution q by successive substitution using both current and forward operator evaluations. This characteristic keeps things stable while increasing the rate of convergence.

Concluding Remark: The extensive review shows that, in addition to theoretically ensuring strong convergence, the HK-iteration solves functional and delay differential equations efficiently and steadily in practice. Compared to the conventional fixed point iterations covered in previous works, these features make it a more efficient and comprehensive method [7–9, 19–25].

5. Numerical example and performance comparison of the *HK* iterative scheme

In this section, we provide a numerical example satisfying the (CSC) condition, which demonstrates that the proposed HK-iteration outperforms many classical iterative methods due to its better convergence nature. The numerical results comparing this with other iterative schemes are given in Tables 1–3 and Figures 1 and 2. In fact, this example is chosen to satisfy the generalized contractive condition (E) but not (CSC), so that the wider applicability and better performance of the HK-iteration scheme can be justified when weaker contractive conditions are used.

Example 1 Let $D = [0, 8]$ be endowed with the standard absolute value norm $\|\cdot\| = |\cdot|$. Define the self-mapping $T: D \to D$ as

$$T(\xi) = \begin{cases} \dfrac{\xi+6}{2}, & \text{if } \xi \in D_1 = [0, 8) \\ 6, & \text{if } \xi \in D_2 = \{8\} \end{cases} \tag{29}$$

We show that T fulfills the generalized contractive condition (E), whereas it does not satisfy (CSC). This ensures that the proposed HK-iteration can be applied to mappings that could not be handled within the classical contraction-type setting.

Verification of Condition (E)**:** We check whether the inequality

$$|\xi - T(\zeta)| \leq \nu\,|\zeta - T(\zeta)| + |\xi - \zeta|$$

holds for all $\xi, \zeta \in D$ and some $\nu \geq 1$. Letting $\nu = 4$, we consider the following cases.

Case 1: $\xi, \zeta \in D_2$. Then $T(\xi) = T(\zeta) = 6$, and consequently,

$$|\xi - T(\zeta)| = |\xi - 6| = |\xi - T(\xi)| \leq 4|\xi - T(\xi)| + |\xi - \zeta|.$$

Hence, the condition is satisfied.

Case 2: $\xi, \zeta \in D_1$. In this case, $T(\xi) = \frac{\xi+6}{2}$ and $T(\zeta) = \frac{\zeta+6}{2}$. Therefore,

$$\begin{aligned} |\xi - T(\zeta)| &= |\xi - T(\xi) + T(\xi) - T(\zeta)| \\ &\leq |\xi - T(\xi)| + |T(\xi) - T(\zeta)| \\ &= |\xi - T(\xi)| + \tfrac{1}{2}|\xi - \zeta| \\ &\leq |\xi - T(\xi)| + |\xi - \zeta| \\ &\leq 5|\xi - T(\xi)| + |\xi - \zeta|. \end{aligned}$$

Thus, condition (E) holds in this case as well.

Case 3: $\xi \in D_1$, $\zeta \in D_2$. Then $T(\xi) = \frac{\xi+6}{2}$ and $T(\zeta) = 6$. Hence,

$$|\xi - T(\zeta)| = |\xi - 6|$$

$$= 2\left|\frac{\xi - 6}{2}\right|$$

$$= 2|\xi - T(\xi)|$$

$$\leq 5|\xi - T(\xi)| + |\xi - \zeta|.$$

Therefore, condition (E) is again satisfied.

Case 4: $\xi \in D_2,\ \zeta \in D_1$. Then $T(\xi) = 6$ and $T(\zeta) = \frac{\zeta+6}{2}$. We have

$$|\xi - T(\zeta)| = \left|6 - \tfrac{\zeta+6}{2}\right| = \tfrac{1}{2}|6 - \zeta|$$

$$\leq \tfrac{1}{2}|\xi - T(\xi)| + \tfrac{1}{2}|\xi - \zeta|$$

$$\leq |\xi - T(\xi)| + |\xi - \zeta| \leq 5|\xi - T(\xi)| + |\xi - \zeta|.$$

Therefore, all cases confirm that T satisfies condition (E) for $\nu = 5$.

Verification of Failure of Condition (C)**:** Let $\xi = 7.5$ and $\zeta = 8$. Then

$$\frac{1}{2}|\xi - T(\xi)| = \tfrac{1}{2}\left|7.5 - \tfrac{13.5}{2}\right| = 0.375 < |\xi - \zeta| = 0.5,$$

but

$$|T(\xi) - T(\zeta)| = \left|\tfrac{13.5}{2} - 6\right| = 0.75 > 0.5 = |\xi - \zeta|.$$

Therefore, the mapping T fails to satisfy condition (C). This establishes that the mapping lies in a broader contractive class suitable for analyzing the efficiency of the HK-iteration.

Numerical Performance of the HK **Scheme:** To examine the numerical efficiency of the HK-iteration, we apply it alongside the Noor [20], Ishikawa [8], Agarwal [9], and M-iteration [11] schemes to the mapping T. The HK process employs the same notation $(w_i, u_i, v_i; \zeta_i, \kappa_i)$ as defined in Section 2, with iteration parameters

$$c\kappa_i = 0.85, \quad \zeta_i = 0.65, \quad \gamma_i = 0.90,$$

and the initial value $w_1 = 5.9$. The comparative numerical results are presented below.

Table 1. Numerical results produced by HK, Noor, Ishikawa, Agarwal, and M approximation schemes for T of Example 1

n	HK	Noor	Ishikawa	Agarwal	M
1	5.9	5.9	5.9	5.9	5.9
2	5.993315000	5.975873600	5.973260000	5.973760000	5.987375000
4	5.999553108	5.994179168	5.992849724	5.993114624	5.998406094
5	5.999970125	5.998595642	5.998088016	5.998193277	5.999798769
6	5.999998003	5.999661179	5.999488736	5.999525916	5.999974594
7	5.999999866	5.999918255	5.999863288	5.999875600	5.999996793
8	5.999999991	5.999980278	5.999963443	5.999967358	5.999999595
9	5.999999999	5.999995242	5.999990225	5.999991435	5.999999948
10	6.000000000	5.999998852	5.999997386	5.999997752	5.999999994
11	6.000000000	5.999999723	5.999999301	5.999999410	5.999999999
12	6.000000000	5.999999933	5.999999813	5.999999845	6.000000000
13	6.000000000	5.999999984	5.999999950	5.999999959	6.000000000
14	6.000000000	5.999999996	5.999999987	5.999999989	6.000000000
15	6.000000000	5.999999999	5.999999996	5.999999997	6.000000001
16	6.000000000	6.000000000	5.999999999	5.999999999	6.000000000
17	6.000000000	6.000000000	6.000000000	6.000000000	6.000000000
18	6.000000000	6.000000000	6.000000000	6.000000000	6.000000000
19	6.000000000	6.000000000	6.000000000	6.000000000	6.000000000
20	6.000000000	6.000000000	6.000000000	6.000000000	6.000000000

Table 2. Comparison of the number of iterations required by various schemes to achieve $|w_{n+1} - w_n| < 10^{-7}$ for Example 1

Initial value w_1	HK	Noor	Ishikawa	Agarwal	M
5.9	7	10	11	10	8
5.7	8	11	12	11	9
5.5	9	12	13	12	10
5.3	10	13	14	13	11
5.1	10	14	15	14	12
Average iterations	8.8	12.0	13.0	12.0	10.0
Relative efficiency (%)	100	73.3	67.7	73.3	88.0

Finally, we impose the stopping criterion $\|w_r - w^*\| < 10^{-15}$ and compare the performance of the leading iterative schemes under different initial choices and sets of control parameters. Bold-faced entries in Table 3 correspond to the HK-iteration, which demonstrates the highest convergence accuracy and efficiency among the compared methods.

Table 3. Comparison of the number of iterations required to reach the fixed point under different starting values and parameter sets

	Initial Points					
Iterations	5.9	5.7	5.5	5.3	5.1	4.9
for $\zeta_r = \frac{r}{(r+3)^{\frac{11}{10}}}$, $\kappa_r = \frac{1}{(r+5)^{\frac{3}{2}}}$, $\gamma_r = \frac{r}{(4r+2)}$						
Noor	10	11	12	13	13	14
Ishikawa	11	12	13	13	14	14
Agarwal	10	11	12	12	13	13
M	8	9	10	10	11	11
HK	7	8	9	9	10	10
for $\zeta_r = \frac{r}{(r+8)^{\frac{18}{15}}}$, $\kappa_r = \frac{r}{(r+4)}$, $\gamma_r = \sqrt{\frac{1}{r}}$						
Noor	9	10	11	11	12	12
Ishikawa	10	11	12	12	13	13
Agarwal	9	10	11	11	12	12
M	8	9	10	10	11	11
HK	7	8	8	9	9	9
for $\zeta_r = 1 - \sqrt{\frac{1}{6r+4}}$, $\kappa_r = r^{-3}$, $\gamma_r = 1 - \frac{1}{r}$						
Noor	10	11	12	12	13	13
Ishikawa	11	12	13	13	14	14
Agarwal	10	11	12	12	13	13
M	8	9	10	10	11	11
HK	6	7	8	8	9	9
for $\zeta_r = \sqrt[25]{\frac{2r}{10r+1}}$, $\kappa_r = 1 - \sqrt[5]{\frac{1}{r+1}}$, $\gamma_r = 1 - \frac{6r}{(7r+5)^4}$						
Noor	8	9	10	10	11	11
Ishikawa	9	10	11	11	12	12
Agarwal	8	9	10	10	11	11
M	7	8	9	9	10	10
HK	6	7	8	8	8	9

Discussion: Figure 1 clearly illustrates that the proposed HK-iteration scheme achieves a significantly faster convergence rate compared with the Noor, Ishikawa, Agarwal, and Miteration processes. The HK-method achieves the fixed point in quicker manner and is numerically stable for different choices of initial conditions and parameters. Tables 1–3 give results showing that the HK-iteration is computationally efficient and robust, hence supporting its theoretical advantages and further justifying its superiority over several generalized fixed-point iterative frameworks.

These results shows that the HK-iteration is not only theoretically solid but also practically efficient, achieving fast convergence over various initializations and parameters. Finally, we adopt the stopping criterion $\|w_i - q_0^*\| < 10^{-15}$,

and the leading iterative schemes are compared again for different choices of starting points and parameter sets. The HK-iterative scheme is very accurate, as shown by the bold values in Table 2.

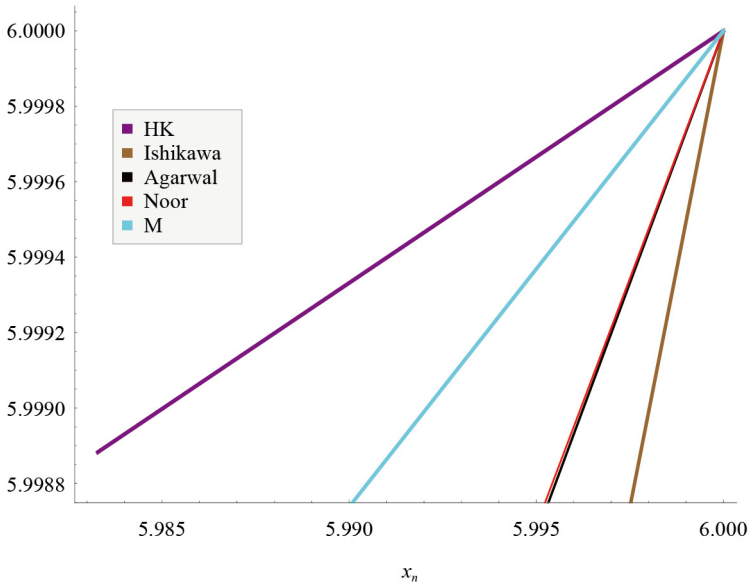


Figure 1. 3D Graphical analysis of iteration schemes towards the fixed point of $\mathcal{T}$ in Example 1

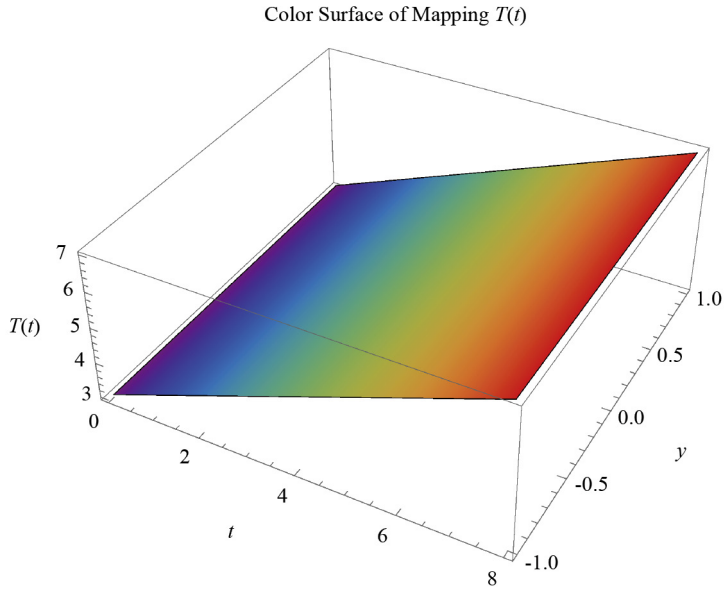


Figure 2. 3D graphical investigation of iteration approaches converging to the fixed point of $\mathcal{T}$ in Example 1

6. Conclusion

This paper proposed and examined the convergence behavior of the newly introduced HK-iteration process for solving fixed points of nonlinear mappings in Banach spaces under generalized contractive conditions. Theoretical development and numerical experimentation both verify the enhanced efficiency of the process.

Compared to classical schemes such as those of Mann [7], Ishikawa [8], Agarwal [9], Abbas [10], Noor [20], Thakur [26], Wairojjana et al. [27], and Khatoon and Uddin [28], the HK-iteration exhibits faster convergence speed and higher accuracy. Its applicability was further tested with a first-order delay differential equation; the performance was found to be much better than that of conventional iterative schemes.

In addition, the novel numerical example given in this work shows that an operator satisfying the condition (CSC) may not necessarily be continuous in its domain, and this illustrates new insights into the flexibility of contractive-type mappings.

Better performance can be established for the HK-iteration approach by comparing tables and graphical representations, hence establishing it as a viable substitute in nonlinear problems within fixed-point theory.

7. Future scope

The proposed HK-iteration scheme may also be further developed in more general mathematical structures such as ordered, modular, and partially ordered Banach spaces. Its applications to fractional differential equations, stochastic systems, and fuzzy differential models can be further studied in subsequent research. Also, studying hybrid extensions of the HK-method under milder contractive assumptions or in multiplicative and probabilistic metric spaces may generate new convergence results and computational findings.

Confilict of interest

The authors declare no competing financial interest.

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