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Research Article

Global Well-Posedness and Dynamics of Two-Component Reaction-Diffusion Systems with Arbitrary-Growing Nonlinearities

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Abstract: This work investigates the global well-posedness and long-term dynamics of two-component reaction-diffusion systems on bounded domains under homogeneous Dirichlet boundary conditions. We introduce a weaker dissipative condition that enables us to prove the global existence and uniqueness of classical solutions to the associated Cauchy problem, without imposing any growth constraints on the nonlinear terms. The admissible nonlinearities include, but are not limited to, polynomial and exponential growth types. Furthermore, we demonstrate that such systems admit both global and exponential attractors, which exhibit finite-dimensional characteristics in appropriate continuous function spaces.

Keywords: reaction-diffusion system, global well-posedness, global attractor, exponential attractor, finite-dimensionality

MSC: 35B40, 35B41, 35B65, 35K20, 35K58

1. Introduction

Many processes in contemporary science–such as heat conduction, chemical kinetics, and mathematical biology–can be described by reaction-diffusion systems of the form:

$$\mathbf{u}_t - D\Delta\mathbf{u} = \mathbf{F}(x, \mathbf{u}), \qquad x \in \Omega,\ t > 0, \tag{1}$$

where $\Omega \subset \mathbb{R}^n$ is a domain, $\mathbf{u} = (u_1, \cdots, u_m) \in \mathbb{R}^m$ is a vector-valued unknown, and $D = \mathrm{diag}(d_1, \cdots, d_m)$ is a diagonal matrix with nonnegative entries. This sparked in-depth research on the global well-posedness and dynamics of such systems from the perspective of dissipativity.

In most cases, such a model may have a superlinear or even supercritical nonlinearity. It is well known that superlinear nonlinearity can cause blow-up phenomenon in the solutions of a system and therefore destroy its global well-posedness. To preclude this, one generally needs to derive suitable a priori global estimates for the solutions. In the scalar case, this may be done successfully by making use of suitable Lyapunov functions or the comparison principle; see Kostianko et

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al. [1] for details. However, such a strategy is not quite effective in the non-scalar case since for a system like (1), it is in general difficult to construct a Lyapunov function; moreover, the comparison principle can fail to be true. Consequently, establishing global existence for reaction-diffusion systems with superlinear nonlinearities remains a nontrivial task.

An alternative and popular way to establish a priori global estimates for reaction-diffusion systems is to find appropriate structure conditions which simultaneously guarantee the dissipativity of the systems in suitable functional spaces. For scalar equation

$$u_t - \Delta u = g(u), \qquad x \in \Omega,\ t > 0, \tag{2}$$

a widely used dissipative-type structure condition in the literature is as below:

$$-N - c_1|u|^q \le g(u)u \le -c_2|u|^q + N, \qquad u \in \mathbb{R}, \tag{3}$$

where q, c_1, c_2, $N > 0$ are constants independent of $u \in \mathbb{R}$. This condition allows f to have a corresponding polynomial growth rate to assure the global well-posedness and dissipativity of the equation, regardless of whether the nonlinearity is subcritical or supercritical. A typical example is a polynomial $g(u) = \sum_{0\le k\le 2p+1} a_k u^k$ with a negative leading coefficient $a_{2p+1} < 0$. Under this hypothesis, it can be shown that for any initial data in $L^2(\Omega)$, the homogeneous Dirichlet (Neumman) boundary value problem of (2) admits a unique weak solution $u \in C([0,\infty); L^2(\Omega))$ with $g(u) \in L^{q/(q-1)}(0,T; L^{q/(q-1)}(\Omega))$ for any $T \ge 0$; see, e.g., Robinson [2] and Temam [3, Chap. III].

Another frequently used dissipativity condition for (2) is

$$g(u)u \le N, \qquad u \in \mathbb{R}; \tag{4}$$

as found in [2, 4]. Using this weaker assumption one usually needs to impose on g some additional restrictions on the growth rate to ensure global existence.

The dissipative hypotheses (3) and (4) for scalar equations can be extended directly to the vector case to study the global well-posedness and dynamics of system (1). For example, one can show that the initial-boundary value problem for (1) has a unique weak solution if $\mathbf{F}$ satisfies

$$-N - c_1|\mathbf{u}|^q \le \mathbf{F}(\mathbf{u})\cdot\mathbf{u} \le -c_2|\mathbf{u}|^q + N, \qquad \mathbf{u} \in \mathbb{R}^m \tag{5}$$

accompanied by the following monotone hypothesis:

$$\nabla_{\mathbf{u}}\mathbf{F}(\mathbf{u}) \le K, \tag{6}$$

where $\nabla_{\mathbf{u}}\mathbf{F}(\mathbf{u})$ denotes the Jaccobi matrix of $\mathbf{F}(\mathbf{u})$, and $\nabla_{\mathbf{u}}\mathbf{F}(\mathbf{u}) \le K$ stands for $\nabla_{\mathbf{u}}\mathbf{F}(\mathbf{u})\xi\cdot\xi \le K|\xi|^2$, $\forall \xi \in \mathbb{R}^m$. Unfortunately many important examples of vector equations from applications do not fulfill such universal structure conditions mentioned above. Compared with scalar equations, vector ones pose significantly greater challenges due to their inherent complexity and multidimensional nature.

Assume the nonlinearity $\mathbf{F}$ in (1) have a polynomial growth rate:

$$|\mathbf{F}(x, \mathbf{u})| \leq C_0(1+|\mathbf{u}|^q), \qquad \mathbf{u} \in \mathbb{R}^m.$$

In Efendiev and Zelik [5, p.674] the authors introduced an *anisotropic dissipativity assumption*: there exist sufficiently large exponents $p_i \geq 0$ $(1 \leq i \leq m)$ depending upon q such that

$$\sum_{1\leq i\leq m} g_i(x, \mathbf{u})|u_i|^{p_i} u_i \leq C_1, \qquad x \in \overline{\Omega},\ \mathbf{u} \in \mathbb{R}^m.$$

For system (1), the global well-posedness of its initial-boundary value problem has been established under the given condition, with the associated solution semigroup exhibiting a global attractor in an appropriately defined phase space.

Another type of important weaker dissipation condition widely used for positive solutions in the literature is as follows:

$$\sum_{1\leq i\leq m} g_i(x, \mathbf{u}) \leq 0,$$

see [6–8] and [9] etc. for more information and recent development along this line.

For reaction-diffusion systems whose nonlinearities do not satisfy standard dissipative structural conditions or exhibit arbitrary growth, the invariant-region technique can sometimes be applied to obtain global existence. The underlying idea is to identify a bounded invariant region in the phase space for the system; see, e.g., Temam [3, Chap. III, Sec. 1.2.3] for detail. This technique was also used recently by Wang and Yi to get global L^∞-estimate for the solutions of the classical Belousov-Zhabotinskii equation in chemical reactions [10].

To the best of our knowledge, there are very few works addressing the global well-posedness and dynamics of reaction-diffusion systems with fully unrestricted nonlinear growth–even in the scalar case–apart from those relying on the invariant-region method as mentioned above. In this paper, we study the following two-component reaction-diffusion system

$$\begin{cases} u_t - d_1\Delta u = f(x, u, v), \\ \\ v_t - d_2\Delta v = g(x, u, v), \end{cases} \qquad t > 0,\ x \in \Omega \tag{7}$$

subject to homogeneous Dirichlet boundary conditions

$$u|_{\partial\Omega} = 0, \qquad v|_{\partial\Omega} = 0, \tag{8}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with sufficiently smooth boundary $\partial\Omega$, and $d_i > 0$ are diffusion coefficients.

Two-component reaction-diffusion systems provide a flexible mathematical framework for modeling interactive dynamics between two substances involving both diffusion and reaction processes. Their applicability is well-recognized in biological systems, particularly in describing predator-prey population dynamics (see, e.g., [11–15]). However, existing theoretical approaches have largely been shaped by the need to address specific nonlinear forms emerging from particular applications. As a result, the establishment of a framework capable of characterizing the dynamical behavior of general two-component reaction-diffusion systems is both necessary and valuable.

This work aims to contribute to a unified analytical framework for reaction-diffusion systems by establishing the existence of global attractors for system (7)-(8) under weaker dissipativity conditions than those typically required in the literature. Specifically, we develop a theoretical approach within the space of continuous functions that is not constrained by domain-specific assumptions and removes the traditional growth restrictions on nonlinear terms. This approach accommodates a broad class of nonlinearities, including but not limited to polynomial and exponential types, and is carried out within the classical solution framework. As a result, our formulation expands the class of admissible nonlinear terms in the analysis of reaction-diffusion systems.

Define $\mathbf{C}_0(\overline{\Omega}) := \left[C_0(\overline{\Omega})\right]^2$, where

$$C_0(\overline{\Omega}) := \{u:\ u \in C(\overline{\Omega}),\ u|_{\partial\Omega} = 0\}.$$

The main results are summarized in the following theorem.

Theorem 1 Suppose $f, g \in C^1(\overline{\Omega} \times \mathbb{R}^2)$ and satisfy the following structure condition:

($\mathbf{F}0$) There are constants $\alpha, \beta, m, l > 0$ and $\Lambda_i, c_i, N_i > 0\,(i = 1, 2)$ with

$$\alpha, \beta \geq 2, \qquad \alpha\beta > ml \tag{9}$$

such that

$$f(x, u, v)u \leq -\Lambda_1|u|^\alpha + c_1|v|^m + N_1, \qquad (x, u, v) \in \overline{\Omega} \times \mathbb{R}^2, \tag{10}$$

and

$$g(x, u, v)v \leq -\Lambda_2|v|^\beta + c_2|u|^l + N_2, \qquad (x, u, v) \in \overline{\Omega} \times \mathbb{R}^2. \tag{11}$$

Then for every initial datum $\mathbf{u}_0 \in \mathbf{C}_0(\overline{\Omega})$, system (7)-(8) admits a unique global classical solution $\mathbf{u} = (u, v)$. Moreover, the solution semigroup $S(t)$ possesses a finite-dimensional global attractor $\mathscr{A}$ and an exponential attractor $\mathscr{M}$ in $\mathbf{C}_0(\overline{\Omega})$.

A distinctive feature of our approach is that the structure condition ($\mathbf{F}0$) permits the non-dissipative components to grow at rates exceeding those of the dissipative components, as illustrated in Section 6, Example 2. This represents an advancement beyond existing literature. Furthermore, we observe that the attractor problem for reaction-diffusion systems remains relatively unexplored within the classical solution framework, even for polynomial-growth nonlinearities. Our work thus addresses this gap while expanding the theoretical understanding of nonlinear dynamics in continuous function spaces.

This study addresses several theoretical challenges in the analysis of semilinear parabolic equations. The main difficulty involves establishing a connection between the energy estimates commonly used for weak solutions and the framework of mild solutions under the structure condition ($\mathbf{F}0$). This approach requires combining these different methodologies while maintaining rigorous regularity analysis to obtain global classical solutions and establish attractor existence in continuous function spaces. The integration of these techniques represents an essential aspect of our analytical framework.

The paper is structured as follows. Section 2 introduces the necessary function spaces and notation. In Section 3, we establish a global a priori L^∞-estimate for classical solutions of the Cauchy problem associated with (7)-(8). Sections 4 and 5 are devoted to proving global existence, uniqueness, and regularity of classical solutions. Section 6 addresses the existence of finite-dimensional global and exponential attractors for (7)-(8) in $\mathbf{C}_0(\overline{\Omega})$. Finally, Section 7 offers some remarks concerning positive solutions and their attractors.

2. Preliminaries

Let $\mathbb{R}^+$ and $\mathbb{Z}^+$ denote the sets of nonnegative real numbers and nonnegative integers, respectively. For convenience, a vector $\mathbf{u} \in \mathbb{R}^2$ with components u_i $(i = 1, 2)$ will be written as (u_1, u_2) or simply (u_i). The Euclidean norm of $\mathbf{u}$ is denoted by $|\mathbf{u}|$, and the scalar product of $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$ by $\mathbf{u} \cdot \mathbf{v}$.

• **Functional spaces.** Let $\Omega \subset \mathbb{R}^n$ be a domain, and $k \in \mathbb{Z}^+$. The Hölder space $C^{k+\delta}(\overline{\Omega})$ $(0 \le \delta < 1)$ consists of real functions whose k-th derivatives are δ-Hölder continuous on $\overline{\Omega}$. Denote for simplicity $C(\overline{\Omega}) = C^0(\overline{\Omega})$ and

$$C_0(\overline{\Omega}) = \{u \in C(\overline{\Omega}) \colon u|_{\partial\Omega} = 0\}.$$

The standard Sobolev spaces are denoted by $W^{k,q}(\Omega)$ and $W_0^{k,q}(\Omega)$ for $q \ge 1$.

For notational convenience, we define the following spaces:

$$\mathbf{C}^{k+\delta}(\overline{\Omega}) = [C^{k+\delta}(\overline{\Omega})]^2; \qquad \mathbf{C}_0(\overline{\Omega}) = [C_0(\overline{\Omega})]^2;$$

$$\mathbf{W}^{k,q}(\Omega) = [W^{k,q}(\Omega)]^2, \qquad \mathbf{W}_0^{k,q}(\Omega) = [W_0^{k,q}(\Omega)]^2;$$

and

$$\mathbf{C}(\overline{\Omega}) = [C(\overline{\Omega})]^2, \qquad \mathbf{L}^q(\Omega) = [L^q(\Omega)]^2.$$

For brevity, we may drop "(Ω)" and "$(\overline{\Omega})$" from the above notations and simply write, say, $\mathbf{W}^{k,q}(\Omega)$ as $\mathbf{W}^{k,q}$.

The notation $\|\cdot\|_X$ denotes the norm of a Banach space X, and $(\cdot, \cdot)$ represents the inner product on either L^2 or $\mathbf{L}^2$.

• **Classical solutions of** (7)-(8)**.** Given $0 < T \le \infty$, denote $\mathbf{C}^{2,1}(Q_T)$ the set of vector-valued functions $\mathbf{u} = (u, v)$ on $Q_T \colon = \Omega \times (0, T)$ which are twice and once continuously differentiable in x and t, respectively.

Definition 1 Let $\mathbf{u}_0 = (u_0, v_0) \in \mathbf{C}_0(\overline{\Omega})$ and $0 < T \le \infty$. A *classical solution* of (7)-(8) on $\overline{Q}_T \colon = \overline{\Omega} \times [0, T)$ is a continuous vector-valued function $\mathbf{u} = (u, v)$ on $\overline{Q}_T$ with $\mathbf{u}(0) = \mathbf{u}_0$ and $\mathbf{u} \in \mathbf{C}^{2,1}(Q_T)$ such that u, v fulfill (7) and (8) in the classical sense.

For convenience in statement, denote

$$\mathbf{u} = \mathbf{u}(x, t; \mathbf{u}_0) = (u(x, t; \mathbf{u}_0),\ v(x, t; \mathbf{u}_0))$$

the classical solution of system (1.7)-(1.8) with initial value $\mathbf{u}_0$. The maximal existence interval of $\mathbf{u}$ is denoted by $[0, T_{\mathbf{u}_0})$. Following standard practice, we may write $\mathbf{u}(x, t; \mathbf{u}_0) =: \mathbf{u}(t; \mathbf{u}_0)$ and regard the solution $\mathbf{u}$ as a mapping from $[0, T_{\mathbf{u}_0})$ to $\mathbf{C}_0(\overline{\Omega})$. When the dependence on the initial value is clear from context, we omit $\mathbf{u}_0$ and simply write $\mathbf{u}(t)$.

• **A fundamental inequality.** The following simple inequality will be used in establishing the $\mathbf{L}^{\infty}$-estimate for system (7)-(8).

Lemma 1 ([16, Appendix B, Lemma B.1]) Let $p > 0$. Then for any $a_1, \cdots, a_m \geq 0$,

$$(a_1 + \cdots + a_m)^p \leq m^p (a_1^p + \cdots + a_m^p).$$

3. A priori L^{∞}-estimate for classical solutions

The main result of this section is the following uniform bound for classical solutions.

Theorem 2 Assume $\mathbf{G} = (f, g)$ satisfies hypothesis $(\mathbf{F}0)$. Then there exist positive constants λ, ρ_*, θ such that for every initial datum $\mathbf{u}_0 = (u_0, v_0) \in \mathbf{C}_0(\overline{\Omega})$, the classical solution $\mathbf{u} = (u, v)$ of (7)-(8) satisfies

$$\|u(t)\|_{L^\infty} + \|v(t)\|_{L^\infty}^{\theta} \leq 2e^{-\lambda_* t}\left(\|u_0\|_{L^\infty} + \|v_0\|_{L^\infty}^{\theta}\right) + \rho_*, \qquad 0 \leq t < T_{\mathbf{u}_0}. \tag{12}$$

Proof. Let $2 \leq p, q < \infty$, where for each p fixed, q is to be further determined. Taking the scalar product of $\mathbf{u}(t; \mathbf{u}_0) := (u(t), v(t))$ in $\mathbf{L}^2(\Omega)$ with $\mathbf{w} := \left(p|u|^{p-2}u,\ q|v|^{q-2}v\right)$, we obtain that

$$\begin{aligned} &\frac{d}{dt}\left(\|u\|_{L^p}^p + \|v\|_{L^q}^q\right) \\ &+ d_1 p(p-1)\int_\Omega |u|^{p-2}|\nabla u|^2 dx + d_2 q(q-1)\int_\Omega |v|^{q-2}|\nabla v|^2 dx \\ =& \int_\Omega \mathbf{F}(\mathbf{u}) \cdot \mathbf{w} dx. \end{aligned} \tag{13}$$

Denote $\theta_1 := m/\alpha$, $\theta_2 := l/\beta$, $\widetilde{p} := p + \alpha - 2$ and $\widetilde{q} := q + \beta - 2$. Applying hypothesis $(\mathbf{F}0)$ and Young's inequality, we derive the estimates

$$\begin{aligned} \int_\Omega f(\mathbf{u})|u|^{p-2}u dx &\leq \int_\Omega \left(-\Lambda_1|u|^{\alpha} + c_1|v|^m + N_1\right)|u|^{p-2}dx \\ &\leq -\frac{\Lambda_1}{2}\|u\|_{L^{\widetilde{p}}}^{\widetilde{p}} + M_1\left(\|v\|_{L^{\theta_1\widetilde{p}}}^{\theta_1\widetilde{p}} + 1\right), \end{aligned}$$

$$\begin{aligned} \int_\Omega g(\mathbf{u})|v|^{q-2}v dx &\leq \int_\Omega \left(-\Lambda_2|v|^{\beta} + c_2|u|^l + N_2\right)|v|^{q-2}dx \\ &\leq -\frac{\Lambda_2}{2}\|v\|_{L^{\widetilde{q}}}^{\widetilde{q}} + M_2\left(\|u\|_{L^{\theta_2\widetilde{q}}}^{\theta_2\widetilde{q}} + 1\right), \end{aligned}$$

where

$$M_1 = \left(\frac{4}{\Lambda_1}\right)^{\frac{p-2}{\alpha}} \left(c_1^{\frac{p+\alpha-2}{\alpha}} + |\Omega| N_1^{\frac{p+\alpha-2}{\alpha}}\right), \qquad M_2 = \left(\frac{4}{\Lambda_2}\right)^{\frac{q-2}{\beta}} \left(c_2^{\frac{q+\beta-2}{\beta}} + |\Omega| N_2^{\frac{q+\beta-2}{\beta}}\right).$$

Therefore

$$\int_\Omega \mathbf{F}(\mathbf{u}) \cdot \mathbf{w} dx \leq -\frac{p\Lambda_1}{2} \|u\|_{L^{\widetilde{p}}}^{\widetilde{p}} + qM_2 \left(\|u\|_{L^{\theta_2 \widetilde{q}}}^{\theta_2 \widetilde{q}} + 1\right)$$

$$-\frac{q\Lambda_2}{2} \|v\|_{L^{\widetilde{q}}}^{\widetilde{q}} + pM_1 \left(\|v\|_{L^{\theta_1 \widetilde{p}}}^{\theta_1 \widetilde{p}} + 1\right).$$

We infer from ($\mathbf{F}0$) that $\theta_1 \theta_2 = \frac{ml}{\alpha\beta} < 1$. Since θ_1, $\theta_2 > 0$, one finds that $0 < \theta_1 \widetilde{p} < \frac{\widetilde{p}}{\theta_2}$. Now for each fixed $p \geq 2$, we take $q = q(p)$ such that

$$\widetilde{q} = \theta \widetilde{p}, \qquad \text{where } \theta = \frac{1}{2}\left(\theta_1 + \frac{1}{\theta_2}\right), \tag{14}$$

then

$$\theta_1 \widetilde{p} < \widetilde{q} < \frac{\widetilde{p}}{\theta_2}.$$

Using the Young's inequality once again, we obtain

$$qM_2 \left(\|u\|_{L^{\theta_2 \widetilde{q}}}^{\theta_2 \widetilde{q}} + 1\right) \leq \frac{p\Lambda_1}{4} \|u\|_{L^{\widetilde{p}}}^{\widetilde{p}} + \left(\frac{4}{p\Lambda_1}\right)^{\frac{1+\theta_1\theta_2}{1-\theta_1\theta_2}} |\Omega| (qM_2)^{\frac{2}{1-\theta_1\theta_2}} + qM_2,$$

and

$$pM_1 \left(\|v\|_{L^{\theta_1 \widetilde{p}}}^{\theta_1 \widetilde{p}} + 1\right) \leq \frac{q\Lambda_2}{4} \|v\|_{L^{\widetilde{q}}}^{\widetilde{q}} + \left(\frac{4}{q\Lambda_2}\right)^{\frac{2\theta_1\theta_2}{1-\theta_1\theta_2}} |\Omega| (pM_1)^{\frac{1+\theta_1\theta_2}{1-\theta_1\theta_2}} + pM_1.$$

Hence

$$\int_\Omega \mathbf{F}(\mathbf{u}) \cdot \mathbf{w} dx \leq -\frac{p\Lambda_1}{4} |u|_{L^{\widetilde{p}}}^{\widetilde{p}} - \frac{q\Lambda_2}{4} |v|_{L^{\widetilde{q}}}^{\widetilde{q}} + \widetilde{M}_{p,q}, \tag{15}$$

where

$$\widetilde{M}_{p,q} = \left(\frac{4}{p\Lambda_1}\right)^{\frac{1+\theta_1\theta_2}{1-\theta_1\theta_2}} |\Omega| (qM_2)^{\frac{2}{1-\theta_1\theta_2}} + qM_2 + \left(\frac{4}{q\Lambda_2}\right)^{\frac{2\theta_1\theta_2}{1-\theta_1\theta_2}} |\Omega| (pM_1)^{\frac{1+\theta_1\theta_2}{1-\theta_1\theta_2}} + pM_1.$$

Note that the condition $\alpha, \beta \geq 2$ implies $\widetilde{p} \geq p$ and $\widetilde{q} \geq q$. Applying Young's inequality then yields the estimates

$$\|u\|_{L^p}^p \leq \|u\|_{L^{\widetilde{p}}}^{\widetilde{p}} + |\Omega|, \quad \|v\|_{L^q}^q \leq \|v\|_{L^{\widetilde{q}}}^{\widetilde{q}} + |\Omega|.$$

Thus by (13) and (15) one concludes that

$$\frac{d}{dt}\left(\|u\|_{L^p}^p + \|v\|_{L^q}^q\right) \leq -\lambda_{p,q}\left(\|u\|_{L^p}^p + \|v\|_{L^q}^q\right) + M_{p,q}, \tag{16}$$

where

$$\lambda_{p,q} := \frac{1}{4}\min\{p\Lambda_1, q\Lambda_2\}, \qquad M_{p,q} = \widetilde{M}_{p,q} + \frac{1}{4}(p\Lambda_1 + q\Lambda_2)|\Omega|.$$

Applying the classical Gronwall's lemma to (16), it yields

$$\|u(t)\|_{L^p}^p + \|v(t)\|_{L^q}^q \leq e^{-\lambda_{p,q}t}\left(\|u_0\|_{L^p}^p + \|v_0\|_{L^q}^q\right) + \frac{M_{p,q}}{\lambda_{p,q}}, \qquad 0 \leq t < T_{\mathbf{u}_0}.$$

By virtue of Lemma 1, we deduce that

$$\begin{aligned}\frac{1}{2}\left(\|u(t)\|_{L^p} + \|v(t)\|_{L^q}^{q/p}\right) &\leq \left(\|u(t)\|_{L^p}^p + \|v(t)\|_{L^q}^q\right)^{1/p} \\ &\leq 2^{1/p} e^{-\frac{\lambda_{p,q}}{p}t}\left(\|u_0\|_{L^p} + \|v_0\|_{L^q}^{q/p}\right) + 2^{1/p}\left(\frac{M_{p,q}}{\lambda_{p,q}}\right)^{1/p}\end{aligned} \tag{17}$$

By the choice of q (see (14)) it is easy to verify that

$$\lim_{p\to\infty}\frac{q}{p} = \lim_{p\to\infty}\frac{\tilde{q}}{\tilde{p}} = \theta, \qquad \lim_{p\to\infty}\frac{\lambda_{p,q}}{p} = \frac{1}{4}\min\left\{\Lambda_1, \theta\Lambda_2\right\} := \lambda_*.$$

To estimate the last term in (17), we decompose $M_{p,q}$ into five distinct terms:

$$T_1 = \left(\frac{4}{p\Lambda_1}\right)^{\frac{1+\theta_1\theta_2}{1-\theta_1\theta_2}} |\Omega|(qM_2)^{\frac{2}{1-\theta_1\theta_2}},$$

$$T_2 = qM_2,$$

$$T_3 = \left(\frac{4}{q\Lambda_2}\right)^{\frac{2\theta_1\theta_2}{1-\theta_1\theta_2}} |\Omega|(pM_1)^{\frac{1+\theta_1\theta_2}{1-\theta_1\theta_2}},$$

$$T_4 = pM_1,$$

$$T_5 = \frac{1}{4}(p\Lambda_1 + q\Lambda_2)|\Omega|.$$

Substituting the explicit expression of M_2 into T_1, we obtain

$$T_1 = \left(\frac{4}{p\Lambda_1}\right)^{\frac{1+\theta_1\theta_2}{1-\theta_1\theta_2}} |\Omega| q^{\frac{2}{1-\theta_1\theta_2}} \left(\frac{4}{\Lambda_2}\right)^{\frac{2(q-2)}{\beta(1-\theta_1\theta_2)}} \left(c_2^{\frac{q+\beta-2}{\beta}} + |\Omega| N_2^{\frac{q+\beta-2}{\beta}}\right)^{\frac{2}{1-\theta_1\theta_2}}.$$

Utilizing the asymptotic relation $\lim_{p\to\infty} \frac{q}{p} = \theta$, we derive the bound

$$\limsup_{p\to\infty} \left(\frac{T_1}{\lambda_{p,\,q}}\right)^{1/p} \le \left(\frac{4}{\Lambda_2}\right)^{\frac{1+\theta_1\theta_2}{\beta\theta_2(1-\theta_1\theta_2)}} \left(c_2^{\frac{1+\theta_1\theta_2}{\beta\theta_2(1-\theta_1\theta_2)}} + N_2^{\frac{1+\theta_1\theta_2}{\beta\theta_2(1-\theta_1\theta_2)}}\right).$$

Through analogous computations, we establish the following bounds for the remaining terms

$$\limsup_{p\to\infty} \left(\frac{T_2}{\lambda_{p,\,q}}\right)^{1/p} \le \left(\frac{4}{\Lambda_2}\right)^{\frac{1+\theta_1\theta_2}{2\beta\theta_2}} \left(c_2^{\frac{1+\theta_1\theta_2}{2\beta\theta_2}} + N_2^{\frac{1+\theta_1\theta_2}{2\beta\theta_2}}\right),$$

$$\limsup_{p\to\infty} \left(\frac{T_3}{\lambda_{p,\,q}}\right)^{1/p} \le \left(\frac{4}{\Lambda_1}\right)^{\frac{1+\theta_1\theta_2}{\alpha(1-\theta_1\theta_2)}} \left(c_1^{\frac{1+\theta_1\theta_2}{\alpha(1-\theta_1\theta_2)}} + N_1^{\frac{1+\theta_1\theta_2}{\alpha(1-\theta_1\theta_2)}}\right),$$

$$\limsup_{p\to\infty} \left(\frac{T_4}{\lambda_{p,\,q}}\right)^{1/p} \le \left(\frac{4}{\Lambda_1}\right)^{1/\alpha} \left(c_1^{1/\alpha} + N_1^{1/\alpha}\right),$$

$$\limsup_{p\to\infty} \left(\frac{T_5}{\lambda_{p,\,q}}\right)^{1/p} \le 2.$$

Combining these estimates through the subadditivity of $\limsup$, we conclude that there exists a positive constant $\rho_* > 0$ such that

$$\limsup_{p\to\infty} \left(\frac{M_{p,\,q}}{\lambda_{p,\,q}}\right)^{1/p} \le \frac{\rho_*}{2}.$$

Finally, taking the limit in equation (17) as p tends to infinity, we immediately obtain the desired estimate (12). □

4. Existence and regularity of mild solutions

To establish the main results stated in Theorem 1, we begin by examining the existence and regularity properties of mild solutions for system (7)-(8) in appropriate function spaces.

4.1 *Mild solutions of abstract evolution equations*

Let X be a Banach space with norm $\|\cdot\|$, and let $A\colon D(A)\subset X\to X$ be a closed linear operator, where the domain $D(A)$ is not necessarily dense in X.

Definition 2 [17] The operator A is said to be sectorial if there exist constants $\omega\in\mathbb{R}$, $\theta\in(0,\pi/2)$, and $M>0$ such that

(i) The resolvent set satisfies $\rho(A)\supset S_{\theta,\omega}$, where

$$S_{\theta,\omega}=\{\lambda\in\mathbb{C}\colon \theta\le|\arg(\lambda-\omega)|\le\pi,\ \lambda\neq\omega\},$$

(ii) The resolvent estimate holds:

$$\|R(\lambda,A)\|_{L(X)}\le\frac{M}{|\lambda-\omega|}\quad \text{for all } \lambda\in S_{\theta,\omega}.$$

Remark 1 This definition of a sectorial operator differs from the standard one found in the literature (see e.g. Henry [18] etc.) in that it does not require A to be densely defined.

Remark 2 According to Lunardi [17, Chap. 2], if A is a sectorial operator in X, then it generates an analytic semigroup ${e^{-At}}_{t\ge0}$ on X. Moreover, the strong continuity property

$$\|e^{-At}x-x\|\to0\qquad\text{as } t\to0^+,\qquad \forall x\in X$$

holds if and only if A is densely defined, i.e., $\overline{D(A)}=X$.

Let U be a domain in a Banach space X and $f\colon U\to X$ a continuous mapping. Consider the Cauchy problem

$$u_t+Au=g(u),\qquad u(0)=u_0. \tag{18}$$

The concept of a mild solution for this problem, as defined in [17, Def. 7.0.2], does not require continuity of u at $t=0$.

Definition 3 A continuous mapping $u\colon(0,T]\to X$ with $g(u(s))\in L^1(0,T;X)$ is called a mild solution of (4.1) on $[0,T]$ if it satisfies

$$u(t)=e^{-At}u_0+\int_0^t e^{-A(t-s)}g(u(s))ds,\qquad 0\le t\le T.$$

The following lemma summarizes fundamental existence and uniqueness results for mild solutions (see, e.g., [17, Thm. 7.1.2]).

Lemma 2 Assume that $g\colon X \to X$ is locally Lipschitz continuous. Then for every $u_0 \in X$, there exists $T > 0$ such that problem (4.1) admits a unique mild solution on $[0, T]$ satisfying

$$u \in L^\infty(0, T; X).$$

4.2 *Mild solutions of system (7)-(8)*

We now turn our attention to system (7)-(8).

Let $C_b(\Omega)$ denote the Banach space of all continuous bounded functions on Ω, equipped with the supremum norm, and let $A_0 := -\Delta$ be the Dirichlet Laplacian. Define its domain as

$$D(A_0) := \left\{ u\colon \; u \in \bigcap_{q \geq 1} W^{2,q}(\Omega), \; \Delta u \in C_b(\Omega), \; u|_{\partial\Omega} = 0 \right\}.$$

Then A_0 is a sectorial operator in $C_b(\Omega)$ in the terminology of Definition 2 ([17, Coro. 3.1.21 (ii)]). Consequently $\mathbf{A}_0 = \mathrm{diag}(d_1 A_0, d_2 A_0)$ is a sectorial operator in $\mathbf{C}_b(\Omega) := [C_b(\Omega)]^2$.

Now we view (7)-(8) as an evolution system on $\mathbf{C}_b(\Omega)$:

$$\mathbf{u}' + \mathbf{A}_0 \mathbf{u} = \mathbf{F}(\mathbf{u}), \qquad \mathbf{u}(0) = \mathbf{u}_0, \tag{19}$$

where $\mathbf{u} = (u, v)$, and $\mathbf{F}(\mathbf{u})$ denotes the Nemistkii operator corresponding to $(f(x, u, v), g(x, u, v))$. If we assume that $f, g \in C^1(\overline{\Omega} \times \mathbb{R}^2)$, then one trivially verifies that $\mathbf{F}$ is a locally Lipschitz mapping on $X := \mathbf{C}_b(\Omega)$. Thus by Lemma 2, we have

Proposition 1 Assume $f, g \in C^1(\overline{\Omega} \times \mathbb{R}^2)$. Then for any $\mathbf{u}_0 \in \mathbf{C}_b(\Omega)$, there exists a $T > 0$ such that (19) possesses a unique mild solution on $[0, T]$ with

$$\mathbf{u} \in L^\infty(0, T; \mathbf{C}_b(\Omega)).$$

4.3 *Regularity of mild solutions*

To obtain some regularity results of mild solutions of (19), we need to view (19) as an evolution system on $\mathbf{L}^q(\Omega) = [L^q(\Omega)]^2$.

4.3.1 *The Dirichlet Laplacian as a sectorial operator in $L^q(\Omega)$*

Let $2 \leq q < \infty$, and let $X_q := L^q(\Omega)$. We infer from [19, Appendix E 51.1] that the Dirichlet Laplacian $A_q = -\Delta$ is a *densely defined* sectorial operator in X_q with domain $D(A_q) := W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega)$. Let X_q^α $(\alpha \geq 0)$ be the fractional power spaces generated by A_q. Since A_q possesses a compact resolvent, the embedding $X_q^\alpha \hookrightarrow X_q^\beta$ is compact (denoted as $X_q^\alpha \hookrightarrow\hookrightarrow X_q^\beta$) whenever $\alpha > \beta$. It is known that

$$X_q^{\frac{1}{2}} = W_0^{1,q}(\Omega), \qquad X_q^1 = W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega).$$

Let $\mathbf{A}_q = \mathrm{diag}(d_1 A_q, d_2 A_q)$. Then $\mathbf{A}_q$ is a densely defined sectorial operator in $\mathbf{X}_q := [X_q]^2$ with $D(\mathbf{A}_q) = \mathbf{W}^{2,q}(\Omega) \cap \mathbf{W}_0^{1,q}(\Omega)$. It is trivial to see that $\mathbf{X}_q^\alpha = [X_q^\alpha]^2$ for all $\alpha \geq 0$. Denote $\|\cdot\|_{\mathbf{L}^q}$ and $\|\cdot\|_\alpha$ ($\alpha > 0$) the norms of $\mathbf{X}_q$ and $\mathbf{X}_q^\alpha$, respectively.

The following fundamental embedding result can be found in [20, Pro. 1.3.10].

Lemma 3 Let $\alpha > 0$ and $q \in [2, \infty)$. Then the continuous embedding $\mathbf{X}_q^\alpha \hookrightarrow \mathbf{C}^{j+\delta}(\overline{\Omega})$ holds provided that

$$2\alpha - \frac{n}{q} \geq j + \delta, \qquad j \in \mathbb{Z}^+, \ \delta \in (0, 1).$$

Remark 3 Since $\mathbf{X}_q^\alpha \hookrightarrow\hookrightarrow \mathbf{X}_q^\beta$ for $\alpha > \beta$, we obtain from Lemma 3 that $\mathbf{X}_q^\alpha$ can be compactly embedded in $\mathbf{C}^{j+\delta}(\overline{\Omega})$ provided that $2\alpha - n/q > j + \delta$. In particular, the compact embedding $\mathbf{X}_q^\alpha \hookrightarrow\hookrightarrow \mathbf{C}^i(\overline{\Omega})$ holds provided that $2\alpha - n/q > i$.

Denote $e^{-\mathbf{A}_q t}$, $t \geq 0$ the *strongly continuous* analytic semigroup generated by $\mathbf{A}_q$ on $\mathbf{X}_q$. Note that $\mathrm{Re}\,\sigma(\mathbf{A}_q) > 0$. Pick a positive number $\mu < \mathrm{Re}\,\sigma(\mathbf{A}_q)$. Then by Henry [18, Theorem 1.4.3] we know that for any $\alpha \geq 0$, there exists $C_\alpha > 0$ such that

$$\|e^{-\mathbf{A}_q t}\mathbf{u}\|_\alpha \leq C_\alpha t^{-\alpha} e^{-\mu t} \|\mathbf{u}\|_{\mathbf{L}^q}, \qquad \forall t > 0, \ \mathbf{u} \in \mathbf{X}_q; \tag{20}$$

and if $\alpha \in (0, 1]$,

$$\|(e^{-\mathbf{A}_q t} - I)\mathbf{u}\|_{\mathbf{L}^q} \leq \alpha^{-1} C_{1-\alpha} t^\alpha \|\mathbf{u}\|_\alpha, \qquad \forall t \geq 0, \ \mathbf{u} \in \mathbf{X}_q^\alpha \tag{21}$$

for some $C_{1-\alpha} > 0$ independent of $\mathbf{u}$.

4.3.2 *Regularity of mild solutions*

Let $\mathbf{u}(t) = \mathbf{u}(t; \mathbf{u}_0)$ be the mild solution of (19) on $[0, T]$ given by Proposition 1. Write $\mathbf{F}(\mathbf{u}(t)) =: \mathbf{h}(t)$. Then $\mathbf{h} \in L^\infty(0, T; \mathbf{C}_b(\Omega)) \subset L^\infty(0, T; \mathbf{X}_q)$. Since $\mathbf{u}_0 \in \mathbf{C}_b(\Omega) \subset \mathbf{X}_q$, one easily sees that $\mathbf{u}$ is also a mild solution of the linear equation in $\mathbf{X}_q$:

$$\mathbf{u}' + \mathbf{A}_q \mathbf{u} = \mathbf{h}(t). \tag{22}$$

That is, $\mathbf{u}$ fulfills

$$\mathbf{u}(t) = e^{-\mathbf{A}_q t}\mathbf{u}_0 + \int_0^t e^{-\mathbf{A}_q (t-s)} \mathbf{F}(\mathbf{u}(s)) ds, \qquad t \in (0, T]. \tag{23}$$

Proposition 2 For every $\alpha \in (0, 1)$ and $\gamma \in (0, 1-\alpha)$, we have $\mathbf{u} \in C^\gamma_{\mathrm{loc}}((0, T]; \mathbf{X}_q^\alpha)$.

Proof. Throughout this proof, C denotes a generic positive constant that may change from line to line.

Since $\mathbf{u} \in L^\infty(0, T; \mathbf{C}_b(\Omega))$, by the continuity of $\mathbf{F}(\cdot)$, one knows $\mathbf{F}(\mathbf{u}) \in L^\infty(0, T; \mathbf{C}_b(\Omega))$. Then by the embedding $\mathbf{C}_b(\Omega) \hookrightarrow \mathbf{L}^q(\Omega)$, one has

$$\max_{\tau \in [0,\, T]} \|\mathbf{F}(\mathbf{u}(\tau))\|_{\mathbf{L}^q} < \infty.$$

Using (20), we readily obtain

$$\|\mathbf{u}(t)\|_\alpha \leq \|\mathbf{A}_q^\alpha e^{-\mathbf{A}_q t}\mathbf{u}_0\|_{\mathbf{L}^q} + \int_0^t \|\mathbf{A}_q^\alpha e^{\mathbf{A}_q(t-\tau)}\mathbf{F}(\mathbf{u}(\tau))\|_{\mathbf{L}^q} d\tau$$

$$\leq C\left(t^{-\alpha} + t^{1-\alpha}\right), \qquad \forall t \in (0,\, T].$$

This establishes that $\mathbf{u}(t) \in \mathbf{X}_q^\alpha$ for all $t \in (0,\, T]$.

To complete the proof, we show that $\mathbf{u}$: $(0,\, T) \to \mathbf{X}_q^\alpha$ is locally γ-Hölder continuous, thus completing the proof of the proposition. Let $0 < T_1 < T_2 \leq T$, and let $T_1 \leq s < t \leq T_2$. Then by (23),

$$\|\mathbf{u}(t) - \mathbf{u}(s)\|_\alpha \leq \left\| \left(e^{-\mathbf{A}_q(t-s)} - I\right) \mathbf{A}_q^\alpha e^{-\mathbf{A}_q s}\mathbf{u}_0 \right\|_{\mathbf{L}^q}$$

$$+ \int_0^s \left\| \left(e^{-\mathbf{A}_q(t-s)} - I\right) \mathbf{A}_q^\alpha e^{-\mathbf{A}_q(s-\tau)}\mathbf{F}(\mathbf{u}(\tau)) \right\|_{\mathbf{L}^q} d\tau$$

$$+ \int_s^t \left\| \mathbf{A}_q^\alpha e^{-\mathbf{A}_q(t-\tau)}\mathbf{F}(\mathbf{u}(\tau)) \right\|_{\mathbf{L}^q} d\tau := I_1 + I_2 + I_3.$$

Using (20) and (21) we find that

$$I_1 + I_2 \leq C|t-s|^\gamma \left(\|\mathbf{A}_q^\alpha e^{-\mathbf{A}_q s}\mathbf{u}_0\|_\gamma + \int_0^s \|\mathbf{A}_q^\alpha e^{-\mathbf{A}_q(s-\tau)}\mathbf{F}(\mathbf{u}(\tau))\|_\gamma d\tau \right)$$

$$\leq C|t-s|^\gamma \left(s^{-(\alpha+\gamma)}\|\mathbf{u}_0\|_{\mathbf{L}^q} + \int_0^s \|\mathbf{A}_q^{\alpha+\gamma} e^{-\mathbf{A}_q(s-\tau)}\| \|\mathbf{F}(\mathbf{u}(\tau))\|_{\mathbf{L}^q} \right) d\tau$$

$$\leq C|t-s|^\gamma \left(T_1^{-(\alpha+\gamma)}\|\mathbf{u}_0\|_{\mathbf{L}^q} + \mu^{\alpha+\gamma-1}\Gamma(1-\alpha-\gamma) \right) \leq C|t-s|^\gamma$$

provided $\gamma < 1 - \alpha$, and

$$I_3 \leq C\int_s^t (t-\tau)^{-\alpha} e^{-\mu(t-\tau)} d\tau \leq C|t-s|^{1-\alpha}.$$

Combined with the above two estimates and noting $\gamma < 1 - \alpha$, it immediately follows that

$$\|\mathbf{u}(t) - \mathbf{u}(s)\|_\alpha \leq C|t-s|^\gamma, \qquad \forall s,\, t \in [T_1,\, T_2]. \tag{24}$$

This completes the proof. $\square$

5. Global well-posedness of (7)-(8)

We employ the same notations in previous sections. In particular, $\mathbf{X}_q$ and $\mathbf{X}_q^\alpha$ denote the spaces given in Section 4.3 with norms $\|\cdot\|_{\mathbf{L}^q}$ and $\|\cdot\|_\alpha$, respectively.

The main result in this section is the following theorem.

Theorem 3 Assume hypothesis (**F**0). Then for any $\mathbf{u}_0 \in \mathbf{C}_0(\overline{\Omega})$, system (7)-(8) has a unique global classical solution $\mathbf{u}(x, t; \mathbf{u}_0) = (u(x, t; \mathbf{u}_0), v(x, t; \mathbf{u}_0))$ with $\mathbf{u}(t) := \mathbf{u}(\cdot, t; \mathbf{u}_0)$. Moreover, the solution satisfies

$$\mathbf{u} \in C\left([0, \infty); \mathbf{C}_0(\overline{\Omega})\right) + C_{\mathrm{loc}}^\gamma\left((0, \infty); \mathbf{C}^{1+\beta}(\overline{\Omega})\right)$$

for any $\beta \in (0, 1)$ and $\gamma \in (0, (1-\beta)/2)$.

Proof. Let $\mathbf{u}_0 \in \mathbf{C}_0(\overline{\Omega})$, and let $\mathbf{u}(t) = \mathbf{u}(t; \mathbf{u}_0)$ be the mild solution of system (19) on some interval $[0, T]$ given by Proposition 1. Given $\beta \in (0, 1)$, by Lemma 3 one can choose an $\alpha \in (1/2, 1)$ and $q > 2$ such that $\mathbf{X}_q^\alpha \hookrightarrow \mathbf{C}^{1+\beta}(\overline{\Omega}) \cap \mathbf{C}_0(\overline{\Omega})$. It then follows by Proposition 2 that

$$\mathbf{u} \in C_{\mathrm{loc}}^\gamma\left((0, T]; \mathbf{C}^{1+\beta}(\overline{\Omega})\right). \qquad \forall \gamma \in (0, (1-\beta)/2). \tag{25}$$

For clarity, we split the remaining argument into several steps.

Step 1. The continuity of $\mathbf{u}(t)$ at $t = 0$.

Specifically, we show that if $\mathbf{u}_0 \in \mathbf{C}_0(\overline{\Omega})$ then

$$\|\mathbf{u}(t) - \mathbf{u}_0\|_{\mathbf{C}_0(\overline{\Omega})} \to 0 \quad \text{as } t \to 0^+. \tag{26}$$

To this end, let $A_c := -\Delta$ be the Dirichlet Laplacian in $C_0(\overline{\Omega})$ with domain

$$D(A_c) = \left\{u:\ u \in \bigcap_{q \ge 1} W^{2,q}(\Omega),\ \Delta u \in C(\overline{\Omega}),\ \Delta u|_{\partial\Omega} = u|_{\partial\Omega} = 0\right\}.$$

We infer from [21, Chap. 7.3] that $-A_c$ generates a *strongly continuous* analytic semigroup $e^{-A_c t}$ on $C_0(\overline{\Omega})$. Consequently there is a strongly continuous analytic semigroup $e^{-\mathbf{A}_c t}$ on $\mathbf{C}_0(\overline{\Omega}) := \left[C_0(\overline{\Omega})\right]^2$ generated by$\mathbf{A}_c := \mathrm{diag}(d_1 A_c, d_2 A_c)$. We observe that

$$\|\mathbf{u}(t) - \mathbf{u}_0\|_{\mathbf{C}_0(\overline{\Omega})} \le \|\mathbf{u}(t) - e^{-\mathbf{A}_c t}\mathbf{u}_0\|_{\mathbf{C}_0(\overline{\Omega})} + \|e^{-\mathbf{A}_c t}\mathbf{u}_0 - \mathbf{u}_0\|_{\mathbf{C}_0(\overline{\Omega})}. \tag{27}$$

Recall that $\mathbf{A}_q$ is the sectorial operator defined by the Dirichlet Laplacian in $\mathbf{L}^q(\Omega)$. Since $\mathbf{A}_q = \mathbf{A}_c$ on $D(\mathbf{A}_c)$ and $\mathbf{A}_c$ generates a semigroup on $\mathbf{C}_0(\overline{\Omega}) = \overline{D(\mathbf{A}_c)}^{\mathbf{C}(\overline{\Omega})}$, it follows that $e^{-\mathbf{A}_q t}$ restricts to a semigroup on $\mathbf{C}_0(\overline{\Omega})$ and satisfies $e^{-\mathbf{A}_c t}\mathbf{u}_0 = e^{-\mathbf{A}_q t}\mathbf{u}_0$ for all $\mathbf{u}_0 \in \mathbf{C}_0(\Omega)$. Let $0 < \alpha < 1$ and $q \ge 2$ satisfy $2\alpha - n/q > 0$. Since $\mathbf{X}_q^\alpha \hookrightarrow \mathbf{C}_0(\overline{\Omega})$, we have

$$
\begin{aligned}
\|\mathbf{u}(t) - e^{-\mathbf{A}_c t}\mathbf{u}_0\|_{\mathbf{C}_0(\overline{\Omega})} &= \|\mathbf{u}(t) - e^{-\mathbf{A}_q t}\mathbf{u}_0\|_{\mathbf{C}_0(\overline{\Omega})} \\
&\leq C\|\mathbf{u}(t) - e^{-\mathbf{A}_q t}\mathbf{u}_0\|_\alpha \\
&\leq \text{ (by (23))} \leq C\int_0^t \|\mathbf{A}_q^\alpha e^{\mathbf{A}_q(t-\tau)}\mathbf{F}(\mathbf{u}(\tau))\|_{\mathbf{L}^q} d\tau \to 0
\end{aligned} \tag{28}
$$

as $t \to 0^+$, where $\|\cdot\|_\alpha$ denotes the norm on $\mathbf{X}_q^\alpha$.

On the other hand, the strong continuity of $e^{-\mathbf{A}_c t}$ on $\mathbf{C}_0(\overline{\Omega})$ implies that

$$
\lim_{t\to 0^+} \|e^{-\mathbf{A}_c t}\mathbf{u}_0 - \mathbf{u}_0\|_{\mathbf{C}_0(\overline{\Omega})} = 0.
$$

Combining this with (27) and (28), one immediately concludes the validity of (26).

It is a simple consequence of (25) and (26) that

$$
\mathbf{u} \in C\left([0, T]; \mathbf{C}_0(\overline{\Omega})\right). \tag{29}
$$

Step 2. u is a classical solution.

We now consider $\mathbf{u}(t)$ as a function of (x, t), namely,

$$
\mathbf{u}(x, t) = \mathbf{u}(t)(x), \qquad (x, t) \in \overline{Q}_T := \overline{\Omega} \times [0, T).
$$

We show that $\mathbf{u} \in \mathbf{C}^{2,1}(Q_T)$ and fulfills (7) in the classical sense on $Q_T = \Omega \times (0, T)$, and hence $\mathbf{u}$ constitutes a classical solution of (7)-(8) on $\overline{Q}_T$.

Let $0 < \varepsilon < T$. By Proposition 2 we have $\mathbf{u} \in C^\gamma\left([\varepsilon, T]; \mathbf{C}^{1+\beta}(\overline{\Omega})\right)$. Assuming $\gamma \leq \beta$, and since $\mathbf{F} \in C^1$, it follows that $\mathbf{h}(x, t) := \mathbf{F}(\mathbf{u}(x, t))$ is γ-Hölder continuous on $\overline{\Omega} \times [\varepsilon, T]$. Hence by [22, Chap. 3, Cor. 2] the initial-boundary value problem

$$
\mathbf{v}_t - \Delta\mathbf{v} = \mathbf{h}(x, t), \qquad (x, t) \in \Omega \times (\varepsilon, T], \tag{30}
$$

$$
\mathbf{v}|_{t=\varepsilon} = \mathbf{u}(\varepsilon), \qquad \mathbf{v}|_{x\in\partial\Omega} = 0 \tag{31}
$$

has a classical solution $\mathbf{v}$ on $\overline{\Omega} \times [\varepsilon, T]$. Of course $\mathbf{v}(t) := \mathbf{v}(\cdot, t)$ is a mild solution of the abstract equation (22) on $(\varepsilon, T]$ with initial value $\mathbf{u}(\varepsilon)$. Since $\mathbf{u}$ is also a mild solution of (22) on $(\varepsilon, T]$ with the same initial value, one concludes that $\mathbf{v} = \mathbf{u}$ on $\overline{\Omega} \times [\varepsilon, T]$. As ε is arbitrary, we conclude that $\mathbf{u} \in \mathbf{C}^{2,1}(Q_T)$ and fulfills (7) in the classical sense.

Step 3. Extending u to a global solution. Using the standard extension theory for semilinear parabolic equations, for any $\mathbf{u}_0 \in \mathbf{C}_b(\Omega)$, there exists a maximal existence time $0 < \mathscr{T} := T_{\mathbf{u}_0} \leq \infty$ such that the abstract Cauchy problem (19) admits a unique mild solution on $[0, \mathscr{T})$ satisfying

$$\mathbf{u} \in L^{\infty}(0, T; \mathbf{C}_b(\Omega)), \qquad \forall T \in (0, \mathscr{T}).$$

The global L^{∞}-estimate given in Theorem 2 ensures that $\|\mathbf{u}\|_{\mathbf{C}(\overline{Q}_{\mathscr{T}})} < \infty,$ from which it follows that

$$\max_{t \in [0, \mathscr{T})} \|\mathbf{F}(\mathbf{u}(t))\|_{\mathbf{L}^q} < \infty.$$

Now choose an arbitrary $0 < T_1 < \mathscr{T}$. Employing the same reasoning as in Proposition 2 with T_2 replaced by T, we establish the existence of a constant $C > 0$ such that

$$\|\mathbf{u}(t) - \mathbf{u}(s)\|_{\alpha} \leq C|t-s|^{\gamma}, \qquad \forall s, t \in [T_1, \mathscr{T}).$$

This implies that in case $\mathscr{T} < \infty$, then the limit $\lim_{t \to \mathscr{T}} \mathbf{u}(t)$ exists in $\mathbf{X}_q^{\alpha}$ (and hence in $\mathbf{C}_0(\overline{\Omega})$). By the continuation principle for mild solutions, this allows us to extend $\mathbf{u}$ to a global mild solution of problem (19). Hence we actually have $\mathscr{T} = \infty$, and $\mathbf{u}$ constitutes a global classical solution of system (19). □

In the rest part of the section, we will check the continuity of the solutions of system (7)-(8) with respect to initial data.

To facilitate the discussion, we introduce the solution operator $S(t)$, $t \geq 0$ defined by $S(t)\mathbf{u}_0 := \mathbf{u}(t; \mathbf{u}_0)$, where $\mathbf{u}(t; \mathbf{u}_0) = \mathbf{u}(\cdot, t; \mathbf{u}_0)$ denotes the classical solution of (7)-(8).

Theorem 4 Given $R > 0$, there exist $C, M_1 > 0$ such that for all $\mathbf{u}_0, \mathbf{v}_0 \in \overline{\mathbf{B}}_{\mathbf{C}_0(\overline{\Omega})}(R)$,

$$\|S(t)\mathbf{u}_0 - S(t)\mathbf{v}_0\|_{\mathbf{C}_0(\overline{\Omega})} \leq Ct^{-\alpha}e^{M_1 t}\|\mathbf{u}_0 - \mathbf{v}_0\|_{\mathbf{C}_0(\overline{\Omega})}, \qquad t > 0.$$

Proof. Let $R > 0$. Theorem 2 provides a constant $M_R > 0$ such that

$$\|\nabla_{\mathbf{u}}\mathbf{F}(S(t)\mathbf{u}_0)\|_{\mathbf{C}(\overline{\Omega})} \leq M_R, \qquad t \geq 0, \ \mathbf{u}_0 \in \overline{\mathbf{B}}_{\mathbf{C}_0(\overline{\Omega})}(R). \tag{32}$$

Given $\mathbf{u}_0, \mathbf{v}_0 \in \overline{\mathbf{B}}_{\mathbf{C}_0(\overline{\Omega})}(R)$, let

$$\mathbf{u}(t) = S(t)\mathbf{u}_0, \quad \mathbf{v}(t) = S(t)\mathbf{v}_0, \quad \mathbf{w} = \mathbf{u} - \mathbf{v}, \quad \mathbf{w}_0 = \mathbf{u}_0 - \mathbf{v}_0.$$

Applying the variation-of-constants formula (23) and utilizing the Lipschitz condition (32), we obtain

$$\|\mathbf{w}(t)\|_{\mathbf{L}^q} \leq \|e^{-\mathbf{A}_q t}\mathbf{w}_0\|_{\mathbf{L}^q} + \int_0^t \|e^{-\mathbf{A}_q(t-s)}(\mathbf{F}(\mathbf{u}(s)) - \mathbf{F}(\mathbf{v}(s)))\|_{\mathbf{L}^q} ds$$

$$\leq C_0 e^{-\mu t}\|\mathbf{w}_0\|_{\mathbf{L}^q} + C_0 M_R \int_0^t e^{-\mu(t-s)}\|\mathbf{w}(s)\|_{\mathbf{L}^q} ds$$

$$\leq M_1\left(e^{-\mu t}\|\mathbf{w}_0\|_{\mathbf{L}^q}+\int_0^t e^{-\mu(t-s)}\|\mathbf{w}(s)\|_{\mathbf{L}^q}ds\right), \qquad t\geq 0,$$

where $M_1 = M_1(\mathbf{F}, M_R, q, \Omega) > 0$. Applying the classical Gronwall lemma to the above inequality, it yields

$$\|\mathbf{w}(t)\|_{\mathbf{L}^q}\leq M_1e^{-\mu t}\left(1+M_1te^{M_1t}\right)\|\mathbf{w}_0\|_{\mathbf{L}^q}\leq M_1\left(1+M_1te^{M_1t}\right)\|\mathbf{w}_0\|_{\mathbf{L}^q}, \qquad t\geq 0. \tag{33}$$

Therefore for $\alpha\in(0, 1)$ and $q > 2$, by (23) one concludes that

$$\|\mathbf{w}(t)\|_\alpha\leq\|e^{-\mathbf{A}_qt}\mathbf{w}_0\|_\alpha+\int_0^t\|e^{-\mathbf{A}_q(t-s)}\left(\mathbf{F}(\mathbf{u}(s))-\mathbf{F}(\mathbf{v}(s))\right)\|_\alpha ds$$

$$\leq C_\alpha t^{-\alpha}e^{-\mu t}\|\mathbf{w}_0\|_{\mathbf{L}^q}+2C_\alpha M_R\int_0^t(t-s)^{-\alpha}e^{-\mu(t-s)}\|\mathbf{w}(s)\|_{\mathbf{L}^q}ds$$

$$\leq(\text{by } (33))\leq Ct^{-\alpha}(1+t+t^2e^{M_1t})\|\mathbf{w}_0\|_{\mathbf{L}^q}$$

$$\leq Ct^{-\alpha}e^{M_1t}\|\mathbf{w}_0\|_{\mathbf{C}_0(\overline{\Omega})}, \qquad t>0.$$

We may fix an $\alpha\in(0,\ 1)$ and $q>n$ so that $\mathbf{X}_q^\alpha\hookrightarrow\mathbf{C}_0(\overline{\Omega})$. Then

$$\|\mathbf{w}(t)\|_{\mathbf{C}_0(\overline{\Omega})}\leq C\|\mathbf{w}(t)\|_\alpha\leq Ct^{-\alpha}e^{M_1t}\|\mathbf{w}_0\|_{\mathbf{C}_0(\overline{\Omega})}, \qquad t>0.$$

This finishes the proof of what we desired. □

6. Global attractors and exponential attractors in $\mathbf{C}_0(\overline{\Omega})$

First, we recall some general results about the existence of global attractors and exponential attractors for semigroups on Banach spaces.

6.1 *Existence of global attractors and exponential attractors for semigroups on Banach spaces*

Let X be a Banach space. A *semigroup* S on X is a family of continuous mappings $S = S(t)$ $(t\geq 0)$ on X satisfying that

(i) $S(0) = \mathrm{id}_X$, where id_X is the identity mapping on X;
(ii) $S(t+s) = S(t)S(s)$ for all $t,\ s\geq 0$;
(iii) the mapping $t\to S(t)u$ is continuous from $[0,\ \infty)$ into X for every $u\in X$.

Let S be a semigroup on a Banach space X.

A set $\mathscr{D}\subset X$ is called an *absorbing set* of S if for any bounded set $B\subset X$, there is a $T := T(B) > 0$ such that

$$S(t)B\subset\mathscr{D}, \qquad t\geq T.$$

Definition 4 A compact set $\mathscr{A} \subset X$ is called a *global attractor* of S, if

(i) it is invariant under S, i.e., $S(t)\mathscr{A} = \mathscr{A}$ for all $t \geq 0$; and

(ii) it attracts every bounded set in X, that is, for any bounded $B \subset X$ and any neighborhood $\mathscr{O}(\mathscr{A})$, there exists a $T := T(B, \mathscr{O}) > 0$ such that

$$S(t)B \subset \mathscr{O}(\mathscr{A}), \qquad t \geq T.$$

The following definition of an exponential attractor is adapted from [23], see also [1, 24–26].

Definition 5 A set $\mathscr{M} \subset X$ is called an exponential attractor for S if

(i) $\mathscr{M}$ is compact in X;

(ii) $\mathscr{M}$ is positively invariant, i.e., $S(t)\mathscr{M} \subset \mathscr{M}$, $\forall t \geq 0$;

(iii) $\mathscr{M}$ has a finite fractal dimension in X, i.e., $\dim_F(\mathscr{M}, X) < \infty$;

(iv) $\mathscr{M}$ attracts every bounded in X set exponentially, i.e., for any bounded set $B \subset X$,

$$\text{dist}(S(t)B, \mathscr{M}) \leq Q(\|B\|_X)e^{-\gamma t}$$

for some $\gamma > 0$ and monotone function Q independent of B, where $\|B\| := \sup_{x\in B} \|x\|$.

Regarding the existence of global attractors, we recall the following result.

Proposition 3 [27] Let $S = S(t)$ $(t \geq 0)$ be a semigroup on a Banach space X. Suppose that S has a compact absorbing set B. Then S possesses a global attractor $\mathscr{A}$ given by

$$\mathscr{A} = \bigcap_{\tau>0} \overline{\bigcup_{t\geq\tau} S(t)B}.$$

The following proposition summarizes some fundamental results from [1, Sec. 8] (see also [25]) regarding the existence of an exponential attractor for S.

Proposition 4 Assume there exist a Banach space Y compactly embedded in X (denoted $Y \hookrightarrow\hookrightarrow X$), a time $T > 0$, and a bounded closed set $B \subset X$ such that the following conditions hold:

(S1) $S(T)B \subset B \cap Y$;

(S2) there is a constant $K > 0$ such that

$$\|S(T)u - S(T)v\|_Y \leq K\|u - v\|_X, \qquad u, v \in B;$$

(S3) there exists $\gamma \in (0, 1]$ and $L > 0$ such that

$$\|S(t)u - S(s)v\|_X \leq L(\|u - v\|_X + |t - s|^\gamma), \qquad u, v \in B, \ t, s \in [T, 2T]. \tag{34}$$

Then S admits an exponential attractor $\mathscr{M} \subset B$.

Remark 4 We point out that the Hölder-continuity hypothesis (S3) above is only used to guarantee the finite dimensionality of $\mathscr{M}$. This means that if we drop it from Proposition 4, then one can still show that there is a compact set $\mathscr{M}$ satisfying all the requirements in Definition 5 except (iii) therein. See, e.g., [1, 25].

6.2 *Global attractor of system (7)-(8) in* $\mathbf{C}_0(\overline{\Omega})$

Now let $S = S(t)$ $(t \geq 0)$ denote the semigroup generated by (7)-(8) on $\mathbf{C}_0(\overline{\Omega})$, namely,

$$S(t)\mathbf{u}_0 = \mathbf{u}(\cdot,\, t;\, \mathbf{u}_0), \qquad t \geq 0,\ \ \mathbf{u}_0 \in \mathbf{C}_0(\overline{\Omega}),$$

where $\mathbf{u}(x,\, t;\, \mathbf{u}_0)$ is the global classical solution of (7)-(8) for given initial value $\mathbf{u}_0$.

Theorem 5 Assume $\mathbf{F}$ satisfies $(\mathbf{F}0)$. Then the semigroup S admits a global attractor $\mathscr{A}$ in the space $\mathbf{C}_0(\overline{\Omega})$. Moreover, finite fractal dimensionality of $\mathscr{A}$ is also established.

Proof. Let $X = \mathbf{C}_0(\overline{\Omega})$. We first verify that the closed ball $\overline{\mathbf{B}}_X(\rho_0)$ is an absorbing set of S in X, where

$$\rho_0 = \rho_* + (\rho_* + 1)^{1/\theta} + 1$$

with $\rho_*,\, \theta > 0$ being the constants from in Theorem 2.

Indeed, let $R > 0$. Then one trivially verifies that

$$\|u_0\|_{\mathbf{C}_0(\overline{\Omega})} + \|v_0\|^{\theta}_{\mathbf{C}_0(\overline{\Omega})} \leq R + R^{\theta}, \qquad \forall\,(u_0,\, v_0) \in \overline{\mathbf{B}}_X(R).$$

Thus by Theorem 2, there exists $t_0 = t_0(R) > 0$ such that

$$\|u(t)\|_{\mathbf{C}_0(\overline{\Omega})} + \|v(t)\|^{\theta}_{\mathbf{C}_0(\overline{\Omega})} < \rho_* + 1, \qquad t \geq t_0,\ \ \mathbf{u}_0 = (u_0,\, v_0) \in \overline{\mathbf{B}}_X(R),$$

where $\mathbf{u}(t) := (u(t),\, v(t))$ denotes the solution of (7)-(8) with initial value $\mathbf{u}_0$, from which it immediately follows that

$$\|\mathbf{u}(t)\|_{\mathbf{C}_0(\overline{\Omega})} = \|u(t)\|_{\mathbf{C}_0(\overline{\Omega})} + \|v(t)\|_{\mathbf{C}_0(\overline{\Omega})} < \rho_0, \qquad t \geq t_0,\ \ \mathbf{u}_0 \in \overline{\mathbf{B}}_X(R). \tag{35}$$

From (35), we deduce the existence of a constant $M_{\mathbf{F}} > 0$, independent of R, such that

$$\|\mathbf{F}(\mathbf{u}(t))\|_{\mathbf{C}(\overline{\Omega})} \leq M_{\mathbf{F}}, \qquad t \geq t_0,\ \ \mathbf{u}_0 \in \overline{\mathbf{B}}_X(R). \tag{36}$$

Let $\mathbf{X}_q$, $\mathbf{X}_q^{\alpha}$ be the spaces introduced in Section 4. We fix an $\alpha \in (0,\, 1)$ and $q > n$ such that $\mathbf{X}_q^{\alpha} \hookrightarrow\hookrightarrow \mathbf{C}(\overline{\Omega})$. For any $\mathbf{u}_0 \in \overline{\mathbf{B}}_X(R)$, let $\mathbf{u}(t) = S(t)\mathbf{u}_0$. Then using (23) one finds that

$$\begin{aligned}\|\mathbf{u}(t)\|_{\alpha} &\leq C_{\alpha}\left(e^{-\mu}\|\mathbf{u}(t-1)\|_{\mathbf{L}^q} + \int_{t-1}^{t}(t-\tau)^{-\alpha}e^{-\mu(t-\tau)}\|\mathbf{F}(\mathbf{u}(\tau))\|_{\mathbf{L}^q}d\tau\right)\\ &\leq (\text{by (35) and (36)}) < \rho_1, \qquad t \geq t_0 + 1,\end{aligned} \tag{37}$$

where ρ_1 is a positive constant independent of R. This shows that the closed ball $\overline{\mathbf{B}}_{\mathbf{X}_q^\alpha}(\rho_1+1)$ is a compact absorbing set of S in the space $\mathbf{C}_0(\overline{\Omega})$. Applying Proposition 3 we conclude that S possesses a unique global attractor $\mathscr{A}$ in $\mathbf{C}_0(\overline{\Omega})$.

The finite-dimensional character of $\mathscr{A}$ is a consequence of the existence of an exponential attractor, which will be established in Theorem 6 below. □

6.3 *Existence of exponential attractors of (7)-(8)*

We now establish the existence of exponential attractors for system (7)-(8).

Theorem 6 Assume $\mathbf{F}$ satisfies (**F**0). Then S possesses an exponential attractor $\mathscr{M}$ in the phase space $\mathbf{C}_0(\overline{\Omega})$.

Proof. Let $X := \mathbf{C}_0(\overline{\Omega})$, and let $Y := \mathbf{X}_q^\alpha$ (whose norm is denoted by $\|\cdot\|_\alpha$). Take a $q > n$ and $\alpha \in (1/2, 1)$ such that $Y \hookrightarrow\hookrightarrow X$. Let $B = \overline{\mathbf{B}}_{\mathbf{X}_q^\alpha}(\rho_1+1)$, where ρ_1 is the constant given in (37). From the proof of Theorem 5, we know that B is a compact absorbing set for S in X. Thus one can take a $T > 0$ such that

$$S(T)B \subset B \subset B \cap Y.$$

Hence hypothesis (S1) in Proposition 4 holds.

For any $\mathbf{u}_0 \in X$, write $\mathbf{u}(t) = S(t)\mathbf{u}_0$. By (36) there exists $M_B > 0$ such that

$$\|\mathbf{F}(\mathbf{u}(t))\|_{\mathbf{C}(\overline{\Omega})} \le M_B, \qquad t \ge 0, \;\; \mathbf{u}_0 \in B. \tag{38}$$

Since $\mathbf{F} \in \mathbf{C}^1$, we consequently have

$$\|\nabla_{\mathbf{u}}\mathbf{F}(\mathbf{u}(t))\|_{\mathbf{C}(\overline{\Omega})} \le M_B', \qquad t \ge 0, \;\; \mathbf{u}_0 \in B. \tag{39}$$

For $\mathbf{u}(t) = S(t)\mathbf{u}_0$ and $\mathbf{v}(t) = S(t)\mathbf{v}_0$, using (38) and (39) one trivially verifies that for all $\mathbf{u}_0, \mathbf{v}_0 \in B$,

$$\begin{aligned}\|\mathbf{F}(\mathbf{u}(t)) - \mathbf{F}(\mathbf{v}(t))\|_{\mathbf{X}_q} &= \|\mathbf{F}(\mathbf{u}(t)) - \mathbf{F}(\mathbf{v}(t))\|_{\mathbf{L}^q} \\ &\le C\|\mathbf{F}(\mathbf{u}(t)) - \mathbf{F}(\mathbf{v}(t))\|_{\mathbf{C}(\overline{\Omega})} \le C\|\mathbf{u}(t) - \mathbf{v}(t)\|_{\mathbf{C}_0(\overline{\Omega})}, \qquad t \ge 0.\end{aligned}$$

Further by (23) we deduce that

$$\begin{aligned}\|\mathbf{u}(T) - \mathbf{v}(T)\|_\alpha \le & \|e^{-\mathbf{A}_q T}(\mathbf{u}_0 - \mathbf{v}_0)\|_\alpha + \int_0^T \|e^{-\mathbf{A}_q(T-s)}[\mathbf{F}(\mathbf{u}(s)) - \mathbf{F}(\mathbf{v}(s))]\|_\alpha ds \\ = & \|\mathbf{A}_q^\alpha e^{-\mathbf{A}_q T}(\mathbf{u}_0 - \mathbf{v}_0)\|_{\mathbf{L}^q} + \int_0^T \|\mathbf{A}_q^\alpha e^{-\mathbf{A}_q(T-s)}[\mathbf{F}(\mathbf{u}(s)) - \mathbf{F}(\mathbf{v}(s))]\|_{\mathbf{L}^q} ds \\ \le & C_T\left(\|\mathbf{u}_0 - \mathbf{v}_0\|_{\mathbf{C}_0(\overline{\Omega})} + \int_0^T (T-s)^{-\alpha} e^{-\mu(T-s)}\|\mathbf{u}(s) - \mathbf{v}(s)\|_{\mathbf{C}_0(\overline{\Omega})} ds\right)\end{aligned}$$

$$\leq (\text{by Thm. 4}) \leq C'_T \|\mathbf{u}_0 - \mathbf{v}_0\|_{\mathbf{C}_0(\overline{\Omega})}, \qquad \forall \mathbf{u}_0, \mathbf{v}_0 \in B.$$

This assures hypothesis (S2) in Proposition 4.

The verification of Proposition 4 (S3) constitutes the final component required for completing the demonstration.

Fix a $\gamma \in (0, 1-\alpha)$. Then using a similar argument as in verifying (24), we establish the existence of a constant $L_1 > 0$ such that

$$\|S(t)\mathbf{u}_0 - S(s)\mathbf{u}_0\|_\alpha \leq L_1 |t-s|^\gamma, \qquad \forall s, t \in [T, 2T], \ \mathbf{u}_0 \in B.$$

Then

$$\begin{aligned} \|S(t)\mathbf{u}_0 - S(s)\mathbf{u}_0\|_{\mathbf{C}_0(\overline{\Omega})} &\leq C\|S(t)\mathbf{u}_0 - S(s)\mathbf{u}_0\|_\alpha \\ &\leq CL_1 |t-s|^\gamma, \qquad \forall s, t \in [T, 2T], \ \mathbf{u}_0 \in B. \end{aligned} \tag{40}$$

Through analytical implementation of Theorem 4, there is necessarily derived a strictly positive constant $L_2 := CT^{-\alpha} e^{2M_1 T} > 0$ such that for any $u_0 \in B$ and $t \in [T, 2T]$,

$$\|\mathbf{u}(t) - \mathbf{v}(t)\|_{\mathbf{C}_0(\overline{\Omega})} \leq L_2 \|\mathbf{u}_0 - \mathbf{v}_0\|_{\mathbf{C}_0(\overline{\Omega})}. \tag{41}$$

Combining (40) and (41) one immediately concludes that (34) in hypothesis (S3) holds true with $L = \max\{CL_1, L_2\}$. □

6.4 *Two examples*

Example 1 Consider the FitzHugh-Nagumo-type reaction-diffusion system (which were used to model excitable systems such as neurons in human physiology [3, 13]):

$$u_t = \varepsilon^2 \Delta u + f(u) - v, \qquad x \in \Omega, \ t > 0, \tag{42}$$

$$v_t = \Delta v - \gamma v + \delta_\varepsilon u, \qquad x \in \Omega, \ t > 0. \tag{43}$$

$$u|_{\partial\Omega} = v|_{\partial\Omega} = 0, \tag{44}$$

where $u(x, t)$ and $v(x, t)$ represent the membrane potential and sodium gating variable, respectively; $\varepsilon > 0$ is a parameter, $\gamma > 0$ is a constant, $\delta_\varepsilon > 0$ depends on ε. Suppose that $f \in C^1$, and that there exist $c_0, \sigma > 0$ such that

$$f(u)u \leq -c_0(|u|^{2+\sigma} + 1), \qquad u \in \mathbb{R}. \tag{45}$$

The satisfaction of condition (**F**0) by the operator $\mathbf{F}(u, v) := (f(u) - v, -\gamma v + \delta_\varepsilon u)$ is readily verifiable. This in turn guarantees the existence of: (i) A global attractor $\mathscr{A}$ (ii) An exponential attractor $\mathscr{M}$ for the solution semigroup S governing the initial value problem in $\mathbf{C}_0(\overline{\Omega})$.

A typical choice of f in practice is that $f(u) = u(u-a)(1-u)$ with the parameter $a \in (0, 1/2)$, which clearly satisfies (45).

Note that we do not impose on f any growth restrictions.

Example 2 Many important phenomena in physics, chemical reaction and biology can be modeled by two-component reaction-diffusion systems like (7) with

$$\begin{cases} f(x, u, v) & = -a(x, u, v)u^{2p+1} + P_m(u, v), \\ g(x, u, v) & = -b(x, u, v)v^{2q+1} + Q_l(u, v), \end{cases} \tag{46}$$

where

$$P_m(u, v) = \sum_{i \le 2p,\ i+j \le m} a_{ij} u^i v^j, \qquad Q_l(u, v) = \sum_{j \le 2q,\ i+j \le l} b_{ij} u^i v^j,$$

$p, q, i, j \in \mathbb{Z}^+$, $a(x, u, v)$ and $b(x, u, v)$ are positive functions, and a_{ij} and b_{ij} are constants.

Under the regularity conditions $a, b \in C^1$ with uniform positivity bounds established in (47), namely

$$a(x, u, v),\ b(x, u, v) \ge \delta > 0, \qquad \forall (u, v) \in \mathbb{R}^2, \tag{47}$$

the verification of hypothesis (**F**0) for $\mathbf{F} = (f, g)$ becomes mathematically tractable, contingent upon satisfaction of at least one subsequent requirement:

(**H**1)

$$m \le 2p+1, \quad l \le 2q+1; \quad ml < (2p+1)(2q+1).$$

(**H**2)

$$m > 2p+1, \quad l < (2q+1)/(m-2p);$$

or,

$$l > 2q+1, \quad m < (2p+1)/(l-2q).$$

Indeed, by (47) we have

$$f(x, u, v)u \le -\delta u^{2(p+1)} + M \sum_{i \le 2p,\ i+j \le m} |u|^{i+1} |v|^j, \tag{48}$$

$M=\max\{|a_{ij}|:\ i\le 2p,\ i+j\le m\}$. Since $i+1<2(p+1)$, by the Young inequality we find that

$$|u|^{i+1}|v|^{j}\le \varepsilon|u|^{2(p+1)}+C_{ij}(\varepsilon)|v|^{2(p+1)j/(2p+1-i)} \tag{49}$$

for any $\varepsilon>0$. Noticing that $i+j\le m$, we have

$$\frac{2(p+1)j}{2p+1-i}\le\frac{2(p+1)(m-i)}{2p+1-i}\le m_1$$

for all $i,\ j$, where

$$m_1=\begin{cases}2(p+1)m/(2p+1), & \text{if } m\le 2p+1;\\[2ex] 2(p+1)(m-2p), & \text{if } m>2p+1.\end{cases}$$

Therefore applying Young's inequality once again, by (48) and (49) one easily deduces that

$$f(x,\,u,\,v)u\le-\left(\delta-\frac{1}{2}M(m+1)(m+2)\varepsilon\right)u^{2(p+1)}+C(\varepsilon)\left(|v|^{m_1}+1\right). \tag{50}$$

Now we take $\varepsilon=\delta/(M(m+1)(m+2))$. Then (50) amounts to say that

$$f(x,\,u,\,v)u\le-\frac{\delta}{2}u^{2(p+1)}+C_0\left(|v|^{m_1}+1\right),\qquad \forall\,(x,\,u,\,v)\in\overline{\Omega}\times\mathbb{R}^2$$

for some $C_0>0$.

Similarly we also have

$$g(x,\,u,\,v)v\le-\frac{\delta}{2}v^{2(q+1)}+C_1\left(|v|^{l_1}+1\right),\qquad \forall\,(x,\,u,\,v)\in\overline{\Omega}\times\mathbb{R}^2,$$

where

$$l_1=\begin{cases}2(q+1)l/(2q+1), & \text{if } l\le 2q+1;\\[2ex] 2(q+1)(l-2q), & \text{if } l>2q+1.\end{cases}$$

Now assume (**H**1) or (**H**2) holds. Then it is trivial to check that

$$m_1l_1<4(p+1)(q+1).$$

Hence we see that hypothesis ($\mathbf{F}0$) is fulfilled by $\mathbf{F}$. As a result, we have

Theorem 7 Suppose the nonlinear functions f and g in system (2.1) take the form specified in (46), and assume the conditions in (47) together with either ($\mathbf{H}1$) or ($\mathbf{H}2$) are satisfied. Under these assumptions, the initial-boundary value problem is globally well-posed in the classical sense. Moreover, the solution semigroup S acting on $\mathbf{C}_0(\overline{\Omega})$ admits both a global attractor $\mathscr{A}$ and an exponential attractor $\mathscr{M}$, each possessing finite fractal dimension.

An illustration example for f and g satisfying (47) and ($\mathbf{H}2$) is that

$$f(x,\,u,\,v)=-u^3+uv^4,\qquad g(x,\,u,\,v)=-v^5+u.$$

Remark 5 We do not know whether the conclusions in Theorem 7 remain true if instead of ($\mathbf{H}1$) and ($\mathbf{H}2$), we only assume that

$$ml<(2p+1)(2q+1).$$

More generally, we have the following open question.

• **Open question** Do the results in previous sections hold true for system (7)-(8) if we replace the dissipative-type hypothesis ($\mathbf{F}0$) by a weaker one as below:

$(\mathbf{F}0)'$ There exist $\alpha,\,\beta\geq 2$ and $\Lambda_1,\,\Lambda_2>0$ such that

$$f(x,\,u,\,v)u\leq-\Lambda_1|u|^{\alpha}+P(u,\,v),\qquad g(x,\,u,\,v)v\leq-\Lambda_2|v|^{\beta}+Q(u,\,v)$$

for all $(x,\,u,\,v)\in\overline{\Omega}\times\mathbb{R}^2$, where

$$P(u,\,v)=N_1\sum_{0\leq i,\,j\leq m}|u|^{\alpha_i}|v|^{\mu_j},\qquad Q(u,\,v)=N_2\sum_{0\leq i,\,j\leq l}|u|^{\gamma_i}|v|^{\beta_j},$$

$N_1,\,N_2,\,\alpha_i,\,\mu_j,\,\gamma_i,\,\beta_j\geq 0$, and

(i)

$$\alpha_i<\alpha\ (i\leq m),\qquad \beta_j<\beta\ (j\leq l);$$

(ii)

$$\max_{0\leq i,\,j\leq m}(\alpha_i+\mu_j)\max_{0\leq i,\,j\leq l}(\gamma_i+\beta_j)<\alpha\beta.$$

7. Positive solutions and attractors

For many mathematical models from applications, only positive solutions are of interest. In this section we make some remarks on this subject.

For $\mathbf{u}\in\mathbb{R}^m$, we will write $\mathbf{u}\geq 0$ ($\mathbf{u}>0$) if all its entries are nonnegative (positive). Denote $\mathbb{R}^m_+=\{\mathbf{u}\in\mathbb{R}^m\colon\ \mathbf{u}\geq 0\}$. The following result concerning positive solutions of reaction-diffusion systems is well known and is a consequence of maximum principle (see e.g. [8]):

Lemma 4 Let $\mathbf{F} = (g_1, g_2, \cdots, g_m) \in C^1(\overline{\Omega} \times \mathbb{R}^m; \mathbb{R}^m)$. Assume that for all $1 \le i \le m$,

$$g_i(x, \mathbf{u}) \ge 0, \qquad \forall (x, \mathbf{u}) \in \overline{\Omega} \times \mathbb{R}^m_+, \ u_i = 0. \tag{51}$$

Then for any initial value $\mathbf{u}_0 \in \mathbf{C}_0(\overline{\Omega})$, $\mathbf{u}_0 \ge 0$, the homogenous Dirichlet boundary value problem of system (1) has on some interval $[0, T)$ a unique nonnegative classical solution $\mathbf{u}$.

We now return back to the two-component system (7)-(8). Let $\mathbf{F} = (f, g)$ be the nonlinearity therein. Combining Theorems 3 and 4, we obtain

Theorem 8 Assume that

($\mathbf{F}1$) $\mathbf{F}$ satisfies hypothesis ($\mathbf{F}0$) with $\mathbb{R}^2$ therein replaced by $\mathbb{R}^2_+$; and

($\mathbf{F}2$) for all $x \in \overline{\Omega}$ and $s \ge 0$,

$$f(x, 0, s) \ge 0, \qquad g(x, s, 0) \ge 0.$$

Then for any nonnegative function $\mathbf{u}_0 \in \mathbf{C}_0(\overline{\Omega})$, system (7)-(8) has a unique nonnegative global classical solution; furthermore, Theorem 4 holds true with these nonnegative solutions.

Theorem 8 allows us to define a solution semigroup $S_+ = S_+(t)$ $(t \ge 0)$ of (7)-(8) on

$$\mathbf{P}_+ := \{\mathbf{u} \in \mathbf{C}_0(\overline{\Omega}) : \ \mathbf{u} \ge 0\}.$$

Repeating the same argument in Section 6 with $\mathbf{C}_0(\overline{\Omega})$ therein replaced by $\mathbf{P}_+$, we obtain

Theorem 9 Assume the hypotheses ($\mathbf{F}1$) and ($\mathbf{F}2$) in Theorem 8. Then S_+ has a unique global attractor $\mathscr{A}$ in $\mathbf{P}_+$ with finite fractal dimension.

Example 3 Consider Beddington-DeAngelis predator-prey model incorporating prey refuge and self-diffusion effects [11]:

$$u_t = d_1 \Delta u + ru\left(1 - \frac{u}{K}\right) - \frac{auv}{b + u + cv}, \qquad x \in \Omega, \ t > 0, \tag{52}$$

$$v_t = d_2 \Delta v - dv + \frac{eauv}{b + u + cv} - hv^2, \qquad x \in \Omega, \ t > 0. \tag{53}$$

$$u|_{\partial\Omega} = v|_{\partial\Omega} = 0, \tag{54}$$

where $\Omega \subset \mathbb{R}^n$ is a smooth bounded domain representing the habitat. The state variables $u(t, x)$ and $v(t, x)$ denote the population densities of prey and predator species, respectively, subject to homogeneous Dirichlet boundary conditions.

The system parameters satisfy the following biologically meaningful constraints:

- Diffusion coefficients: $d_1 > 0$ (prey motility), $d_2 > 0$ (predator dispersal)
- Prey dynamics: $r > 0$ (intrinsic growth rate), $K > 0$ (environmental carrying capacity)
- Predation parameters: $a > 0$ (maximum predation rate), $b > 0$ (saturation constant), $c > 0$ (predator interference scaling)
- Conversion and mortality: $e > 0$ (biomass conversion efficiency), $d > 0$ (predator mortality rate)
- Competition term: $h > 0$ (intra-specific competition among predators)

The functional response term $\frac{auv}{b+u+cv}$ incorporates both prey saturation and predator interference effects, while the term hv^2 models density-dependent regulation within the predator population. The homogeneous Dirichlet boundary conditions reflect hostile environmental conditions at the habitat boundaries.

It is easy to see that all the hypotheses in the above theorems are fulfilled by the nonlinearities in the system. Hence all the conclusions in Theorems 8 and 9 hold true with (52).

Remark 6 This paper focuses on two-component reaction-diffusion systems. For systems with three or more components, the analysis becomes more complex. In the case of three-component reaction-diffusion systems modeling autocatalytic chemical reactions, such as the Oregonator system, the nonlinear terms face estimation barriers arising from coefficient disparities between different equations. To overcome these obstacles, You [28] introduced techniques such as rescaling and grouping estimation. Therefore, research on dissipative structures and long-term dynamics in multi-component reaction-diffusion systems remains a substantial and challenging area worthy of further investigation.

8. Concluding remarks

This study presents an analytical framework for investigating the long-term behavior of two-component reaction-diffusion systems formulated in continuous function spaces. The approach developed here successfully decouples the analysis of system dynamics from the conventional growth limitations typically required in L^2-based theories.

The principal outcomes of our investigation include:

1. The introduction of condition $(\mathbf{F}0)$, a relaxed dissipativity requirement that expands the scope of admissible nonlinearities beyond polynomial growth constraints, while maintaining guarantees for global solution existence.

2. The accommodation of asymmetric growth patterns between dissipative and non-dissipative components within the nonlinear terms, addressing a previously underexplored aspect of reaction-diffusion theory.

3. The demonstration that within the classical solution context, the system supports both finite-dimensional global attractors and exponential attractors under the proposed framework.

4. The successful adaptation of these theoretical developments to the positive solution regime, with corresponding attractor existence established in the nonnegative cone $\mathbf{P}_+$.

Several research trajectories emerge naturally from this work. The generalization of our framework to multi-component systems or equations incorporating nonlocal operators represents an immediate extension. Furthermore, elucidating the connections between condition $(\mathbf{F}0)$ and specific applied contexts in the physical and biological sciences would strengthen the practical relevance of these theoretical developments. The construction of effective numerical methods leveraging these continuous-space results presents another valuable direction for implementation-focused research.

While the current formulation substantially widens the range of tractable nonlinearities, the structural assumptions embedded in condition $(\mathbf{F}0)$ necessarily define its operational boundaries and cannot be directly extended to reaction-diffusion systems with more than two components. Subsequent investigations might explore further relaxations of these requirements or alternative analytical paradigms capable of encompassing even more diverse nonlinear phenomena.

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Conflict of interest

The authors declare no competing financial interest.

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