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Research Article

Method of Computer Analogy and Its Applications

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Abstract: This paper further develops a previously proposed method of computer analogy, based on the formalization of core computational properties: a finite number of digits and the transfer from one rank to another. Aimed at constructing new analytical solutions for nonlinear differential systems, the key outcome of this method is semi-analytical solution. We obtained new results compared to our earlier publications. Namely, the construction of an analytical solution was demonstrated for Carleman kinetic system. Two semi-analytical approximations are proposed for the Van der Pol equation, reduced to a system of two nonlinear equations. The semi-analytical approximation derived for the Shimizu-Morioka system—based on the probabilistic properties of the coefficients of the highest-degree terms—demonstrates the presence of a strange attractor.

Keywords: computer analogy method, solution of systems of differential equations, Van der Pol equation, Shimizu-Morioka system, Carleman kinetic system

MSC: 34A45, 34A34, 65L12

1. Introduction

Modern computing technology makes it possible to solve many complex problems. However, the pursuit of analytical (or semi-analytical) approaches remains highly relevant. The solution presented explicitly form has undeniable advantages. Its clear analytical expression provides deeper insight into the problem, offering a holistic view of the solution with its qualitative features and asymptotic behavior, which is important for physical problems. Moreover, it is known that the class of differential equations solvable analytically in the closed form is limited [1].

Advancements in computing technology have driven the development of increasingly powerful computing devices [2–6], and well-known computational models such as Turing machines [7] and Lambda calculus [8]. A computing device makes it possible to obtain an answer in a complete form, but the results are provided numerically. However, the process of computation points to a way of folding up the intermediate steps that emerge when "erasing" intermediate data. We are thus faced with the question: can we formulate a mathematical procedure (ideally analytical) to obtain the solution without a computer, while formalizing the operations that a computer would perform? To address this issue, a Method of Computer Analogy (MCA) was proposed in [9, 10]. This method is used to construct solutions to differential equations and

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employs the formalization of conventional computer operation methods. A noteworthy methodology direction: instead of proposing a new computer architecture based on mathematical model, we propose a mathematical model derived from modern computers for solving differential equations (or obtaining approximations) without using computers. MCA utilizes two main principles of storing numbers in digital computers: 1) the numbers are represented as segments of a power series, and 2) there is a digit shifting procedure. At each level, the numerical solution is presented as a segment of a series in terms of the powers of the step τ of the independent variable. The digit shifting procedure is applied at each level to change the values of the coefficients of the terms such that each coefficient does not exceed τ^{-1}. It has been demonstrated that digit shifting can produce quasi-random numbers that have been used to exclude intermediate levels of computations. This method can provide a solution in an explicit form (because a computer provides a solution in a numerical form after executing many intermediate and "hidden" operations).

There exist a multitude of numerical, analytical and semi-analytical methods [11, 12]. The present study is devoted to the development of a new method for obtaining semi-analytical approximations for complex nonlinear systems of differential equations. This goal is a step towards the broader objective of building explicit solutions for these systems using MCA. A problem is considered to be solved analytically if it can be resolved in terms of special functions (or elementary functions) [13]. Similarly, the authors obtain a solution in the form of a combination of certain functions they introduce. In this study, we considered nonlinear systems with a physical sense. The Carleman kinetic system, Van der Pol equation (equivalent to a system of two Ordinary Differential Equations (ODEs)), and Shimizu-Morioka system.

Note that the MCA possesses unique approximation properties. In a sense, it combines the features of the finite difference method and the representation of a solution as a superposition of functions with specified properties. This method considers a convergent finite-difference scheme, termed the guiding scheme, and the MCA representation approximates this difference solution. The degree of the highest-order term required to ensure the convergence of the approximation to the solution depends on the order of nonlinearity of the corresponding equation. An analog of the digit grid of a digital computer arises, which is why this scheme can be called a τ-computer. It is important to emphasize that, unlike known methods for representing a solution as a superposition of given functions (e.g., the collocation method, the Bubnov-Galerkin method, the Galerkin method, etc. [14]), where convergence to the solution is generally achieved by increasing the number of functions in the superposition, while in MCA the number of terms in the sum is fixed. Refinement is achieved by decreasing the time step τ. The primary role in the approximation is played by the lower digits-the zeroth and the first. The higher digits become vanishingly small as the step size tends to zero, yet their role is crucial because of digit transfer. In a certain sense, the process of obtaining an analytical solution in the MCA can be referred to as the inverse Monte Carlo method. While Monte Carlo methods leverage function information to compute an average [15], MCA utilizes the average to derive information about the desired function. This is done by binding the change of the variables to the expected (average) values of lower digit shifts.

This study presents an analytical solution for the Carleman kinetic system, which is treated as a model problem (The method is designed for kinetic equations with a collision integral approximated by a quadratic form). This study further demonstrated the dual utility of a linear approximation of the desired solution. First, it offers qualitative insights into the behavior of the solution. Second, it yields quantitative data, specifically providing an approximation of the transfers from the first to the zeroth rank. This information is crucial for constructing a complete solution. The study also provides an approximation for the solution of the Van der Pol equation (reduced to a system of two ordinary differential equations), demonstrating its cyclic behavior, and an approximation for the solution of the Shimizu-Morioka system, which reproduces a strange attractor.

2. Method overview

In most cases, nonlinear differential equations cannot be solved analytically, but they can usually be solved numerically. Consider the following Cauchy problem (here, we consider a nonlinear equation or a system of such equations):

$$\frac{dy}{dt} = f(y), \quad y(0) = A \tag{1}$$

Let there be a q-th order explicit finite-difference scheme that approximates the solution, which can be written as

$$y_{n+1} = y_n + \tau G(y_n) + O\left(\tau^{q+1}\right),\ y_0 = A,$$

where τ is a step of the independent variable t, and $G(y)$ is determined by the function f and by the chosen finite difference method. We assume that τ is small enough such that the scheme is stable; hence, it converges to the solution of problem (1). Note that, for the first-order method, G is equivalent to f. Although a description is provided for the general case, we primarily focus on the first-order Euler scheme.

When using the numerical solution, the computer acts as a black box that runs the algorithm. After performing a large number of steps, the algorithm provides the result as an array of numbers. Computer operations are fairly simple, but owing to the nonlinearity of the considered problem (in general), it is difficult to understand how the values will change after a few operations.

From a certain point of view, the computer solves the numerical analog of the analytical problem, and instead of analysis and analytical computations, it executes an algorithm. However, we can consider a numerical solution to be the independent problem and search for analytical computations (or approximation) for this particular problem. First, we formalize the number representation on computers. The following aspects are essential: (1) the numbers are stored as segments of the power series and (2) there is a digit shifting procedure that shifts a part of the value to the left digit.

The power series in powers of τ can be obtained from the explicit finite difference scheme and Taylor expansion. If $G(y)$ is an infinitely differentiable function with bounded derivatives, y_n can be represented by a series of powers of τ:

$$y_n = \sum_{m=0}^{\infty} a_{m,\,n}\tau^m. \tag{2}$$

This follows from the Taylor expansion of $G(y_n)$ at $y^* = a_{0,\,n}$:

$$G(y_n) = G(y^*) + \sum_{m=1}^{\infty} \frac{G^{(m)}(y^*)}{m!}(y_n - y^*)^m.$$

Using this expansion, we obtained y_{n+1}. A comparison of the coefficients in the sequential layers yields the following recurrent relations:

$$a_{0,\,n+1} = a_{0,\,n},\ a_{1,\,n+1} = a_{1,\,n} + G(y^*),\ a_{2,\,n+1} = a_{2,\,n} + a_{1,\,n}G'(y^*), \tag{3}$$

and so on. The expansion of y_n cannot be truncated after a fixed number of terms, because the truncated value will be of a greater order than the accuracy of the finite-difference method, and such a solution would diverge.

Let us construct a procedure of redistribution of the values a_i so that

(1) the value of y_n remains unchanged;

(2) for any i: $|a_i| < 1/\tau$.

Consider the following example. Let $\tau = 0.1$ and in n-th layer y_n is given:

$$y_n = 5\tau^0 + 2\tau^1 + 6\tau^2 = 5.26.$$

Let after applying the finite-difference formula we got y_{n+1}

$$y_{n+1} = 5\tau^0 + 2\tau^1 + 15\tau^2 = 5.35.$$

The coefficient at τ^2 is greater than $1/\tau = 10$, which leads to an eventual divergence. To avoid this problem, we applied a digit shifting procedure. This can be done in different ways, but the following is how we do it in common positional numeral systems. We take 10 out of 15 and add $10\tau^2 = \tau$ to 2 (at τ^1):

$$y_{n+1} = 5\tau^0 + 3\tau^1 + 5\tau^2 = 5.35.$$

As can be seen, the value does not change, but the structure of the value changes. The digit shifting procedure allows us to hold values when applying the numerical scheme, that would otherwise be truncated.

Next, we demonstrate the construction of a shifting procedure. To guarantee that series (2) is convergent, we restrain the absolute values of a_i by τ^{-1}. Let a piecewise function $\varphi(x)$ map x to segment $[-1,\ 1]$. Consider a function $\psi(x) = \tau^{-1}\varphi(\tau x)$. This is referred to as a shifting function.

Consider the expansion y_n where the absolute values of the coefficients can be greater than τ^{-1}:

$$y_n = \widetilde{a}_{0,\,n}\tau^0 + \widetilde{a}_{1,\,n}\tau^1 + \widetilde{a}_{2,\,n}\tau^2 + \ldots$$

Let $\varphi(x)$ be the shifting function. This should be applied to each coefficient. Let us consider its application to the k-th coefficient. First, we must add the value shifted from the $k+1$-th coefficient, denoted as $\delta_{k+1,\,n}$. We add and subtract the value of $\psi\left(\widetilde{a}_{k,\,n} + \delta_{n,\,k+1}\right)$.

$$\psi\left(\widetilde{a}_{k,\,n} + \delta_{k+1,\,n}\right) + \widetilde{a}_{k,\,n} + \delta_{k+1,\,n} - \psi\left(\widetilde{a}_{k,\,n} + \delta_{k+1,\,n}\right).$$

Since $|\psi(x)| \leq 1/\tau$, we imply the following:

$$a_{k,\,n} \equiv \psi\left(\widetilde{a}_{k,\,n} + \delta_{k+1,\,n}\right),\ \delta_{k,\,n} \equiv \tau\left(\widetilde{a}_{k,\,n} + \delta_{k+1,\,n} - \psi\left(\widetilde{a}_{k,\,n} + \delta_{k+1,\,n}\right)\right).$$

The shifting procedure guarantees that the absolute values of the coefficients in (2) are limited by the value of τ^{-1}, implying that the series may be truncated. Let the shifting function be $\psi(x) = x \bmod \tau^{-1}$. This particular shifting function was used further.

3. Probabilistic features of the method

Consider $a_{m,\,n}$:

$$a_{m,\,n} = (\widetilde{a}_{m,\,n} + \delta_{m+1,\,n}) \mod \tau^{-1}.$$

This expression conforms to the random number generator formula (see [16]):

$$x_m = (bx_{m-1} + c) \mod P,\ 0 \le b,\ c < P.$$

We can consider the coefficients $a_{q+1},\ \ldots,\ a_p$ to be quasi-random integer numbers. The experiments show that the values are distributed uniformly (see Figure 1). Therefore, the solution can be written as the sum of the following two parts: deterministic: $a_{0,\,n} + \ldots + a_{q,\,n}$, random: $a_{q+1,\,n} + \ldots + a_{p,\,n}$.

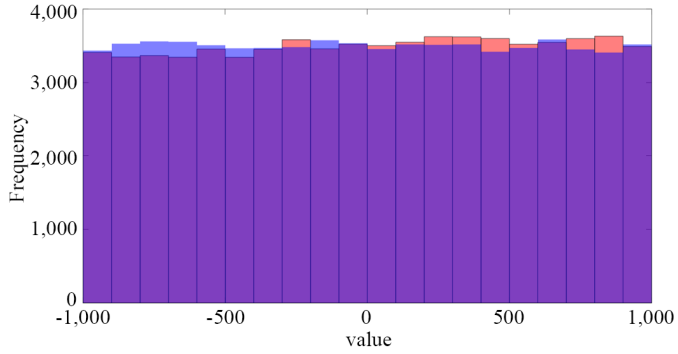

Figure 1. Distribution of the values of a_2 for Shimizu-Morioka system (red columns) compared to uniformly distributed random generated numbers (blue columns, values obtained by using Octave function randi in the range $[-1/\tau + 1,\ 1/\tau - 1]$, the intersection of red and blue columns is marked in purple)

These two parts are linked by δ_{q+1}—an integer value that is shifted from random part to the deterministic. This is a key value for the method, as it defines the change of deterministic part. It can be thought of as an aggregated value of composition of several random number generators which are hidden in random part. If the probabilities of the δ_{q+1} values are known, we can exclude the intermediate layers of the independent variable on which the deterministic part remains unchanged.

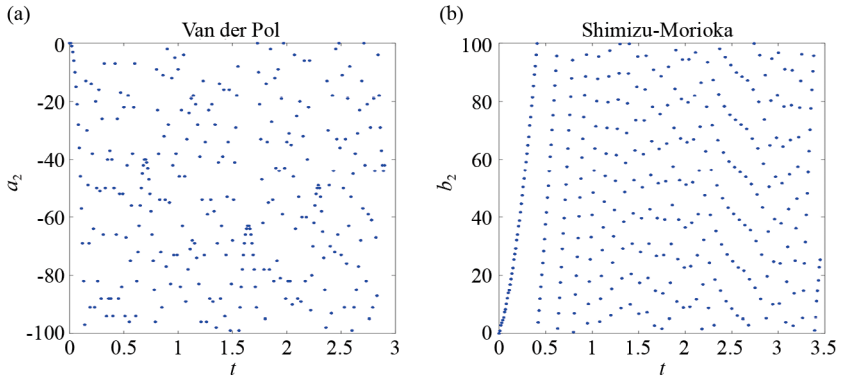

Figure 2. Coefficient a_2 of the solution of the Van der Pol equation (left) and coefficient b_2 of the solution of the Shimizu-Morioka system (right), $\tau = 0.01$ for both graphs

The coefficients of higher-order terms demonstrate probabilistic properties. The probabilistic behavior of the solution coefficient a_2 of the Van der Pol equation [17] and b_2 of the Shimizu-Morioka [18] system are shown in Figure 2. These systems were considered further in more detail.

4. Approximation properties

Let $y_n = y_0 + \sum_{i=1}^{\infty} a_i \tau^i$ and y_{n+1} is given by the following expression

$$y_{n+1} = y_n + \tau G(y_n) + O\left(\tau^{q+1}\right) \tag{4}$$

Suppose that on each layer we truncate the terms with powers of τ greater than k.

Theorem 1 Let m be a smallest integer for which

$$\frac{d^k G(y)}{dy^k} = 0, k > m$$

Then (4) holds true for the segment of the series

$$y_n = a_0 + \sum_{i=1}^{p} a_i \tau^i$$

with minimal value of $p = m + q$, and $|a_i| < \tau^{-1}$.

Proof. Consider (4):

$$y_{n+1} - (y_n + \tau G(y_n)) = (y_{n+1} - y_n) - G(y_n) = O\left(\tau^{q+1}\right)$$

To maintain the truncated term in $y_{n+1} - y_n$ of order of $O\left(\tau^{q+1}\right)$, p should be at least $q+1$ because $a_{q+1}\tau^{q+1} = O(\tau^q)$.

Now consider $G(y_n)$. Decompose it into series near $y = a_0$:

$$y_n = G(a_0) + G'(a_0) \sum_{i=1}^{p} a_i \tau^i + \ldots + \frac{G^{(m)}(a_0)}{m!} \left(\sum_{i=1}^{p} a_i \tau^i \right)^m.$$

Note that m and p are constants with respect to τ; thus, τ can be selected so small that any nonlinear combination of m and p multiplied by τ is of the order of τ.

Consider expression

$$\left(\sum_{i=1}^{p} a_i \tau^i \right)^m \tag{5}$$

Let us determine the order of the multiplier of τ^{m+k}. For this, we will raise the linear term $a_1\tau$ to the power of $m-1$ and multiply it by $a_{k+1}\tau^{k+1}$.

$$(a_1\tau)^{m-1}\cdot a_{k+1}\tau^{k+1} = \left(a_1{}^{m-1}\cdot a_{k+1}\right)\cdot\tau^{m+k} = O\left(\tau^k\right)$$

It is noteworthy that the other combinations of coefficients are less than $1/\tau^m$, and their quantity does not depend on τ. Thus, the greatest order of multiplier of τ^{m+k} is $1/\tau^m$.

Let $k=q$; the multiplier of τ^{m+q} is $O(\tau^{-m})$, which yields the $O(\tau^q)$ value. In (4) it is multiplied by τ and truncated as it is $O\left(\tau^{q+1}\right)$.

For $k=q-1$, we have $O\left(\tau^{q-1}\right)$ multiplied by τ in (4), yielding the $O(\tau^q)$ value, that cannot be truncated: thus, $p=m+q$.

Let us consider an important case with a polynomial right-hand side by examining the Shimizu-Morioka system and the first-order finite-difference scheme:

$$\begin{cases} \dfrac{dx}{dt} = y, \\ \dfrac{dy}{dt} = x - 0.81y - xz, \\ \dfrac{dz}{dt} = -0.375z + x^2. \end{cases} \tag{6}$$

Euler's scheme for the system has the form

$$\begin{cases} x_{n+1} = x_n + \tau y_n \\ y_{n+1} = y_n + \tau\left(x_n - 0.81y_n - x_n z_n\right) \\ z_{n+1} = z_n + \tau\left(-0.375z_n + x_n^2\right) \end{cases}$$

The approximations were constructed based on the degrees of nonlinearity of the right-hand sides of the corresponding equations. For a linear equation, an approximation up to the second degree was used.

$$\begin{cases} x_n = a_{0,\,n} + a_{1,\,n}\tau + a_{2,\,n}\tau^2 \\ y_n = b_{0,\,n} + b_{1,\,n}\tau + b_{2,\,n}\tau^2 + b_{3,\,n}\tau^3 \\ z_n = c_{0,\,n} + c_{1,\,n}\tau + c_{2,\,n}\tau^2 + c_{3,\,n}\tau^3 \end{cases}$$

Let us substitute the obtained expressions into the difference scheme, discarding terms of higher than second degree in the formulas for x and y, and terms of higher than third degree-in the formulas for z. Hence, the values Δx_{n+1}, Δy_{n+1}, and Δz_{n+1} are discarded at every step.

$$\begin{cases} \Delta x_{n+1} = b_{2,\,n}\tau^3 + b_{3,\,n}\tau^4 \\ \Delta y_{n+1} = -\left(0.81 b_{3,\,n} + a_{0,\,n}c_{3,\,n} + a_{1,\,n}c_{2,\,n} + a_{2,\,n}c_{1,\,n}\right)\tau^4 - \left(a_{1,\,n}c_{3,\,n} + a_{2,\,n}c_{2,\,n} + a_{3,\,n}c_{1,\,n}\right)\tau^5 - a_{2,\,n}c_{3,\,n}\tau^6 \\ \Delta z_{n+1} = \left(-0.375 c_{3,\,n} + 2a_{1,\,n}a_{2,\,n}\right)\tau^4 + {a_{2,\,n}}^2\tau^5 \end{cases}$$

In all three cases, the error per step did not exceed $O\left(\tau^2\right)$, which ensured convergence to the solution of the guiding Euler scheme.

Let us examine the stability of the MCA using Carleman system [19] (this system will be discussed in more detail in the next section)

$$\begin{cases} \dfrac{du}{dt} = v^2 - u^2 \\ \dfrac{dv}{dt} = u^2 - v^2 \end{cases}, \quad t \in [0;\ +\infty),$$

The stability condition for explicit Euler scheme is

$$\tau \le \frac{1}{u_n + v_n}$$

If the condition $0 \le u_n + v_n \le 1$ is satisfied at each step (numerical solution confirms this condition), then the stability condition becomes $\tau \le 1$. This condition is certainly satisfied for the values we use ($\tau = 0.1$ and smaller). The MCA introduces an error of $O\left(\tau^2\right)$ at each step. For small τ this constitute a minor perturbation. Since the stability condition of the finite-difference scheme is met, this perturbation does not amplify. Therefore, the MCA is stable and, combined with its approximation properties, ensures convergence. All our numerical calculations confirm this conclusion (The same assertions hold for the Shimizu-Morioka system, though their proof is more complicated).

5. Construction of a solution to the Carleman kinetic system

This method is prospectively aimed at solving kinetic equations with a collision integral approximated by a quadratic form. In this framework, a simple Carleman equation acts as a model prototype.

Consider the Carleman system, which is the simplest example of a kinetic system with discrete velocities.

$$\begin{cases} \dfrac{du}{dt} = v^2 - u^2 \\ \dfrac{dv}{dt} = u^2 - v^2 \end{cases}, \quad t \in [0;\ \ +\infty),$$

with following initial conditions $u(0) = 1,\ v(0) = 0$. The analytical solution of the corresponding Cauchy problem is

$$u(t) = \frac{1+e^{-2t}}{2}, \quad v(t) = \frac{1-e^{-2t}}{2}.$$

Euler's scheme for the system has the form

$$\begin{cases} u_{n+1} = u_n + \tau v_n^2 - \tau u_n^2, & u_0 = 1, \\ v_{n+1} = v_n + \tau u_n^2 - \tau v_n^2, & v_0 = 0. \end{cases}$$

By virtue of the law of conservation and considering the initial conditions, we obtain:

$$v_n = 1 - u_n.$$

We substituted v_n into the first equality

$$u_{n+1} = \tau + u_n(1-2\tau).$$

We used a first-order explicit Euler scheme. If the equation is linear, it is sufficient to approximate the solution to the term τ^2. Then the solution should be sought in the form (the minus sign is used for convenience)

$$u_n = a_{0,\,n}\tau^0 - a_{1,\,n}\tau + a_{2,\,n}\tau^2.$$

We get an approximation with an accuracy $O\left(\tau^2\right)$

$$u_{n+1} = a_{0,\,n}\tau^0 - \left(a_{1,\,n} - 1 + 2a_{0,\,n}\right)\tau + \left(a_{2,\,n} + 2a_{1,\,n}\right)\tau^2.$$

We apply digit shifting procedure

$$\begin{aligned} u_{n+1} &= \left(a_{0,\,n} - \delta_{1,\,n+1}\right)\tau^0 - \left(\left(a_{1,\,n} - 1 + 2a_{0,\,n} - \delta_{2,\,n+1}\right) \bmod \frac{1}{\tau}\right)\tau + \left(\left(a_{2,\,n} + 2a_{1,\,n}\right) \bmod \frac{1}{\tau}\right)\tau^2, \\ \delta_{1,\,n+1} &= \left[\left(a_{1,\,n} - 1 + 2a_{0,\,n} - \delta_{2,\,n+1}\right)\tau\right], \quad \delta_{2,\,n+1} = \left[\left(a_{2,\,n} + 2a_{1,\,n}\right)\tau\right]. \end{aligned} \tag{7}$$

By applying the above formulas and considering the relationship between u and v, we can obtain the values of these functions in any layer. We determine the conditions under which a transfer to a term with zero-degree τ^0 is possible. Note that while $\delta_1 = 0$, the value of a_0 does not change and is equal to the initial value, that is, 1. Due to chosen form of approximation, $a_{1,\,n} \geq 0$, $a_{2,\,n} \geq 0$, $\delta_{1,\,n+1} \geq 0$, $\delta_{2,\,n+1} \geq 0$ and the transfer occurs when $\delta_{1,\,n+1} \geq 1$. Hence

$$(a_{1,\,n}-1+2a_{0,\,n}-\delta_{2,\,n+1})\,\tau=(a_{1,\,n}+1-\delta_{2,\,n+1})\,\tau\geq 1,$$

then

$$a_{1,\,n}+1-\delta_{2,\,n+1}\geq\frac{1}{\tau}.$$

Given that $a_{1,\,n}<1/\tau$ (which follows from $a_{1,\,n}\geq 0$ and $|a_{1,\,n}|\leq 1/\tau$), the only case in which this holds is

$$\begin{cases} a_{1,\,n} & =\dfrac{1}{\tau}-1 \\ \delta_{2,\,n+1} & =0 \end{cases}$$

but this is impossible since substituting $a_{1,\,n}=1/\tau-1$ in (7) shows that $\delta_{2,\,n+1}>0$:

$$\delta_{2,\,n+1}=\left[\left(a_{2,\,n}+2\left(\frac{1}{\tau}-1\right)\right)\tau\right]=[a_{2,\,n}\tau+2-2\tau]=2+[a_{2,\,n}\tau-2\tau].$$

Then the presentation of the solution is significantly simplified

$$\begin{aligned} u_{n+1} &= 1-(a_{1,\,n}+1-\delta_{2,\,n+1})\,\tau+\left((a_{2,\,n}+2a_{1,\,n})\bmod\frac{1}{\tau}\right)\tau^2, \\ \delta_{2,\,n+1} &= [(a_{2,\,n}+2a_{1,\,n})\,\tau]. \end{aligned} \tag{8}$$

Let's find the probability that $\delta_{2,\,n}=0$.

$$(a_{2,\,n}+2a_{1,\,n})\,\tau<1\Rightarrow a_{2,\,n}<\frac{1}{\tau}-2a_{1,\,n}.$$

The expression obtained on the right-hand side is the number of different values of $a_{2,\,n}$ for which $\delta_{2,\,n}=0$ holds for a given $a_{1,\,n}$. By dividing the right side of the inequality by the total number of possible cases $1/\tau$, we obtain the probability that $\delta_{2,\,n}=0$

$$P\{\delta_{2,\,n}=0\}=1-2a_{1,\,n}\tau.$$

It can be shown that $\delta_{2,\,n}$ only takes the values 0 and 1, then the probability of $\delta_{2,\,n}=1$ is

$$P\{\delta_{2,\,n}=1\}=1-P\{\delta_{2,\,n}=0\}=2a_{1,\,n}\tau.$$

Then solution u_{n+1} can be represented as a sum of two terms

$$u_{n+1} = 1 - (a_{1,\,n} + 1 - \delta_{2,\,n+1})\,\tau,$$

where $\delta_{2,\,n}$ is obtained as a random variable with a Bernoulli distribution with probability $P\{\delta_{2,\,n} = 1\}$.

The exact solution, the solution using MCA, and the solution using probability are compared in Figure 3. The solutions converge as the step size τ decreases.

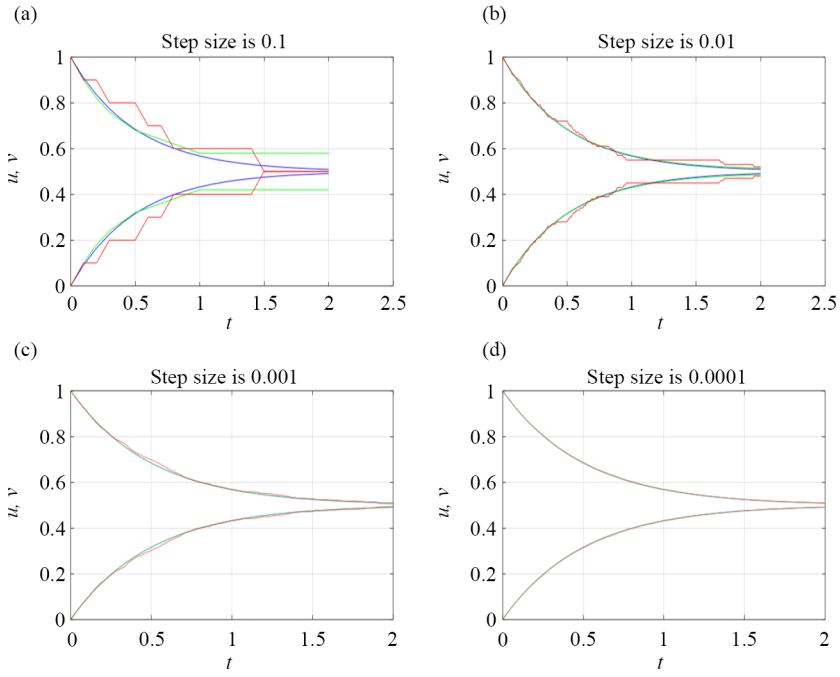

Figure 3. The exact solution (blue line), the solution using MCA (green line) and the solution using probability (red line) for different step sizes

We find the dependence of n_i on $a_1 \equiv i$ as $n_i = f(i,\ \tau)$. This establishes the inverse relationship required to solve the original problem; therefore, direct dependence can be restored at intervals of monotonicity of the desired function. Since $a_{1,\,n+1} = a_{1,\,n} + 1 - \delta_{2,\,n+1}$, and $\delta_{2,\,n+1} \in \{0,\ 1\}$ is a binary random value, the probability that $a_{1,\,n+1} = a_{1,\,n} + 1$ is expected value of $a_{1,\,n}$ change is $P\{\delta_{2,\,n} = 0\} = 1 - 2a_{1,\,n}\tau$. On the other hand, let $a_1 \equiv i$ be some integer value. Denote Δ_i the number of steps where a_1 remains unchanged. Then $\Delta_i \cdot P\{\delta_2 = 0\} = 1$ which provides the estimation of a segment of persistency for a_1, that is $\Delta_i = (1 - 2i\tau)^{-1}$. Thus the inverse dependency can be obtained:

$$n_i = 1 + \sum_{m=1}^{i-1} \Delta_m = 1 + \sum_{m=1}^{i-1} \frac{1}{1 - 2m\tau}.$$

It is noteworthy that this technique may be considered as inverse Monte Carlo method—the function is restored by treating average value, that is obtained as expectation of random value, as an independent variable. We refer to this technique as the "inverse" Monte Carlo method because it replaces the sum of random variables with their theoretically computed expected value. Conversely, standard Monte Carlo method estimates the expectation by a sum of random values.

The solution to the original problem is obtained by taking the limit in the constructed approximation as the step size τ tends to zero. The ordinate $(a_{1,\,n})$ is partitioned into equal segments, hence $y = i\tau$, and correspondingly, $t = n_i\tau$. We fixed the point y and considered the limit $t(y) = \lim_{i\to\infty} f(i,\ y/i)$. See [9].

6. Linear approximation

Let us examine how MCA can be used to construct linear analytical approximations, which in certain cases can be used to obtain qualitative (or, in some cases, quantitative) information about the solution. Let the leading exponent of step τ be equal to one. Thus, the solution contains only two terms:

$$y_n = a_{0,\,n} + a_{1,\,n}\tau.$$

Using (3) we get the value in the next layer:

$$y_n = a_{0,\,n} + (a_{1,\,n} + G(a_{0,\,n}))\,\tau.$$

Now we apply the digit shifting procedure:

$$y_{n+1} = (a_{0,\,n} + \delta_n) + \left((a_{1,\,n} + G(a_{0,\,n})) \bmod \tau^{-1}\right)\tau,\ \delta_n = \left[(a_{1,\,n} + G(a_{0,\,n}))\,\tau\right].$$

Comparing the coefficients on sequential layers we get the following relations:

$$a_{0,\,n+1} = a_{0,\,n} + \delta_n,\ a_{1,\,n+1} = (a_{1,\,n} + G(a_{0,\,n})) \bmod \tau^{-1}.$$

We see that a_0 can be changed only when nonzero digit shifting occurs. Suppose that for several sequential layers δ equals zero; therefore, no values are shifted to a_0. This implies that for these layers $G(a_{0,\,n})$ is constant, and a_1 grows linearly. When digit shifting occurs a_0 changes, leading to a change in the angular coefficient of a_1, which is $G(a_0)$. When the inequality $|a_1| < \tau^{-1}$ is violated, δ_n takes on a non-zero value, which changes a_0 and, consequently, the angular coefficient G.

Let $y_0 = a_{0,\,0} + a_{1,\,0}\tau$ correspond to some moment of time t_0. Suppose $\delta_0,\ \delta_1,\ \ldots,\ \delta_{M-1}$ are equal to zero; therefore, there are no shifts in the first M layers, and there is a shift in the $M+1$-th layer. Then, we imply $a_{0,\,m} = a_{0,\,0}$ for any m that $1 \le m \le M$.

This yields the recurrent relation for a_1, which can be written as follows:

$$a_{1,\,m+1} = (a_{1,\,m} + G(a_{0,\,0})) \bmod \tau^{-1} = a_{1,\,m} + G(a_{0,\,0}) = a_{1,\,0} + mG(a_0).$$

Notice that $m\tau$ is the value of time t_m, thus:

$$y_m = y_0 + (t_m - t_0)\,G(a_0).$$

This relation holds true for any value of t such that $t_0 \le t \le t_M$, thus

$$y = y_0 + (t - t_0)\, G(a_0)\,.$$

We denote $G \equiv G(a_0)$ and consider two cases: $G \ge 0$ and $G < 0$. The first case implies that a_1 grows linearly, whereas the second case implies that a_1 decreases linearly. Let $G \ge 0$; then, digit shifting occurs when $a_{1,\,n} \ge \tau^{-1}$:

$$a_{1,\,0} + nG \ge \tau^{-1}.$$

For n we used the smallest integer value of the right-hand part; thus, $n \equiv \left[\dfrac{1 - a_{1,\,0}\tau}{G\tau}\right]$. By analogy, we obtain the following value for n in case, when $G < 0$: $n \equiv \left[-\dfrac{1 + a_{1,\,0}\tau}{G\tau}\right]$. Now, we need to show how to obtain new values of $a_{0,\,0}$ and $a_{1,\,0}$ after shifting the digit. We denote them as $a^*_{0,\,0}$ and $a^*_{1,\,0}$. If digit shifting is triggered, a_1 changes according to the shifting function; therefore,

$$a^*_{1,\,0} = a_{1,\,n} \bmod \tau^{-1},\; a^*_{0,\,0} = a_{0,\,0} + \left[(a_{1,\,0} + nG(a_{0,\,0}))\,\tau\right].$$

7. Approximation of the complete solution using a linear approximation

Let us consider Van der Pol equation:

$$\frac{d^2u}{dt^2} - \mu\left(1 - u^2\right)\frac{du}{dt} + u = 0,$$

that is equivalent to a system of two equations

$$\begin{cases} \dfrac{du}{dt} = v, \\ \dfrac{dv}{dt} = -u + \mu\left(1 - u^2\right)v. \end{cases}$$

The parameter μ was considered to be equal to 1. The following Cauchy problem is posed

$$\begin{cases} \dfrac{du}{dt} = v, & u(0) = 1; \\ \dfrac{dv}{dt} = -u + \left(1 - u^2\right)v, & v(0) = 1. \end{cases} \tag{9}$$

Euler's scheme for the system has the form

$$\begin{cases} u_{n+1} = u_n + \tau v_n, & u_0 = 1 \\ v_{n+1} = v_n - \tau u_n + \left(1 - u_n^2\right) v_n, & v_0 = 1. \end{cases}$$

The solution should be sought in the following form (according to the above statement, because the nonlinearity is of the third degree we use the fourth degree approximation in step τ)

$$u_n = a_{0,\,n} + a_{1,\,n}\tau + a_{2,\,n}\tau^2,$$

$$v_n = b_{0,\,n} + b_{1,\,n}\tau + b_{2,\,n}\tau^2 + b_{3,\,n}\tau^3 + b_{4,\,n}\tau^4.$$

This solution can be obtained in a simpler way. Instead of calculating the transfers δ_1 and ω_1 (for u and v, respectively) that occur in the case of a complete solution, one can use the analogous transfers of the linear approximation. The remaining transfers were calculated using the same algorithm as before. The solution using MCA and the solution with the replacement of the transfers δ_1 and ω_1 with analogous of the linear approximation are compared in Figure 4. The solutions converge as the step size τ decreases. It is important to note that despite the convergence of the solutions, the conditions $|a_1| < 1/\tau$ and $|b_1| < 1/\tau$ are violated for the second one.

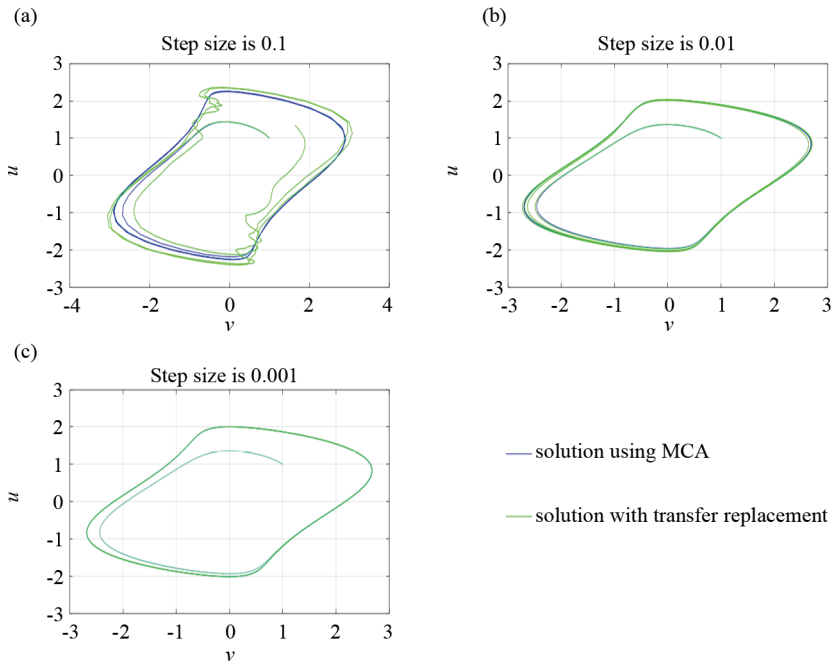

Figure 4. The solution using MCA and the solution with the replacement of the transfers δ_1 and ω_1 with analogous from the linear approximation for different step sizes

8. Approximation of the complete solution using probability

Let us look for a solution in the form

$$u_n = a_{0,\,n} + a_{1,\,n}\tau + a_{2,\,n}\tau^2,$$

$$v_n = b_{0,\,n} + b_{1,\,n}\tau + b_{2,\,n}\tau^2 + b_{3,\,n}\tau^3 + b_{4,\,n}\tau^4.$$

After applying the Euler scheme, we obtain an approximation

$$u_{n+1} = a_{0,\,n} + (a_{1,\,n} + b_{0,\,n})\,\tau + (a_{2,\,n} + b_{1,\,n})\,\tau^2,$$

$$\begin{aligned} v_{n+1} = {} & b_{0,\,n} + \left(b_{1,\,n} - a_{0,\,n} + b_{0,\,n} - a_{0,\,n}^2 b_{0,\,n}\right)\tau + \left(b_{2,\,n} - a_{1,\,n} + b_{1,\,n} - a_{0,\,n}^2 b_{1,\,n} - 2a_{0,\,n}a_{1,\,n}b_{0,\,n}\right)\tau^2 \\ & + \left(b_{3,\,n} - a_{2,\,n} + b_{2,\,n} - a_{0,\,n}^2 b_{2,\,n} - a_{1,\,n}^2 b_{0,\,n} - 2a_{0,\,n}a_{1,\,n}b_{1,\,n} - 2a_{0,\,n}a_{2,\,n}b_{0,\,n}\right)\tau^3 \\ & + \left(b_{4,\,n} + b_{3,\,n} - a_{0,\,n}^2 b_{3,\,n} - a_{1,\,n}^2 b_{1,\,n} - 2a_{0,\,n}a_{1,\,n}b_{2,\,n} - 2a_{1,\,n}a_{2,\,n}b_{0,\,n} - 2a_{0,\,n}a_{2,\,n}b_{1,\,n}\right)\tau^4. \end{aligned}$$

$$u_{n+1} = (a_{0,\,n} + \delta_{1,\,n+1}) + \left((a_{1,\,n} + b_{0,\,n} + \delta_{2,\,n+1}) \bmod \frac{1}{\tau}\right)\tau + \left((a_{2,\,n} + b_{1,\,n}) \bmod \frac{1}{\tau}\right)\tau^2,$$

$$\begin{aligned} v_{n+1} = {} & (b_{0,\,n} + \omega_{1,\,n+1}) + \left(\left(b_{1,\,n} - a_{0,\,n} + b_{0,\,n} - a_{0,\,n}^2 b_{0,\,n} + \omega_{2,\,n+1}\right) \bmod \frac{1}{\tau}\right)\tau \\ & + \left(\left(b_{2,\,n} - a_{1,\,n} + b_{1,\,n} - a_{0,\,n}^2 b_{1,\,n} - 2a_{0,\,n}a_{1,\,n}b_{0,\,n} + \omega_{3,\,n+1}\right) \bmod \frac{1}{\tau}\right)\tau^2 \\ & + \left(\left(b_{3,\,n} - a_{2,\,n} + b_{2,\,n} - a_{0,\,n}^2 b_{2,\,n} - a_{1,\,n}^2 b_{0,\,n} - 2a_{0,\,n}a_{1,\,n}b_{1,\,n} - 2a_{0,\,n}a_{2,\,n}b_{0,\,n} + \omega_{4,\,n+1}\right) \bmod \frac{1}{\tau}\right)\tau^3 \\ & + \left(\left(b_{4,\,n} + b_{3,\,n} - a_{0,\,n}^2 b_{3,\,n} - a_{1,\,n}^2 b_{1,\,n} - 2a_{0,\,n}a_{1,\,n}b_{2,\,n} - 2a_{1,\,n}a_{2,\,n}b_{0,\,n} - 2a_{0,\,n}a_{2,\,n}b_{1,\,n}\right) \bmod \frac{1}{\tau}\right)\tau^4, \end{aligned}$$

$$\delta_{1,\,n+1} = \left[(a_{1,\,n} + b_{0,\,n} + \delta_{2,\,n+1})\,\tau\right],$$

$$\delta_{2,\,n+1} = \left[(a_{2,\,n} + b_{1,\,n})\,\tau\right],$$

$$\omega_{1,\,n+1} = \left[\left(b_{1,\,n} - a_{0,\,n} + b_{0,\,n} - a_{0,\,n}^2 b_{0,\,n} + \omega_{2,\,n+1}\right)\tau\right],$$

$$\omega_{2,\,n+1} = \left[\left(b_{2,\,n} - a_{1,\,n} + b_{1,\,n} - a_{0,\,n}^2 b_{1,\,n} - 2a_{0,\,n}a_{1,\,n}b_{0,\,n} + \omega_{3,\,n+1}\right)\tau\right],$$

$$\omega_{3,\,n+1} = \left[\left(b_{3,\,n} - a_{2,\,n} + b_{2,\,n} - a_{0,\,n}^2 b_{2,\,n} - a_{1,\,n}^2 b_{0,\,n} - 2a_{0,\,n}a_{1,\,n}b_{1,\,n} - 2a_{0,\,n}a_{2,\,n}b_{0,\,n} + \omega_{4,\,n+1}\right)\tau\right],$$

$$\omega_{4,\,n+1} = \left[\left(b_{4,\,n} + b_{3,\,n} - a_{0,\,n}^2 b_{3,\,n} - a_{1,\,n}^2 b_{1,\,n} - 2a_{0,\,n} a_{1,\,n} b_{2,\,n} - 2a_{1,\,n} a_{2,\,n} b_{0,\,n} - 2a_{0,\,n} a_{2,\,n} b_{1,\,n}\right)\tau\right].$$

We apply digit shifting procedure. Let us find the probability that $\delta_{2,\,n} = 0$ (within the interval where $a_{2,\,n}$ and $b_{1,\,n}$ are non-negative) in the same way as we found for the Carleman system.

$$(a_{2,\,n} + b_{1,\,n})\,\tau < 1 \Rightarrow a_{2,\,n} < \frac{1}{\tau} - b_{1,\,n}.$$

$$P\{\delta_{2,\,n} = 0\} = 1 - b_{1,\,n}\tau.$$

When $a_{2,\,n}$ and $b_{1,\,n}$ are non-negative, it is easy to show that $\delta_{2,\,n}$ is either 0 or 1. In this case, the probability of $\delta_{2,\,n} = 1$ is

$$P\{\delta_{2,\,n} = 1\} = 1 - P\{\delta_{2,\,n} = 0\} = b_{1,\,n}\tau.$$

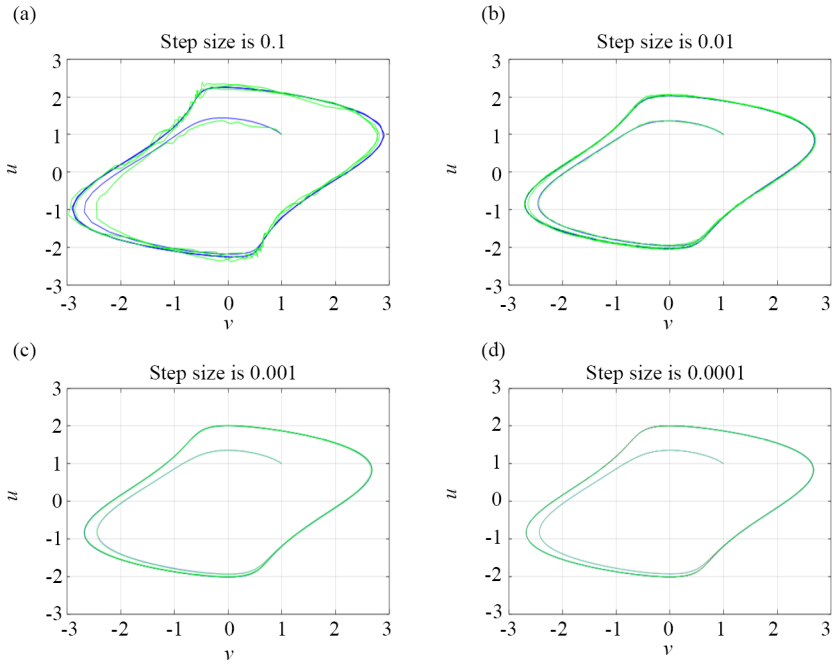

Figure 5. The solution using MCA (blue line) and the solution using probability (green line) for different step sizes

Then solution u_{n+1} can be represented in the form

$$u_{n+1} = (a_{0,\,n} + \delta_{1,\,n+1}) + \left((a_{1,\,n} + b_{0,\,n} + \delta_{2,\,n+1}) \bmod \frac{1}{\tau}\right)\tau + a_{2,\,n+1}\tau^2,$$

where $\delta_{2,\,n}$ is obtained as a random variable with a Bernoulli distribution and probability $P\{\delta_{2,\,n}=1\}$. In the case of non-positive $a_{2,\,n}$ and $b_{1,\,n}$, $\delta_{2,\,n}$ can only be 0 or -1. Then, $\delta_{2,\,n}$ is generated as -1 with probability $|b_{1,\,n}\tau|$. $a_{2,\,n+1}$ is obtained as a random variable with a uniform distribution in the interval $-1/\tau+1;\ 1/\tau-1$. We need to calculate $a_{2,\,n+1}$ to calculate changes in the coefficients $b_{0,\,n+1}$, $b_{1,\,n+1}$, $b_{2,\,n+1}$, $b_{3,\,n+1}$, $b_{4,\,n+1}$, which are calculated using the same formulas as before. The solution using MCA and the solution using probability are compared in Figure 5. The solutions converge as the step size τ decreases.

9. Construction of a solution to the Shimizu-Morioka system

Consider the following Cauchy problem for the Shimizu-Morioka system. There is a range of parameters for which a chaotic attractor arises. We adopt the common parameters $\mu=0.81$, $\alpha=0.375$, mentioned in [18], under which this system produces a "Lorenz-like" attractor. Note that at $\alpha=0.19145$ the system generates a Burke-Shaw-like attractor, though this case is not considered here. The present study is limited to a standard parameter set. In the future, the method can be applied to other chaotic systems, particularly those like the one discussed in [20].

$$\begin{cases} \dfrac{dx}{dt}=y, & x(0)=0.1 \\ \dfrac{dy}{dt}=x-0.81y-xz, & y(0)=0.1 \\ \dfrac{dz}{dt}=-0.375z+x^2, & z(0)=0.1 \end{cases} \tag{10}$$

This system is of particular interest since its behavior exhibits features of deterministic chaos.

The solution is sought in form (because the nonlinearity is of the second degree, we use the third degree approximation in step τ)

$$x_n=a_{0,\,n}+a_{1,\,n}\tau+a_{2,\,n}\tau^2$$

$$y_n=b_{0,\,n}+b_{1,\,n}\tau+b_{2,\,n}\tau^2+b_{3,\,n}\tau^3$$

$$z_n=c_{0,\,n}+c_{1,\,n}\tau+c_{2,\,n}\tau^2+c_{3,\,n}\tau^3$$

After applying the Euler scheme, we obtain an approximation

$$x_{n+1}=a_{0,\,n}+(a_{1,\,n}+b_{0,\,n})\,\tau+(a_{2,\,n}+b_{1,\,n})\,\tau^2,$$

$$y_{n+1}=b_{0,\,n}+(b_{1,\,n}+a_{0,\,n}-0.81b_{0,\,n}-a_{0,\,n}c_{0,\,n})\,\tau+(b_{2,\,n}+a_{1,\,n}-0.81b_{1,\,n}-a_{0,\,n}c_{1,\,n}-a_{1,\,n}c_{0,\,n})\,\tau^2$$

$$+(b_{3,\,n}+a_{2,\,n}-0.81b_{2,\,n}-a_{0,\,n}c_{2,\,n}-a_{1,\,n}c_{1,\,n}-a_{2,\,n}c_{0,\,n})\,\tau^3,$$

$$z_{n+1}=c_{0,\,n}+\left(c_{1,\,n}-0.375c_{0,\,n}+a_{0,\,n}^2\right)\tau+(c_{2,\,n}-0.375c_{1,\,n}+2a_{0,\,n}a_{1,\,n})\,\tau^2$$

$$+\left(c_{3,\,n}-0.375c_{2,\,n}+a_{1,\,n}^{2}+2a_{0,\,n}a_{2,\,n}\right)\tau^{3}.$$

We apply digit shifting procedure

$$x_{n+1}=(a_{0,\,n}+\delta_{1,\,n+1})+\left((a_{1,\,n}+b_{0,\,n}+\delta_{2,\,n+1})\bmod\frac{1}{\tau}\right)\tau+\left((a_{2,\,n}+b_{1,\,n})\bmod\frac{1}{\tau}\right)\tau^{2},$$

$$\begin{aligned}y_{n+1}=(b_{0,\,n}+\omega_{1,\,n+1})&+\left((b_{1,\,n}+a_{0,\,n}-0.81b_{0,\,n}-a_{0,\,n}c_{0,\,n}+\omega_{2,\,n+1})\bmod\frac{1}{\tau}\right)\tau\\&+\left((b_{2,\,n}+a_{1,\,n}-0.81b_{1,\,n}-a_{0,\,n}c_{1,\,n}-a_{1,\,n}c_{0,\,n}+\omega_{3,\,n+1})\bmod\frac{1}{\tau}\right)\tau^{2}\\&+\left((b_{3,\,n}+a_{2,\,n}-0.81b_{2,\,n}-a_{0,\,n}c_{2,\,n}-a_{1,\,n}c_{1,\,n}-a_{2,\,n}c_{0,\,n})\bmod\frac{1}{\tau}\right)\tau^{3},\end{aligned}$$

$$\begin{aligned}z_{n+1}=(c_{0,\,n}+\gamma_{1,\,n+1})&+\left(\left(c_{1,\,n}-0.375c_{0,\,n}+a_{0,\,n}^{2}+\gamma_{2,\,n+1}\right)\bmod\frac{1}{\tau}\right)\tau\\&+\left((c_{2,\,n}-0.375c_{1,\,n}+2a_{0,\,n}a_{1,\,n}+\gamma_{3,\,n+1})\bmod\frac{1}{\tau}\right)\tau^{2}\\&+\left(\left(c_{3,\,n}-0.375c_{2,\,n}+a_{1,\,n}^{2}+2a_{0,\,n}a_{2,\,n}\right)\bmod\frac{1}{\tau}\right)\tau^{3},\end{aligned}$$

$$\delta_{1,\,n+1}=\left[(a_{1,\,n}\ \ +b_{0,\,n}+\delta_{2,\,n+1})\,\tau\right],$$

$$\delta_{2,\,n+1}=\left[(a_{2,\,n}+b_{1,\,n})\,\tau\right],$$

$$\omega_{1,\,n+1}=\left[(b_{1,\,n}+a_{0,\,n}-0.81b_{0,\,n}-a_{0,\,n}c_{0,\,n}+\omega_{2,\,n+1})\,\tau\right]$$

$$\omega_{2,\,n+1}=\left[(b_{2,\,n}+a_{1,\,n}-0.81b_{1,\,n}-a_{0,\,n}c_{1,\,n}-a_{1,\,n}c_{0,\,n}+\omega_{3,\,n+1})\,\tau\right],$$

$$\omega_{3,\,n+1}=\left[(b_{3,\,n}+a_{2,\,n}-0.81b_{2,\,n}-a_{0,\,n}c_{2,\,n}-a_{1,\,n}c_{1,\,n}-a_{2,\,n}c_{0,\,n})\,\tau\right],$$

$$\gamma_{1,\,n+1}=\left[\left(c_{1,\,n}-0.375c_{0,\,n}+a_{0,\,n}^{2}+\gamma_{2,\,n+1}\right)\tau\right],$$

$$\gamma_{2,\,n+1}=\left[(c_{2,\,n}-0.375c_{1,\,n}+2a_{0,\,n}a_{1,\,n}+\gamma_{3,\,n+1})\,\tau\right],$$

$$\gamma_{3,\,n+1}=\left[\left(c_{3,\,n}-0.375c_{2,\,n}+a_{1,\,n}^{2}+2a_{0,\,n}a_{2,\,n}\right)\tau\right].$$

Within the first-order scheme (given known transfers from the first to the zeroth rank), the transfer from the second to the first rank effectively defines function [9]. Therefore, the coefficients of the first and second powers of τ, along with the corresponding transfers, are of primary interest. The behaviors of $a_{1,\,n}$, $a_{2,\,n}$, $b_{1,\,n}$, $b_{2,\,n}$, $c_{1,\,n}$, $c_{2,\,n}$ and $\delta_{2,\,n}$ ($\tau = 0.01$) as a function of discrete time are shown in Figures 6 and 7.

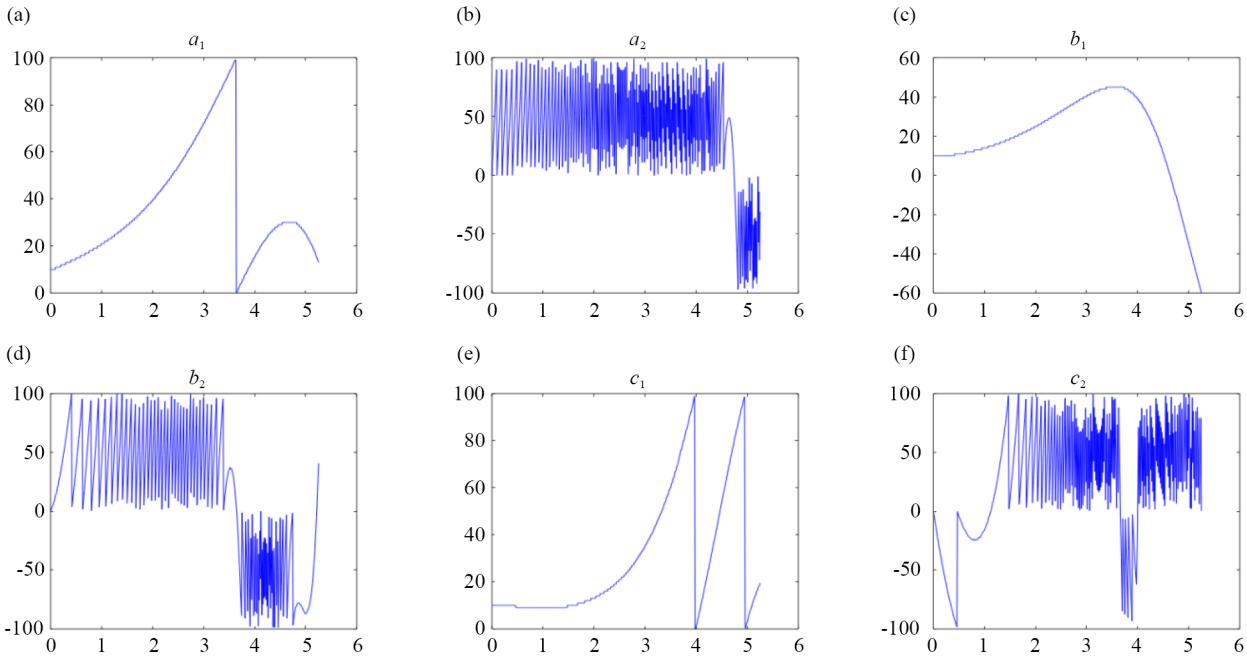

Figure 6. Coefficients at the first and second powers of step size for (10)

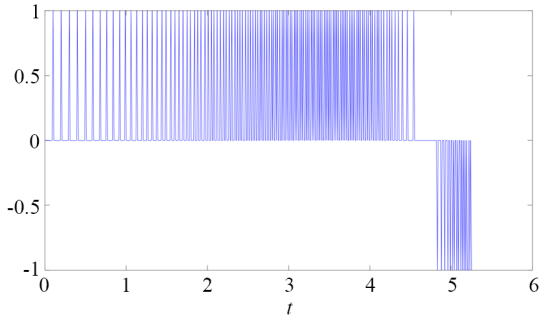

Figure 7. Transfer $\delta_{2,\,n}$ for (10)

As shown in Figure 7, for the time interval under consideration, $\delta_{2,\,n}$ only takes values of -1, 0, and 1. Moreover, at some intervals $\delta_{2,\,n}$ takes the values of 0 and 1, and at others, 0 and -1. This is because of the signs of the coefficient $a_{2,\,n}$, as shown in Figure 6 (you can notice the similarities between the graphs for $a_{2,\,n}$ and $\delta_{2,\,n}$). It can be shown that transfer $\delta_{2,\,n}$ does not take other values.

Consider non-negative $a_{2,\,n}$ and $b_{1,\,n}$. Due to $|a_i| < 1/\tau$

$$\delta_{2,\,n+1} = [(a_{2,\,n} + b_{1,\,n})\,\tau] \le \left[\left(\frac{1}{\tau} + b_{1,\,n}\right)\tau\right] = 1 + [b_{1,\,n}\tau].$$

Since $|b_i| < 1/\tau$,

$$[b_{1,\,n}\tau] = 0.$$

Then $\delta_{2,\,n+1} \leq 1$. Consider non-positive $a_{2,\,n}$ and $b_{1,\,n}$. Due to $|a_i| < 1/\tau$

$$\delta_{2,\,n+1} = -\left[|a_{2,\,n} + b_{1,\,n}|\,\tau\right] \geq -\left[\left|-\frac{1}{\tau} + b_{1,\,n}\right|\tau\right] = -1 - \left[|b_{1,\,n}|\,\tau\right].$$

Since $|b_i| < 1/\tau$,

$$\left[|b_{1,\,n}|\,\tau\right] = 0.$$

Then $\delta_{2,\,n+1} \geq -1$. When $a_{2,\,n}$ and $b_{1,\,n}$ have different signs, $|a_{2,\,n} + b_{1,\,n}| \leq 1/\tau$, hence $|\delta_{2,\,n+1}| \leq 1$.

To obtain a semi-analytical approximation, we use the probabilistic properties of $\delta_{2,\,n}$. The behavior of $\delta_{2,\,n}$ allows it to be generated as a random variable with values of 0 and 1 (for non-negative values of $a_{2,\,n}$ and $b_{1,\,n}$) and a random variable with values of 0 and -1 (for non-positive values of $a_{2,\,n}$ and $b_{1,\,n}$). In the second case, the formula for the first case under modulus is used to determine the probability. Consider the time intervals with non-negative $a_{2,\,n}$ and $b_{1,\,n}$.

Let us find the probability that $\delta_{2,\,n} = 0$ in the same way as we found it for the Carleman system.

$$(a_{2,\,n} + b_{1,\,n})\,\tau < 1 \Rightarrow a_{2,\,n} < \frac{1}{\tau} - b_{1,\,n}.$$

$$P\{\delta_{2,\,n} = 0\} = 1 - b_{1,\,n}\tau.$$

When $a_{2,\,n}$ and $b_{1,\,n}$ are non-negative, it is easy to show that $\delta_{2,\,n}$ is either 0 or 1. In this case, the probability of $\delta_{2,\,n} = 1$ is

$$P\{\delta_{2,\,n} = 1\} = 1 - P\{\delta_{2,\,n} = 0\} = b_{1,\,n}\tau.$$

Then solution x_{n+1} can be represented in the form

$$x_{n+1} = (a_{0,\,n} + \delta_{1,\,n+1}) + \left((a_{1,\,n} + b_{0,\,n} + \delta_{2,\,n+1}) \bmod \frac{1}{\tau}\right)\tau + a_{2,\,n+1}\tau^2,$$

where $\delta_{2,\,n}$ is obtained as a random variable with a Bernoulli distribution with probability $P\{\delta_{2,\,n} = 1\}$ and $a_{2,\,n+1}$ is obtained as a random variable with uniform distribution in the interval $\left[-1/\tau + 1;\ 1/\tau - 1\right]$. We need to calculate $a_{2,\,n+1}$ to calculate the changes in the solutions y_{n+1} and z_{n+1}, which are calculated using the same formulas as before.

The solution using MCA and the solution using probability are compared in Figure 8 for different time intervals ($\tau = 0.001$). The first graph (for the interval [0, 35]) shows convergence of the solutions. The second graph (for the interval [0, 700]) clearly shows a strange attractor. The difference in solution values at large time values is a consequence

of the chaotic behavior of the solution (the slightest difference in values leads to a significant divergence in the solution trajectories).

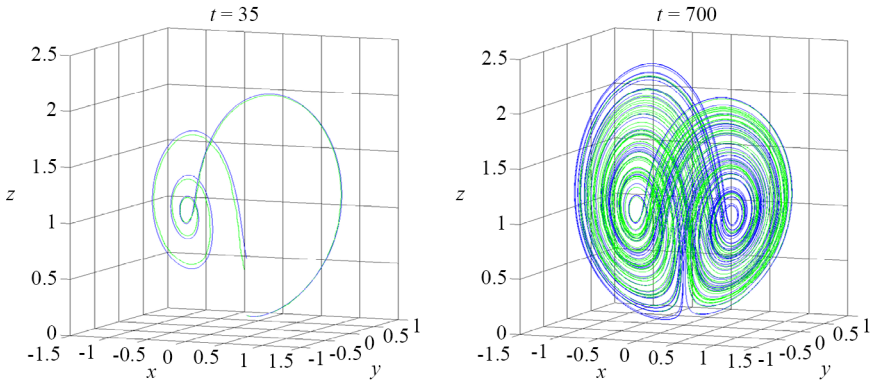

Figure 8. The solution using MCA (blue line) and the solution using probability (green line) for different time intervals, $\tau = 0.001$

10. Conclusion

In this study, we developed the Method of Computer Analogy (MCA). MCA yields generally semi-analytical solutions and an exact solution in cases where the transfer functions from the first rank to the zeroth and from the second rank to the first rank can be successfully replicated. This approach leads to an exact solution for the simple case of the Carleman kinetic system. Modeling the probabilistic properties of transfers makes it possible to construct semi-analytical approximations for other cases. By simulating transfers from the second rank to the first for one of the variables, it is possible to replicate the behavior characteristic of the Van der Pol equation solution, namely, its cyclicity. Another approximation idea involves replacing the transfer from the first rank to the zeroth with an analogous transfer for a (deterministic) linear approximation. Furthermore, by modeling transfers from the second rank to the first for one of the variables, it is possible to reproduce complex behavioral patterns (a strange attractor) for the Shimizu-Morioka system. We propose that if the transfers from the second rank to the first rank can be defined analytically for each variable in the systems under consideration, an analytical solution can be obtained.

Conflict of interest

The authors declare no competing financial interest.

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