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Research Article

Inverse Problem: Simultaneous Reconstruction of Multiple Coefficients in Tridiagonal Competitive Reaction-Diffusion Systems with Positive Feedback

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Abstract: This paper establishes uniqueness results for solving inverse problems involving multiple non-constant coefficients in a trio of parabolic equations modeling tridiagonal competitive diffusion. We demonstrate that six coefficients in the system-namely, the reaction coefficients r_1, r_2, r_3 and the interaction coefficients a_{11}, a_{22}, a_{33}-can be uniquely determined under the following conditions: 1. The coefficients belong to the $C^n([a,\ b])$ function space. 2. Pointwise measurements of the solution components $(u_1,\ u_2,\ u_3)$ along with their spatial derivatives $\frac{\partial u_1}{\partial x}$ and $\frac{\partial u_2}{\partial x}$ are available at a single spatial point x_0. Notably, these measurements only need to be observed over an arbitrarily small time interval $(0,\ \varepsilon)$.

***Keywords*:** inverse problem, tridiagonal competitive system, uniqueness, reaction-diffusion equation, positive feedback

MSC: 35K20, 35K40, 35K35, 35K57

1. Introduction

This study investigates the following three-dimensional competitive tridiagonal reaction-diffusion model with positive feedback [1, 2] for $t > 0$ and $x \in (a,\ b) \subset \mathbb{R}$,

$$\begin{cases} \dfrac{\partial u_1}{\partial t} = D_1 \dfrac{\partial^2 u_1}{\partial x^2} + u_1(r_1 - a_{11}u_1 - a_{12}u_2 + a_{13}u_3), \\ \dfrac{\partial u_2}{\partial t} = D_2 \dfrac{\partial^2 u_2}{\partial x^2} + u_2(r_2 - a_{21}u_1 - a_{22}u_2 - a_{23}u_3), \\ \dfrac{\partial u_3}{\partial t} = D_3 \dfrac{\partial^2 u_3}{\partial x^2} + u_3(r_3 + a_{31}u_1 - a_{32}u_2 - a_{33}u_3), \end{cases} \tag{1}$$

where $a_{ij} > 0$, $r_i > 0$, $D_i > 0$, $i,\ j = 1,\ 2,\ 3$.

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Model (1) has extensive applications spanning biology, chemistry, and physics. In biology, Fisher derived the equation $u_t - ku_{xx} = bu(1-cu)$ by integrating the Verhulst logistic model with diffusion mechanisms to describe species migration [3], later extended as the Fisher-Kolmogorov-Petrovsky-Piskunov population model. In chemistry, reaction-diffusion systems similar to (1) are fundamental in studying combustion dynamics and chemical pattern formation. For example, cubic autocatalytic reactions can be modeled by generalized forms of (1), where higher-order polynomial terms capture nonlinear reaction kinetics [4]. These models are crucial in combustion theory, where diffusion balances exothermic heat release, enabling analysis of flame propagation and stability [5]. The Belousov-Zhabotinsky reaction, a classic oscillatory chemical system, can also be approximated by competitive tridiagonal forms of (1), with diffusion regulating feedback-driven oxidation cycles [6]. Additionally, problem (1) integrates experimental data and neural networks to predict reservoir permeability changes due to calcium sulfate precipitation during waterflooding, offering a theoretical framework for optimizing injection processes and scale inhibition [7]. Notably, Turing identified a specialized version of (1) capable of simulating spontaneous pattern formation-such as stripes and spots-emerging from homogeneous conditions [8].

The investigation of model (1) is driven by both its significant physical applications and inherent mathematical importance. Recent studies [9, 10] utilize Carleman estimates at fixed time t for spatial domains $x \in \Omega \subset \mathbb{R}^n$ to examine the nonlocal equation (1) and its relationship with scalar parabolic equations, where coefficient determination can be approached through direct estimation and boundary measurement techniques. Existing literature has thoroughly explored the inverse problem of recovering spatially homogeneous nonlinear terms $f(u)$ in scalar reaction-diffusion equations using boundary measurements (see, e.g., [11–17]). For one-dimensional cases, earlier works [18, 19] established uniqueness in determining single or multiple spatially varying coefficients. The findings are based on knowing the solution $u(t,\ x_0)$ and its spatial derivative $\dfrac{\partial u}{\partial x}(t,\ x_0)$ for all $t \in (0,\ \varepsilon)$, as well as $u(0,\ x)$ for $x \in \Omega$. These achievements build upon foundational contributions by Roques and Cristofol. In [20], the authors study the following Lotka-Volterra competition-diffusion system for $t > 0$ and $x \in (a,\ b) \subset \mathbb{R}$

$$\begin{cases} \dfrac{\partial u_1}{\partial t} = D_1 \dfrac{\partial^2 u_1}{\partial x^2} + u_1(r_1 - a_{11}u_1 - a_{12}u_2), \\ \dfrac{\partial u_2}{\partial t} = D_2 \dfrac{\partial^2 u_2}{\partial x^2} + u_2(r_2 - a_{21}u_1 - a_{22}u_2). \end{cases} \tag{2}$$

Assuming D_1, D_2, r_1, a_{11}, and a_{22} are known constants, while a_{12}, $r_2(x)$, and $a_{21}(x)$ are unknown, they derive uniqueness results for multiple coefficients under piecewise analyticity conditions using only local or boundary measurements. Related work by Li and Yao [21] examines a Partial Differential Equation (PDE) system modeling two competing species with diffusion and mutual repulsive chemotaxis, establishing global well-posedness and species coexistence in weak competition scenarios. The study further analyzes long-term behavior, employing analytical methods for weakly nonlinear regimes and numerical approaches for strongly nonlinear cases. Additionally, Esmailpour et al. [22] developed a numerical framework combining finite difference and B-spline collocation methods, incorporating Tikhonov regularization and the ABC algorithm to address ill-posedness. Their comparative studies demonstrate the efficacy of solving inverse problems for such competition-diffusion models.

This study, inspired by [20–22], investigates an inverse problem for determining spatially heterogeneous coefficients in a three-species Lotka-Volterra parabolic system. We establish a novel theoretical framework that advances ecological modeling and inverse problem-solving by: (1) extending Lotka-Volterra dynamics to a 3D system with nonlinear feedback, and (2) enabling unique identification of eight coefficients through single-point measurements-significantly reducing data requirements and regularity assumptions. The details are as follows:

1. Unlike classical two-species competition models [21, 22], this work develops a rigorous framework for analyzing three-species competitive interactions with non-standard feedback. The proposed tridiagonal system (1) introduces:

• A generalization of Lotka-Volterra dynamics to three dimensions, capturing more complex ecological interactions.

• Nonlinear feedback mechanisms modeling mutual repulsion and competitive pressures, yielding richer dynamics absent in two-species systems.

The methodology advances inverse problem solutions by drastically reducing measurement needs and regularity assumptions compared to [20, 23, 24].

Key improvements:

• **Measurement efficiency:** Where [20] required interval measurements to recover two coefficients, our method recovers eight coefficients from single-point data. We prove that eight spatially varying coefficients (r_2, a_{12}, a_{13}, a_{21}, a_{22}, a_{23}, a_{31}, a_{32}) can be uniquely determined using only u_1, u_2, u_3 and their derivatives $\frac{\partial u_1}{\partial x}$, $\frac{\partial u_3}{\partial x}$ measured at a single point x_0 over $(0,\ \varepsilon)$.

• **Regularity relaxation:** Our techniques may extend to Lipschitz stability under additional coefficient structure assumptions, complementing frameworks like [23, 24]. Unlike previous 2D studies requiring piecewise analyticity, we only assume piecewise C^n regularity on $[a,\ b]$, substantially relaxing smoothness requirements. The single-point measurement adaptation warrants further investigation.

These results are particularly valuable for ecological monitoring, chemical process control, and biological pattern formation, where comprehensive measurements are often impractical. The outline of the paper is as follows. In section 2, we states our assumptions and main theorem. In section 3, we present some technical lemmas and proofs of theorems.

2. Hypotheses and main conclusions

The single-species model is described by the PDE for $t > 0$ and $x \in (a,\ b) \subset \mathbb{R}$

$$\frac{\partial u}{\partial t} = D_1 \frac{\partial^2 u}{\partial x^2} - a_{11}(x)u^2 + r_1(x)u, \tag{3}$$

where $D_1 > 0$, $r_1 \in C^{0,\ \alpha}([a,\ b])$ and $a_{11} \in C^{0,\ \alpha}([a,\ b])$ with $a_{11} > 0$. Here $\alpha \in (0,\ 1)$, and $C^{0,\ \alpha}$ denotes the space of Hölder's continuous functions with exponent α [25]. We consider the three-species tridiagonal competitive reaction-diffusion model (1) with positive feedback, which satisfies the following assumptions

(H_1) **Model coefficients and competition structure.** Consider the following system parameters and functions:

• Constants: As in equation (3), the constants $D_2 > 0$ and $D_3 > 0$.

• Functions:

1. $r_2,\ r_3 \in C^{0,\ \alpha}([a,\ b])$,

2. $a_{22},\ a_{33} \in C^{0,\ \alpha}([a,\ b])$, with $a_{22} > 0$ and $a_{33} > 0$,

3. $a_{12},\ a_{13},\ a_{21},\ a_{23},\ a_{31},\ a_{32} \in C^{0,\ \alpha}([a,\ b])$, where each of these coefficients is strictly positive on $[a,\ b]$.

Here, the coefficient a_{ij} (for $i \neq j$, $i,\ j = 1,\ 2,\ 3$) quantifies the competitive effect of species j on species i.

(H_2) **Unknown coefficients.** This study assumes all coefficients related to Species 2 are unknown quantities: $r_2(x)$, $a_{21}(x)$, $a_{22}(x)$, $a_{23}(x)$, as well as $a_{12}(x)$, $a_{13}(x)$, $a_{31}(x)$, and $a_{32}(x)$.

2.1 *Initial and boundary conditions*

Let $u_1,\ u_2,\ u_3$ satisfy the initial conditions

$$u_1(0,\ x) = u_{10},\ u_2(0,\ x) = u_{20},\ u_3(0,\ x) = u_{30}.$$

For $t > 0$, the boundary conditions are given by

$$\alpha_i u_i(t,\ a) - \beta_i \frac{\partial u_i}{\partial x}(t,\ a) = 0,\ \gamma_i u_i(t,\ b) + \delta_i \frac{\partial u_i}{\partial x}(t,\ b) = 0, \tag{4}$$

where

$$\alpha_i,\ \beta_i,\ \gamma_i,\ \delta_i \geq 0,\ \alpha_i^2 + \beta_i^2 > 0 \text{ and } \delta_i^2 + \gamma_i^2 > 0,\ i = 1,\ 2,\ 3. \tag{5}$$

This general form encompasses classical cases for each $i = 1,\ 2,\ 3$.
(i) Dirichlet case when $\beta_i = \delta_i = 0$.
(ii) Neumann case when $\alpha_i = \gamma_i = 0$.
In establish classical solutions $(u_1,\ u_2,\ u_3)$ of problem (1), we assume the initial data satisfy

$$u_{10},\ u_{20},\ u_{30} \in C^{2,\ \alpha}([a,\ b]).$$

Further, the following compatibility conditions hold at the boundaries

$$\begin{cases} \alpha_1 u_{10}(a) - \beta_1 (u_{10})'(a) = 0 \text{ and } (u_{10})''(a) = 0, \\ \gamma_1 u_{10}(b) - \delta_1 (u_{10})'(b) = 0 \text{ and } (u_{10})''(b) = 0, \\ \alpha_2 u_{20}(a) - \beta_2 (u_{20})'(a) = 0 \text{ and } (u_{20})''(a) = 0, \\ \gamma_2 u_{20}(b) - \delta_2 (u_{20})'(b) = 0 \text{ and } (u_{20})''(b) = 0, \\ \alpha_3 u_{30}(a) - \beta_3 (u_{30})'(a) = 0 \text{ and } (u_{30})''(a) = 0, \\ \gamma_3 u_{30}(b) - \delta_3 (u_{30})'(b) = 0 \text{ and } (u_{30})''(b) = 0. \end{cases} \tag{6}$$

Under the above assumptions and standard parabolic theory (see [26], Chapter 8), problem (1) admits a unique classical solution

$$u_1,\ u_2,\ u_3 \in C_2^1([0,\ \infty) \times [a,\ b]),$$

with guaranteed existence, uniqueness, and regularity.

2.2 *Proposed hypotheses about the undetermined coefficients*

We assume the coefficients satisfy

$$r_2,\ a_{12},\ a_{13},\ a_{22},\ a_{23},\ a_{31},\ a_{32} \in \mathscr{M},$$

where the function class $\mathscr{M}$ is defined as

$$\mathscr{M} := \left\{ \psi \in C^{0,\,\alpha}([a,\ b]) : \psi \text{ is piecewise } C^n([a,\ b]) \text{ for some } n \geq 1 \right\}. \tag{7}$$

A function ψ is piecewise C^n (for $n \geq 1$) if:

1. ψ is continuous on $[a,\ b]$, and
2. There exists a partition $a = k_1 < k_2 < \ldots < k_m = b$ such that

$$\psi(x) := \sum_{j=1}^{m-1} \chi_{[k_j,\ k_{j+1})}(x)\varphi_j(x),\ \forall\, x \in (a,\ b),$$

where:

• Each φ_j is C^n on $[k_j,\ k_{j+1}]$,

• $\chi_{\ [k_j,\ k_{j+1})}$ denotes the characteristic function of the interval $[k_j,\ k_{j+1})$.

Consequently, if $\psi \in M$, for every $x \in [a,\ b)$ (respectively, $x \in (a,\ b]$), there exists $r = r_x > 0$ such that ψ is C^n on $[x,\ x+r]$ (respectively, $[x-r,\ x]$). The following inequality holds for $x \in [a,\ b]$ with $x \neq 0$ (if $0 \in [a,\ b]$) and some constant $C > 0$

$$|f^{(n)}(x)| \leq C \sum_{k=0}^{n-1} \frac{|f^{(k)}(x)|}{|x|^{n-k}}. \tag{8}$$

Note that if a function g is piecewise analytic, then it satisfies inequality (8) piecewise. The key significance of inequality (8) lies in its ability to relax smoothness requirements: instead of needing C^∞ smoothness, we only require C^n differentiability. However, we emphasize that the vanishing of a function and its first n derivatives in some region does not, by itself, guarantee the function's identical vanishing there. To address this, we introduce the control condition (8), whose mathematical justification is provided in Lemma 3. This condition plays a pivotal role in our proof that the interspecific competition coefficient vanishes identically in Theorem 1.

2.3 *Measurement of the solution*

We perform measurements on solutions of problem (1) for four distinct initial conditions. Specifically, consider initial data

$$(u_{10}^1,\ u_{20}^1,\ u_{30}^1),\ (u_{10}^2,\ u_{20}^2,\ u_{30}^2),\ (u_{10}^3,\ u_{20}^3,\ u_{30}^3),\ (u_{10}^4,\ u_{20}^4,\ u_{30}^4)$$

satisfying

$$u_{i0}^j > 0 \in (a,\ b),\ i = 1,\ 2,\ 3,\ 4,\ j = 1,\ 2,\ 3,\ 4 \tag{9}$$

and

$$\det\begin{pmatrix} u_{10}^1 & u_{20}^1 & u_{30}^1 & 1 \\ u_{10}^2 & u_{20}^2 & u_{30}^2 & 1 \\ u_{10}^3 & u_{20}^3 & u_{30}^3 & 1 \\ u_{10}^4 & u_{20}^4 & u_{30}^4 & 1 \end{pmatrix} \neq 0. \tag{10}$$

Under the assumption that $x_0 \in (a,\ b)$ exists, the solution $(u_1,\ u_2,\ u_3)$ to problem (1) and its spatial derivatives are measurable with respect to the Lebesgue measure. For a fixed $\varepsilon > 0$ and four distinct sets of initial conditions $(u_{10}^k,\ u_{20}^k,\ u_{30}^k)$, where $k = 1,\ 2,\ 3,\ 4$, we assume that the values $(u_1,\ u_2,\ u_3)(t,\ x_0)$ as well as the spatial derivatives $\partial_x u_1(t,\ x_0)$ and $\partial_x u_3(t,\ x_0)$ (as in Theorem 1) are measurable for all $t \in (0,\ \varepsilon)$. As shown in Theorem 1, these measurements uniquely determine the eight coefficients

$$r_2(x),\ a_{12},\ a_{13},\ a_{21},\ a_{22},\ a_{23},\ a_{31},\ a_{32},$$

thereby reconstructing the solution $(u_1,\ u_2,\ u_3)$ on $(0,\ \infty) \times (a,\ b)$.

Main results. Assume that $\tilde{r}_2,\ \tilde{a}_{12},\ \tilde{a}_{21},\ \tilde{a}_{22},\ \tilde{a}_{23},\ \tilde{a}_{32} \in C^{n,\ \alpha}([a,\ b])$ for some $n \geq 0$ and $\alpha \in (0,\ 1)$. Suppose the differences

$$r_2 - \tilde{r}_2,\ a_{12} - \tilde{a}_{12},\ a_{21} - \tilde{a}_{21},\ a_{22} - \tilde{a}_{22},\ a_{23} - \tilde{a}_{23} \in \mathscr{M}$$

and satisfy inequality (8). Furthermore, assume

$$a_{13} - \tilde{a}_{13},\ a_{31} - \tilde{a}_{31},\ a_{32} - \tilde{a}_{32} \in \mathscr{M}$$

and satisfy (8) with $n = 1$ and a constant $C \in (0,\ 1)$. Let $(\tilde{u}_1,\ \tilde{u}_2,\ \tilde{u}_3)$ denote the solution to the following system of PDEs for $t > 0$ and $x \in (a,\ b) \subset \mathbb{R}$,

$$\begin{cases} \dfrac{\partial \tilde{u}_1}{\partial t} = D_1 \dfrac{\partial^2 \tilde{u}_1}{\partial x^2} + \tilde{u}_1\left(r_1 - a_{11}\tilde{u}_1 - \tilde{a}_{12}\tilde{u}_2 + \tilde{a}_{13}\tilde{u}_3\right), \\ \dfrac{\partial \tilde{u}_2}{\partial t} = D_2 \dfrac{\partial^2 \tilde{u}_2}{\partial x^2} + \tilde{u}_2\left(\tilde{r}_2 - \tilde{a}_{21}\tilde{u}_1 - \tilde{a}_{22}\tilde{u}_2 - \tilde{a}_{23}\tilde{u}_3\right), \\ \dfrac{\partial \tilde{u}_3}{\partial t} = D_3 \dfrac{\partial^2 \tilde{u}_3}{\partial x^2} + \tilde{u}_3\left(r_3 + \tilde{a}_{31}\tilde{u}_1 - \tilde{a}_{32}\tilde{u}_2 - a_{33}\tilde{u}_3\right), \end{cases} \tag{11}$$

subject to the same boundary conditions (4) as $(u_1,\ u_2,\ u_3)$, and with $\tilde{u}_1(0,\ x) = u_{10}(x)$, $\tilde{u}_2(0,\ x) = u_{20}(x)$, $\tilde{u}_3(0,\ x) = u_{30}(x)$.

Theorem 1 Let $x_0,\ x_1 \in (a,\ b)$ be two distinct points and $\varepsilon > 0$. Assume that for each $k = 1,\ 2,\ 3,\ 4$, the initial condition $(u_{10}^k,\ u_{20}^k,\ u_{30}^k)$ satisfies (9) and (10). If for all $t \in (0,\ \varepsilon)$, the following conditions hold

$$\begin{cases} (u_1,\, u_2,\, u_3)(t,\, x_i) = (\tilde{u}_1,\, \tilde{u}_2,\, \tilde{u}_3)(t,\, x_i), & i = 0,\, 1, \\ \dfrac{\partial u_1}{\partial x}(t,\, x_i) = \dfrac{\partial \tilde{u}_1}{\partial x}(t,\, x_i), & i = 0,\, 1, \\ \dfrac{\partial u_3}{\partial x}(t,\, x_i) = \dfrac{\partial \tilde{u}_3}{\partial x}(t,\, x_i), & i = 0,\, 1, \end{cases} \tag{12}$$

and the matrices satisfy the non-degeneracy conditions

$$\det \begin{pmatrix} u_1(t,\, x_0) & u_2(t,\, x_0) \\ u_1(t,\, x_1) & u_2(t,\, x_1) \end{pmatrix} \cdot \det \begin{pmatrix} u_2(t,\, x_0) & u_3(t,\, x_0) \\ u_2(t,\, x_1) & u_3(t,\, x_1) \end{pmatrix} \neq 0, \tag{13}$$

then we have

$$\tilde{r}_2 = r_2 \text{ and } \tilde{a}_{ij} = a_{ij},\ \forall (i,\ j) \in \mathscr{I} \text{ and } \forall x \in (a,\, b), \tag{14}$$

where $\mathscr{I} := \{(1,\, 2),\, (1,\, 3),\, (2,\, 1),\, (2,\, 2),\, (2,\, 3),\, (3,\, 1),\, (3,\, 2)\}$.

Knowing the second derivatives $\dfrac{\partial^2 u_1}{\partial x_2^2}(t,\, x_0)$ and $\dfrac{\partial^2 u_3}{\partial x_1^2}(t,\, x_0)$ allows determining the $r_2(x)$, a_{12}, a_{13}, a_{21}, a_{22}, a_{23}, a_{31}, and a_{32} without measuring the component u_2, leading to the following key result.

Theorem 2 Let $x_0 \in (a,\, b)$ be a fixed spatial point and $\varepsilon > 0$ a time parameter. Assume that for each $k = 1,\, 2,\, 3,\, 4$, the initial condition $(u_{10}^k,\, u_{20}^k,\, u_{30}^k)$ satisfies (9) and (10). If the following conditions are satisfied for all $t \in (0,\, \varepsilon)$

$$\begin{cases} (u_1,\, u_3)(t,\, x_0) = (\tilde{u}_1,\, \tilde{u}_3)(t,\, x_0), \\ \dfrac{\partial u_1}{\partial x}(t,\, x_0) = \dfrac{\partial \tilde{u}_1}{\partial x}(t,\, x_0), \\ \dfrac{\partial u_3}{\partial x}(t,\, x_0) = \dfrac{\partial \tilde{u}_3}{\partial x}(t,\, x_0), \\ \dfrac{\partial^2 u_1}{\partial x^2}(t,\, x_0) = \dfrac{\partial^2 \tilde{u}_1}{\partial x^2}(t,\, x_0), \\ \dfrac{\partial^2 u_3}{\partial x^2}(t,\, x_0) = \dfrac{\partial^2 \tilde{u}_3}{\partial x^2}(t,\, x_0), \end{cases} \tag{15}$$

and the inequality (13) is satisfied, then the equation (14) holds.

3. Proof of Theorem 1 and Theorem 2

First we give some lemmas which are useful for the proofs of Theorems 1 and 2.

Lemma 1 (cf. [27]) Let $f(x) \in C^n([0,\ b])$ with $n \geq 1$, and suppose $\alpha \geq 0$. If $f^{(k)}(0) = 0$ for all $k = 0,\ \ldots,\ n + \left[\frac{\alpha}{2}\right]$, then for each $k = 0,\ ..,\ n-1$

$$\int_0^b \frac{\left(f^{(k)}(x)\right)^2}{x^{\alpha+2(n-k)}}\mathrm{dx} \leq \frac{4}{(\alpha+1)^2}\int_0^b \frac{\left(f^{(n)}(x)\right)^2}{x^{\alpha}}\mathrm{dx}.$$

Lemma 2 (cf. [27]) Let $f(x) \in C^n([0,\ b])$ $n \geq 1$. Assume $\alpha \geq 0,\ f^{(k)}(0) = 0,\ k = 0,\ ...,\ n + \left[\frac{\alpha}{2}\right]$. Then

$$\int_0^b \frac{1}{x^{\alpha}}\sum_{k=0}^{n-1}\left(\frac{\left(f^{(k)}(x)\right)}{x^{n-k}}\right)^2 \mathrm{dx} \leq \frac{4n}{(\alpha+1)^2}\int_0^b \frac{1}{x^{\alpha}}\left(f^{(n)}(x)\right)^2 \mathrm{dx}.$$

Lemma 3 (cf. [27]) Let $f(x) \in C^n([0,\ b])$ $(n \geq 1)$. Assume

$$|f^{(n)}(x)| \leq C\sum_{k=0}^{n-1}\frac{|f^{(k)}(x)|}{|x|^{n-k}}\mathrm{dx},\ x \in [0,\ b] \tag{16}$$

for some $C > 0$. Then $f^{(k)}(0) = 0, k = 0,\ ...,\ n + \left[(n+1)\ C - \frac{1}{2}\right]$ implies $f \equiv 0$ on $[0,\ b]$.

Lemma 4 (cf. [27]) Let $f(x) \in C^n([0,\ b])$ $(n \geq 1)$. Assume (16) holds for some $C > 0$. Then $f \not\equiv 0$ implies the zero set $\{f^{-1}(0)\} \subset [0,\ b]$ is finite.

Lemma 5 (cf. [27]) Let $f(x) \in C^n([0,\ b])$ $(n \geq 1)$. Assume $\alpha \geq 0,\ f^{(k)}(0) = 0,\ k = 0,\ ...,\ n + \left[\frac{\alpha}{m}\right]$. Then

$$\int_0^b \frac{\left(f^{(k)}(x)\right)^m}{x^{\alpha+2(n-k)}}\mathrm{dx} \leq \left(\frac{2}{\alpha+1}\right)^m (m-1)^{m-1}\int_0^b \frac{\left(f^{(n)}(x)\right)^m}{x^{\alpha+2-m}}\mathrm{dx} \text{ for } k = 0,\ ..,\ n-1, \text{ and } m > 1.$$

Next, we give the proof of Theorem 1 and Theorem 2 as follows.

Proof of Theorem 1 Define the difference variables

$$U := u_1 - \tilde{u}_1,\ V := \tilde{u}_2 - u_2,\ W := u_3 - \tilde{u}_3,\ R := r_2 - \tilde{r}_2$$

and

$$A_{ij} := a_{ij} - \tilde{a}_{ij},\ \forall\ i,\ j \in \{1,\ 2,\ 3\}^2 \backslash (1,\ 1).$$

For arbitrary initial condition $(u_{10}^k,\ u_{20}^k,\ u_{30}^k)$ with $k = 1,\ 2,\ 3,\ 4$, the varibles $(U,\ V,\ W)$ satisfies for all $t > 0$ and $x \in (a,\ b)$

$$\begin{cases} \dfrac{\partial U}{\partial t} = D_1 \dfrac{\partial^2 U}{\partial x^2} + U\left(r_1 - a_{11}(u_1+\tilde{u}_1) - \tilde{a}_{12}u_2 - \tilde{a}_{13}u_3\right) + \tilde{a}_{12}\tilde{u}_1 V + \tilde{a}_{13}\tilde{u}_1 W - A_{12}u_1u_2 - A_{13}u_1u_3, \\ \dfrac{\partial V}{\partial t} = D_2 \dfrac{\partial^2 V}{\partial x^2} + V\left(\tilde{r}_2 - \tilde{a}_{22}(u_2+\tilde{u}_2) - \tilde{a}_{21}\tilde{u}_1 - \tilde{a}_{23}\tilde{u}_3\right) + \tilde{a}_{21}\tilde{u}_2 U + \tilde{a}_{23}\tilde{u}_2 W \\ \qquad\qquad -u_2\left(R - A_{12}u_1 - A_{12}u_1 - A_{12}u_1\right) \\ \dfrac{\partial W}{\partial t} = D_3 \dfrac{\partial^2 W}{\partial x^2} + W\left(r_3 - a_{33}(u_3+\tilde{u}_3) - \tilde{a}_{31}u_1 - \tilde{a}_{32}u_2\right) + \tilde{a}_{31}\tilde{u}_3 U + \tilde{a}_{32}\tilde{u}_3 V - A_{31}u_1u_3 - A_{32}u_2u_3, \end{cases} \tag{17}$$

subject to the same boundary condition as $(u_1,\ u_2,\ u_3)$. When we evaluate the first equation of (1) at the points $(0,\ x_0)$ and $(0,\ x_1)$ under the assumptions of Theorem 1 and from the fact that

$$U(0,\ x) = V(0,\ x) = W(0,\ x) = 0,\ \forall\, x \in [a,\ b],$$

we have that

$$A_{12}(x_0)u_1(x_0)u_2(x_0) + A_{13}(x_0)u_1(x_0)u_3(x_0) = 0,$$

and

$$A_{12}(x_1)u_1(x_1)u_2(x_1) + A_{13}(x_1)u_1(x_1)u_3(x_1) = 0.$$

Employing the fact that $(u_{10},\ u_{20},\ u_{30}) = (u_{10}^1,\ u_{20}^1,\ u_{30}^1)$ and invoking (13), we deduce

$$A_{12}(x_0) = A_{13}(x_0) = 0 \Rightarrow \tilde{a}_{12}(x_0) = a_{12}(x_0),\ \tilde{a}_{13}(x_0) = a_{13}(x_0).$$

Differentiation of the equation (17) with respect to x yields

$$A_{12}'(x_0) = A_{13}'(x_0) = 0.$$

The Lemma 3 tells

$$A_{12} \equiv A_{13} \equiv 0 \text{ on } [a,\ b]. \tag{18}$$

Similar to (18), we have

$$A_{31}(x) \equiv A_{32}(x) \equiv 0 \Rightarrow \tilde{a}_{31} = a_{31},\ \tilde{a}_{32} = a_{32} \text{ on } [a,\ b].$$

The model specified in (17) can be reformulated as follows for $t > 0$, $x \in (a,\ b) \subset \mathbb{R}$

$$\begin{cases} \dfrac{\partial U}{\partial t} = D_1 \dfrac{\partial^2 U}{\partial x^2} + U[r_1 - a_{11}(u_1 + \tilde{u}_1) - \tilde{a}_{12}u_2 - \tilde{a}_{13}u_3] + \tilde{a}_{12}\tilde{u}_1 V + \tilde{a}_{13}\tilde{u}_1 W, \\ \dfrac{\partial V}{\partial t} = D_2 \dfrac{\partial^2 V}{\partial x^2} + V[\tilde{r}_2 - \tilde{a}_{22}(u_2 + \tilde{u}_2) - \tilde{a}_{21}\tilde{u}_1 - \tilde{a}_{23}\tilde{u}_3] \\ \qquad\qquad + \tilde{a}_{21}\tilde{u}_2 U + \tilde{a}_{23}\tilde{u}_2 W - u_2[R - A_{12}u_1 - A_{12}u_1 - A_{12}u_1] \\ \dfrac{\partial W}{\partial t} = D_3 \dfrac{\partial^2 W}{\partial x^2} + W[r_3 - a_{33}(u_3 + \tilde{u}_3) - \tilde{a}_{31}u_1 - \tilde{a}_{32}u_2] + \tilde{a}_{31}\tilde{u}_3 U + \tilde{a}_{32}\tilde{u}_3 V. \end{cases} \tag{19}$$

We define

$$A_+ := \left\{ x \geq x_0 : \underbrace{R(y) \equiv A_{21}(y) \equiv A_{22}(y) \equiv A_{23}(y)}_{:=T_1} \equiv 0,\ \forall y \in (x_0,\ x] \right\}.$$

and

$$x_1 := \begin{cases} \sup(A_+), & \text{if } A_+ \neq \phi, \\ x_0, & \text{if } A_+ = \phi. \end{cases}$$

When $x_1 = b$, we have that for all $y \in (x_0,\ b]$

$$T_1 \equiv 0. \tag{20}$$

For the subsequent analysis, we assume, by way of contradiction, that $x_1 < b$.

Step 1. We claim that there exist initial conditions $(u_{10}^{k^*},\ u_{20}^{k^*},\ u_{30}^{k^*})$ and $k^* \in \{1,\ 2,\ 3,\ 4\}$ such that for all $(t,\ x) \in [0,\ \theta] \times [x_1,\ x_2]$ (where $\theta \in (0,\ \varepsilon)$ and $x_2 \in (x_1,\ b)$), the following holds:

$$R(x) \neq \sum_{i=1}^{3} A_{2i}(x) u_i(t,\ x). \tag{21}$$

Since $x_1 < b$, it follows from (7) and Lemma 4 that there exists $\delta > 0$ such that

$$x_1 + \delta < b,$$

and the functions R, A_{21}, A_{22}, and A_{23} are all well-defined on the interval $[x_1,\ x_1+\delta]$ with at least one of them being non-zero on this interval. It follows that

$$(t,\ x) \mapsto R(x) - \sum_{i=1}^{3} A_{2i}(x) u_i(t,\ x)$$

admits the factorization

$$(t,\ x) \mapsto (x-x_1)^{\rho} \left(\hat{R}(x) - \sum_{i=1}^{3} \hat{A}_{2i}(x) u_i(t,\ x) \right),\ \forall (t,\ x) \in \mathbb{R}^{+} \times [x_1,\ x_1+\delta],$$

where the functions $\hat{R}$, $\hat{A}_{21}$, $\hat{A}_{22}$, $\hat{A}_{23}$ do not all vanish at $x = x_1$. Hence there exists a $k^* \in \{1,\ 2,\ 3,\ 4\}$ such that

$$\hat{R}(x_1) - \sum_{i=1}^{3} \hat{A}_{2i}(x_i) u_{i0}^{k^*}(x_1) \neq 0.$$

The solution $(u_1,\ u_2,\ u_3)$ of problem (1) with initial data $(u_{10}^{k^*},\ u_{20}^{k^*},\ u_{30}^{k^*})$ is continuous, which implies that the mapping

$$(t,\ x) \mapsto H(x,\ t) := \hat{R}(x) - \sum_{i=1}^{3} \hat{A}_{2i}(x) u_i(t,\ x)$$

remains continuous over $[0,\ \infty) \times [x_1,\ x_1+\delta)$. As a result, we can find

$$H(x,\ t) \neq 0,\ \forall (t,\ x) \in [0,\ \theta] \times (x_1,\ x_2].$$

This guarantees that (21) holds.

Step 2. We assert that when $(u_{10},\ u_{20},\ u_{30}) = (u_{10}^{k^*},\ u_{20}^{k^*},\ u_{30}^{k^*})$, there exists $\varepsilon' > 0$ such that the function U, V and W satisfy either

$$U > 0,\ V > 0 \ \text{ and } \ W > 0,\ \forall (t,\ x) \in (0,\ \varepsilon') \times (x_0,\ x_2) \tag{22}$$

or

$$U < 0,\ V < 0 \ \text{ and } \ W < 0,\ \forall (t,\ x) \in (0,\ \varepsilon') \times (x_0,\ x_2). \tag{23}$$

The proof proceed by analysis two distinct cases (I) and (II).
Cases (I): Assume the inequality

$$R(x) - \sum_{i=1}^{3} A_{2i}(x)u_i(t,\, x) < 0,\ \forall\ (t,\, x) \in [0,\ \theta] \times (x_1,\, x_2].$$

Given that

$$U(0,\, x) = V(0,\, x) = W(0,\, x) = 0,\ \forall x \in (a,\, b). \tag{24}$$

We first establish the initial time derivatives

$$U_t(0,\, x_2) = 0 \text{ and } U_{tt}(0,\, x_2) > 0. \tag{25}$$

From (16) and $U(0,\, x) = 0$ in $(a,\, b)$, we have

$$U_t(0,\, x) = 0,\ \forall x \in (a,\, b).$$

Ultilizing (24) and (25), we derive

$$U_{tt}(0,\, x_2) = a_{12}u_{10}^{k^*}(x_2)V_t(0,\, x_2) > 0.$$

From (24), it follows that for some $\varepsilon' \in (0,\ \varepsilon)$, all three functions $U(t,\, x_2)$, $V(t,\, x_2)$, and $W(t,\, x_2)$ are positive whenever $t \in (0,\, \varepsilon')$. By Theorem 1, the boundary condition at x_0 satisties

$$U(t,\, x_0) = V(t,\, x_0) = W(t,\, x_0) = 0,\ \forall t \in (0,\ \varepsilon').$$

Consequently, the $(U,\, V,\, W)$ satisfies for $(t,\, x) \in (0,\ \varepsilon') \times (x_0,\, x_2)$

$$\begin{cases} \dfrac{\partial U}{\partial t} = D_1 \dfrac{\partial^2 U}{\partial x^2} + U[r_1 - a_{11}(u_1 + \tilde{u}_1) - \tilde{a}_{12}u_2 - \tilde{a}_{13}u_3] + \tilde{a}_{12}\tilde{u}_1 V + \tilde{a}_{13}\tilde{u}_3 W, \\ \dfrac{\partial V}{\partial t} > D_1 \dfrac{\partial^2 V}{\partial x^2} + V[\tilde{r}_2 - \tilde{a}_{22}(u_2 + \tilde{u}_2) - \tilde{a}_{21}\tilde{u}_1 - \tilde{a}_{23}\tilde{u}_3] + \tilde{a}_{21}\tilde{u}_2 U + \tilde{a}_{23}\tilde{u}_2 W, \\ \dfrac{\partial W}{\partial t} = D_1 \dfrac{\partial^2 W}{\partial x^2} + W[r_3 - \tilde{a}_{31}u_1 - a_{33}(u_3 + \tilde{u}_3) - \tilde{a}_{32}u_2] + \tilde{a}_{31}\tilde{u}_3 U + \tilde{a}_{32}\tilde{u}_3 V, \end{cases} \tag{26}$$

subject to initial boundary conditions

$$\begin{cases} U(t,\, x_0) = 0,\, U(t,\, x_2) > 0,\, t \in (0,\, \varepsilon'); \\ \\ V(t,\, x_0) = 0,\, V(t,\, x_2) > 0,\, t \in (0,\, \varepsilon'); \\ \\ W(t,\, x_0) = 0,\, W(t,\, x_2) > 0,\, t \in (0,\, \varepsilon'); \\ \\ U(0,\, x) = V(0,\, x) = W(0,\, x) = 0,\, x \in (x_0,\, x_2). \end{cases} \tag{27}$$

From the continuity property of the solutions $(\tilde{u}_1,\, \tilde{u}_2,\, \tilde{u}_3)$ in t, we derive that

$$\tilde{u}_1(t,\, x) > 0,\, \tilde{u}_2(t,\, x) > 0,\, \tilde{u}_3(t,\, x) > 0 \ \text{ for all } \ x \in (x_0,\, x_2) \ \text{ and } \ t > 0 \text{ enough small.}$$

We assume that

$$\tilde{u}_1(t,\, x) > 0,\, \tilde{u}_2(t,\, x) > 0,\, \tilde{u}_3(t,\, x) > 0 \text{ in } (0,\, \varepsilon') \times (x_0,\, x_2).$$

Given that the coefficients a_{12}, $\tilde{a}_{21} > 0$, system (1) satisfies monotonicity assumption. Specifically, the following partial derivatives are stictly positive

$$\begin{cases} \dfrac{\partial}{\partial V}(a_{12}\tilde{u}_1 V) = a_{12}\tilde{u}_1 > 0, \\ \\ \dfrac{\partial}{\partial U}(\tilde{a}_{21}\tilde{u}_2 U + \tilde{a}_{23}\tilde{u}_3 W) = a\tilde{a}_{21}\tilde{u}_2 > 0, \\ \\ \dfrac{\partial}{\partial W}(\tilde{a}_{21}\tilde{u}_2 U + \tilde{a}_{23}\tilde{u}_3 W) = a\tilde{a}_{23}\tilde{u}_3 > 0, \\ \\ \dfrac{\partial}{\partial V}(a_{32}\tilde{u}_3 V) = a_{32}\tilde{u}_3 > 0, \end{cases} \tag{28}$$

which implies that

1. $(U,\, V,\, W) > 0$ throughout the domain $(0,\, \varepsilon') \times (x_0,\, x_2)$.

2. There exists a $t_0 \in (0,\, \varepsilon')$ for which at least one of the components U, V, or W is identically zero on $(0,\, t_0) \times (x_0,\, x_2)$.

However, given the boundary conditions at x_2, the functions U, V, and W remain positive for all $(t,\, x) \in (0,\, \varepsilon') \times (x_0,\, x_2)$.

Cases (II): Suppose there exist $\theta > 0$ such that

$$R(x) - A_{21}(x)u_1(t,\, x) - A_{22}(x)u_2(t,\, x) - A_{23}(x)u_3(t,\, x) > 0$$

holds on $[0,\ \theta] \times (x_1,\ x_2]$. Based on the initial conditions $U(0,\ x) = V(0,\ x) = W(0,\ x) = 0$ for all $x \in (a,\ b)$, and considering the symmetry under the transformation $(-U,\ -V,\ -W)$, we conclude that for all $(t,\ x) \in (0,\ \varepsilon') \times (x_0,\ x_2)$

$$(U,\ V,\ W) < 0.$$

Step 3. Using Hopf's Lemma and the condition of Theorem 1, we arrive at a contradiction based on the assumption made in Step 1. We now consider the case where U, V, and W satisfy (22) (the case where U, V, and W satisfy (23) can be handled similarly). From (26), we derive the system of inequality

$$\begin{cases} \dfrac{\partial U}{\partial t} > D_1 \dfrac{\partial^2 U}{\partial x^2} + U\left(r_1 - a_{11}(u_1 + \tilde{u}_1) - \tilde{a}_{12}u_2 - \tilde{a}_{13}u_3\right),\ (x,\ t) \in (0,\ \varepsilon') \times (x_0,\ x_2), \\ \dfrac{\partial V}{\partial t} > D_1 \dfrac{\partial^2 V}{\partial x^2} + V\left(\tilde{r}_2 - \tilde{a}_{22}(u_2 + \tilde{u}_2) - \tilde{a}_{21}\tilde{u}_1 - \tilde{a}_{23}\tilde{u}_3\right),\ (x,\ t) \in (0,\ \varepsilon') \times (x_0,\ x_2), \\ \dfrac{\partial W}{\partial t} > D_1 \dfrac{\partial^2 W}{\partial x^2} + W\left(r_3 - \tilde{a}_{31}u_1 - a_{33}(u_3 + \tilde{u}_3) - \tilde{a}_{32}u_2\right),\ (x,\ t) \in (0,\ \varepsilon') \times (x_0,\ x_2). \end{cases} \tag{29}$$

Observe that the inequality (22) satisfies

$$U(t,\ x_0) = V(t,\ x_0) = W(t,\ x_0) = 0.$$

By Hopf's Lemma, we have

$$\frac{\partial U}{\partial x}(t,\ x_0) > 0 \ \text{ and } \ \frac{\partial W}{\partial x}(t,\ x_0) > 0,$$

contradicting the assumption in Theorem 1. Therefore, the hypothesis $x_1 < b$ in Step 1 cannot hold.

Step 4. The Step 3 tells

$$x_1 = b,$$

which implies that the functions $R(y), A_{21}(y), A_{22}(y)$, and $A_{23}(y)$ vanish identically on the interval $(x_0,\ b]$. We now define

$$A_- := \{x \le x_0 : T_1 \equiv 0,\ y \in (x,\ x_0]\}$$

and

$$y_1 := \begin{cases} \inf(A_-),\ \text{if } A_- \neq \phi, \\ x_0,\ \ \text{if } A_- = \phi. \end{cases}$$

This yields the equality (20) holds for $a \leq y \leq x_0$. Therefore the equality (20) still holds for $a \leq y \leq b$.

Proof of Theorem 2 Assume that u_1 satisfies the conditions of Theorem 2. For arbitrary initial date $(u_{10}^k,\ u_{20}^k,\ u_{30}^k)$, hence

$$\tilde{a}_{12}(x) = a_{12}(x),\ \tilde{a}_{13}(x) = a_{13}(x),\ x \in [a,\ b],$$

and similarly,

$$\tilde{a}_{31}(x) = a_{31}(x),\ \tilde{a}_{32}(x) = a_{32}(x),\ x \in [a,\ b].$$

Reformulating the equation satisfied by u_1 and $\tilde{u}_1$, we obatin the system for $t > 0,\ x \in (a,\ b) \subset \mathbb{R}$,

$$\begin{cases} \dfrac{\partial u_1}{\partial t} = D_1 \dfrac{\partial^2 u_1}{\partial x^2} + u_1(r_1 - a_{11}u_1 - a_{12}u_2 + a_{13}u_3), \\ \dfrac{\partial \tilde{u}_1}{\partial t} = D_1 \dfrac{\partial^2 \tilde{u}_1}{\partial x^2} + \tilde{u}_1(r_1 - a_{11}\tilde{u}_1 - a_{12}\tilde{u}_2 + a_{13}\tilde{u}_3), \end{cases} \tag{30}$$

which implies that for $t \in (0,\ \varepsilon)$ and $x = x_0$,

$$\begin{cases} \dfrac{\partial u_1}{\partial t} = D_1 \dfrac{\partial^2 u_1}{\partial x^2} + u_1(r_1 - a_{11}u_1 - a_{12}u_2 + a_{13}u_3), \\ \dfrac{\partial u_1}{\partial t} = D_1 \dfrac{\partial^2 u_1}{\partial x^2} + u_1(r_1 - a_{11}u_1 - a_{12}\tilde{u}_2 + a_{13}u_3). \end{cases} \tag{31}$$

By the continuity of u_1 in the variable t, we have

$$u_1(t,\ x_0) > 0 \text{ for sufficiently small } t.$$

This implies

$$a_{12}u_1u_2 = a_{12}u_1\tilde{u}_2,\ u_1(t,\ x_0)u_2(t,\ x_0) = u_1(t,\ x_0)\tilde{u}_2(t,\ x_0).$$

Threfore

$$u_2(t,\ x_0) = \tilde{u}_2(t,\ x_0) \text{ for all } t \in (0,\ \varepsilon') \text{ and some } \varepsilon' \in (0,\ \varepsilon).$$

Since all conditions of Theorem 1 are satisfied, the result follows directly from Theorem 1.

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Conflict of interest

The authors declare no competing financial interest.

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