Contemporary Mathematics
https://ojs.wiserpub.com/index.php/CM/

Research Article

Existence and Multi-Stability of a Generalized ABC Fractional-Order Neural Network Model

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Received: 18 August 2025; **Revised:** 15 September 2025; **Accepted:** 22 September 2025

Abstract: This study presents a fractional-order neural network model formulated using the Atangana-Baleanu-Caputo Fractional Derivative (ABC-FD) defined with respect to a generalized kernel function $\vartheta(t)$. The primary objective is to establish rigorous results on the existence, uniqueness, and stability of solutions under minimal regularity assumptions. By employing Banach's and Krasnoselskii's fixed point theorems, we prove existence and uniqueness. The stability analysis compares three regimes: Mittag-Leffler, asymptotic, and finite-time, showing that they form a hierarchy of convergence strength: asymptotic stability ensures gradual decay, Mittag-Leffler stability provides algebraic convergence, and finite-time stability guarantees exact quenching within a bounded interval. Numerical simulations of two- and three-neuron systems confirm these theoretical distinctions, illustrating the role of both the fractional order and $\vartheta(t)$ in shaping the rate and type of convergence.

Keywords: Atangana-Baleanu-Caputo Fractional Derivative (ABC-FD), neural network system, existence and stability analysis, numerical simulation

MSC: 26A33, 34A12, 34D20

1. Introduction

Fractional-order systems have found successful applications in various domains, including viscoelasticity, anomalous diffusion, and control systems, due to their ability to capture long-term dependencies and complex dynamic behaviors. In neural networks, fractional differential equations provide a natural framework to model processes such as synaptic plasticity, adaptive coupling, and memory effects [1–5].

The development of fractional calculus has significantly advanced the modeling and analysis of complex dynamical systems, particularly in neuroscience, where memory-dependent processes and non-local interactions are pervasive [6]. In this context, Mechee et al. [7] introduced novel α-fractional operators for general functions, providing new perspectives for defining and analyzing fractional integrals and derivatives. With its non-singular Mittag-Leffler kernel, the Atangana-Baleanu-Caputo Fractional Derivative (ABC-FD) has subsequently emerged as a powerful tool for capturing hereditary

DOI: https://doi.org/10.37256/cm.7120268596

effects in neural networks. Unlike the classical Caputo or Riemann-Liouville FDs (Cap-FDs or RL-FDs), the Atangana-Baleanu-Caputo (ABC) operator avoids kernel singularities while preserving the ability to model power-law memory decay, making it particularly suitable for neurodynamic systems [8, 9].

Fractional-order neural networks have attracted increasing attention for their ability to capture memory, hereditary properties, and delayed signal transmission inherent in biological and artificial neural systems. Various studies have established asymptotic, Mittag-Leffler, and finite-time stability results for delayed and inertial fractional-order networks, contributing to global convergence and robustness analyses [10–18].

Recent developments in fractional-order artificial neural networks have further explored the influence of fractional activation functions and fractional orders on synchronization and stability, with several reviews and analyses highlighting their advantages in system stabilization and parameter optimization [19–21]. Variable-order and tempered fractional neural networks have also been investigated to address complex dynamic behaviors and enhance modeling flexibility, establishing new existence and stability results using fixed-point and Mittag-Leffler stability approaches [22–24].

Since time delays are intrinsic to neural communication, many works have focused on their impact on the stability of fractional systems. Notable contributions include stability criteria for delayed and fuzzy neural networks using Lyapunov, Razumikhin, and fractional techniques, improving both theoretical and computational tractability [25–29].

Building upon these advances, the present study introduces a fractional neural network model governed by a time-scaling function $\vartheta(t)$ and Mittag-Leffler-type kernel, providing a unified framework that generalizes existing non-singular fractional models. The paper establishes the existence and uniqueness of solutions and investigates three types of stability: asymptotic, Mittag-Leffler, and finite-time, under explicit conditions, thus deepening the understanding of how $\vartheta(t)$ modulates the memory and decay characteristics of the system. Specifically, we consider the fractional model as follows:

$$ {}^{\mathscr{ABC}}\mathscr{D}_{0,\vartheta}^{\beta_k}\rho_i(t) = -\gamma_i\rho_i(t) + \sum_{j=1}^{N} w_{ij}\mathscr{K}(\rho_j(t)) + \eta_i(t) f(\rho_i(t), t), \quad \rho_i(0) = \rho_i^0, \tag{1} $$

where ${}^{\mathscr{ABC}}\mathscr{D}_{0,\vartheta}^{\beta_k}$ is the ABC-FD of order $\beta_k \in (0, 1)$ with respect to increasing function $\vartheta(t)$, $\rho_i(t)$ is the state of the i-th neuron $i = 1, 2, \ldots, N$, γ_i is the damping coefficient of the i-th neuron, w_{ij} represents the synaptic weight from the j-th to the i-th neuron, $\mathscr{K}(\cdot)$ is a kernel function modeling coupling between neurons, $\eta_i(t)$ is a modulation function dependent on time, and $f(\rho_i(t), t)$ is an activation function for the i-th neuron.

The central question of this study is how the existence, uniqueness, and stability of a fractional-order neural network are affected by employing the ABC derivative with respect to a time-scaling function $\vartheta(t)$. We investigate whether this formulation enhances dynamical flexibility and ensures convergence across asymptotic, Mittag-Leffler, and finite-time stability regimes.

The main contributions of this work can be summarized as follows. First, we establish existence and uniqueness results for the fractional neural system (1) using Banach's and Krasnoselskii's fixed-point theorems under minimal assumptions. Second, we derive explicit damping conditions characterizing three distinct stability regimes: asymptotic, Mittag-Leffler, and finite-time, and clarify their hierarchical relationship in terms of convergence strength. Third, we provide numerical validation through Python-based simulations for two- and three-neuron networks, which confirm the theoretical findings by illustrating algebraic decay under Mittag-Leffler stability and exact quenching under finite-time stability.

The remainder of this paper is organized as follows: Section 2 provides a review of ABC fractional calculus, key lemmas, and the study's methodology. Section 3 introduces the equivalent integral of the model. Section 4 establishes the existence and uniqueness results, and further provides detailed analyses of asymptotic, Mittag-Leffler, and finite-time stability. Section 5 presents numerical examples and simulations. Section 6 discusses the conclusion and future work.

2. Preliminaries

This section presents the background and methodology pertinent to the study.

2.1 *Background*

Some fundamental introductions to fractional calculus and the proposed model are provided in this section.

As usual, $\mathscr{C} := C(\mathbb{J}, \mathbb{R}^N)$ is the Banach space of all continuous functions from $\mathbb{J}$ to $\mathbb{R}^N$, with the norm $\|\cdot\|_\infty$ defined by $\|\omega_i\| = \max_{1\le i\le N}\sup_{\nu\in\mathbb{J}}|\omega(\nu)|$.

Definition 1 [6] Let $\mu \in (0, 1]$ and $\omega \in H^1(0, T)$. Then, the ABC-FD for a function ω in the sense of Caputo is expressed as

$$^{\mathscr{ABC}}\left(\mathscr{D}_0^\mu \omega\right)(t) = \frac{\Delta(\mu)}{1-\mu}\int_0^t E_\mu\left[-\frac{\mu}{1-\mu}(t-\nu)^\mu \omega'(\nu)\right]d\nu, \quad t\in\mathbb{J},$$

where $\Delta(\mu) = 1-\mu+\frac{\mu}{\Gamma(\mu)}$ is a normalization function satisfying $\Delta(0) = \Delta(1) = 1$, and $E_\mu(\cdot)$ is the Mittag-Leffler function, defined as

$$E_\mu(\omega) = \sum_{i=0}^{\infty}\frac{\omega^i}{\Gamma(\mu i+1)}, \qquad \mathrm{Re}(\mu) > 0,\ r\in\mathbb{C},$$

where $\Gamma(\mu) = \int_0^\infty e^{-x}x^{\mu-1}dx, \quad \mu > 0.$

Definition 2 [6] For $\mu \in (0, 1]$ and $\omega \in H^1(0, T)$, the Atangana-Baleanu (AB) fractional integral of order μ for a function ω is given by

$$^{\mathscr{AB}}\mathscr{I}_0^\mu \omega(t) = \frac{1-\mu}{\Delta(\mu)}\omega(t) + \frac{\mu}{\Delta(\mu)\Gamma(\mu)}\int_0^t (t-\nu)^{\mu-1}\omega(s)\,ds, \quad t\in\mathbb{J}.$$

Definition 3 [30] The ϑ-RL fractional integral of an integrable function $\omega(t)$ with respect to another function $\vartheta(t)$ is defined by

$$^{\mathscr{RL}}\mathscr{I}_{0,\vartheta}^\mu \omega(t) = \frac{1}{\Gamma(\mu)}\int_0^t (\vartheta(t)-\vartheta(s))^{\mu-1}\vartheta'(s)\omega(s)\,ds. \tag{2}$$

Definition 4 [31, 32] The ϑ-ABC-FD for a function $\omega(t)$ with respect to another function $\vartheta(t)$ is defined as

$$^{\mathscr{ABC}}\mathscr{D}_{0,\vartheta}^\mu \omega(t) = \frac{\Delta(\mu)}{1-\mu}\int_0^t \vartheta'(s)E_\mu\left(-\frac{\mu}{1-\mu}(\vartheta(t)-\vartheta(s))^\mu\right)\omega'_\vartheta(\nu)\,d\nu,$$

where $\vartheta'(t) = \frac{d}{dt}\vartheta(t)$ and $\omega'_\vartheta(t) = \frac{\omega'(t)}{\vartheta'(t)}$. Moreover, the corresponding AB fractional integral is

$$^{\mathscr{AB}}\mathscr{I}_{0;\vartheta}^\mu \omega(t) = \frac{1-\mu}{\Delta(\mu)}\omega(t) + \frac{\mu}{\Delta(\mu)}\,{}^{RL}\mathscr{I}_{0,\vartheta}^\mu \omega(t), \quad t\in\mathbb{J}.$$

Lemma 1 [30] Let μ, $\rho > 0$ and ω: $\mathbb{J} \to \mathbb{R}$. Then
1) ${}^{\mathscr{RL}}\mathscr{I}^{\mu}_{0;\,\vartheta}\,[\vartheta(t)-\vartheta(0)]^{\rho-1} = \frac{\Gamma(\rho)}{\Gamma(\mu+\rho)}\,[\vartheta(u)-\vartheta(0)]^{\mu+\rho-1}$;
2) ${}^{\mathscr{RL}}\mathscr{I}^{\mu}_{0;\,\vartheta}\;{}^{\mathscr{RL}}\mathscr{I}^{\rho}_{0;\,\vartheta}\;\omega(t) = {}^{\mathscr{RL}}\mathscr{I}^{\mu+\rho;\,\varphi}_{0;\,\vartheta}\,\omega(t)$;
3) $\left(\frac{1}{\varphi(t)}\frac{d}{dt}\right)\,{}^{\mathscr{RL}}\mathscr{I}^{\mu}_{0;\,\vartheta}\;\omega(t) = \omega(t)$.
Lemma 2 [33] For $\mu \in (0,\, 1]$, the following relations hold:
i) $\left({}^{\mathscr{AB}}\mathscr{I}^{\mu}_{0;\,\vartheta}\;{}^{\mathscr{ABC}}\mathscr{D}^{\mu}_{0;\,\vartheta}\,\omega\right)(t) = \omega(t)-\omega(0)$;
ii) $\left({}^{\mathscr{ABC}}\mathscr{D}^{\mu}_{0;\,\vartheta}\;{}^{\mathscr{AB}}\mathscr{I}^{\mu}_{0;\,\vartheta}\,\omega\right)(t) = \omega(t)-\omega(0)\,E_{\mu}\left(\frac{-\mu}{1-\mu}(\vartheta(u)-\vartheta(0))^{\mu}\right)$.

Lemma 3 [34] Let ψ, η: $[\alpha,\,\beta] \to \mathbb{R}_+$ be Lebesgue integrable functions and κ: $[\alpha,\,\beta] \to \mathbb{R}_+$ a continuous, non-decreasing function. Let $\varphi \in C^1[\alpha,\,\beta]$ be a strictly increasing time-scaling function with $\varphi'(t) > 0$ for all $t \in [\alpha,\,\beta]$. Suppose the inequality

$$\psi(t) \le \eta(t)+\kappa(t)\int_{\alpha}^{t}\varphi'(s)\,(\varphi(t)-\varphi(s))^{\nu-1}\,\psi(s)\,ds, \qquad \nu \in (0,\,1)$$

holds for all $t \in [\alpha,\,\beta]$. Then, the following bound applies:

$$\psi(t) \le \eta(t)+\int_{\alpha}^{t}\mathbb{G}_{\nu}\left(\kappa(t)\Gamma(\nu),\, \varphi(t)-\varphi(s)\right)\varphi'(s)\eta(s)\,ds,$$

where

$$\mathbb{G}_{\nu}(z,\,\tau) = \sum_{k=1}^{\infty}\frac{z^k\tau^{\nu k-1}}{\Gamma(\nu k)}$$

is the generalized fractional resolvent kernel.

Corollary 1 [34] Under the hypothesis of Lemma 3, let η be a nondecreasing function on $[\alpha,\,\beta]$. Then, we have

$$\psi(t) \le \eta(t)E_{\nu}\left(\kappa(t)\Gamma(\nu)\right)[\varphi(t)-\varphi(\alpha)]^{\nu}, \qquad \forall t \in [\alpha,\,\beta].$$

Finally, since fundamental results like Banach's and Krasnoselskii's fixed-point theorems and the Lipschitz condition are standard in most texts, we refer to them in the reference [11].

2.2 *Methodology*

In this subsection, we present the methodological framework adopted to analyze the proposed fractional-order neural network model formulated using the Atangana-Baleanu-Caputo Fractional Derivative (ABC-FD) with respect to a time-scaling function $\vartheta(t)$. The study begins with the formulation of a generalized neural network system that incorporates memory effects through a nonsingular Mittag-Leffler kernel. The differential model is then transformed into an equivalent integral equation by employing the properties of the ABC fractional integral operator, allowing the use of fixed-point theory. Existence and uniqueness of solutions are rigorously established under continuity, Lipschitz, and boundness conditions on the nonlinear activation function $f(x,\,t)$ and kernel $\mathscr{K}(x)$, using Banach's and Krasnoselskii's fixed-point theorems. Subsequently, we investigate three types of stability: asymptotic, Mittag-Leffler, and finite-time, by deriving appropriate fractional inequalities and damping conditions that characterize the system's convergence behavior. Finally, a numerical scheme based on a fractional Adams-Bashforth-Moulton approach is implemented to approximate the solutions.

Python simulations for two- and three-neuron systems are conducted to illustrate the theoretical predictions and compare the distinct stability behaviors.

3. Equivalent integral of model (1)

In this section, the generalized ABC fractional model and its equivalent integral form are introduced. The dynamics of the i-th neural network are governed by:

$$ {}^{\mathscr{ABC}}\mathscr{D}_{0,\,\vartheta}^{\beta_k}\rho_i(t) = -\gamma_i\rho_i(t) + \sum_{j=1}^{N} w_{ij}\mathscr{K}(\rho_j(t)) + \eta_i(t)f(\rho_i(t),\, t), \quad t \in [0,\, T], $$

with the initial conditions

$$ \rho_i(0) = \rho_i^0, $$

where $i = 1, 2, \ldots, N$, γ_i, ρ_i, w_{ij}, $\mathscr{K}(\cdot)$, η_i, and f are defined as above.

Based on the preceding results, we now derive the equivalent integral form of the proposed model. By applying Lemma 2 and using the fractional integral operator ${}^{\mathscr{AB}}\mathscr{I}_{0;\,\vartheta}^{\beta_k}$ on both sides of the system, we obtain

$$ \rho_i(t) - \rho_i(0) = {}^{\mathscr{AB}}\mathscr{I}_{0;\,\vartheta}^{\beta_k}\left(-\gamma_i\rho_i(t) + \sum_{j=1}^{N} w_{ij}\mathscr{K}(\rho_j(t)) + \eta_i(t)f(\rho_i(t),\, t)\right). $$

Using Definition 4 for ${}^{\mathscr{AB}}\mathscr{I}_{0;\,\vartheta}^{\beta_k}$, we have

$$ \begin{aligned} \rho_i(t) =& \rho_i(0) + \frac{1-\beta_k}{\Delta(\beta_k)}\left[-\gamma_i\rho_i(t) + \sum_{j=1}^{N} w_{ij}\mathscr{K}(\rho_j(t)) + \eta_i(t)f(\rho_i(t),\, t)\right] \\ &+ \frac{\beta_k}{\Delta(\beta_k)}\, {}^{RL}\mathscr{I}_{0;\,\vartheta}^{\beta_k}\left[-\gamma_i\rho_i(t) + \sum_{j=1}^{N} w_{ij}\mathscr{K}(\rho_j(t)) + \eta_i(t)f(\rho_i(t),\, t)\right]. \end{aligned} $$

By Eq. (2),

$$ \begin{aligned} \rho_i(t) =& \rho_i(0) + \frac{1-\beta_k}{\Delta(\beta_k)}\left[-\gamma_i\rho_i(t) + \sum_{j=1}^{N} w_{ij}\mathscr{K}(\rho_j(t)) + \eta_i(t)f(\rho_i(t),\, t)\right] \\ &+ \frac{\beta_k}{\Gamma(\beta_k)\Delta(\beta_k)}\int_0^t [\vartheta(t) - \vartheta(s)]^{\beta_k - 1}\,\vartheta'(s)\Big[-\gamma_i\rho_i(s) + \sum_{j=1}^{N} w_{ij}\mathscr{K}(\rho_j(s)) \\ &+ \eta_i(s)f(\rho_i(s),\, s)\Big]ds. \end{aligned} $$

Note that the first term is a function that includes $\rho_i(t)$ as an argument. Then

$$\rho_i(t) = \frac{1}{\left(1+\gamma_i \frac{1-\beta_k}{\Delta(\beta_k)}\right)} \left[\rho_i(0) + \frac{1-\beta_k}{\Delta(\beta_k)} \left(\sum_{j=1}^{N} w_{ij}\mathscr{K}(\rho_j(t)) + \eta_i(t) f(\rho_i(t), t)\right)\right]$$

$$+ \frac{1}{\left(1+\gamma_i \frac{1-\beta_k}{\Delta(\beta_k)}\right)} \frac{\beta_k}{\Gamma(\beta_k)\Delta(\beta_k)} \int_0^t [\vartheta(t)-\vartheta(s)]^{\beta_k-1} \vartheta'(s)$$

$$\times \left[-\gamma_i\rho_i(s) + \sum_{j=1}^{N} w_{ij}\mathscr{K}(\rho_j(s)) + \eta_i(s) f(\rho_i(s), s)\right] ds.$$

For convenience, let us denote

$$\vartheta_{\beta_k}(t, s) := [\vartheta(t)-\vartheta(s)]^{\beta_k}, \quad \vartheta^*_{\beta_k}(t, s) := [\vartheta(t)-\vartheta(s)]^{\beta_k-1} \vartheta'(s),$$

$$a := \frac{1-\beta_k}{\Delta(\beta_k)}, \quad b := \frac{\beta_k}{\Gamma(\beta_k)\Delta(\beta_k)},$$

$$\mathscr{F}_i(\rho_i(t), t) := \sum_{j=1}^{N} w_{ij}\mathscr{K}(\rho_j(t)) + \eta_i(t) f(\rho_i(t), t),$$

and

$$\mathscr{G}_i(\rho_i(t), t) := -\gamma_i\rho_i(t) + \sum_{j=1}^{N} w_{ij}\mathscr{K}(\rho_j(t)) + \eta_i(t) f(\rho_i(t), t).$$

Thus, the model (1) has a solution given by

$$\rho_i(t) = \frac{1}{(1+a\gamma_i)} [\rho_i(0) + a\mathscr{F}_i(\rho_i(t), t)] + \frac{b}{(1+a\gamma_i)} \int_0^t \vartheta^*_{\beta_k}(t, s)\, \mathscr{G}_i(\rho_i(s), s) ds. \tag{3}$$

4. Main results

In this section, we establish the existence and uniqueness of solutions, and discuss three different types of stability. Moreover, we provide some numerical examples to justify our main results.

4.1 *Existence and uniqueness results*

This subsection aims to discuss the existence and uniqueness of solutions to system (1). To facilitate the establishment of our findings, the following assumptions are necessary.

(H1) The function $f: \mathbb{R} \times [0, T] \to \mathbb{R}$ and kernel $\mathscr{K}: \mathbb{R} \to \mathbb{R}$ are continuous and satisfy:

$$|f(x, t) - f(y, t)| \leq L_f|x-y|, \quad |\mathscr{K}(x) - \mathscr{K}(y)| \leq L_{\mathscr{K}}|x-y|,$$

for all $x, y \in \mathbb{R}$, $t \in [0, T]$, and constants $L_f, L_{\mathscr{K}} > 0$.

(H2) The coefficients are bounded as

$$|\eta_i(t)| \leq \eta_{\max, i}, \quad |w_{ij}| \leq w_{\max, i}, \quad \gamma_i \geq \gamma_{\min, i} > 0, \quad \mathscr{K}(0) = \mathscr{K}_0, \quad f(0, 0) = f_0.$$

(H3) For each neuron i,

$$\gamma_i > \eta_{\max, i} L_f + \sum_{j=1}^{N} w_{\max, i} L_{\mathscr{K}}.$$

In view of Eq. (3), a map $\Phi: \mathscr{C} \to \mathscr{C}$ is defined as

$$\Phi\rho_i(t) = \frac{1}{(1+a\gamma_i)}\left[\rho_i(0) + a\mathscr{F}_i(\rho_i(t), t)\right] + \frac{b}{(1+a\gamma_i)}\int_0^t \vartheta^*_{\beta_k}(t, s)\,\mathscr{G}_i(\rho_i(s), s)ds. \tag{4}$$

Rewrite Eq. (4) using two operators Φ_1 and Φ_2, where

$$\Phi_1\rho_i(t) = \frac{1}{(1+a\gamma_i)}\left[\rho_i(0) + a\mathscr{F}_i(\rho_i(t), t)\right],$$

$$\Phi_2\rho_i(t) = \frac{b}{(1+a\gamma_i)}\int_0^t \vartheta^*_{\beta_k}(t, s)\,\mathscr{G}_i(\rho_i(s), s)ds.$$

Now, we prove the existence theorems by Krasnoselskii's fixed point theorem [35].

Theorem 1 (Existence of Solutions) Let $f: \mathbb{J} \times \mathbb{R} \to \mathbb{R}$ is a continuous. Assume that conditions (H1)-(H3) hold. Then, there exists at least one solution $\rho_i(t) \in \mathscr{C}$ to the system (1). Provided that

$$\nabla =: \frac{2\gamma_i\,\vartheta_{\beta_k}(T, 0)\,b}{(1+a\gamma_i)\,\beta_k} < 1, \quad i = 1, 2, \ldots, N. \tag{5}$$

Proof. Let $B_\kappa = \{\rho \in \mathscr{C}: \|\rho\| \leq \kappa\}$ be a closed, convex, and bounded subset with $\kappa > 0$ and $\kappa > \frac{\Omega}{1-\nabla}$ where

$$\Omega := \frac{1}{(1+a\gamma_i)}\left(|\rho_i(0)| + a(\gamma_i\,\kappa + f_0) + \frac{\beta_k\,\vartheta_{\beta_k}(T, 0)\mathscr{G}_{\max, i}(0, 0)}{\Gamma(\beta_k+1)\Delta(\beta_k)}\right),$$

and $\mathscr{G}_{\max, i}(0, 0) := \sum_{j=1}^{N} w_{\max, i}\,\mathscr{K}_0 + \eta_{\max, i} f_0$.

Step 1: Φ_1 is a contraction.

Let $\rho,\ \rho' \in B_\kappa$. Then

$$\left|\ \Phi_1\rho_i(t)-\Phi_1\rho_i'(t)\right|\ \leq \frac{a}{(1+a\gamma_i)}\left|\ \mathscr{F}_i(\rho_i(t),\,t)-\mathscr{F}_i(\rho_i'(t),\,t)\right|.$$

From assumptions $(H1)$, $(H2)$, and the definition of $\mathscr{F}_i$, we have

$$\left|\ \mathscr{F}_i(\rho_i(t),\,t)-\mathscr{F}_i(\rho_i'(t),\,t)\right|=\left|\ \sum_{j=1}^{N}w_{ij}\mathscr{K}(\rho_j(t))+\eta_i(t)f(\rho_i(t),\,t)-\sum_{j=1}^{N}w_{ij}\mathscr{K}(\rho_j'(t))+\eta_i(t)f(\rho_i'(t),\,t)\right|$$

$$\leq \sum_{j=1}^{N}\left|w_{ij}\right|\left|\mathscr{K}(\rho_j(t))-\mathscr{K}(\rho_j'(t))\ \right|+\left|\eta_i(t)\right|\left|\ f(\rho_i(t),\,t)-\eta_i(t)f(\rho_i'(t),\,t)\right|$$

$$\leq \sum_{j=1}^{N}\left|w_{ij}\right|L_{\mathscr{K}}\left|\ \rho_j(t)-\rho_j'(t)\right|+\left|\eta_i(t)\right|L_f\left|\ \rho_i(t)-\rho_i'(t)\right|$$

$$\leq \left(\sum_{j=1}^{N}w_{\max,\,i}\,L_{\mathscr{K}}+\eta_{\max,\,i}L_f\right)\left|\ \rho_i(t)-\rho_i'(t)\right|$$

$$\leq \gamma_i\left|\ \rho_i(t)-\rho_i'(t)\right|.$$

Thus,

$$\left\|\Phi_1\rho_i-\Phi_1\rho_i'\right\|\ \leq \frac{a\gamma_i}{(1+a\gamma_i)}\left\|\rho_i-\rho_i'\right\|.$$

By condition Eq. (5), Φ_1 is a contraction.
Step 2: Φ_2 is compact.
Now, we have to show that Φ_2 is equicontinuous and uniformly bounded.
Clearly, Φ_2 is continuous, as is $\mathscr{G}_i$. Moreover, we have

$$\left|\ \mathscr{G}_i(\rho_i(s),\,s)-\mathscr{G}_i(0,\,0)\right|=\gamma_i\left|\rho_i(t)\right|+\sum_{j=1}^{N}\left|w_{ij}\right|\left|\mathscr{K}(\rho_j(t))-\mathscr{K}(0)\right|+\left|\eta_i(t)\right|\left|\ f(\rho_i(t),\,t)-f(0,\,0)\right|$$

$$\leq \gamma_i\left|\rho_i(t)\right|+\sum_{j=1}^{N}w_{\max\,.i}\,L_{\mathscr{K}}\left|\rho_j(t)\right|+\eta_{\max,\,i}\,L_f\ \left|\rho_i(t)\right| \tag{6}$$

$$\leq \gamma_i\left|\rho_i(t)\right|+\sum_{j=1}^{N}w_{\max\,.i}\,L_{\mathscr{K}}\left|\rho_j(t)\right|+\eta_{\max,\,i}\,L_f\ \left|\rho_i(t)\right|,$$

and

$$|\,\mathscr{G}_i(0,0)| \leq \sum_{j=1}^{N} w_{\max,\,i}\,\mathscr{K}_0 + \eta_{\max,\,i} f_0 = \mathscr{G}_{\max,\,i}(0,0). \tag{7}$$

Hence, for all of $\rho_i \in B_\kappa$ one has

$$\begin{aligned}
|\Phi_2\rho_i(t)| \ &\leq \frac{b}{(1+a\gamma_i)}\int_0^t \vartheta^*_{\beta_k}(t,s)\ |\,\mathscr{G}_i(\rho_i(s),s)|\,ds\\
&\leq \frac{b}{(1+a\gamma_i)}\int_0^t \vartheta^*_{\beta_k}(t,s)\,|\,\mathscr{G}_i(\rho_i(s),s)-\mathscr{G}_i(0,0)|+|\,\mathscr{G}_i(0,0)|\\
&\leq \frac{b}{(1+a\gamma_i)}\int_0^t \vartheta^*_{\beta_k}(t,s)\ \Bigg[\gamma_i\,|\rho_i(t)|+\sum_{j=1}^{N} w_{\max\,.i}\,L_{\mathscr{K}}\left|\rho_j(t)\right|+\eta_{\max,\,i}\,L_f\ \left|\rho_j(t)\right|\\
&\quad+\sum_{j=1}^{N} w_{\max,\,i}\,\mathscr{K}_0+\eta_{\max,\,i} f_0\Bigg]\,ds\\
&\leq \frac{b}{(1+a\gamma_i)}\int_0^t \vartheta^*_{\beta_k}(t,s)\ \Bigg[\gamma_i\|\rho_i\|+\sum_{j=1}^{N} w_{\max\,.i}\,L_{\mathscr{K}}\|\rho_i\|+\eta_{\max,\,i}\,L_f\ \|\rho_i\|\\
&\quad+\sum_{j=1}^{N} w_{\max,\,i}\,\mathscr{K}_0+\eta_{\max,\,i} f_0\Bigg]\,ds\\
&\leq \frac{b}{(1+a\gamma_i)}\int_0^t \vartheta^*_{\beta_k}(t,s)\ \Bigg[\gamma_i\kappa+\left(\sum_{j=1}^{N} w_{\max\,.i}\,L_{\mathscr{K}}+\eta_{\max,\,i}\,L_f\right)\kappa\\
&\quad+\sum_{j=1}^{N} w_{\max,\,i}\,\mathscr{K}_0+\eta_{\max,\,i} f_0\Bigg]\,ds\\
&\leq \frac{1}{(1+a\gamma_i)}\,\frac{\beta_k}{\Gamma(\beta_k+1)\Delta(\beta_k)}\,\vartheta_{\beta_k}(T,0)\,[\,2\gamma_i\kappa+\mathscr{G}_{\max,\,i}(0,0)]\,.
\end{aligned}$$

Consequently

$$\|\Phi_2\rho_i\| \leq \frac{\beta_k\ \vartheta_{\beta_k}(T,0)\,[2\gamma_i\kappa+\mathscr{G}_{\max,\,i}(0,0)]}{(1+a\gamma_i)\,\Gamma(\beta_k+1)\Delta(\beta_k)}. \tag{8}$$

Therefore, it follows from Eq. (8) that Φ_2 is uniformly bounded.

To prove the equicontinuity of Φ_2, let $\rho_i \in B_\kappa$. Then, by assumption (H3) together with Eqs. (6) and (7), we obtain

$$|\mathscr{G}_i(\rho_i(s),\, s)| \leq |\mathscr{G}_i(\rho_i(s),\, s) - \mathscr{G}_i(0,\, 0)\ | + |\mathscr{G}_i(0,\, 0)|$$

$$\leq \gamma_i \|\rho_i\| + \sum_{j=1}^{N} w_{\max\,.i}\, L_{\mathscr{K}}\, \|\rho_i\| + \eta_{\max,\, i}\, L_f\ \ \|\rho_i\| + |\ \mathscr{G}_i(0,\, 0)\ | \leq 2\gamma_i \kappa + \mathscr{G}_{\max,\, i}(0,\, 0).$$

For $t_1,\, t_2 \in \mathbb{J}$ with $t_1 < t_2,$ we have

$$\begin{aligned}
|\Phi_2\rho_i(t_2) - \Phi_2\rho_i(t_1)\ | =& \left| \frac{b}{(1+a\gamma_i)} \int_0^{t_2} \vartheta^*_{\beta_k}(t_2,\, s)\mathscr{G}_i(\rho_i(s),\, s)ds \right. \\
& \left. - \frac{b}{(1+a\gamma_i)} \int_0^{t_1} \vartheta^*_{\beta_k}(t_1,\, s)\mathscr{G}_i(\rho_i(s),\, s)ds \right| \\
\leq & \frac{b}{(1+a\gamma_i)} \int_0^{t_1} \left| \vartheta^*_{\beta_k}(t_1,\, s) - \vartheta^*_{\beta_k}(t_2,\, s) \right| \mathscr{G}_i(\rho_i(s),\, s)ds \\
& + \frac{b}{(1+a\gamma_i)} \int_{t_1}^{t_2} \vartheta^*_{\beta_k}(t_2,\, s)\, |\mathscr{G}_i(\rho_i(s),\, s)|\, ds \\
\leq & \frac{\beta_k\, [2\gamma_i\kappa + \mathscr{G}_{\max,\, i}(0,\, 0)\ ]}{\Delta(\beta_k)\, (1+a\gamma_i)} \frac{1}{\Gamma(\beta_k)} \int_0^{t_1} \left| \vartheta^*_{\beta_k}(t_1,\, s) - \vartheta^*_{\beta_k}(t_2,\, s) \right| ds \\
& + \frac{\beta_k\, [2\gamma_i\kappa + \mathscr{G}_{\max,\, i}(0,\, 0)\ ]}{\Delta(\beta_k)\, (1+a\gamma_i)} \frac{1}{\Gamma(\beta_k)} \int_{t_1}^{t_2} \vartheta^*_{\beta_k}(t_2,\, s)ds \\
\leq & \frac{\beta_k\, [2\gamma_i\kappa + \mathscr{G}_{\max,\, i}(0,\, 0)\ ]}{\Delta(\beta_k)\, (1+a\gamma_i)} \left[\vartheta_{\beta_k}(t_1,\, 0) - \vartheta_{\beta_k}(t_2,\, 0) + 2\vartheta_{\beta_k}(t_2,\, t_1) \right] \\
\leq & \frac{2\beta_k\, [2\gamma_i\kappa + \mathscr{G}_{\max,\, i}(0,\, 0)\ ]}{\Delta(\beta_k)\, (1+a\gamma_i)\, \Gamma(\beta_k+1)} \left[\vartheta(t_2) - \vartheta(t_1) \right]^{\beta_k}.
\end{aligned}$$

As $t_1 \to t_2$, $|\Phi_2\rho_i(t_2) - \Phi_2\rho_i(t_1)\ | \to 0$. Hence, Φ_2 is equicontinuous. By the Arzelà–Ascoli theorem, Φ_2 is compact.
Step 3: $\Phi_1\rho_i + \Phi_2\rho_i' \in B_\kappa$.
Let $\rho_i,\, \rho_i' \in B_\kappa$. Then

$$|\ \mathscr{F}_i(\rho_i(s),\, s)| \leq |\ \mathscr{F}_i(\rho_i(s),\, s) - \mathscr{F}_i(0,\, 0)| + |\ \mathscr{F}_i(0,\, 0)|$$

$$\leq \sum_{j=1}^{N} \left| w_{ij} \right| \left| \mathscr{K}(\rho_j(t)) - \mathscr{K}(0) \right| + |\eta_i(t)|\, |f(\rho_i(t),\, t) - f(0,\, 0)| + f_0$$

$$\leq \sum_{j=1}^{N} w_{\max.i}\, L_{\mathscr{K}} \left|\rho_j(t)\right| + \eta_{\max,\,i}\, L_f \, \left|\rho_j(t)\right| + f_0$$

$$\leq \sum_{j=1}^{N} w_{\max.i}\, L_{\mathscr{K}} \left\|\rho_j\right\| + \eta_{\max,\,i}\, L_f \, \left\|\rho_j\right\| + f_0$$

$$\leq \left(\sum_{j=1}^{N} w_{\max.i}\, L_{\mathscr{K}} + \eta_{\max,\,i}\, L_f\right) \kappa + f_0 \leq \gamma_i\, \kappa + f_0.$$

Hence

$$|\Phi_1 \rho_i(t)| \leq \frac{1}{(1+a\gamma_i)} \left[|\, \rho_i(0)| + a\,|\, \mathscr{F}_i(\rho_i(t),\, t)|\right]$$

$$\leq \frac{|\, \rho_i(0)| + a\,(\gamma_i\, \kappa + f_0)}{(1+a\gamma_i)}.$$

It follows from Eq. (8) that

$$\left\|\Phi_2 \rho_i'\right\| \leq \frac{\beta_k\, \vartheta_{\beta_k}(T,\, 0)\, [2\gamma_i \kappa + \mathscr{G}_{\max,\, i}(0,\, 0)]}{(1+a\gamma_i)\,\Gamma(\beta_k+1)\Delta(\beta_k)}.$$

Therefore

$$\left\|\Phi_1 \rho_i + \Phi_2 \rho_i'\right\| \leq \left\|\Phi_1 \rho_i\right\| + \left\|\Phi_2 \rho_i'\right\|$$

$$\leq \frac{|\, \rho_i(0)| + a\,(\gamma_i\, \kappa + f_0)}{(1+a\gamma_i)} + \frac{\beta_k\, \vartheta_{\beta_k}(T,\, 0)\, [2\gamma_i \kappa + \mathscr{G}_{\max,\, i}(0,\, 0)]}{(1+a\gamma_i)\,\Gamma(\beta_k+1)\Delta(\beta_k)}$$

$$= \Omega + \nabla \kappa$$

$$\leq (1-\nabla)\kappa + \nabla \kappa = \kappa,$$

which implies

$$\left\|\Phi_1 \rho_i + \Phi_2 \rho_i'\right\| \leq k.$$

Since all statements of Krasnoselskii's theorem are satisfied, the system (1) has at least one solution. □

Subsequently, the uniqueness result will be established by applying Banach's fixed point theorem [35].

Theorem 2 (Uniqueness of solutions) Let $f: \mathbb{J}\times\mathbb{R}\to\mathbb{R}$ is continuous. Assume that conditions (H1) and (H2) hold, then there exists a unique solution for the model (1) on $\mathbb{J}$, provided that

$$\left(\frac{a\,\gamma_i}{(1+a\gamma_i)}+\nabla\right)<1, \quad i=1,\,2,\,...,\,N, \tag{9}$$

where ∇ is defined as in Theorem 1.

Proof. Consider the map Φ: $\mathscr{C}\to\mathscr{C}$ defined by

$$\Phi\rho_i(t)=\frac{1}{(1+a\gamma_i)}\left[\,\rho_i(0)+a\mathscr{F}_i(\rho_i(t),\,t)\right]+\frac{b}{(1+a\gamma_i)}\int_0^t \vartheta^*_{\beta_k}(t,\,s)\,\mathscr{G}_i(\rho_i(s),\,s)ds.$$

Let $\rho,\ \rho'\in\mathscr{C}$, we have

$$\begin{aligned}\left|\,\Phi\rho_i(t)-\Phi\rho_i'(t)\right| \leq&\frac{a}{(1+a\gamma_i)}\left|\,\mathscr{F}_i(\rho_i(t),\,t)-\mathscr{F}_i(\rho_i'(t),\,t)\right|\\&+\frac{b}{(1+a\gamma_i)}\int_0^t\vartheta^*_{\beta_k}(t,\,s)\,\left|\,\mathscr{G}_i(\rho_i(s),\,s)-\mathscr{G}_i(\rho_i'(s),\,s)\right|ds.\end{aligned}$$

From assumptions $(H1)$, $(H2)$, and definitions of $\mathscr{F}_i$ and $\mathscr{G}_i$, we have

$$\left|\,\mathscr{F}_i(\rho_i(t),\,t)-\mathscr{F}_i(\rho_i'(t),\,t)\right|\leq\gamma_i\left|\,\rho_i(t)-\rho_i'(t)\right|,$$

and

$$\begin{aligned}\left|\,\mathscr{G}_i(\rho_i(t),\,t)-\mathscr{G}_i(\rho_i'(t),\,t)\right| =&\gamma_i\left|\rho_i(t)-\rho_i'(t)\right|+\sum_{j=1}^N\left|w_{ij}\right|\left|\mathscr{K}(\rho_j(t))-\mathscr{K}(\rho_j'(t))\right|\\&+\left|\eta_i(t)\right|\left|f(\rho_i(t),\,t)-f(\rho_i'(t),\,t)\,\right|\\ \leq&\gamma_i\left|\rho_i(t)-\rho_i'(t)\right|+\sum_{j=1}^N w_{\max.i}\,L_{\mathscr{K}}\left|\rho_j(t)-\rho_j'(t)\right|\\&+\eta_{\max,\,i}\,L_f\,\left|\rho_i(t)-\rho_i'(t)\right|\\ \leq&\left(\gamma_i+\sum_{j=1}^N w_{\max.i}\,L_{\mathscr{K}}+\eta_{\max,\,i}\,L_f\right)\left|\rho_i(t)-\rho_i'(t)\right|\end{aligned}$$

$$\leq 2\gamma_i \left|\rho_i(t) - \rho_i'(t)\right|.$$

It follows that

$$\begin{aligned}
\left|\, \Phi\rho_i(t) - \Phi\rho_i'(t)\right| \ \leq & \frac{a\ \gamma_i}{(1+a\gamma_i)} \left|\ \rho_i(t) - \rho_i'(t)\right| \\
& + \frac{2\gamma_i\ \beta_k}{(1+a\gamma_i)\Delta(\beta_k)} \frac{1}{\Gamma(\beta_k)} \int_0^t \vartheta^*_{\beta_k}(t,\ s)\ \left|\rho_i(s) - \rho_i'(s)\right| ds \\
& \leq \left(\frac{a\ \gamma_i}{(1+a\gamma_i)} + \frac{2\gamma_i\ \beta_k}{(1+a\gamma_i)\Delta(\beta_k)} \frac{\vartheta_{\beta_k}(t,\ 0)}{\Gamma(\beta_k+1)} \right) \|\ \rho_i - \rho_i'\| \\
& \leq \left(\frac{a\ \gamma_i}{(1+a\gamma_i)} + \frac{2\gamma_i\ \ \vartheta_{\beta_k}(T,\ 0)b}{(1+a\gamma_i)\,\beta_k} \right) \|\ \rho_i - \rho_i'\|,
\end{aligned}$$

which implies

$$\|\ \Phi\rho_i - \Phi\rho_i'\| \ \leq \left(\frac{a\ \gamma_i}{(1+a\gamma_i)} + \nabla \right) \|\ \rho_i - \rho_i'\|.$$

From Eq. (9), Φ is contraction. So, Banach's fixed point theorem implies that system (1) admits a unique solution. □

4.2 *Stability analysis*

This subsection examines three types of stability for the model (1): asymptotic stability, Mittag-Leffler stability, and finite-time stability.

Theorem 3 (Asymptotic Stability) Assume that $\mathscr{K}$ and f are bounded, i.e,

$$|\mathscr{K}(\rho(t))| \leq L_K|\rho(t)|, \quad |f(\rho(t),\,t)| \leq L_f|\rho(t)|,$$

and $|\eta_i(t)| \leq M_\eta$. Then, the zero solution of system (1) is asymptotically stable ($\lim_{t\to\infty} \|\rho(t)\| = 0$). Provided that

$$\sup_{t\geq 0}[\vartheta(t) - \vartheta(0)] < \infty, \quad \gamma_i > S + \frac{\beta_k}{\Gamma(\beta_k)(1-\beta_k)},$$

where $S = \sum_{j=1}^{N} |w_{ij}|L_K + M_\eta L_f$.

Proof. Rewrite the solution for $\rho_i(t)$ as follows

$$\rho_i(t) = \frac{\rho_i(0) + a\mathscr{F}_i(\rho(t), t)}{1 + a\gamma_i} + \frac{b}{1 + a\gamma_i} \int_0^t \vartheta^*_{\beta_k}(t, s)\mathscr{G}_i(\rho(s), s)ds,$$

where $a = \frac{1-\beta_k}{\Delta(\beta_k)}$, $b = \frac{\beta_k}{\Gamma(\beta_k)\Delta(\beta_k)}$, $\mathscr{F}_i = \sum_j w_{ij}\mathscr{K}(\rho_j) + \eta_i f(\rho_i, t)$, and $\mathscr{G}_i = -\gamma_i\rho_i + \mathscr{F}_i$.

Applying Lipschitz conditions, we obtain

$$|\rho_i(t)| \leq \frac{|\rho_i(0)| + a\left(\sum_j |w_{ij}|L_K|\rho_j(t)| + M_\eta L_f|\rho_i(t)|\right)}{1 + a\gamma_i}$$

$$+ \frac{b}{1 + a\gamma_i} \int_0^t |\vartheta^*_{\beta_k}(t, s)| \left(-\gamma_i|\rho_i(s)| + \sum_j |w_{ij}|L_K|\rho_j(s)| + M_\eta L_f|\rho_i(s)|\right) ds.$$

Since $\|\rho\| = \max_i |\rho_i(t)|$,

$$\|\rho(t)\| \leq \frac{\|\rho(0)\| + aS\|\rho(t)\|}{1 + a\gamma_i} + \frac{b}{1 + a\gamma_i} \int_0^t \vartheta^*_{\beta_k}(t, s)(-\gamma_i + S)\|\rho(s)\|ds,$$

which implies

$$\left(\frac{1 + a\gamma_i - aS}{1 + a\gamma_i}\right)\|\rho(t)\| \leq \frac{\|\rho(0)\|}{1 + a\gamma_i} + \frac{b}{1 + a\gamma_i} \int_0^t \vartheta^*_{\beta_k}(t, s)(-\gamma_i + S)\|\rho(s)\|ds.$$

Since $\gamma_i > S + \frac{\beta_k}{\Gamma(\beta_k)(1-\beta_k)}$, we have

$$1 + a\gamma_i - aS > 0 \quad \text{and} \quad -\gamma_i + S < 0.$$

Hence

$$\|\rho(t)\| \leq C_1\|\rho(0)\| + C_2 \int_0^t \vartheta^*_{\beta_k}(t, s)\|\rho(s)\|ds,$$

where $C_1 = \frac{1}{1+a\gamma_i-aS}$, $C_2 = \frac{b(-\gamma_i+S)}{1+a\gamma_i-aS}$.

Using the fractional Grönwall inequality (Lemma 3 and Corollary 1), we obtain

$$\|\rho(t)\| \leq C_1\|\rho(0)\|E_{\beta_k}\left(C_2\Gamma(\beta_k)[\vartheta(t) - \vartheta(0)]^{\beta_k}\right).$$

Since $C_2 < 0$ and $\sup_{t\geq 0}[\vartheta(t) - \vartheta(0)]^{\beta_k} < \infty$,

$$E_{\beta_k}(-z) \sim \frac{1}{z\Gamma(1-\beta_k)} \quad \text{as } z \to \infty.$$

As $t \to \infty$, $z = |C_2|\Gamma(\beta_k)[\vartheta(t) - \vartheta(0)]^{\beta_k} \to$ const. $< \infty$, giving

$$\|\rho(t)\| \leq \frac{C_1\|\rho(0)\|}{|C_2|\Gamma(1-\beta_k)z} \to 0.$$

i.e, $\|\rho(t)\| \to 0$ as $t \to \infty$. Therefore, all neuron states $\rho_i(t)$ converges asymptotically to zero, establishing asymptotic stability. □

Theorem 4 (Mittag-Leffler Stability) Under the conditions of Theorem 3, there exist C, $\lambda > 0$ such that

$$\|\rho(t)\| \leq C\|\rho(0)\|E_{\beta_k}\left(-\lambda[\vartheta(t) - \vartheta(0)]^{\beta_k}\right).$$

Proof. From Theorem 3, the error vector $\rho(t)$ satisfies the integral inequality

$$\|\rho(t)\| \leq C_1\|\rho(0)\| + C_2\int_0^t \vartheta^*_{\beta_k}(t, s)\|\rho(s)\|ds,$$

where

$$C_1 = \frac{1}{1+a\gamma_i - aS}, \qquad C_2 = \frac{b(-\gamma_i + S)}{1+a\gamma_i - aS}, \quad S = \sum_{j=1}^{N}|w_{ij}|L_{\mathscr{K}} + M_\eta L_f.$$

Since $C_2 < 0$ whenever $\gamma_i > S$, the above inequality becomes

$$\|\rho(t)\| \leq C_1\|\rho(0)\| - |C_2|\int_0^t \vartheta^*_{\beta_k}(t, s)\|\rho(s)\|ds.$$

Applying the fractional Grönwall inequality (Lemma 3 and Corollary 1), we obtain

$$\|\rho(t)\| \leq C_1\|\rho(0)\|E_{\beta_k}\left(-|C_2|\Gamma(\beta_k)[\vartheta(t) - \vartheta(0)]^{\beta_k}\right).$$

Letting

$$\lambda = |C_2|\Gamma(\beta_k) = \Gamma(\beta_k)\frac{b(\gamma_i - S)}{1+a\gamma_i - aS},$$

and $C = C_1$, we derive the Mittag-Leffler decay estimate

$$\|\rho(t)\| \leq C\|\rho(0)\| E_{\beta_k}\left(-\lambda[\vartheta(t)-\vartheta(0)]^{\beta_k}\right), \qquad \lambda = \Gamma(\beta_k)\frac{b(\gamma_i - S)}{1+a\gamma_i - aS}.$$

The condition $\gamma_i > S = \sum_{j=1}^{N} |w_{ij}| L_{\mathscr{K}} + M_\eta L_f$ guarantees $\lambda > 0$, ensuring asymptotic Mittag-Leffler decay, i.e.,

$$E_{\beta_k}(-z) \sim \frac{1}{z\Gamma(1-\beta_k)} \quad \text{as } z \to \infty,$$

and therefore $\|\rho(t)\| \to 0$ as $t \to \infty$. □

Theorem 5 (Finite-Time Stability) Under the conditions of Theorem 3. If for some $T > 0$, the damping coefficients satisfy

$$\gamma_i > \frac{\Gamma(\beta_k)\Delta(\beta_k)}{[\vartheta(T)-\vartheta(0)]^{\beta_k}} - S,$$

where $\Delta(\beta_k) = 1 - \beta_k + \frac{\beta_k}{\Gamma(\beta_k)}$, then $\rho(t) \equiv 0$ for $t \geq T$.

Proof. By Theorem 3, the error vector $\rho(t)$ satisfies the integral inequality

$$\|\rho(t)\| \leq C_1\|\rho(0)\| + C_2 \int_0^t \vartheta^*_{\beta_k}(t, s)\,\|\rho(s)\|\,ds,$$

where

$$C_1 = \frac{1}{1+a\gamma_i - aS}, \qquad C_2 = \frac{b(-\gamma_i + S)}{1+a\gamma_i - aS}, \qquad S = \sum_{j=1}^{N} |w_{ij}| L_{\mathscr{K}} + M_\eta L_f.$$

From the hypothesis $\gamma_i > S$, we have $C_2 < 0$. Rewriting the inequality with $|C_2| = -C_2$ gives

$$\|\rho(t)\| \leq C_1\|\rho(0)\| - |C_2| \int_0^t \vartheta^*_{\beta_k}(t, s)\,\|\rho(s)\|\,ds.$$

Applying the fractional Grönwall inequality (Lemma 3/Corollary 1) yields the Mittag-Leffler estimate

$$\|\rho(t)\| \leq C_1\|\rho(0)\|\, E_{\beta_k}\Big(-|C_2|\Gamma(\beta_k)[\vartheta(t)-\vartheta(0)]^{\beta_k}\Big).$$

At $t = T$ and denote $L\colon = [\vartheta(T)-\vartheta(0)]^{\beta_k}$ and $z_T\colon = |C_2|\Gamma(\beta_k)L$. Then, using the first-order bound for the Mittag-Leffler function:

$$E_{\beta_k}(-z) \leq 1 - \frac{\Delta(\beta_k)}{\Gamma(\beta_k)} z \qquad (z \geq 0), \qquad \Delta(\beta_k)\colon = 1 - \beta_k + \frac{\beta_k}{\Gamma(\beta_k)},$$

we obtain

$$\|\rho(T)\| \leq C_1 \|\rho(0)\| \Big(1 - \Delta(\beta_k)\,|C_2|\,L\Big). \tag{10}$$

Substituting $|C_2| = \dfrac{b(\gamma_i - S)}{1 + a\gamma_i - aS}$ and $C_1 = \dfrac{1}{1 + a\gamma_i - aS}$, we get

$$1 - \Delta(\beta_k)\,|C_2|\,L = 1 - \Delta(\beta_k)\,\frac{b(\gamma_i - S)L}{1 + a\gamma_i - aS}.$$

Multiplying both sides by $b/(1 + a\gamma_i - aS)$ and rearranging, the condition $\gamma_i > \frac{\Gamma(\beta_k)\Delta(\beta_k)}{L} - S$ is algebraically equivalent to $1 - \Delta(\beta_k)\,|C_2|\,L \leq 0$. Hence, the right-hand side of Eq. (10) is nonpositive. Since $C_1 > 0$ and $\|\rho(0)\| \geq 0$, it follows that

$$\|\rho(T)\| = 0, \quad \text{i.e.} \quad \rho(T) = 0.$$

Finally, according to (Theorem 3), the solution that vanishes at time T remains identically zero for all subsequent times. Hence

$$\rho(t) \equiv 0 \qquad \text{for all } t \geq T.$$

□

4.3 *Comparison of stability conditions*

Asymptotic, Mittag-Leffler, and finite-time stability types vary in their convergence properties and damping needs. Table 1 shows that asymptotic stability ensures long-term convergence by imposing the least amount of damping. In contrast, the Mittag-Leffler stability conclusion employs the same damping requirement as the asymptotic case but provides a more precise characterization of the rate of convergence using the Mittag-Leffler function, which depicts a decay of fractional order. On the other hand, finite-time stability ensures that the state achieves zero within a finite period and requires the most stringent constraint on γ_i. Therefore, the stronger the required convergence rate, the less feasible it is in practice.

Table 1. Comparison of the stability conditions and their practical implications

Stability Type	Damping Condition	Convergence Behavior	Restrictiveness
Asymptotic	$\gamma_i > S + \dfrac{\beta_k}{\Gamma(\beta_k)(1-\beta_k)}$	$\rho(t) \to 0$ as $t \to \infty$	Low
Mittag-Leffler	Same as asymptotic	$E_{\beta_k}\left(-\lambda[\vartheta(t) - \vartheta(0)]^{\beta_k}\right) \to 0$	Low
Finite-Time	$\gamma_i > \dfrac{\Gamma(\beta_k)\Delta(\beta_k)}{[\vartheta(T) - \vartheta(0)]^{\beta_k}} - S$	$\rho(t) = 0$ for $t \geq T$	High

5. Implementations

Numerical simulations of two- and three-neuron systems are conducted to demonstrate and validate the theoretical results.

Example 1 (2-neuron system) Consider the following fractional model

$$ {}^{ABC}\mathscr{D}^{\beta_k}_{0,\,\vartheta}\rho_1(t) = -\gamma_1\rho_1(t) + \frac{1}{4}\left(w_{11}\sin(\rho_1(t)) + w_{12}\sin(\rho_2(t))\right) + \eta_1(t)\left(\frac{t}{1+t}\right)\rho_1(t), $$

$$ {}^{ABC}\mathscr{D}^{\beta_k}_{0,\,\vartheta}\rho_2(t) = -\gamma_2\rho_2(t) + \frac{1}{4}\left(w_{21}\sin(\rho_1(t)) + w_{22}\sin(\rho_2(t))\right) + \eta_2(t)\left(\frac{t}{1+t}\right)\rho_2(t), \tag{11} $$

with initial conditions $\rho_1(0) = 0$ and $\rho_2(0) = 1$, where $i = 1, 2$, $t \in [0, 1]$, $w_{11} = 0.2$, $w_{12} = 0.3$, $w_{21} = 0.4$, $w_{22} = -0.1$, $\gamma_1 = \gamma_2 = 0.5$, $\eta_1(t) = 0.1\cos(t)$, $\eta_2(t) = 0.2\cos(t)$, $\beta_0 = 0.8$, $\vartheta(t) = \ln(1+t)$, and $\vartheta'(t) = \frac{1}{1+t}$. The system is based on the following nonlinear terms: $\mathscr{K}(\rho(t)) = \frac{1}{4}\sin(\rho(t))$and $f(\rho(t), t) = \frac{t}{1+t}\rho(t)$. For $x, y \in \mathbb{R}$, $|\mathscr{K}(x) - \mathscr{K}(y)| \le \frac{1}{4}|x-y|$ and $|f(x, t) - f(y, t)| \le \frac{1}{2}||x-y|$. Thus, $\mathscr{K}$, f are Lipschitz continuous with constants $L_{\mathscr{K}} = \frac{1}{4}$, $L_f = \frac{1}{2}$ on $\mathbb{R}$. Hence, the condition $(H1)$ holds. Also, $\gamma_1 = \gamma_2 = 0.5$, which satisfies $|\gamma_i| \ge \gamma_{\min,\,i} = 0.5 > 0$, $|\eta_1(t)| \le 0.1$ and $|\eta_2(t)| \le 0.2$, which are bounded by $\eta_{\max,\,i}$, $|w_{ij}| = 0.2, 0.3, 0.4, -0.1$, which are bounded by $w_{\max,\,i}$. Thus, the conditions in $(H2)$ are satisfied with the bounds $\gamma_{\min,\,i} = 0.5$, $\eta_{\max,\,i} = 0.2$, and $w_{\max,\,i} = 0.4$.

Furthermore, condition (H3) is satisfied, that is

$$ \eta_{\max,\,i}L_f + \sum_{j=1}^{N} w_{\max,\,i}\,L_{\mathscr{K}} = 0.2\left(\frac{1}{2}\right) + 2\,(0.4)\,\frac{1}{4} = 0.3 < 0.5 = \gamma_i. $$

For existence, the condition of Theorem 1 is

$$ \nabla = \frac{2\gamma_i\ \vartheta_{\beta_k}(T, 0)\ b}{(1+a\gamma_i)\,\beta_k} = \frac{2\gamma_i\ \vartheta_{\beta_k}(T, 0)}{\left(1 + \frac{1-\beta_k}{\Delta(\beta_k)}\gamma_i\right)\Delta(\beta_k)\Gamma(\beta_k)} < 1. $$

We have $T = 1$, $k = 0$, $\gamma_i = 0.5$, $\beta_0 = 0.8$, and $\vartheta(t) = \ln(1+t)$, which gives $\vartheta_{\beta_0}(T, 0) = [\vartheta(T) - \vartheta(0)]^{\beta_0} = 0.8\ln(2) \approx 0.733$. Substituting the known values and $\Delta(\beta_0) = 1 - \beta_0 + \frac{\beta_0}{\Gamma(\beta_0)} = 0.887$, we conclude that $\nabla \approx 0.638 < 1$. Thus, all the hypotheses of Theorem 1 are fulfilled.

For uniqueness, the condition of Theorem 2 is

$$ \Lambda\colon = \left(\frac{a\ \gamma_i}{(1+a\gamma_i)} + \nabla\right) < 1. $$

Substituting the above values with $a \approx 0.225$, we obtain

$$ \Lambda \approx \left(\frac{(0.225)\,(0.5)}{1+(0.225)\,(0.5)} + 0.638\right) \approx 0.739 < 1. $$

Consequently, by Theorem 2, the system (11) admits a unique solution. Figure 1 shows the simulation for Example 1.

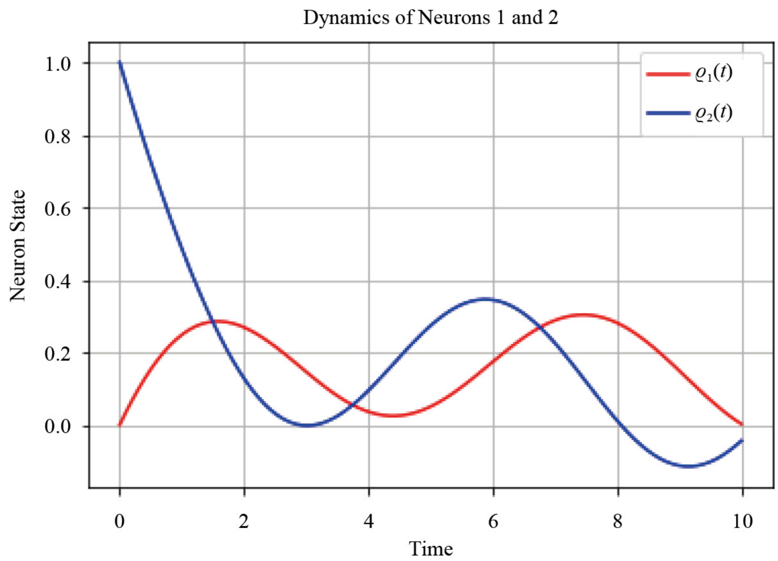

Figure 1. Numerical simulation of 2-neuron system

Example 2 (3-neuron system) The model is defined as

$$ {}^{ABC}\mathscr{D}^{\beta_k}_{0,\,\vartheta}\rho_i(t) = -\gamma_i\rho_i(t) + \frac{1}{6}\sum_{j=1}^{3} w_{ij}\tanh(\rho_j(t)) + \eta_i(t)(1+\frac{t}{8}\sin(\rho_i)), \quad i = 1, 2, 3, \tag{12} $$

where $t \in [0, 1]$, $\gamma_i = 0.3$, $\eta_i(t) = 0.5\cos(t)$, for all $i = 1, 2, 3$, initial conditions $\rho(0) = (0.1, 0.1, 0.1)$, $\beta_k = 0.8$, $\vartheta(t) = \ln(1+t)$, $\vartheta'(t) = \frac{1}{1+t}$, and the weight matrix is

$$ W = \begin{bmatrix} -0.4 & -0.1 & -0.2 \\ 0.1 & 0.4 & 0.1 \\ 0.4 & 0.1 & 0.2 \end{bmatrix}. $$

Here, diagonal matrix D is $D = \text{diag}(0.1, 0.1, 0.1)$. To verify condition $(H1)$, we have $\mathscr{K}(\rho(t)) = \frac{1}{6}\tanh(\rho(t))$, and $f(\rho, t) = 1+\frac{t}{8}\sin(\rho_i)$. For $\rho, y \in \mathbb{R}$, $|\mathscr{K}(\rho) - \mathscr{K}(y)| = \frac{1}{6}|\tanh(\rho) - \tanh(y)| \leq \frac{1}{6}|\rho - y|$, and $|f(\rho, t) - f(y, t)| \leq \frac{1}{8}|t||\sin(\rho) - \sin(y)| \leq \frac{1}{8}|x-y|$. Thus, $\mathscr{K}$, f are Lipschitz continuous with constants $L_{\mathscr{K}} = \frac{1}{6}$, $L_f = \frac{1}{8}$ on $\mathbb{R}$. Hence, the condition $(H1)$ holds. Also, for $(H2)$, $|\gamma_1| = |\gamma_2| = |\gamma_3| = 0.3$, which satisfies $\gamma_i \geq \gamma_{\min,\,i} = 0.3$, $|\eta_1(t)| \leq 0.5 = \eta_{\max,\,i}$, which are bounded by 0.5, $|w_{ij}| \leq 0.4 = w_{\max,\,i}$, which are bounded by 0.4. Thus, the conditions in $(H2)$ are satisfied with the bounds $\gamma_{\min,\,i} = 0.3$, $\eta_{\max,\,i} = 0.5$, and $w_{\max,\,i} = 0.4$.

The condition $(H3)$ is satisfied too, i.e,

$$\eta_{\max,\,i}L_f + \sum_{j=1}^{N} w_{\max,\,i}\, L_{\mathscr{K}} = 0.5\left(\frac{1}{8}\right) + 3\,(0.4)\,\frac{1}{6} = 0.2 < 0.3 = \gamma_i.$$

For the existence, the condition of Theorem 1 is

$$\nabla = \frac{2\gamma_i\ \vartheta_{\beta_k}(T,\,0)\ b}{(1+a\gamma_i)\,\beta_k} = \frac{2\gamma_i\ \vartheta_{\beta_k}(T,\,0)}{\left(1+\frac{1-\beta_k}{\Delta(\beta_k)}\gamma_i\right)\Delta(\beta_k)\Gamma(\beta_k)} < 1.$$

Substituting the above values with $\vartheta_{\beta_0}(T,\,0) \approx 0.733$, $a = \frac{1-\beta_k}{\Delta(\beta_k)} \approx 0.225$, $b = \frac{1}{\Delta(\beta_k)\Gamma(\beta_k)} \approx 0.968$ and $\Delta(\beta_0) \approx 0.887$, we conclude that $\nabla \approx 0.499 < 1$. Thus, all the hypotheses of Theorem 1 are fulfilled.

For uniqueness, the condition of Theorem 2 is

$$\Lambda\colon = \left(\frac{a\ \gamma_i}{(1+a\gamma_i)} + \nabla\right) < 1.$$

Substituting the above values with $a \approx 0.225$, we obtain,

$$\Lambda \approx \left(\frac{(0.225)\,(0.3)}{1+(0.225)\,(0.3)} + 0.499\right) \approx 0.562 < 1.$$

Therefore, Theorem 2 shows that the model (12) has a unique solution. Figure 2 shows the simulation for Example 2.

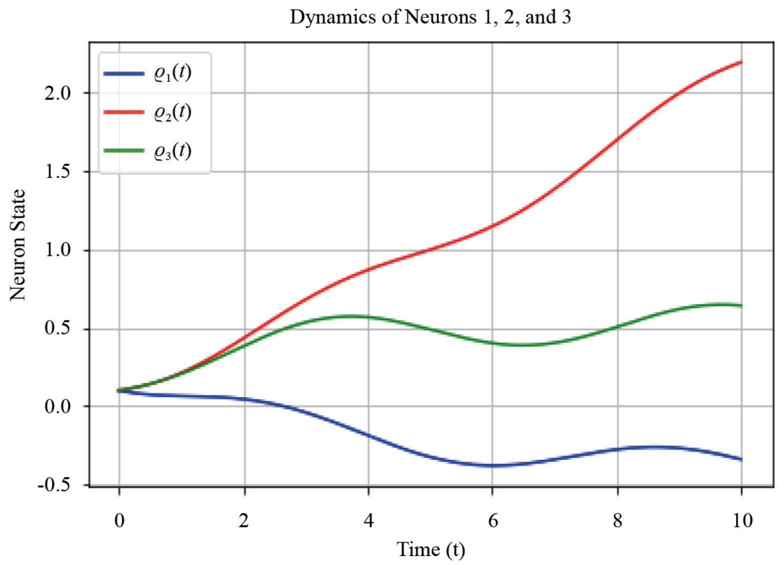

Figure 2. Numerical simulation of 3-neuron system

Example 3 Consider the following fractional model with two neurons ($N = 2$) as

$$\begin{cases} {}^{\mathscr{ABC}}\mathscr{D}_{0,t}^{\beta_1}\rho_1(t) = -\gamma_1\rho_1(t) + w_{11}\mathscr{K}(\rho_1(t)) + w_{12}\mathscr{K}(\rho_2(t)) + \eta_1(t)f(\rho_1(t), t), \\ {}^{\mathscr{ABC}}\mathscr{D}_{0,t}^{\beta_2}\rho_2(t) = -\gamma_2\rho_2(t) + w_{21}\mathscr{K}(\rho_1(t)) + w_{22}\mathscr{K}(\rho_2(t)) + \eta_2(t)f(\rho_2(t), t), \end{cases} \tag{13}$$

with initial conditions $\rho_1(0) = \rho_1^0$ and $\rho_2(0) = \rho_2^0$.

For a general Fractional Differential Equation (FDE) of the form ${}^{\mathscr{ABC}}\mathscr{D}_{0,t}^{\beta}\rho(t) = g(t, x);\ x(0) = x_0,$ the numerical approximation using the two-step Newton's method is given by

$$\begin{aligned} x^{p+1} =& x(0) + \frac{1-\beta}{\Delta(\beta)}g(t_p, x^p) \\ &+ \frac{\beta h^\beta}{\Delta(\beta)\Gamma(\beta+1)}\sum_{n=2}^{p} g(t_{n-2}, x^{n-2})\left[(p-n+1)^\beta - (p-n)^\beta\right] \\ &+ \frac{\beta h^\beta}{\Delta(\beta)\Gamma(\beta+2)}\sum_{n=2}^{p}\left[g(t_{n-1}, x^{n-1}) - g(t_{n-2}, x^{n-2})\right]A_1 \\ &+ \frac{\beta h^\beta}{2\Delta(\beta)\Gamma(\beta+3)}\sum_{n=2}^{p}\left[g(t_n, x^n) - 2g(t_{n-1}, x^{n-1}) + g(t_{n-2}, x^{n-2})\right]A_2, \end{aligned}$$

where

$$A_1 = \left[(p-n+1)^\beta(p-n+3+2\beta) - (p-n)^\beta(p-n+3+3\beta)\right],$$

$$\begin{aligned} A_2 = \Big[&(p-n+1)^\beta\left(2(p-n)^2 + (3\beta+10)(p-n) + 2\beta^2 + 9\beta + 12\right) \\ &-(p-n)^\beta\left(2(p-n)^2 + (5\beta+10)(p-n) + 6\beta^2 + 18\beta + 12\right)\Big], \end{aligned}$$

and $\Delta(\beta) = 1 - \beta + \frac{\beta}{\Gamma(\beta)}$.

For our specific model Eq. (13), we define the functions $g_i(t, \rho_1, \rho_2)$ for $i = 1, 2$ as

$$g_1(t, \rho_1, \rho_2) = -\gamma_1\rho_1(t) + w_{11}\mathscr{K}(\rho_1(t)) + w_{12}\mathscr{K}(\rho_2(t)) + \eta_1(t)f(\rho_1(t), t),$$

$$g_2(t, \rho_1, \rho_2) = -\gamma_2\rho_2(t) + w_{21}\mathscr{K}(\rho_1(t)) + w_{22}\mathscr{K}(\rho_2(t)) + \eta_2(t)f(\rho_2(t), t).$$

The iterative scheme for each neuron state is then given by

$$
\begin{aligned}
\rho_1^{p+1} =& \rho_1^0 + \frac{1-\beta_1}{\Delta(\beta_1)} g_1(t_p, \rho_1^p, \rho_2^p) \\
&+ \frac{\beta_1 h^{\beta_1}}{\Delta(\beta_1)\Gamma(\beta_1+1)} \sum_{n=2}^{p} g_1(t_{n-2}, \rho_1^{n-2}, \rho_2^{n-2}) \left[(p-n+1)^{\beta_1} - (p-n)^{\beta_1}\right] \\
&+ \frac{\beta_1 h^{\beta_1}}{\Delta(\beta_1)\Gamma(\beta_1+2)} \sum_{n=2}^{p} \left[g_1(t_{n-1}, \rho_1^{n-1}, \rho_2^{n-1}) - g_1(t_{n-2}, \rho_1^{n-2}, \rho_2^{n-2})\right] A_{1,1} \\
&+ \frac{\beta_1 h^{\beta_1}}{2\Delta(\beta_1)\Gamma(\beta_1+3)} \sum_{n=2}^{p} \left[g_1(t_n, \rho_1^n, \rho_2^n) - 2g_1(t_{n-1}, \rho_1^{n-1}, \rho_2^{n-1}) + g_1(t_{n-2}, \rho_1^{n-2}, \rho_2^{n-2})\right] A_{2,1},
\end{aligned}
$$

and

$$
\begin{aligned}
\rho_2^{p+1} =& \rho_2^0 + \frac{1-\beta_2}{\Delta(\beta_2)} g_2(t_p, \rho_1^p, \rho_2^p) \\
&+ \frac{\beta_2 h^{\beta_2}}{\Delta(\beta_2)\Gamma(\beta_2+1)} \sum_{n=2}^{p} g_2(t_{n-2}, \rho_1^{n-2}, \rho_2^{n-2}) \left[(p-n+1)^{\beta_2} - (p-n)^{\beta_2}\right] \\
&+ \frac{\beta_2 h^{\beta_2}}{\Delta(\beta_2)\Gamma(\beta_2+2)} \sum_{n=2}^{p} \left[g_2(t_{n-1}, \rho_1^{n-1}, \rho_2^{n-1}) - g_2(t_{n-2}, \rho_1^{n-2}, \rho_2^{n-2})\right] A_{1,2} \\
&+ \frac{\beta_2 h^{\beta_2}}{2\Delta(\beta_2)\Gamma(\beta_2+3)} \sum_{n=2}^{p} \left[g_2(t_n, \rho_1^n, \rho_2^n) - 2g_2(t_{n-1}, \rho_1^{n-1}, \rho_2^{n-1}) + g_2(t_{n-2}, \rho_1^{n-2}, \rho_2^{n-2})\right] A_{2,2},
\end{aligned}
$$

where $A_{1,1}$ and $A_{2,1}$ are calculated using β_1, $A_{1,2}$ and $A_{2,2}$ are computed using β_2.

With $g_i(t, \rho_1, \rho_2) = -\gamma_i \rho_i(t) + \sum_{j=1}^{2} w_{ij} \tanh(\rho_j(t)) + \eta_i \rho_i(t) \sin(t)$, where $\mathscr{K}(\rho_j(t)) = \tanh(\rho_j(t))$ and $f(\rho_i(t), t) = \rho_i(t)\sin(t)$, the above numerical scheme enables the simulation of the fractional model Eq. (13). Figures 3-4 show the simulation of the model (13) according to the parameters selected from Table 2.

Table 2. Parameters and functions used in the numerical simulation

Symbol	Description	Value/Expression	Notes
β_k	FD order	0.8	ABC derivative
γ_i	Damping coefficient	8	$i = 1, 2$
w_{ij}	Coupling weights	$\begin{bmatrix} 0 & 0.3 \\ 0.3 & 0 \end{bmatrix}$	Excitatory coupling
$\eta_i(t)$	Modulation function	0.5	Constant ($i = 1, 2$)
$\vartheta(t)$	Time scaling function	$1 - e^{-t}$	$\vartheta'(t) = e^{-t}$
$\mathscr{K}(x)$	Kernel function	$\tanh(x)$	$L_K = 1$
$f(x, t)$	Activation function	$x\sin(t)$	$L_f = 1$

Table 2. (cont.)

Symbol	Description	Value/Expression	Notes
T	Simulation time	10	Units: seconds
h	Step size	0.1	Newton's method
$\rho_1(0)$	Initial state (neuron 1)	1	–
$\rho_2(0)$	Initial state (neuron 2)	-1	–

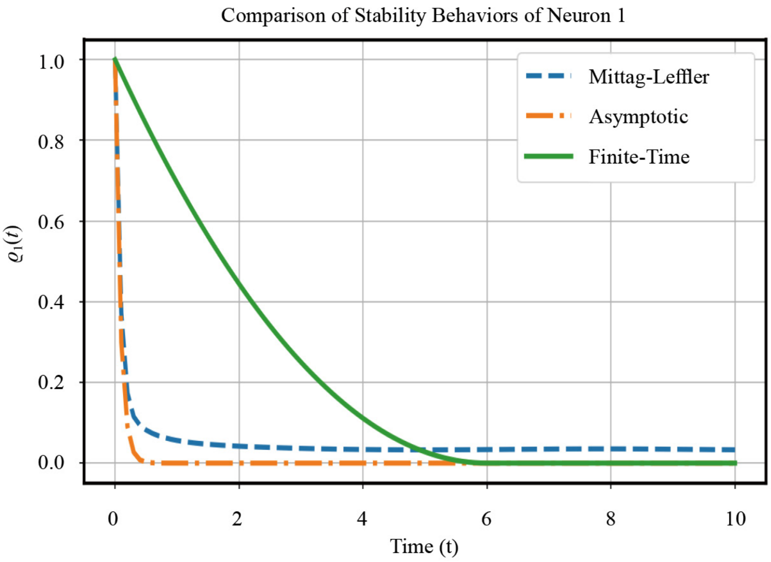

Figure 3. Stability dynamics of neuron 1 ($\rho_1(t)$) under three stability regimes over $t \in [0, 10]$ with $\rho_1(0) = 1$

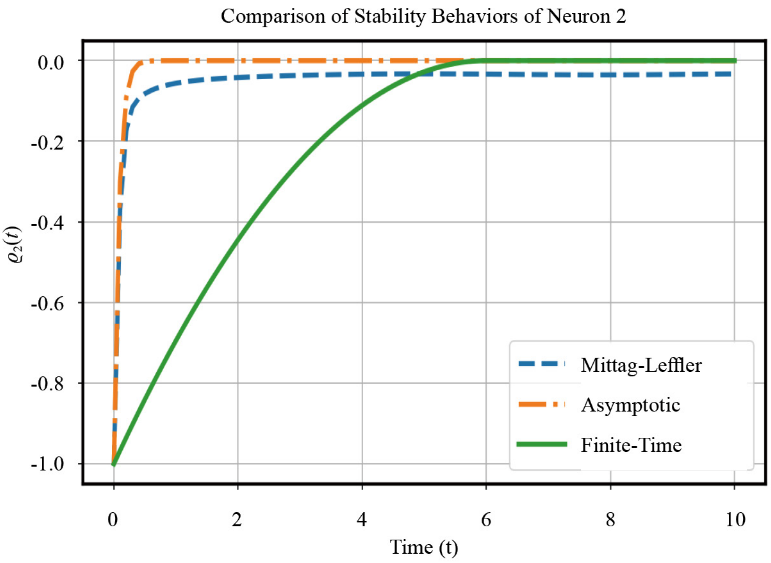

Figure 4. Stability dynamics of neuron 2 ($\rho_2(t)$) under three stability regimes over $t \in [0, 10]$ with $\rho_2(0) = -1$

5.1 *Discussion*

Here, we discuss and interpret the figures obtained from numerical simulations.

Figure 1 displays the dynamics of two interacting neurons governed by a logarithmic kernel $\vartheta(t) = \ln(1+t)$. Coupled sine activations and periodic external inputs generate smooth oscillations that gradually stabilize. The system remains bounded due to damping and balanced coupling, illustrating stable, synchronized behavior over time.

Figure 2 presents the dynamic responses of a three-neuron network with mixed excitatory and inhibitory couplings. Damped oscillations appear due to the balance between the coupling matrix W and the damping coefficients $\gamma_i = 0.3$. The trajectories $\rho_1(t)$, $\rho_2(t)$, and $\rho_3(t)$ converge to steady states, confirming bounded synchronization in the system.

Figures 3 and 4 present the time-domain behavior of Neuron 1 and Neuron 2, respectively, under three distinct stability regimes: Mittag–Leffler, Asymptotic, and Finite-Time. Both neurons are initialized with opposite signs, $\rho_1(0) = 1$ and $\rho_2(0) = -1$, time scaling function $\vartheta(t) = 1 - e^{-t}$, $\beta = 0.8$, $\gamma_i = 8$ and interact symmetrically through the coupling matrix w. The results demonstrate how fractional-order operators modify the stability characteristics of coupled neural systems compared to their integer-order counterparts.

Neuron 1 (Figure 3) exhibits a slow, non-exponential convergence to equilibrium in the Mittag-Leffler trajectory, demonstrating the fractional kernel's ability to sustain memory effects. At $t = T_f = 6$, the finite-time curve decays smoothly to zero, but the asymptotic profile converges more quickly but stays non-zero for $t \to \infty$. Neuron 2 (Figure 4) shows essentially comparable dynamics, but because of its negative initial state, its polarity is flipped. While introducing small amplitude and phase deviations during transients, the connection maintains stability symmetry, as confirmed by the simultaneous convergence of both neurons. The results reveal that asymptotic stability requires the mildest damping condition, while the Mittag-Leffler case provides a precise fractional-order decay rate under the same assumptions. In contrast, finite-time stability ensures complete state extinction within a finite interval but demands the strongest damping threshold. Overall, these findings demonstrate that, depending on the fractional parameters and coupling strengths, fractional-order neural systems can display a range of stability characteristics, from memory-driven Mittag-Leffler decay to exponential and finite-time stabilization.

Remark 1 The function $\vartheta(t)$ governs the time-scaling of the fractional operator and thus directly affects the decay rate and memory strength in the Mittag-Leffler estimate. If $\vartheta(t)$ grows faster (e.g., t^2 or $e^t - 1$), the argument of the Mittag-Leffler function increases rapidly, leading to faster decay and stronger stability. Conversely, a slower function (e.g., $\ln(1+t)$ or $t^{1/2}$) produces slower decay and longer memory. The classical choice $\vartheta(t) = t$ yields the standard fractional case. Hence, $\vartheta(t)$ acts as a tunable factor controlling both the convergence rate and the memory depth of the system.

6. Conclusion

In this work, we have investigated a fractional-order neural system governed by the ABC operator with respect to the function $\vartheta(t)$. Using Banach's and Krasnoselskii's fixed-point theorems, we established rigorous conditions for the existence and uniqueness of solutions. The stability analysis revealed a clear hierarchy among the three stability types: asymptotic stability corresponds to gradual long-term decay, Mittag-Leffler stability describes algebraic-type convergence, and finite-time stability ensures complete quenching of trajectories within a finite horizon. The explicit damping thresholds derived for the finite-time regime guarantee positivity and physical consistency of decay rates. The function $\vartheta(t)$ further enables adaptive temporal scaling, enhancing numerical stability near $t = 0$. Simulations of two- and three-neuron systems validated these results, confirming the theoretical predictions and demonstrating how fractional parameters and damping intensity control the transition between the three stability modes.

Future work may focus on extending the present analysis to larger neural structures and incorporating external perturbations or delays. Moreover, a promising extension of this work involves generalizing the model to higher-order ABC derivatives with $\beta_k \in (m-1, m)$, enabling the incorporation of all m initial conditions $\rho_i^{(k)}(0)$ for $k = 0, 1, , m-1$.

Conflict of interest

The authors declare no competing financial interest.

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