Contemporary Mathematics
https://ojs.wiserpub.com/index.php/CM/

Research Article

A Novel Family of Discrete Variant Inequality for Jensen and Hermite-Hadamard Types in the *p*-Harmonic Convex Function Setting

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Received: 10 September 2025; **Revised:** 9 October 2025; **Accepted:** 21 October 2025

Abstract: This paper introduces a novel form of the discrete Jensen-type inequality specifically designed for p-harmonic convex functions, and extend it to the broader family of (p, h)-harmonic convex functions. Moreover, we have established several majorization-type inequalities in the setting of (p, h)-harmonic convexity. In addition, certain refinements of these inequalities are provided through Hermite-Hadamard-type results for p-harmonically convex functions.

***Keywords*:** harmonic convex functions, p-harmonic convex functions, $(p; h)$-harmonic convex functions, Jensen's type inequality, majorization inequality, Hermite-Hadamard-type inequality

MSC: 26D10, 26D15, 39B62

1. Introduction

Integral inequalities play a crucial role in advancing both pure and applied sciences [1]. They are widely used in complex mathematical problems, including spectral analysis, approximation theory, distribution theory, and statistical analysis. In recent years, increasing attention has been directed toward the study of classical inequalities within the framework of fractional calculus [2–4] and integral operators. Such inequalities, together with their applications remain fundamental in differential equations and applied mathematics, underpinning important theoretical and practical developments [5].

Key topics in this area include Gronwall's inequality [6], Jensen's inequality [7], Grüss-type [8], Chebyshev-type [9], Ostrowski-type [10], Hermite-Hadamard-type [11], Opial-type [12], Hardy-type [13]. These inequalities serve as essential tools for analyzing and solving problems involving fractional calculus and its related operators.

DOI: https://doi.org/10.37256/cm.7120268590

The study of convexity has been widely extended using several novel approaches. Varosanec [14] proposed the idea of h-convexity, which not only include the classical convex functions but also encompass other families such as Godunova-Levin, s-convex, and p-convex functions [15]. Additionally, harmonic convex functions was first investigated by Anderson et al. [16] and further explored by İşcan [17], broadening the scope of convexity in mathematical analysis. Butt et al. [18] introduced novel fractional Hermite-Hadamard-Mercer inequalities tailored for harmonically convex functions. Building on these foundations, Baloch et al. [19–22] developed Jensen-type inequalities within the context of harmonic convexity, marking significant progress in the field of mathematical inequalities. Their work not only refines existing results but also explores various properties and applications, emphasizing the importance of further investigations into harmonic convex functions and related functionals. These papers offer enhancements and extensions of previously established findings, encouraging continued investigations, and developments within this field of research.

Noor et al. [23] established the concept of p-harmonic means, a generalization that encompasses the arithmetic, harmonic, and geometric means as particular instances. Building on this framework, they defined and explored p-harmonic convex sets and functions. It has been demonstrated that p-harmonic convex functions includes both harmonic convex functions and classical convex functions as particular instances. This broader class brings together several known and novel types of convexity under a single framework. For more recent developments and broader generalizations, the reader is referred to [20, 24].

The existing literature reveals a gap in the study of Hermite-Hadamard and Jensen-type inequalities for harmonic convex functions. The present literature mainly focused on harmonic and h-harmonic convex functions for Jensen and Hermite-Hadamard type inequalities. On the other hand, the present study develops a broader framework by extending these inequalities to p-harmonic and $(p,\ h)$-harmonic convex functions, which provide a natural generalization of harmonic convexity. Within this framework, we establish modified Jensen-type inequalities adapted to p-harmonic convex functions and further derive new Hermite-Hadamard-type inequalities as an application. These advancements emphasize the novelty of our contribution, as it goes beyond the restrictions of earlier studies and offers a unified, more general perspective on convexity with wider theoretical and applied significance.

Sections 1 and 2 cover the foundational concepts necessary for this study. In section 3, we introduce a novel variant of Jensen-type inequalities tailored to p-harmonic convex functions and develop related results for $(p;\ h)$-harmonic convex functions. Section 4 presents additional findings for p-harmonic convex functions derived through the application of Majorization-type inequalities. Finally, section 5 offers several applications of the Hermite Hadamard inequality for p-harmonic convex functions, accompanied by graphical and numerical illustrations that highlight and validate the obtained results.

2. Preliminaries and some basic results

The following section, we introduced several novel categories of harmonic convex functions.

Definition 1 [22] Let $\mathscr{G} : \mathscr{J} \subseteq \mathbb{R} \to \mathbb{R}$ be a real-valued function. The function $\mathscr{G}$ is defined as convex (or concave) on the interval $\mathscr{J}$ if, $\forall x,\ y \in \mathscr{I}$ and every $\zeta \in [0,\ 1]$, the following relation is satisfied

$$\mathscr{G}(\zeta x + (1-\zeta)y) \leq (\geq) \zeta(x) + (1-\zeta)(y).$$

Jensen offered a fundamental description of convex functions through the following characterization.

Theorem 1 [25] Consider a function $\mathscr{G}$ that is convex on $\mathscr{I} \subseteq \mathbb{R}$. It satisfies the inequality

$$\mathscr{G}\left(\sum_{i=1}^{m} \Lambda_i x_i\right) \leq \sum_{i=1}^{m} \Lambda_i \mathscr{G}(x_i) \tag{1}$$

valid for every $x_1,\ x_2,\ \ldots,\ x_m \in \mathscr{J}$ and $\Lambda_1,\ \Lambda_2,\ \ldots,\ \Lambda_m \geq 0$ with $\sum_{i=1}^{m} \Lambda_i = 1$.

When $\mathscr{G}$ is concave, the inequality (1) is reversed. The study of inequalities for convex functions plays a central role in mathematical analysis [26, 27], as it encompasses many classical results, including Minkowski's inequality, Hölder inequality, and the arithmetic & geometric mean inequality. Additionally, this inequality is intimately linked with several important inequalities, including the reverse Minkowski inequality [28], Ostrowski inequality [29, 30], Hermite-Hadamard inequality [31], and inequalities related to Bessel functions [32].

Definition 2 [24] A set $H_p \subseteq \mathbb{R}^n \backslash \{0\}$ is called p-harmonic convex set, if

$$\left[\frac{x^p y^p}{\zeta x^p + (1-\zeta) y^p}\right]^{\frac{1}{p}} \in \mathscr{J},\ \forall x,\ y \in H_p,\ \zeta \in [0,\ 1],\ p \neq 0.$$

Definition 3 [23] Let $\mathscr{G}: H_p \to \mathbb{R}$ is defined as p-harmonic convex function, if

$$\mathscr{G}\left(\left[\frac{x^p y^p}{\zeta x^p + (1-\zeta)^p y^p}\right]^{\frac{1}{p}}\right) \leq (1-\zeta)\mathscr{G}(x) + \zeta \mathscr{G}(y),\ \forall x,\ y \in H_p,\ \zeta \in [0,\ 1]. \tag{2}$$

For $\zeta = \dfrac{1}{2}$, this reduces to

$$\mathscr{G}\left(\left[\frac{2x^p y^p}{x^p + y^p}\right]^{\frac{1}{p}}\right) \leq \frac{f(x) + f(y)}{2}, \qquad \forall x,\ y \in H_p.$$

The function $\mathscr{G}$ referred to as a Jensen-p-harmonic convex function.

For $p = 1$, the notion of a p-harmonic convex set coincides with the classical harmonic convex set. Conversely, for $p = -1$, the p-harmonic convex set reduces to the standard convex set. Hence, the framework of p-harmonic convexity serves as a unifying concept that extends and connects these established notions.

Next, several examples of p-harmonic convex functions are presented. In particluar the natural logarithm function, defined as $\mathscr{G}(x) = lnx$ exhibits p-harmonic convexity. It is harmonically convex over the interval $(0,\ \infty)$, yet it does not satisfy classical convexity conditions and shown as Figure 1.

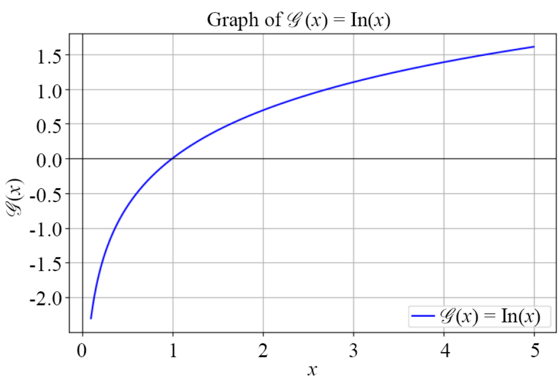


Figure 1. Harmonic and p-harmonic convex function but not classical convex

The following functions serve as examples of functions that are both p-harmonic convex and harmonic convex on the interval $(0, \infty)$ are provided in Figure 2.

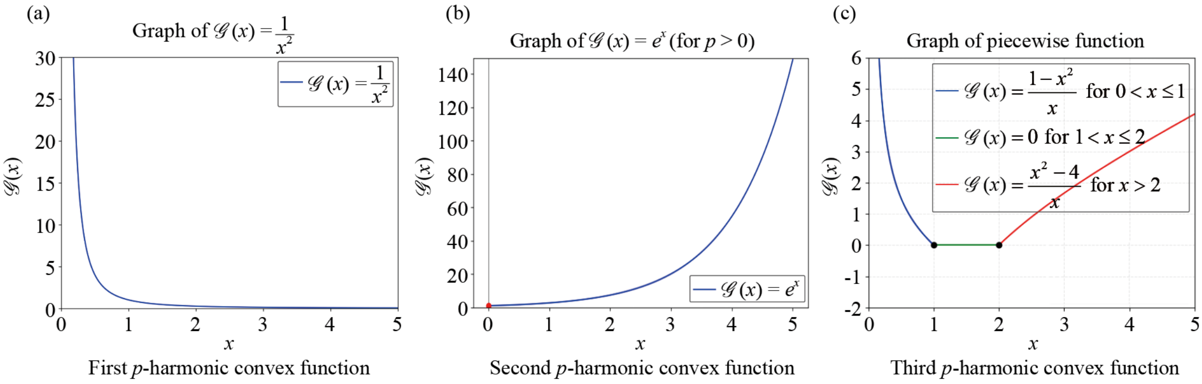


Figure 2. Visualization of three distinct p-harmonic convex functions

Definition 4 [33] Let $\hbar : \mathscr{K} = [0, 1] \to \mathbb{R}$ be a non-negative function. A mapping $\mathscr{G} : K_h \subseteq \mathbb{R} \backslash \{0\} \to \mathbb{R}$ is called $(p; \hbar)$-harmonic convex function, if

$$\mathscr{G}\left(\left[\frac{x^p y^p}{\zeta x^p + (1-\zeta) y^p}\right]^{\frac{1}{p}}\right) \le \hbar(\zeta)\mathscr{G}(x) + \hbar(1-\zeta)\mathscr{G}(y) \quad \forall x, y \in K_h, \zeta \in [0, 1]. \tag{3}$$

Recently, Dragomir derived a Jensen-type inequality applicable to harmonically convex functions, which is stated as.

Theorem 2 [34] Suppose $\mathscr{J}$ is an interval contained in $(0, \infty)$ and $\mathscr{G} : \mathscr{J} \to \mathbb{R}$ is a harmonically convex. Under these conditions the Jensen's inequality can be expressed as follows

$$\mathscr{G}\left(\frac{1}{\sum_{i=1}^{m} \frac{\zeta_i}{x_i}}\right) \le \sum_{i=1}^{m} \zeta_i \mathscr{G}(x_i) \tag{4}$$

where $x_1, \ldots, x_m \in \mathscr{J}$ and the weights $\zeta_1, \ldots, \zeta_m \ge 0$ with $\sum_{i=1}^{m} \zeta_i = 1$.

Definition 5 [14] Let $\hbar : \mathscr{J} \to \mathbb{R}$ is called supermultiplicative function if

$$\hbar(xy) \ge \hbar(x)\hbar(y) \tag{5}$$

for every $x, y \in \mathscr{J}$, if the inequality (5) is reversed, then $\hbar$ is submultiplicative function. In the case of equality, $\hbar$ is referred to as multiplicative.

Example 1 [14] Consider $\hbar(x) = (c+x)^{p-1}$ for $x \ge 0$.

• If $c = 0$, the function $\hbar$ is multiplicative.

• If $c \ge 1$, then on the interval $p \in (0, 1)$, $\hbar$ is supermultiplicative and for $p > 1$ it becomes submultiplicative.

Definition 6 [22] Let $m \geq 2$, and consider two m-tuples $a = (a_1, a_2, \ldots, a_m)$ and $b = (b_1, b_2, \ldots, b_m)$. The vector b is known as majorized by a, or equivalently, a majorizes b (denotes by $a \succ b$) if

- $\sum_{i=1}^{k} a_i \geq \sum_{i=1}^{k} b_i$, for $i = 1, 2, \ldots, m-1$.
- $\sum_{i=1}^{m} a_i = \sum_{i=1}^{m} b_i$.

In this context, a_i denotes the component ranked as the ith largest.

The classical majorization theorem is commonly formulated as follows.

Theorem 3 [18] Let $\mathscr{J} \in \mathbb{R} \setminus \{0\}$, and consider two m-tuples $a_1, a_2, \ldots, a_m$, and $b_1, b_2, \ldots, b_m$ with components in $\mathscr{J}$, $a = (a_1, a_2, \ldots, a_m)$, $b = (b_1, b_2, \ldots, b_m)$ such that $a \succ b$, and $\mathscr{G} : \mathscr{J} \to \mathbb{R}$ is a p-harmonic convex function. Then the inequality

$$\sum_{i=1}^{m} (a_i)^p \mathscr{G}(a_i) \geq \sum_{i=1}^{m} (b_i)^p \mathscr{G}(b_i), \tag{6}$$

holds.

Theorem 4 [21] Let $\mathscr{J} \subset \mathbb{R} \setminus \{0\}$ be an interval, and let $\mathscr{G} : \mathscr{J} \to \mathbb{R}$ be a harmonically p-convex. For any $a, b \in \mathscr{J}$ with $a < b$, and assuming $\mathscr{G}$ is integrable on $[a, b]$, then following inequalities holds

$$\mathscr{G}\left(\frac{2^{\frac{1}{p}} ab}{[a^p + b^p]^{\frac{1}{p}}}\right) \leq \frac{p(ab)^p}{b^p - a^p} \int_a^b \frac{\mathscr{G}(x)}{x^{p+1}} dx \leq \frac{\mathscr{G}(a) + \mathscr{G}(b)}{2}.$$

3. Main results

Lemma 1 Let $\mathscr{J}$ be an interval contained in $\mathbb{R} \setminus \{0\}$, and consider a finite sequence $\{x_k\}_{k=1}^{m} \in \mathscr{J}$ that is strictly increasing and positive. Suppose $\mathscr{G}$ is a p-harmonic convex function defined on the interval $\mathscr{I}$. Then, the following inequality

$$\mathscr{G}\left(\frac{1}{\left(\frac{1}{x_1^p} + \frac{1}{x_m^p} - \frac{1}{x_k^p}\right)^{\frac{1}{p}}}\right) \leq \mathscr{G}(x_1) + \mathscr{G}(x_m) - \mathscr{G}(x_k), \tag{7}$$

holds $\forall\ 1 \leq k \leq m$.

Proof. Define the sequence $\{y_k^p\}$ by the relation

$$\frac{1}{y_k^p} = \frac{1}{x_1^p} + \frac{1}{x_m^p} - \frac{1}{x_k^p}.$$

From this, we obtain that

$$\frac{1}{y_k^p} + \frac{1}{x_k^p} = \frac{1}{x_1^p} + \frac{1}{x_m^p}.$$

This implies that the pairs x_1^p, x_m^p and y_k^p, x_k^p shares the same p-harmonic mean. Consequently, there exist constants κ, $\Lambda \in [0,\ 1]$ with

$$\kappa + \Lambda = 1, \tag{8}$$

such that

$$x_k = \left(\frac{x_1^p x_m^p}{\kappa x_1^p + \Lambda x_m^p} \right)^{\frac{1}{p}},$$

and

$$y_k = \left(\frac{x_1^p x_m^p}{\Lambda x_1^p + \kappa x_m^p} \right)^{\frac{1}{p}}.$$

By applying the property of p-harmonic convexity of the function $\mathscr{G}$, we deduce that

$$\begin{aligned}
\mathscr{G}(y_k) &= \mathscr{G}\left(\frac{x_1^p x_m^p}{\Lambda x_1^p + \kappa x_m^p} \right)^{\frac{1}{p}} \\
&\leq \kappa \mathscr{G}(x_1) + \Lambda \mathscr{G}(x_m) \\
&= (1-\Lambda)\mathscr{G}(x_1) + (1-\kappa)\mathscr{G}(x_m) \qquad \text{using (8)} \\
&= \mathscr{G}(x_1) + \mathscr{G}(x_m) - [\Lambda \mathscr{G}(x_1) + \kappa \mathscr{G}(x_m)] \\
&\leq \mathscr{G}(x_1) + \mathscr{G}(x_m) - \mathscr{G}\left(\frac{x_1^p x_m^p}{\kappa x_1^p + \Lambda x_m^p} \right)^{\frac{1}{p}} \\
&= \mathscr{G}(x_1) + \mathscr{G}(x_m) - \mathscr{G}(x_k).
\end{aligned}$$

Hence, inequality (7) is derived by utilizing the relation

$$\frac{1}{y_k^p} = \frac{1}{x_1^p} + \frac{1}{x_m^p} - \frac{1}{x_k^p}. \qquad \square$$

Example 2 1. Let's take a increasing sequence $\{2,\ 4,\ 6,\ 8\}$ with $x_1 = 2,\ x_k = 4,\ x_m = 8,\ p = 1$. If

$$\frac{1}{y_k^p} = \frac{1}{x_1^p} + \frac{1}{x_m^p} - \frac{1}{x_k^p}.$$

Then

$$\frac{1}{y_k^p} + \frac{1}{x_k^p} = \frac{1}{x_1^p} + \frac{1}{x_m^p}.$$

After substituting the values we find,

$$\frac{1}{y_k} + \frac{1}{x_k} = \frac{5}{8}$$

$$\frac{1}{x_1} + \frac{1}{x_m} = \frac{5}{8},$$

so that the pairs $x_1^p,\ x_m^p$ and $x_k^p,\ y_k^p$ shares the same p-harmonic mean.

2. For the increasing sequence $\{2,\ 4,\ 6,\ 8\}$ and $p = 2$, we found

$$\frac{1}{y_k^2} + \frac{1}{x_k^2} = \frac{17}{64}$$

$$\frac{1}{x_1^2} + \frac{1}{x_n^2} = \frac{17}{64},$$

so that the pairs $x_1^p,\ x_m^p$ and $x_k^p,\ y_k^p$ shares the same p-harmonic mean.

The expression of p-harmonic mean which has been satisfied by taking two finite increasing sequence. The same expression would be used in our first main result.

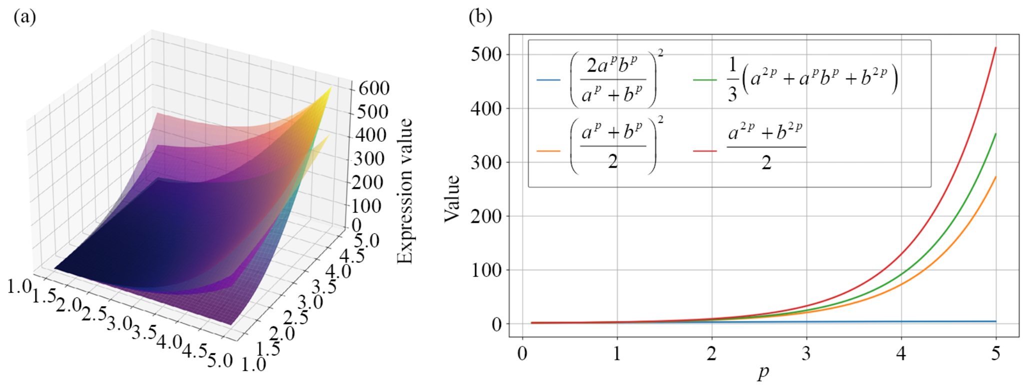
Remark 1 From Lemma 1, we deduce that

- Since $\mathscr{G}(x) = lnx$ is p-harmonic convex on $(0,\ \infty)$, so using inequality (7) we get $\dfrac{2^{\frac{1}{p}} ab}{(a^p + b^p)^{\frac{1}{p}}} \leq \dfrac{(a^p + b^p)^{\frac{1}{p}}}{2^{\frac{1}{p}}}$ for all $a,\ b \in (0,\ \infty)$.
- Since $\mathscr{G}(x) = \sqrt{x}$ is p-harmonic convex on $(0,\ \infty)$, so using inequality (7) we get $\sqrt{\dfrac{2^{\frac{1}{p}} ab}{(a^p + b^p)^{\frac{1}{p}}}} \leq \dfrac{\sqrt{a} + \sqrt{b}}{2}$ for all $a,\ b \in (0,\ \infty)$.
- Since $\mathscr{G}(x) = \dfrac{1}{x^{2p}}$ is p-harmonic convex for any $p > 0$, so using inequality (7) we get $\left(\dfrac{a^p + b^p}{2}\right)^2 \leq \dfrac{a^{2p} + b^{2p}}{2}$ for all $a,\ b \in (0,\ \infty)$.
- Since $\mathscr{G}(x) = x^{2p}$ is p-harmonic convex for any $p > 0$, so using inequality (7) we get $\left(\dfrac{2a^p b^p}{a^p + b^p}\right)^2 \leq \dfrac{a^{2p} + b^{2p}}{2}$ for all $a,\ b \in (0,\ \infty)$.

• It was proven in [22] that $\left(\frac{2ab}{a+b}\right)^2 \leq \left(\frac{a+b}{2}\right)^2 \leq \frac{1}{3}(a^2+ab+b^2) \leq \frac{a^2+b^2}{2}$ for all $a,\ b \in (0,\ \infty)$. Therefore, in the light of previous remarks, we get its generalization as,

$$\left(\frac{2a^p b^p}{a^p+b^p}\right)^2 \leq \left(\frac{a^p+b^p}{2}\right)^2 \leq \frac{1}{3}(a^{2p}+a^p b^p+b^{2p}) \leq \left(\frac{a^{2p}+b^{2p}}{2}\right). \tag{9}$$

This above result implies that the relationship between p-harmonic mean, p-arithmetic mean, symmetric p-quadratic mean and p-quadratic mean. The graphical representation of above inequality is shown as Figure 3. The Table 1, shows the data about inequality (9).



3D graph illustrating the inequality visualization

2D graph illustrating the inequality visualization

Figure 3. Inequality visualization for $a = 1$, $b = 2$, and $p \in [0.1,\ 5]$

Table 1. Numerical verification of inequality (9) for selected values

p	a	b	$\left(\frac{2a^p b^p}{a^p+b^p}\right)^2$	$\left(\frac{a^p+b^p}{2}\right)^2$	$\frac{1}{3}(a^{2p}+a^p b^p+b^{2p})$	$\left(\frac{a^{2p}+b^{2p}}{2}\right)$	Holds
1	0.5	0.8	0.378698	0.422500	0.430000	0.445000	Yes
1	1	2	1.777778	2.250000	2.333333	2.500000	Yes
1	0.2	0.9	0.107107	0.302500	0.343333	0.425000	Yes
1	1.5	2.5	3.515625	4.000000	4.083333	4.250000	Yes
1	0.3	0.4	0.117551	0.122500	0.123333	0.125000	Yes
2	0.5	0.8	0.129277	0.198025	0.210700	0.236050	Yes
2	1	2	2.560000	6.250000	7.000000	8.500000	Yes
2	0.2	0.9	0.005812	0.180625	0.230033	0.328850	Yes
2	1.5	2.5	10.948313	18.062500	19.395833	22.062500	Yes
2	0.3	0.4	0.013271	0.015625	0.016033	0.016850	Yes
3	0.5	0.8	0.040378	0.101442	0.113923	0.138885	Yes
3	1	2	3.160494	20.250000	24.333333	32.500000	Yes
3	0.2	0.9	0.000250	0.135792	0.179112	0.265753	Yes
3	1.5	2.5	30.813455	90.250000	102.755208	127.765625	Yes
3	0.3	0.4	0.001442	0.002070	0.002184	0.002413	Yes
4	0.5	0.8	0.011762	0.055720	0.065759	0.085839	Yes
4	1	2	3.543253	72.250000	91.000000	128.500000	Yes
4	0.2	0.9	0.000010	0.108142	0.143840	0.215235	Yes
4	1.5	2.5	80.341641	486.753906	583.087240	775.753906	Yes
4	0.3	0.4	0.000151	0.000284	0.000309	0.000360	Yes

• Since $\mathscr{G}(x) = e^x$ is p-harmonic convex on $(0, \infty)$, so using inequality (7) we get $e^{\frac{2^{\frac{1}{p}}ab}{(a^p+b^p)^{\frac{1}{p}}}} \leq \dfrac{e^a + e^b}{2}$.

Theorem 5 Let $\mathscr{G} \subseteq \mathbb{R}\backslash\{0\}$ be an interval, and consider a function $\mathscr{G} : \mathscr{J} \to \mathbb{R}$ that is p-harmonic convex. Under these conditions, the following inequality

$$\mathscr{G}\left(\frac{1}{\left(\frac{1}{x_1^p} + \frac{1}{x_m^p} - \sum_{i=1}^{m} \frac{\hat{p}_i}{x_i^p}\right)^{\frac{1}{p}}}\right) \leq \mathscr{G}(x_1) + \mathscr{G}(x_m) - \sum_{i=1}^{m} \hat{p}_i \mathscr{G}(x_i), \tag{10}$$

holds for any finite positive sequence $\{x_i\}_{i=1}^m \in \mathscr{J}$ and $\hat{p}_1,\ .....,\ \hat{p}_m \geq 0$ with $\sum_{i=1}^{m} \hat{p}_i = 1$.

Proof. By combining the results of Theorem 2 and Lemma 1, along with the p-harmonic convexity of $\mathscr{G}$ on the interval $\mathscr{J}$, it can be deduced that

$$\mathscr{G}\left(\frac{1}{\left(\frac{1}{x_1^p} + \frac{1}{x_m^p} - \sum_{i=1}^{m} \frac{\hat{p}_i}{x_i^p}\right)^{\frac{1}{p}}}\right) = \mathscr{G}\left(\frac{1}{\left[\sum_{i=1}^{m} \hat{p}_i \left(\frac{1}{x_1^p} + \frac{1}{x_m^p} - \frac{1}{x_i^p}\right)\right]^{\frac{1}{p}}}\right).$$

Let us consider,

$$\frac{1}{\aleph_i} = \frac{1}{x_1^p} + \frac{1}{x_m^p} - \frac{1}{x_i^p}.$$

So,

$$\mathscr{G}\left(\frac{1}{\left(\frac{1}{x_1^p} + \frac{1}{x_m^p} - \sum_{i=1}^{m} \frac{\hat{p}_i}{x_i^p}\right)^{\frac{1}{p}}}\right) = \mathscr{G}\left(\frac{1}{\left[\sum_{i=1}^{m} \frac{\hat{p}_i}{\aleph_i}\right]^{\frac{1}{p}}}\right)$$

$$= \mathscr{G}\left(\frac{1}{\left[\sum_{i=1}^{m} \frac{\hat{p}_i}{(\aleph_i^{\frac{1}{p}})^p}\right]^{\frac{1}{p}}}\right)$$

$$\leq \sum_{i=1}^{m} \hat{p}_i \mathscr{G}(\aleph_i^{\frac{1}{p}}) \qquad \text{using (4) for } p\text{-harmonic convex}$$

$$= \sum_{i=1}^{m} \hat{p}_i \mathscr{G}\left(\frac{1}{\left[\frac{1}{x_1^p}+\frac{1}{x_m^p}-\frac{1}{x_i^p}\right]^{\frac{1}{p}}}\right)$$

$$\leq \sum_{i=1}^{m} \hat{p}_i (\mathscr{G}(x_1)+\mathscr{G}(x_m)-\mathscr{G}(x_i)) \qquad \text{using (7)}$$

$$= \mathscr{G}(x_1)+\mathscr{G}(x_m)-\sum_{i=1}^{m} \hat{p}_i \mathscr{G}(x_i).$$

This concludes the proof of Theorem 5. □

Theorem 6 Let $m \geq 2$, and let $\mathscr{K} \subseteq (0,\ 1)$ be an interval. Consider a finite sequence $\{x_k\}_{k=1}^{m} \in \mathscr{J} \subseteq \mathbb{R} \backslash \{0\}$ and positive weights $q_1,\ q_2,\ \ldots\ldots,\ q_m > 0$ with total sum $Q_m = \sum_{i=1}^{m} q_i$. Suppose $\hbar : \mathscr{K} \to \mathbb{R}$ is a non-negative super-multiplicative function and $\mathscr{G}$ is a non-negative $(p;\ \hbar)$ -harmonic convex function defined on $\mathscr{J}$. Then one has

$$\mathscr{G}\left(\frac{1}{\frac{1}{Q_m}\sum_{i=1}^{m}\frac{q_i}{x_i^p}}\right)^{\frac{1}{p}} \leq \sum_{i=1}^{m} \hbar\left(\frac{q_i}{Q_m}\right)\mathscr{G}(x_i^p). \tag{11}$$

Proof. We establish Theorem 6 by mathematical induction. If $m = 2$, inequality (11) reduces to inequality (3) with $\zeta = \frac{q_1}{Q_2}$ and $1-\zeta = \frac{q_2}{Q_2}$.

Assume that inequality 11 holds true for the case $m-1$. Then, considering the m-tuples $(x_1,\ \ldots,\ x_n)$ and $(q_1,\ \ldots,\ q_n)$, we obtain

$$\mathscr{G}\left(\frac{1}{\frac{1}{Q_m}\sum_{i=1}^{m}\frac{q_i}{x_i^p}}\right)^{\frac{1}{p}} = \mathscr{G}\left(\frac{1}{\frac{q_m}{Q_m x_m^p}+\sum_{i=1}^{m-1}\frac{q_i}{Q_m x_i^p}}\right)^{\frac{1}{p}}$$

$$= \mathscr{G}\left(\frac{1}{\frac{q_m}{Q_m x_m^p}+\frac{Q_{m-1}}{Q_m}\sum_{i=1}^{m-1}\frac{q_i}{Q_{m-1} x_i^p}}\right)^{\frac{1}{p}}$$

$$\leq \hbar\left(\frac{q_m}{Q_m}\right)\mathscr{G}(x_m)+\hbar\left(\frac{Q_{m-1}}{Q_m}\right)\mathscr{G}\left(\frac{1}{\sum_{i=1}^{m-1}\frac{q_i}{Q_{m-1}x_i}}\right)$$

$$\leq \hbar\left(\frac{q_m}{Q_m}\right)\mathscr{G}(x_m)+\hbar\left(\frac{Q_{m-1}}{Q_m}\right)\sum_{i=1}^{m-1}\hbar\left(\frac{q_i}{Q_{m-1}}\right)\mathscr{G}(x_i).$$

Using super-multiplicativity of $\hbar$,

$$\leq \hbar\left(\frac{q_m}{Q_m}\right)\mathscr{G}(x_m)+\sum_{i=1}^{m-1}\hbar\left(\frac{q_i}{Q_m}\right)\mathscr{G}(x_i)$$

$$=\sum_{i=1}^{m}\hbar\left(\frac{q_i}{Q_m}\right)\mathscr{G}(x_i).$$

Which completes the proof. □

Before presenting the forthcoming result, we first rely on Lemma 2, which serves as an extension of Lemma 1.

Lemma 2 Assume $\hbar:\mathscr{K}\supseteq(0,\ 1)\to\mathbb{R}$ be a non-negative supermultiplicative function defined on $\mathscr{K}$. Suppose $\kappa,\ \Lambda\in[0,\ 1]$, satisfying $\kappa+\Lambda=1$ and the condition $\hbar(\kappa)+\hbar(\Lambda)\leq 1$. Let $\mathscr{G}:\mathscr{J}\subseteq\mathbb{R}\backslash\{0\}\to\mathbb{R}$ be a non-negative $(p;\ \hbar)$-harmonic convex function. Then for any finite positive and increasing sequence $\{x_k\}_{k=1}^{m}\in\mathscr{J}$, the inequality (7) is satisfied.

Proof. Following is the approach used in the proof of Lemma 1, assume that $\frac{1}{y_k^p}=\frac{1}{x_1^p}+\frac{1}{x_m^p}-\frac{1}{x_k^p}$. This implies $\frac{1}{y_k^p}+\frac{1}{x_k^p}=\frac{1}{x_1^p}+\frac{1}{x_m^p}$, indicating that the pairs $x_1^p,\ x_m^p$ and $x_k^p,\ x_k^p$ shares the same p-harmonic mean. Consequently, there exist $\kappa,\ \Lambda\in[0,\ 1]$ with $\kappa+\Lambda=1$ such that

$$x_k=\left(\frac{x_1^p x_m^p}{\kappa x_1^p+\Lambda x_m^p}\right)^{\frac{1}{p}},$$

and

$$y_k=\left(\frac{x_1^p x_m^p}{\Lambda x_1^p+\kappa x_m^p}\right)^{\frac{1}{p}},$$

where $\kappa+\Lambda=1$ and $1\leq k\leq n$. By the p-harmonic convexity of f, we conclude that

$$\mathscr{G}(y_k)=\mathscr{G}\left(\frac{x_1^p x_m^p}{\Lambda x_1^p+\kappa x_m^p}\right)^{\frac{1}{p}}$$

$$\leq \hbar(\kappa)\mathscr{G}(x_1)+\hbar(\Lambda)\mathscr{G}(x_m)$$

$$\leq (1-\hbar(\Lambda))\mathscr{G}(x_1)+(1-\hbar(\kappa))\mathscr{G}(x_m)$$

$$=\mathscr{G}(x_1)+\mathscr{G}(x_m)-[\hbar(\Lambda)\mathscr{G}(x_1)+\hbar(\kappa)\mathscr{G}(x_m)]$$

$$\leq \mathscr{G}(x_1)+\mathscr{G}(x_m)-\mathscr{G}\left(\frac{x_1^p x_m^p}{\kappa x_1^p+\Lambda x_m^p}\right)^{\frac{1}{p}}$$

$$=\mathscr{G}(x_1)+\mathscr{G}(x_m)-\mathscr{G}(x_k).$$

Thus, Lemma 2 has been established. □

Theorem 7 Let $\hbar:\mathscr{K}\subseteq(0,\ 1)\to\mathbb{R}$ be a non-negative supermultiplicative function. Consider positive real numbers $q_1,\ \ldots,\ q_m$ with $(m\geq 2)$ and define $Q_m=\sum_{k=1}^{m} q_i$ such that $\sum_{k=1}^{m}\hbar\left(\frac{q_k}{Q_m}\right)\leq 1$. If $\mathscr{G}$ is a non-negative $(p;\ \hbar)$-harmonic convex function defined on $\mathscr{J}\subseteq\mathbb{R}\backslash\{0\}$, then for any finite positive increasing sequence $\{x_k\}_{k=1}^{m}\in\mathscr{J}$, we have

$$\mathscr{G}\left(\frac{1}{\left(\frac{1}{x_1^p}+\frac{1}{x_m^p}-\frac{1}{Q_m}\sum_{k=1}^{m}\frac{q_k}{x_k^p}\right)^{\frac{1}{p}}}\right)\leq \mathscr{G}(x_1)+\mathscr{G}(x_m)-\sum_{k=1}^{m}\hbar\left(\frac{q_k}{Q_m}\right)\mathscr{G}(x_k). \tag{12}$$

If $\hbar$ is sub-multiplicative function and satisfies $\sum_{k=1}^{m}\hbar\left(\frac{q_k}{Q_m}\right)\geq 1$, and if $\mathscr{G}$ is $(p;\ \hbar)$-harmonic concave, then the inequality stated above holds in the opposite direction.

Proof. The result obtained for the $(p;\ \hbar)$ harmonic convex function is derived in the manner analogous to the approach used in Theorem 2.8 of [22]. □

4. Related results

In this section, we have provided a generalization of Theorem 5 along with several associated results.

Theorem 8 Let $\mathscr{G};\ [a,\ b]\to\mathbb{R}$ be a continuous function that is both p-harmonic and convex on the interval $[a,\ b]\subset(0,\ \infty)$. Consider a vector $a=(a_1,\ \ldots,\ a_m)$ with each component $a_j\in[a,\ b]$ and a real $m\times n$ matrix $\aleph=(x_{ij})$ whose entries satisfy $x_{ij}\in[a,\ a]$ for all $i=1,\ 2,\ 3,\ \ldots,\ m$ and $j=1,\ 2,\ 3,\ \ldots,\ n$. Suppose that the vector a majorizes every row of $\aleph$, that is

$$x_{\mathbf{i}}=(x_{i1},\ x_{i2},\ x_{i3},\ \ldots,\ x_{in})\prec(a_1,\ \ldots,\ a_n)=a \ \text{ for each }\ i=1,\ 2,\ 3,\ \ldots,\ m,$$

and

$$\sum_{j=1}^{n}\frac{1}{a_j^p}=\sum_{j=1}^{n}\frac{1}{x_{ij}^p}, \tag{13}$$

consequently, the inequality can be stated as

$$\mathscr{G}\left(\frac{1}{\left(\sum_{j=1}^{n}\frac{1}{a_j^p}-\sum_{j=1}^{n-1}\sum_{i=1}^{m}\frac{w_i}{x_{ij}^p}\right)^{\frac{1}{p}}}\right)\leq\sum_{j=1}^{n}\frac{a_j^p\mathscr{G}(a_j)}{x_{in}^p}-\sum_{j=1}^{n-1}\sum_{i=1}^{m}\frac{w_ix_{ij}^p\mathscr{G}(x_{ij})}{x_{in}^p},$$

where $w_i,\ p\geq 0$ with $\sum_{i=1}^{m}w_i=1$.

Proof. Given that $\mathscr{G}$ is p-harmonic convex, applying the Jensen-type inequality that holds for such functions yields

$$\begin{aligned}\mathscr{G}\left(\frac{1}{\left(\sum_{j=1}^{n}\frac{1}{a_j^p}-\sum_{j=1}^{n-1}\sum_{i=1}^{n}\frac{w_i}{x_{ij}^p}\right)^{\frac{1}{p}}}\right)&=\mathscr{G}\left(\frac{1}{\left(\sum_{j=1}^{n}\sum_{i=1}^{m}w_i\frac{1}{a_j^p}-\sum_{j=1}^{n-1}\sum_{i=1}^{m}w_i\frac{1}{x_{ij}^p}\right)^{\frac{1}{p}}}\right)\\&=\mathscr{G}\left(\frac{1}{(\sum_{i=1}^{m}w_i)^{\frac{1}{p}}\left[\sum_{j=1}^{n}\frac{1}{a_j^p}-\sum_{j=1}^{n-1}\frac{1}{x_{ij}^p}\right]^{\frac{1}{p}}}\right)\\&\leq\sum_{i=1}^{m}w_i^{\frac{1}{p}}\mathscr{G}\left(\frac{1}{\left(\sum_{j=1}^{n}\frac{1}{a_j^p}-\sum_{j=1}^{n-1}\frac{1}{x_{ij}^p}\right)^{\frac{1}{p}}}\right)\qquad\text{using (4)}\\&=\sum_{i=1}^{m}\frac{w_i^{\frac{1}{p}}}{x_{im}^p}\left[x_{im}^p\mathscr{G}\left(\frac{1}{\left(\sum_{j=1}^{n}\frac{1}{a_j^p}-\sum_{j=1}^{n-1}\frac{1}{x_{ij}^p}\right)^{\frac{1}{p}}}\right)\right].\end{aligned}\tag{14}$$

By utilizing Theorem 3, which pertains to majorization, together with the result stated in (13), we derive the following

$$x_{in}^{p}\mathscr{G}\left(\frac{1}{\left(\sum_{j=1}^{n}\frac{1}{a_j^p}-\sum_{j=1}^{n-1}\frac{1}{x_{ij}^p}\right)^{\frac{1}{p}}}\right)=x_{in}^{p}\mathscr{G}(x_{in})$$

$$\leq\sum_{j=1}^{n}a_j^p\mathscr{G}(a_j)-\sum_{j=1}^{n-1}x_{ij}^p\mathscr{G}(x_{ij}). \tag{15}$$

Combining inequalities (14) and (15) leads to the desired conclusion. □

We now state an equivalent version of the above theorem as follows:

Theorem 9 Assuming the conditions of Theorem 8 are met, we obtain

$$\mathscr{G}\left(\frac{1}{\left(\sum_{j=1}^{n}\frac{1}{a_j^p}-\sum_{j=1}^{k-1}\sum_{i=1}^{m}\frac{w_i}{x_{ij}^p}-\sum_{j=k+1}^{n-1}\sum_{i=1}^{m}\frac{w_i}{x_{ij}^p}\right)^{\frac{1}{p}}}\right)$$

$$\leq\sum_{j=1}^{n}\frac{a_j^p\mathscr{G}(a_j)}{x_{in}^p}-\sum_{j=1}^{k-1}\sum_{i=1}^{m}\frac{w_ix_{ij}^p\mathscr{G}(x_{ij})}{x_{in}^p}-\sum_{j=k+1}^{n-1}\sum_{i=1}^{m}\frac{w_ix_{ij}^p\mathscr{G}(x_{ij})}{x_{in}^p}.$$

Proof. By employing the approach outlined in Theorem 8, the proof becomes straightforward. □

Theorem 10 Suppose that $\mathscr{G}:[a,\,b]\subseteq(0,\,\infty)\to\mathbb{R}$ is a p-harmonic convex function, and let $x_1,\,\ldots,\,x_m$ be elements in the interval $[a,\,b]$. Then

$$\frac{m-1}{m}\sum_{k=1}^{m}\mathscr{G}(x_k)+\mathscr{G}(x_1)+\mathscr{G}(x_n-\mathscr{G}\left(\frac{1}{\left(\frac{1}{x_1^p}+\frac{1}{x_m^p}-\sum_{k=1}^{m}\frac{\frac{1}{m}}{x_k^p}\right)^{\frac{1}{p}}}\right) \tag{16}$$

$$\geq\mathscr{G}\left(\frac{2x_1^px_2^p}{x_1^p+x_2^p}\right)^{\frac{1}{p}}+\cdots+\mathscr{G}\left(\frac{2x_1^px_{m-1}^p}{x_{m-1}^p+x_m^p}\right)^{\frac{1}{p}}+\mathscr{G}\left(\frac{2x_m^px_1^p}{x_m+x_1}\right)^{\frac{1}{p}}$$

Proof. Substituting $\zeta=\dfrac{1}{2}$ into inequality (2) yields

$$\mathscr{G}\left(\frac{2x_1^p x_2^p}{x_1^p+x_2^p}\right)^{\frac{1}{p}}+\cdots+\mathscr{G}\left(\frac{2x_1^p x_{m-1}^p}{x_{m-1}^p+x_m^p}\right)^{\frac{1}{p}}+\mathscr{G}\left(\frac{2x_m^p x_1^p}{x_m+x_1}\right)^{\frac{1}{p}}$$

$$\leq \frac{1}{2}\left[\mathscr{G}(x_1)+\mathscr{G}(x_2)\right]+\cdots+\frac{1}{2}\left[\mathscr{G}(x_{m-1})+\mathscr{G}(x_m)\right]+\frac{1}{2}\left[\mathscr{G}(x_m)+\mathscr{G}(x_1)\right]$$

$$=\mathscr{G}(x_1)+\cdots+\mathscr{G}(x_m)=\sum_{k=1}^{m}\mathscr{G}(x_k). \tag{17}$$

Observe that

$$\sum_{k=1}^{m}\mathscr{G}(x_k)=\sum_{m-1}^{m}\sum_{k=1}^{m}\mathscr{G}(x_k)+\sum_{k=1}^{m}\frac{1}{m}\mathscr{G}(x_k)$$

$$=\frac{m-1}{m}\sum_{k=1}^{m}\mathscr{G}(x_k)+\mathscr{G}(x_1)+\mathscr{G}(x_m)$$

$$-\left[\mathscr{G}(x_1)+\mathscr{G}(x_m)-\sum_{k=1}^{m}\frac{1}{m}\mathscr{G}(x_k)\right]. \tag{18}$$

Hence, the desired conclusion is obtained by combining inequality (10) with equalities (17) and (18). □

Theorem 11 Let $\mathscr{G}:[a,\, b]\subseteq(0,\, \infty)\to\mathbb{R}$ be a p-harmonic convex function, and suppose that $x_1,\, \ldots,\, x_m$ are elements within the interval $[a,\, b]$. Then

$$\sum_{k=1}^{m}\mathscr{G}(y_k)\leq\frac{m-1}{m}\sum_{k=1}^{m}\mathscr{G}(x_k)+\mathscr{G}(x_1)+\mathscr{G}(x_m)$$

$$-\mathscr{G}\left(\frac{1}{\left(\frac{1}{x_1^p}+\frac{1}{x_m^p}-\sum_{k=1}^{m}\frac{\frac{1}{m}}{x_k^p}\right)^{\frac{1}{p}}}\right) \tag{19}$$

where $y_k=\left(\frac{m}{(m-1)\alpha^{-1}+(x_k^{-1})^p}\right)^{\frac{1}{p}}$ and $\alpha=\left(\frac{m}{(x_1^{-1})^p+\cdots+(x_m^{-1})^p}\right)^{\frac{1}{p}}$.

Proof. By applying the Jensen-type inequality associated with p-harmonic convex functions, it follows that

$$\sum_{k=1}^{m} \mathscr{G}(y_k) = \mathscr{G}(y_1) + \cdots + \mathscr{G}(y_m)$$

$$= \mathscr{G}\left(\frac{m}{(m-1)\alpha^{-1} + (y_1^{-1})^p}\right)^{\frac{1}{p}} + \cdots + \mathscr{G}\left(\frac{m}{(m-1)\alpha^{-1} + (x_m^{-1})^p}\right)^{\frac{1}{p}}$$

$$\leq \left[\frac{1}{m}\mathscr{G}(\alpha) + \frac{m-1}{m}\mathscr{G}(x_1)\right] + \cdots + \left[\frac{1}{m}\mathscr{G}(\alpha) + \frac{m-1}{m}\mathscr{G}(x_m)\right]$$

$$= \mathscr{G}(\alpha) + \frac{m-1}{m}\sum_{k=1}^{m}\mathscr{G}(x_k)$$

$$= \mathscr{G}\left(\frac{m}{(x_1^{-1})^p + \cdots + (x_m^{-1})^p}\right)^{\frac{1}{p}} + \frac{m-1}{m}\sum_{k=1}^{m}\mathscr{G}(x_k)$$

$$\leq \frac{1}{m}\sum_{k=1}^{m}\mathscr{G}(x_k) + \frac{m-1}{m}\sum_{k=1}^{m}\mathscr{G}(x_k).$$

Hence, the desired conclusion is obtained by utilizing inequalities (10) and (18). □

Theorem 12 Consider $\mathscr{G}$ as a p-harmonic convex function defined over the interval $[n, N]$. Then

$$\mathscr{G}\left(\frac{1}{\left[\frac{1}{n^p} + \frac{1}{N^p} - \left(\frac{2x^py^p}{x^p+y^p}\right)^{\frac{1}{p}}\right]^{\frac{1}{p}}}\right) \leq \mathscr{G}(n) + \mathscr{G}(N) - \int_0^1 \mathscr{G}\left(\frac{x^py^p}{\zeta y^p + (1-\zeta)x^p}\right)^{\frac{1}{p}} d\zeta$$

$$\leq \mathscr{G}(n) + \mathscr{G}(N) - \mathscr{G}\left(\frac{2x^py^p}{x^p+y^p}\right)^{\frac{1}{p}}. \tag{20}$$

Proof. Inequality (10) implies that

$$\mathscr{G}\left(\frac{1}{\left[\frac{1}{n^p} + \frac{1}{N^p} - \left(\frac{2a^pb^p}{a^p+b^p}\right)^{\frac{1}{p}}\right]^{\frac{1}{p}}}\right) \leq \mathscr{G}(n) + \mathscr{G}(N) - \frac{\mathscr{G}(a) + \mathscr{G}(b)}{2}, \tag{21}$$

for all $a,\ b \in [n,\ N]$.

Let $\zeta \in [0,\ 1]$ and $x,\ y \in [n,\ N]$. By substituting a and b in inequality (21) with $\left(\frac{x^p y^p}{\zeta x^p + (1-\zeta) y^p}\right)^{\frac{1}{p}}$ and $\left(\frac{x^p y^p}{\zeta y^p + (1-\zeta) x^p}\right)^{\frac{1}{p}}$ respectively, we have arrived at the following result

$$\mathscr{G}\left(\frac{1}{\left[\frac{1}{n^p} + \frac{1}{N^p} - \left(\frac{2x^p y^p}{x^p + y^p}\right)^{\frac{1}{p}}\right]^{\frac{1}{p}}}\right) \leq \mathscr{G}(n) + \mathscr{G}(N)$$

$$-\frac{\mathscr{G}\left(\frac{x^p y^p}{\zeta x^p + (1-\zeta) y^p}\right)^{\frac{1}{p}} + \mathscr{G}\left(\frac{x^p y^p}{\zeta y^p + (1-\zeta) x^p}\right)^{\frac{1}{p}}}{2}. \tag{22}$$

By integrating inequality (22) over the interval $[0,\ 1]$ with respect to ζ, we obtain

$$\mathscr{G}\left(\frac{1}{\left[\frac{1}{n^p} + \frac{1}{N^p} - \left(\frac{2x^p y^p}{x^p + y^p}\right)^{\frac{1}{p}}\right]^{\frac{1}{p}}}\right) \leq \mathscr{G}(n) + \mathscr{G}(N) - \frac{1}{2}$$

$$\times \int_0^1 \left[\mathscr{G}\left(\frac{x^p y^p}{\zeta x^p + (1-\zeta) y^p}\right)^{\frac{1}{p}}\right.$$

$$\left. + \mathscr{G}\left(\frac{x^p y^p}{\zeta y^p + (1-\zeta) x^p}\right)^{\frac{1}{p}}\right] d\zeta. \tag{23}$$

Owing to

$$\int_0^1 \mathscr{G}\left(\frac{x^p y^p}{\zeta x^p + (1-\zeta) y^p}\right)^{\frac{1}{p}} d\zeta = \int_0^1 \mathscr{G}\left(\frac{x^p y^p}{\zeta y^p + (1-\zeta) x^p}\right)^{\frac{1}{p}} d\zeta = \frac{p x^p y^p}{y^p - x^p} \int_x^y \frac{\mathscr{G}(\zeta)}{\zeta^{p+1}} d\zeta. \tag{24}$$

The inequality (23) leads to the first inequality in (20). Meanwhile, the second inequality in (20) can be directly derived from the Hermite-Hadamard type inequality applicable to p-harmonic convex function. □

5. Applications

• In [20] establishes a connection among the harmonic, logarithmic, and arithmetic means as shown below

$$\frac{2ab}{a+b} \leq \frac{ab}{b-a}(\ln b - \ln a) \leq \frac{a+b}{2} \quad \text{(Harmonic, Logarithmic, Arithmetic mean inequality).}$$

Moreover, the inequality presented in Theorem 4 yields the following results when applied to the function $\mathscr{G}(x) = x^p,\ p > 0,\ \forall x \in (0,\ \infty)$.

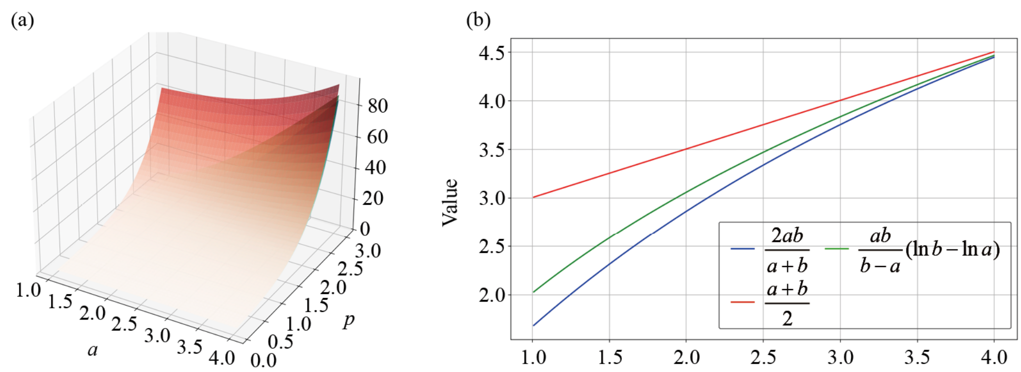
$$\frac{2a^p b^p}{a^p + b^p} \leq \frac{pa^p b^p}{b^p - a^p}(\ln b - \ln a) \leq \frac{a^p + b^p}{2}. \tag{25}$$

Inequality (25) is relation between p-harmonic maen, p-logarithmic and p-arithmetic means. The Table 2, shows the data about inequality (25).

Table 2. Verified cases where the inequality holds for $b = 5$, $a \in [1,\ 4]$, and $p = 1,\ 2$

$(p,\ a)$	Left Hand Side (LHS) of inequality (25)	Middle	Right Hand Side (RHS) of inequality (25)	Inequality holds
(1, 1)	1.667	2.012	3.000	Yes
(1, 2)	2.299	2.500	3.500	Yes
(1, 3)	2.371	3.000	4.000	Yes
(1, 4)	2.293	3.333	4.500	Yes
(2, 1)	1.923	2.563	13.000	Yes
(2, 2)	6.154	6.869	14.500	Yes
(2, 3)	9.000	9.019	17.000	Yes
(2, 4)	10.794	11.111	20.500	Yes

The graphical representation of above inequality is shown as Figure 4.



3D plot illustrating the inequality visualization

2D plot illustrating the inequality visualization

Figure 4. Inequality visualization for $p = 1$ and $b = 5$, with $a \in [1,\ 4]$

• For $\mathscr{G}(x) = \frac{1}{x^p}$, we have obtained a direct consequences of the inequality stated in Theorem 4, which pertains to a p-harmonic convex function. Hence, $\forall\ x \in (0,\ \infty)$ and $p \neq 0$, the following result holds

$$\frac{a^p+b^p}{2a^pb^p} \leq \frac{1}{b^p-a^p}\left(\frac{b^{2p}-a^{2p}}{2a^pb^p}\right) \leq \frac{b^p+a^p}{2a^pb^p}. \tag{26}$$

The graphical representation of above inequality is shown as Figure 5.

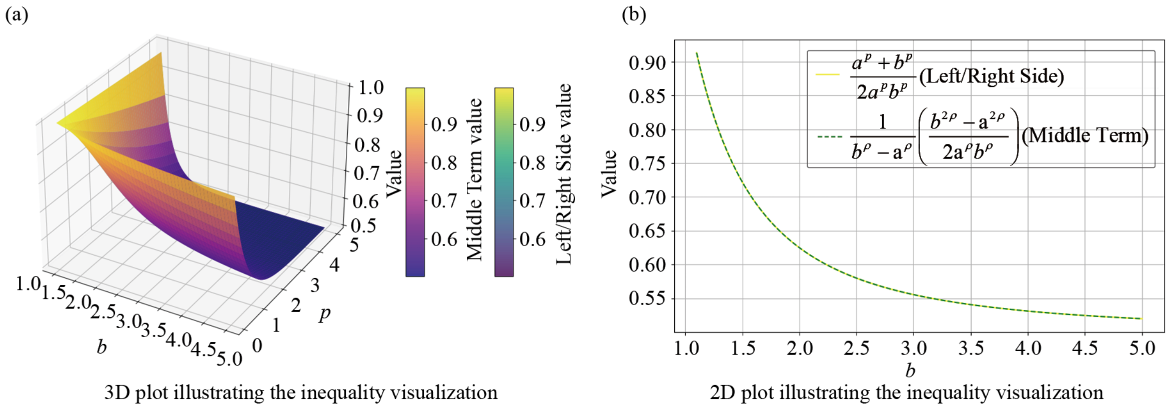


Figure 5. Inequality visualization for $p = 1$, $b = 2$, and $a = 1$

• For $a, b \in (0, \infty)$ with $a \neq b$. By choosing $\mathscr{G}(x) = e^x$ in Theorem 4, we have observed that this function is not only harmonically convex but also p-harmonically convex. Consequently, for $p \geq 1$, we have

$$e^{\frac{2^{\frac{1}{p}}ab}{[a^p+b^p]^{\frac{1}{p}}}} \leq \frac{p(ab)^p}{b^p-a^p}\int_a^b \frac{e^x}{x^{p+1}}dx \leq \frac{e^a+e^b}{2}. \tag{27}$$

Based on the preceding discussion, it follows that a reliable estimate for the integral $\int_a^b \frac{e^x}{x^m}dx$ can be obtained for every $m \in \mathbb{N}$. The Table 3, shows the data about inequality (27).

Table 3. Numerical verification of above inequality for $a = 1$, $b = 2$, and $p = 1, 2, 3, 4$

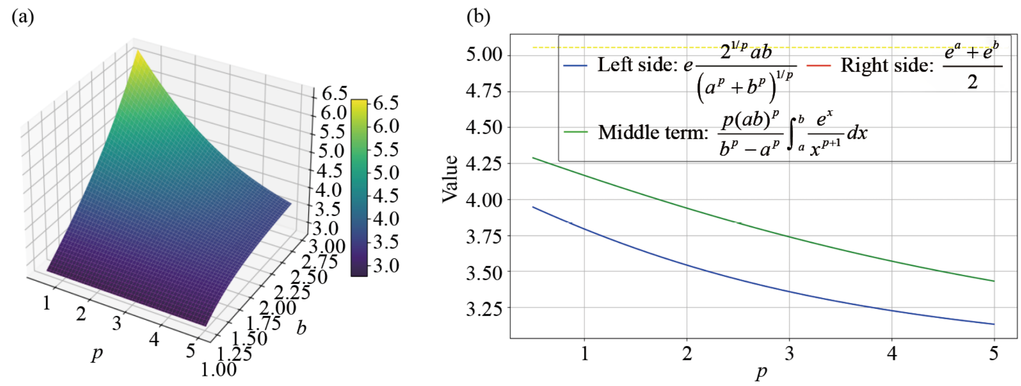
p	LHS	Middle	RHS
1	$e^{4/3} \approx 3.7936678947$	$2\int_1^2 \frac{e^x}{x^2}\,dx \approx 4.1657406373$	$\frac{e+e^2}{2} \approx 5.0536689637$
2	$e^{1.2649110641} \approx 3.5427776419$	$\frac{8}{3}\int_1^2 \frac{e^x}{x^3}\,dx \approx 3.9385174965$	$\frac{e+e^2}{2} \approx 5.0536689637$
3	$e^{1.2114137286} \approx 3.3582289223$	$\frac{24}{7}\int_1^2 \frac{e^x}{x^4}\,dx \approx 3.7389644312$	$\frac{e+e^2}{2} \approx 5.0536689637$
4	$e^{1.1713192055} \approx 3.2262459137$	$\frac{128}{30}\int_1^2 \frac{e^x}{x^5}\,dx \approx 3.5701302557$	$\frac{e+e^2}{2} \approx 5.0536689637$

The Table 4, shows the data about inequality (27).

Table 4. Numerical verification of above inequality for $a = 0.5$, $b = 0.8$

p	$e^{\frac{2^{1/p}b}{(a^p+b^p)^{1/p}}}$	Middle value	$\frac{e^a+e^b}{2}$
1	1.850368142769	1.878257365512	1.937131099596
2	1.821436273624	1.856890423042	1.937131099596
3	1.796284148316	1.836666996212	1.937131099596
4	1.775112873934	1.817896691019	1.937131099596

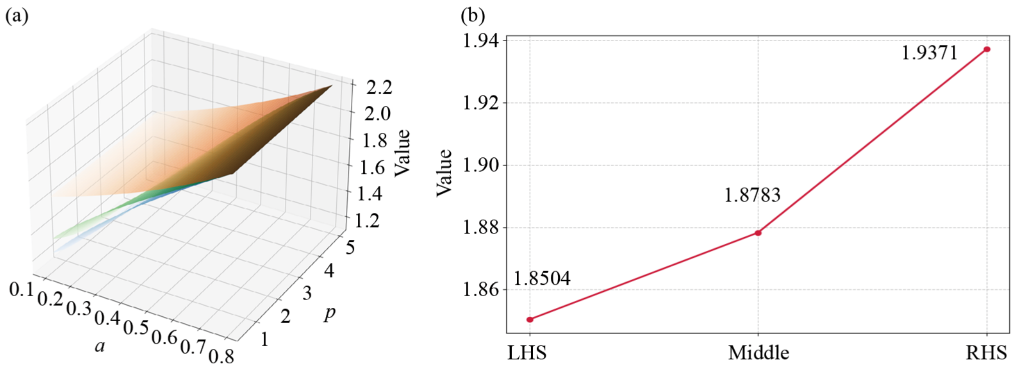
A graphical comparison of this inequality is presented as Figure 6 and 7.



3D graph illustrating the inequality visualization

2D graph illustrating the inequality visualization

Figure 6. Inequality visualization for $p = 1$, $b = 2$, and $a = 1$



3D plot illustrating the inequality visualization

2D plot illustrating the inequality visualization

Figure 7. Inequality visualization for $p = 1$, $a = 0.5$, and $b = 0.8$

• Since the function $\mathscr{G}(x) = x^{p+1}e^{x^{p+1}}$ is non-decreasing convex function defined over $(0,\ 1)$, it qualifies as a p-harmonic convex function. Consequently, applying the inequality from Theorem 4 for $a,\ b \in (0,\ \infty)$ and $a \neq b$, we obtain

$$\left(\frac{2^{\frac{1}{p}}ab}{[a^p+b^p]^{\frac{1}{p}}}\right)^{p+1} e^{\left(\frac{2^{\frac{1}{p}}ab}{[a^p+b^p]^{\frac{1}{p}}}\right)^{p+1}} \leq \frac{p(ab)^p}{b^p-a^p}\int_a^b e^{x^{p+1}}dx \leq \frac{a^{p+1}e^{a^{p+1}}b^{p+1}e^{b^{p+1}}}{2}, \tag{28}$$

for all $a, b \in (0, \infty), p > 0$. The graphical comparison of the inequality as shown as Figures 8 and 9

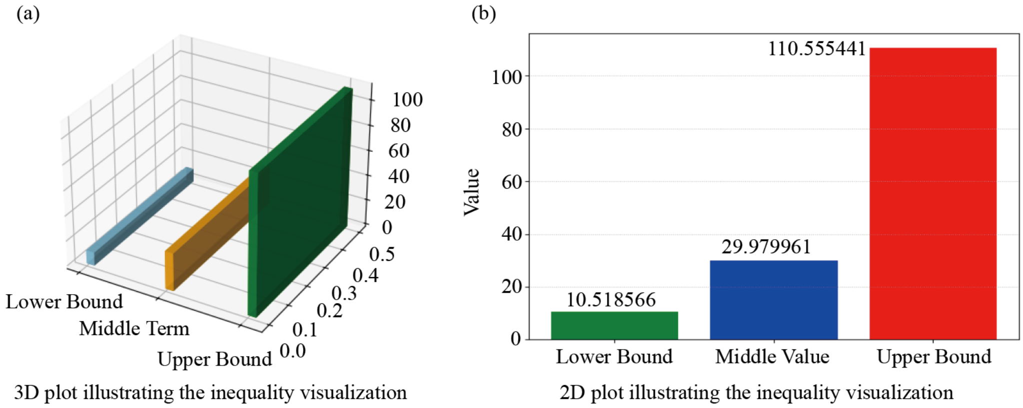


Figure 8. Inequality visualization for $p = 1$, $b = 2$, and $a = 1$

The Table 5, shows the data about inequality (28).

Table 5. Numerical verification of the inequality for $p = 1$, $b = 0.5$, and $a \in [0.1, 0.4]$

a	LHS	Middle	RHS	Inequality holds
0.10	0.021890	0.064969	0.162250	Yes
0.15	0.065677	0.084074	0.168982	Yes
0.20	0.082297	0.124854	0.180286	Yes
0.25	0.137287	0.148746	0.193967	Yes
0.30	0.162199	0.187160	0.217432	Yes
0.35	0.200000	0.223443	0.231688	Yes
0.40	0.236532	0.243212	0.263741	Yes

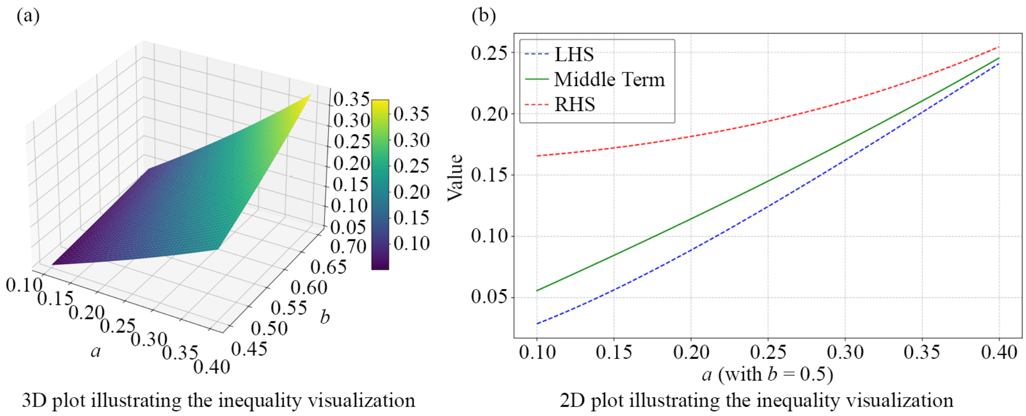


Figure 9. Inequality visualization for $p = 1$, $b = 0.5$, and $a \in [0.1, 0.4]$

• Since $\mathscr{G}(x) = \sin(-x)$ is convex and non-decreasing function on the interval $\left(0,\ \frac{\pi}{2}\right)$, it qualifies as both harmonic and p-harmonic convex function as well $\forall x \in \left(0,\ \frac{\pi}{2}\right)$. Therefore, applying the inequality from Theorem 4 for $a,\ b \in (0,\ \infty)$, we derive the following result

$$\sin\left(\frac{2^{\frac{1}{p}}ab}{[a^p+b^p]^{\frac{1}{p}}}\right) \le \frac{p(ab)^p}{b^p-a^p}\int_a^b \frac{\sin x}{x^{p+1}}dx \le \frac{\sin a + \sin b}{2}. \tag{29}$$

In a similar manner, we can approximate the integrals $\int_a^b \frac{\sin x}{x^m}$ and $\int_a^b \frac{\cos x}{a^m}$ for all $m \in \mathbb{N}$ where $a,\ b \in (0,\ \infty)$. The Table 6, shows the data about inequality (29).

Table 6. Numerical check of $\sin\left(\frac{2^{\frac{1}{p}}ab}{(a^p+b^p)^{\frac{1}{p}}}\right) \le \frac{p(ab)^p}{b^p-a^p}\int_a^b \frac{\sin x}{x^{p+1}}dx \le \frac{\sin a+\sin b}{2}$ for $p = 2$, small $a,\ b$

a	b	LHS $= \sin\left(\frac{2^{\frac{1}{p}}ab}{(a^p+b^p)^{\frac{1}{p}}}\right)$	$\int_a^b \frac{\sin x}{x^{p+1}}dx$	Middle	RHS	Holds
0.01	0.02	0.012649	49.998333	0.013333	0.014999	Yes
0.01	0.05	0.013867	79.993334	0.016665	0.029990	Yes
0.01	0.10	0.014071	89.985003	0.018179	0.054917	Yes
0.01	0.20	0.014124	94.968356	0.019041	0.104335	Yes
0.01	0.30	0.014134	96.618408	0.019345	0.152760	Yes
0.01	0.40	0.014137	97.435177	0.019499	0.199709	Yes
0.01	0.50	0.014139	97.918679	0.019592	0.244713	Yes
0.05	0.10	0.063203	9.991669	0.066611	0.074906	Yes
0.05	0.20	0.068546	14.975022	0.079867	0.124324	Yes
0.05	0.30	0.069692	16.625075	0.085500	0.172750	Yes
0.05	0.40	0.070107	17.441844	0.088593	0.219699	Yes
0.05	0.50	0.070302	17.925346	0.090532	0.264702	Yes
0.10	0.20	0.126154	4.983353	0.132889	0.149251	Yes
0.10	0.30	0.133762	6.633405	0.149252	0.197677	Yes
0.10	0.40	0.136769	7.450175	0.158937	0.244626	Yes
0.10	0.50	0.138231	7.933677	0.165285	0.289629	Yes
0.20	0.30	0.233173	1.650053	0.237608	0.247095	Yes
0.20	0.40	0.250292	2.466822	0.263128	0.294044	Yes
0.20	0.50	0.259605	2.950324	0.280983	0.339047	Yes
0.30	0.40	0.332932	0.816769	0.336042	0.342469	Yes
0.30	0.50	0.355831	1.300271	0.365701	0.387473	Yes
0.40	0.50	0.427501	0.483502	0.429780	0.434422	Yes

A graphical comparison illustrating these inequalities is provided as Figure 10.

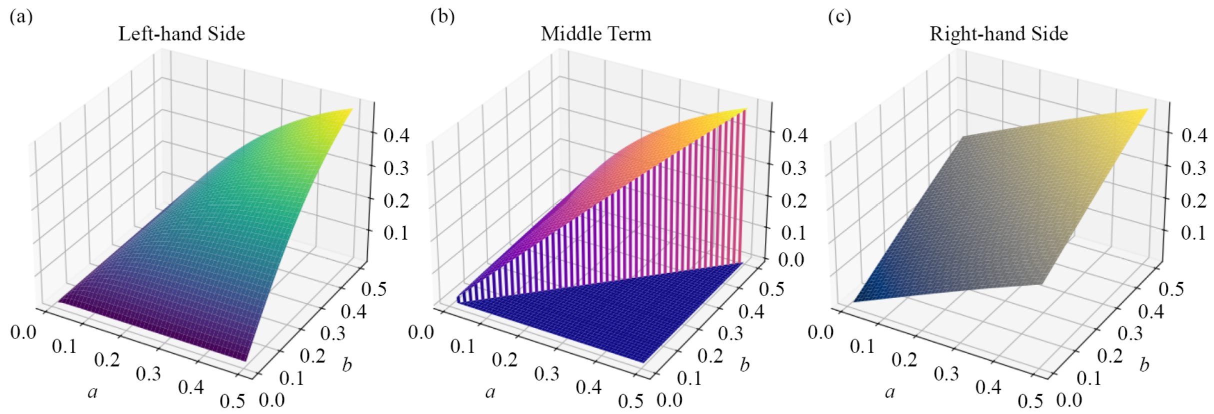


Figure 10. 3D visualization on small values of a and b in the interval $(0.01,\ 0.5)$ with $p = 2$

6. Conclusion

We have developed a new variant of the discrete Jensen-type inequality tailored for p-harmonic and $(p,\ h)$-harmonic convex functions. Numerous examples of p-harmonic convex functions are provided to illustrate these concepts. Moreover, we explored applications of the Hermite-Hadamard inequality within the context of p-harmonic convex functions and generalized $\left(\frac{2a^p b^p}{a^p + b^p}\right)^2 \leq \left(\frac{a^p + b^p}{2}\right)^2 \leq \frac{1}{3}(a^{2p} + a^p b^p + b^{2p}) \leq \frac{a^{2p} + b^{2p}}{2}$ for all $a,\ b \in (0,\ \infty)$, which implies that the relationship between p-harmonic mean, p-arithmetic mean, symmetric p-quadratic mean and p-quadratic mean. Additionally, to get some analytical insights some graphs and tables are constructed to visualize the obtained results. Atlast, the present work represent a significant generalization, improvement and advancement over many previously established results.

Author contributions

"Conceptualization, F.A., M.I.A., A.A.L., M.A.Y., I.A.B., and P.O.M.; Funding acquisition, A.A.L.; Investigation, F.A., M.I.A., M.A.Y., I.A.B., and P.O.M.; Methodology, F.A., M.I.A., M.A.Y., I.A.B., and P.O.M.; Software, M.A.Y.; Supervision, M.I.A and I.A.B; Writing-original draft, F.A., M.A.Y., and P.O.M.; Writing-review & editing, A.A.L., M.I.A., and I.A.B. All authors have read and approved the final manuscript".

Funding

The publication of this paper is supported by University of Oradea, Romania.

Data availability statement

Data is contained within the article or supplementary material.

Conflict of interest

The authors declare no competing financial interest.

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