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Research Article

Exploring Solutions' Families of the Fractional Hirota-Satsuma Coupled KdV Equation via Extended Direct Algebraic Method

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Abstract: An extended direct algebraic method is used in this work to examine the soliton solutions of the fractional Hirota-Satsuma coupled Korteweg-de Vries equation. Understanding the dynamic behaviour of solitons in nonlinear systems using analytical solutions is our goal. To obtain precise soliton solutions, we utilise a logistic technique. These solutions are then shown graphically in three dimensions, two dimensions, and contours. Soliton interactions' complex dynamics and stability in fractional nonlinear systems are demonstrated by the results. This study clarifies the underlying dynamics and possible uses of soliton behaviour in complex systems, advancing our understanding of this phenomenon.

Keywords: fractional calculus, soliton solutions, nonlinear dynamics, Hirota-Satsuma equation, complex systems

MSC: 35C08, 35Q51, 37K40

1. Introduction

Since many events are inherently complex, line-based systems are frequently theoretical approximations of simpler nonlinear systems. On the other hand, nonlinear systems better represent the objective world. Thus, it is essential for modern science and technology to understand and research nonlinear phenomena. Nonlinear differential equations can be used to solve many nonlinear phenomena from the perspective of mathematical physics [1–4]. Nonlinear Fractional Differential Equations (NLFDEs) have attracted a lot of interest among them. NLFDEs, a type of partial differential equation, permit derivative orders with fractional values as opposed to integer values. These equations are helpful in many fields, including biology, engineering, physics, finance, and many more, particularly so expressing systems with memory

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effects, nonlocal events, or complex dynamics. It is typical practice in NLFDEs to move fractional derivatives with respect to geographical or temporal factors. Riemann-Liouville [5, 6], Beta [7], Grünwald-Letnikov [8], and Caputo [9] are the definitions of fractional derivatives that are most often employed. NLFDEs are a rapidly developing field of study that provides powerful tools for elucidating the dynamics. We may observe that increasingly more innovative applications and solution strategies for NLFDEs are appearing as computational theories and methods continue to advance.

Several efficient methods for identifying analytical solutions for NLFDEs have been developed in the literature. They include the fractional subequation method [10–12], the F-expansion method [13], the first integral method, and the generalised Riccati equation mapping approach [14–16]. Kudryashov methods [17, 18], $\exp(-\phi(z))$-expansion method [19, 20], (G'/G)-expansion method [21–23], tanh-function method, truncated Painlevé expansion method [24–26], modified simple equation method [27–29], Sine-Gordon expansion method [30], complex method [31–33], and so forth. Two long waves with distinct dispersion relations usually interact according to the Hirota-Satsuma Coupled KdV (HScKdV) equation [34]. The fractional Hirota-Satsuma Coupled KdV (fHScKdV) equation is extended to the fractional form, capturing extra levels of complexity that integer-order derivatives are unable to adequately represent. Here is the fHScKdV equation [35]:

$$\begin{cases} D_t^{\sigma}\varpi = \dfrac{1}{4}\varpi_{xxx} + 3\varpi\varpi_x + 3(-V^2 + W)_x \\ \\ D_t^{\sigma}V = -\dfrac{1}{2}V_{xxx} - 3\varpi V_x \\ \\ D_t^{\sigma}W = -\dfrac{1}{2}W_{xxx} - 3\varpi W_x. \end{cases} \tag{1}$$

Applications of the fHScKdV equation may be found in physics, engineering, and finance, among other domains. It can be used, for instance, to represent fractional-property wave and turbulence phenomena in fluid mechanics. It can be used in materials science to explain how materials with memory effects behave. It may analyse market dynamics and show long-term consequences in the finance industry. Because the fHScKdV equation is an interesting system, researchers are interested in it. Both the (G'/G)-expansion technique and the Riemann-Liouville derivative were used by Alam et al. [35] to obtain the closed-form solutions. Yin et al. [36] used the unified technique, the enhanced F-expansion method, and the homogeneous balancing approach to extract different analytical solutions. To get accurate results, Ganji et al. [37] used the homotopy perturbation technique. By using the extended Jacobian elliptic function expansion method, Yan [38] generated exact doubly periodic solutions. Kaplan and Bekir [39] used local fractional derivatives and the $\exp(-\phi(z))$-expansion method to obtain exact solutions for the fHScKdV problem.

According to our findings, the Extended Direct Algebraic Method is efficient and straightforward. Researchers in related fields might find this helpful as it enhances the analysis of the fHScKdV equation. This study investigates the fHScKdV equation and looks at its bifurcation, sensitivity, and chaotic behaviours using a novel approach called the logistic method.

1.1 *Key characteristics of the modified riemann-liouville derivative*

This study uses the Extended Differential Algebraic Method (EDAM) approach to construct exact analytical solutions for fractional differential equations that are nonlinear and where the expression σ is present.

$$
D_z^{\sigma}\varphi(z) = \begin{cases} \dfrac{1}{\Gamma(-\sigma)}\displaystyle\int_0^z (z-\psi)^{-\sigma-1}[\varphi(\psi)-\varphi(0)],\ \sigma<0 \\ \\ \dfrac{1}{\Gamma(1-\sigma)}\dfrac{d}{dz}\displaystyle\int_0^z (z-\psi)^{-\sigma}[\varphi(\psi)-\varphi(0)]dz,\ 0<\sigma<1 \\ \\ [\varphi^{(\sigma-\nu)}(z)]^{(\nu)},\ \nu\leq\sigma<\nu+1,\ \nu>1. \end{cases}
$$

where the Gamma function is defined by, $\Gamma(\sigma)$:

$$
\Gamma(\sigma) = \lim_{\nu\to\infty}\frac{\nu!\nu^{\sigma}}{(\sigma+1)(\sigma+2)(\sigma+3)...(\sigma+\nu)}.
$$

Some results are derived:

$$
D_z^{\sigma}z^{\kappa} = \frac{\Gamma(\kappa+1)}{(\kappa-\sigma+1)}z^{\kappa-\sigma},\ \kappa>0
$$

$$
D_z^{\sigma}[\varphi(z)\omega(z)] = \omega(z)D_z^{\sigma}\varphi(z)+\varphi(z)D_z^{\sigma}\omega(z)
$$

$$
D_z^{\sigma}\varphi^{[\omega(z)]} = \varphi'\omega[\omega(z)]D_z^{\sigma}\omega(z) = D\omega^{\sigma}\varphi[\omega(z)](\omega_z')^{\sigma}.
$$

Mittag-Leffler Function
The representation of the two-parameter Mittag-Leffler function is as follows:

$$
E_{\sigma,\ \tau}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\sigma k+\tau)},\quad \mathrm{Re}(\sigma)>0,\quad \mathrm{Re}(\tau)>0.
$$

Notable Characteristics of Fractional Derivatives
The following properties hold:
1.

$$
D^{\sigma}z^{\lambda} = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\sigma+1)}z^{\lambda-\sigma},\quad \lambda>0.
$$

2.

$$
D^{\sigma}(c\Phi(z)) = cD^{\sigma}\Phi(z).
$$

3.

$$D^{\sigma}(a\Phi(z)+b\Psi(z)) = aD^{\sigma}\Phi(z)+bD^{\sigma}\Psi(z).$$

2. EDAM operational procedure

To examine the Fractional Partial Differential Equation (FPDE) [40]:

$$E(w,\ D_t^{\alpha}w,\ D_{h_1}^{\beta}w,\ D_{h_2}^{\gamma}w,\ wD_{h_1}^{\beta}w,\ \ldots) = 0,\ \ 0 < \alpha,\ \beta,\ \gamma \le 1, \tag{2}$$

where $w = w(t,\ h_1,\ h_2,\ h_3,\ \ldots,\ h_n)$ is the given function, we employ the EDAM. The solution process involves the following steps:

Step 1: Transformation.

We transform the FPDE into a Nonlinear Ordinary Differential Equation (NODE) using a variable transformation $w(t,\ h_1,\ h_2,\ h_3,\ \ldots,\ h_n) = W(\tau)$, where τ is a function of $t,\ h_1,\ h_2,\ h_3,\ \ldots,\ h_n$. This yields a NODE of the form:

$$F(W,\ W',\ WW',\ \ldots) = 0. \tag{3}$$

Step 2: Solution Assumption.

We assume a solution to (3) in the form:

$$W(\tau) = \sum_{l=-F}^{F} s_j(\zeta(\tau))^j, \tag{4}$$

where s_j are parameters to be determined, and $\zeta(\tau)$ satisfies the NODE:

$$\zeta'(\tau) = \ln(N)(\omega+\delta\zeta(\tau)+\upsilon(\zeta(\tau))^2), \tag{5}$$

with $\omega,\ \delta,\ \upsilon$ and $N \neq 1,\ 0$ being constants.

Step 3: Homogeneous Balance.

We determine the positive integer F by balancing the highest-order derivative and the most dominant nonlinear term in (3).

Step 4: Algebraic Equation System.

Substituting (4) into (3) and equating coefficients of like powers of $\zeta(\tau)$ to zero, we obtain a system of nonlinear algebraic equations.

Step 5: Solution of Algebraic Equations.

We solve the system of nonlinear algebraic equations using Maple or other computational tools.

Step 6: Travelling Wave Solutions.

Using (4) and the solution $\zeta(\tau)$ from (5), we obtain the travelling wave solutions to (2) along with the unknown parameters.

The solution of (5) yields families of travelling wave solutions, which can be used to analyze the dynamics of the FPDE.

Family 1 If $\upsilon \neq 0$ and $\Upsilon < 0$ are present, where $\Upsilon = \delta^2 - 4\omega\upsilon$.

$$\zeta_1(\tau) = -\frac{1}{2}\frac{\delta}{\upsilon} + \frac{1}{2}\frac{\sqrt{-\Upsilon}\tan\left(\frac{1}{2}\sqrt{-\Upsilon}\tau\right)}{\upsilon},$$

$$\zeta_2(\tau) = -\frac{1}{2}\frac{\delta}{\upsilon} - \frac{1}{2}\frac{\sqrt{-\Upsilon}\cot\left(1/2\sqrt{-\Upsilon}\tau\right)}{\upsilon},$$

$$\zeta_3(\tau) = -\frac{1}{2}\frac{\delta}{\upsilon} + \frac{1}{2}\frac{\sqrt{-\Upsilon}\left(\tan\left(\sqrt{-\Upsilon}\tau\right) \pm \left(\sec\left(\sqrt{-\Upsilon}\tau\right)\right)\right)}{\upsilon},$$

$$\zeta_4(\tau) = -\frac{1}{2}\frac{\delta}{\upsilon} - \frac{1}{2}\frac{\sqrt{-\Upsilon}\left(\cot\left(\sqrt{-\Upsilon}\tau\right) \pm \left(\csc\left(\sqrt{-\Upsilon}\tau\right)\right)\right)}{\upsilon},$$

and

$$\zeta_5(\tau) = -\frac{1}{2}\frac{\delta}{\upsilon} + \frac{1}{4}\frac{\sqrt{-\Upsilon}\left(\tan\left(\frac{1}{4}\sqrt{-\Upsilon}\tau\right) - \cot\left(\frac{1}{4}\sqrt{-\Upsilon}\tau\right)\right)}{\upsilon}.$$

Family 2 $\upsilon \neq 0$ and $\Upsilon > 0$ are present, where $\Upsilon = \delta^2 - 4\omega\upsilon$.

$$\zeta_6(\tau) = -\frac{1}{2}\frac{\delta}{\upsilon} - \frac{1}{2}\frac{\sqrt{\Upsilon}\tanh\left(\frac{1}{2}\sqrt{\Upsilon}\tau\right)}{\upsilon},$$

$$\zeta_7(\tau) = -\frac{1}{2}\frac{\delta}{\upsilon} - \frac{1}{2}\frac{\sqrt{\Upsilon}\coth\left(\frac{1}{2}\sqrt{\Upsilon}\tau\right)}{\upsilon},$$

$$\zeta_8(\tau) = -\frac{1}{2}\frac{\delta}{\upsilon} - \frac{1}{2}\frac{\sqrt{\Upsilon}\left(\tanh\left(\sqrt{\Upsilon}\tau\right) \pm \left(sech\left(\sqrt{\Upsilon}\tau\right)\right)\right)}{\upsilon},$$

$$\zeta_9(\tau) = -\frac{1}{2}\frac{\delta}{\upsilon} - \frac{1}{2}\frac{\sqrt{\Upsilon}\left(\coth\left(\sqrt{\Upsilon}\tau\right) \pm \left(csch\left(\sqrt{\Upsilon}\tau\right)\right)\right)}{\upsilon},$$

and

$$\zeta_{10}(\tau) = -\frac{1}{2}\frac{\delta}{\upsilon} - \frac{1}{4}\frac{\sqrt{\Upsilon}\left(\tanh\left(\frac{1}{4}\sqrt{\Upsilon}\tau\right) - \coth\left(\frac{1}{4}\sqrt{\Upsilon}\tau\right)\right)}{\upsilon}.$$

Family 3 For $\omega\upsilon > 0$ and $\delta = 0$, we have:

$$\zeta_{11}(\tau) = \sqrt{\frac{\omega}{\upsilon}} \tan\left(\sqrt{\omega\,\upsilon}\,\tau\right),$$

$$\zeta_{12}(\tau) = -\sqrt{\frac{\omega}{\upsilon}} \cot\left(\sqrt{\omega\,\upsilon}\,\tau\right),$$

$$\zeta_{13}(\tau) = \sqrt{\frac{\omega}{\upsilon}} \left(\tan\left(2\sqrt{\omega\,\upsilon}\,\tau\right) \pm \left(\sec\left(2\sqrt{\omega\,\upsilon}\,\tau\right)\right)\right),$$

$$\zeta_{14}(\tau) = -\sqrt{\frac{\omega}{\upsilon}} \left(\cot\left(2\sqrt{\omega\,\upsilon}\,\tau\right) \pm \left(\csc\left(2\sqrt{\omega\,\upsilon}\,\tau\right)\right)\right),$$

and

$$\zeta_{15}(\tau) = \frac{1}{2}\sqrt{\frac{\omega}{\upsilon}} \left(\tan\left(\frac{1}{2}\sqrt{\omega\,\upsilon}\,\tau\right) - \cot\left(\frac{1}{2}\sqrt{\omega\,\upsilon}\,\tau\right)\right).$$

Family 4 For $\omega\upsilon < 0$ and $\delta = 0$, we have:

$$\zeta_{16}(\tau) = -\sqrt{-\frac{\omega}{\upsilon}} \tanh\left(\sqrt{-\omega\,\upsilon}\,\tau\right),$$

$$\zeta_{17}(\tau) = -\sqrt{-\frac{\omega}{\upsilon}} \coth\left(\sqrt{-\omega\,\upsilon}\,\tau\right),$$

$$\zeta_{18}(\tau) = -\sqrt{-\frac{\omega}{\upsilon}} \left(\tanh\left(2\sqrt{-\omega\,\upsilon}\,\tau\right) \pm \left(isech\left(2\sqrt{-\omega\,\upsilon}\,\tau\right)\right)\right),$$

$$\zeta_{19}(\Omega) = -\sqrt{-\frac{\omega}{\upsilon}} \left(\coth\left(2\sqrt{-\omega\,\upsilon}\,\tau\right) \pm \left(csch\left(2\sqrt{-\omega\,\upsilon}\,\tau\right)\right)\right),$$

and

$$\zeta_{20}(\tau) = -1/2\sqrt{-\frac{\omega}{\upsilon}} \left(\tanh\left(1/2\sqrt{-\omega\,\upsilon}\,\tau\right) + \coth\left(1/2\sqrt{-\omega\,\upsilon}\,\tau\right)\right).$$

Family 5 For $\upsilon = \omega$ and $\delta = 0$, we have:

$$\zeta_{21}(\tau) = \tan(\omega\,\tau),$$

$$\zeta_{22}(\eta) = -\cot(\omega\,\tau),$$

$$\zeta_{23}(\tau) = \tan(2\,\omega\,\tau) \pm (\sec(2\,\omega\,\tau)),$$

$$\zeta_{24}(\tau) = -\cot(2\,\omega\,\tau) \pm (\csc(2\,\omega\,\tau)),$$

and

$$\zeta_{25}(\tau) = \frac{1}{2}\tan\left(\frac{1}{2}\omega\,\tau\right) - \frac{1}{2}\cot(1/2\,\omega\,\tau).$$

Family 6 For $\upsilon = -\omega$ and $\delta = 0$, we have:

$$\zeta_{26}(\tau) = -\tanh(\omega\,\tau),$$

$$\zeta_{27}(\tau) = -\coth(\omega\,\tau),$$

$$\zeta_{28}(\tau) = -\tanh(2\,\omega\,\tau) \pm (isech(2\,\omega\,\tau)),$$

$$\zeta_{29}(\tau) = -\coth(2\,\omega\,\tau) \pm (csch(2\,\omega\,\tau)),$$

and

$$\zeta_{30}(\tau) = -\frac{1}{2}\tanh\left(\frac{1}{2}\omega\,\tau\right) - \frac{1}{2}\coth\left(\frac{1}{2}\omega\,\tau\right).$$

Family 7 For $\Upsilon = 0$, we have:

$$\zeta_{31}(\tau) = -2\,\frac{\omega\,(\delta\,\tau\ln(N)+2)}{\delta^2\tau\ln(N)}.$$

Family 8 For $\upsilon = 0$, $\delta = \varsigma$ and $\omega = n\varsigma$ (with $n \neq 0$), we have:

$$\zeta_{32}(\tau) = N^{\varsigma\tau} - n.$$

Family 9 For $\upsilon = \delta = 0$, we have:

$$\zeta_{33}(\Omega) = \omega\tau\ln(N).$$

Family 10 For $\delta = \omega = 0$, we have:

$$\zeta_{34}(\tau) = -\frac{1}{\upsilon\tau ln(N)}.$$

Family 11 For $\delta \neq 0$, $\upsilon \neq 0$ and $\omega = 0$, we have:

$$\zeta_{35}(\tau) = -\frac{\delta}{\upsilon\,(\cosh(\delta\,\tau) - \sinh(\delta\,\tau) + 1)},$$

and

$$\zeta_{36}(\tau) = -\frac{\delta\,(\cosh(\delta\,\tau) + \sinh(\delta\,\tau))}{\upsilon\,(\cosh(\delta\,\tau) + \sinh(\delta\,\tau) + 1)}.$$

Family 12 As $\delta = \varsigma$, $\upsilon = n\varsigma$, $\omega = 0$:

$$\zeta_{37}(\tau) = \frac{N^{\varsigma\tau}}{1 - nN^{\varsigma\tau}}.$$

3. Implementation of the method

For the fractional Hirota-Satma coupled KdV problem in space-time, we apply the new approach, to illustrate its effectiveness and adaptability. The complex interaction between two long waves is described by this nonlinear model, which has a clearly defined dispersion relation. By putting the new technique into practice, we want to better understand the mechanics of communication between these two waves and provide insight into the intricate phenomena that result from their interaction.

$$\begin{cases} D_t^\sigma \varpi = \dfrac{1}{4}\varpi_{xxx} + 3\varpi\varpi_x + 3(-V^2 + W)_x \\ \\ D_t^\sigma V = -\dfrac{1}{2}V_{xxx} - 3\varpi V_x \\ \\ D_t^\sigma W = -\dfrac{1}{2}W_{xxx} - 3\varpi W_x. \end{cases} \tag{6}$$

The transformations we use are based on the space-time fractional Hirota-Satsuma linked KdV equation, to facilitate the solution process: $\varpi = \varpi(x,\ t)$, $V = V(x,\ t)$, and $W = W(x,\ t)$ are functions of space x and time t, where $t > 0$ and $0 < \sigma \leq 1$.

$$\begin{cases} \varpi = \dfrac{1}{\Xi}\varpi(\tau)^2 \\ \\ V = -\Xi + \varpi(\tau) \\ \\ W = 2\Xi^2 - 2\Xi\varpi(\tau). \end{cases} \tag{7}$$

where $\tau = x - \dfrac{\Xi}{\sigma}\left(t + \dfrac{1}{\Gamma(1+\sigma)}\right)^{\sigma}$. Then (6) is reduced to the following ODE

$$\Xi\varpi'' - 2\Xi^2\varpi^2 + 2\varpi^3 = 0. \tag{8}$$

$F = 1$ is obtained by applying the homogeneous balancing rule to (8). Consequently,

$$\varpi(\tau) = \sum_{i=-1}^{1} d_i(\varpi(\tau))^i. \tag{9}$$

When (9) is added to (8), all terms with the same orders of $\varpi(\tau)$ are gathered to create an equation in $\varpi(\tau)$. If you set all of the formula's coefficients to zero, you can reduce it to a system of nonlinear algebraic equations. The following problem can be solved in two different ways using Maple:

Case 1

$$\Xi = -\frac{1}{4}(\ln(H))^2\Upsilon,\ d_{-1} = \frac{1}{2}\sqrt{\Upsilon}(\ln(H))^2\omega,\ d_0 = 1/4\sqrt{\Upsilon}(\ln(H))^2(\eta),\ d_1 = 0. \tag{10}$$

Case 2

$$\Xi = -\frac{1}{4}(\ln(H))^2\Upsilon,\ d_{-1} = 0,\ d_0 = \frac{1}{4}\sqrt{\Upsilon}(\ln(H))^2(\eta),\ d_1 = \frac{1}{2}\sqrt{\Upsilon}\upsilon\,(\ln(H))^2. \tag{11}$$

Under the assumption of scenario 1, the following families of soliton solutions for (8) are obtained.

1.1 When $\Upsilon > 0;\ \upsilon \neq 0$, we have:

$$\varpi_{1,\,1}(x,\,t) = \frac{1}{\Xi}\left(\frac{1}{2}\sqrt{\Upsilon}(\ln(H))^2\omega\left(-\frac{1}{2}\frac{\eta}{\upsilon} - \frac{1}{2}\frac{\sqrt{\Upsilon}\tanh\left(\frac{1}{2}\sqrt{\Upsilon}\tau\right)}{\upsilon}\right)^{-1} + \frac{1}{4}\sqrt{\Upsilon}(\ln(H))^2(\eta)\right)^2, \tag{12}$$

$$V_{1,\,1}(x,\,t)=-\Xi+\frac{1}{2}\sqrt{\Upsilon}\left(\ln\left(H\right)\right)^{2}\omega\left(-\frac{1}{2}\frac{\eta}{\upsilon}-\frac{1}{2}\frac{\sqrt{\Upsilon}\tanh\left(\frac{1}{2}\sqrt{\Upsilon}\tau\right)}{\upsilon}\right)^{-1}+\frac{1}{4}\sqrt{\Upsilon}\left(\ln\left(H\right)\right)^{2}\left(\eta\right), \tag{13}$$

$$W_{1,\,1}(x,\,t)=2\Xi^{2}-2\Xi\left(\frac{1}{2}\sqrt{\Upsilon}\left(\ln\left(H\right)\right)^{2}\omega\left(-\frac{1}{2}\frac{\eta}{\upsilon}-\frac{1}{2}\frac{\sqrt{\Upsilon}\tanh\left(\frac{1}{2}\sqrt{\Upsilon}\tau\right)}{\upsilon}\right)^{-1}+\frac{1}{4}\sqrt{\Upsilon}\left(\ln\left(H\right)\right)^{2}\left(\eta\right)\right), \tag{14}$$

$$\varpi_{1,\,2}(x,\,t)=\frac{1}{\Xi}\left(\frac{1}{2}\sqrt{\Upsilon}\left(\ln\left(H\right)\right)^{2}\omega\left(-\frac{1}{2}\frac{\eta}{\upsilon}-\frac{1}{2}\frac{\sqrt{\Upsilon}\coth\left(\frac{1}{2}\sqrt{\Upsilon}\tau\right)}{\upsilon}\right)^{-1}+\frac{1}{4}\sqrt{\Upsilon}\left(\ln\left(H\right)\right)^{2}\left(\eta\right)\right)^{2}, \tag{15}$$

$$V_{1,\,2}(x,\,t)=-\Xi+\frac{1}{2}\sqrt{\Upsilon}\left(\ln\left(H\right)\right)^{2}\omega\left(-\frac{1}{2}\frac{\eta}{\upsilon}-\frac{1}{2}\frac{\sqrt{\Upsilon}\coth\left(\frac{1}{2}\sqrt{\Upsilon}\tau\right)}{\upsilon}\right)^{-1}+\frac{1}{4}\sqrt{\Upsilon}\left(\ln\left(H\right)\right)^{2}\left(\eta\right), \tag{16}$$

$$W_{1,\,2}(x,\,t)=2\Xi^{2}-2\Xi\left(\frac{1}{2}\sqrt{\Upsilon}\left(\ln\left(H\right)\right)^{2}\omega\left(-\frac{1}{2}\frac{\eta}{\upsilon}-\frac{1}{2}\frac{\sqrt{\Upsilon}\coth\left(\frac{1}{2}\sqrt{\Upsilon}\tau\right)}{\upsilon}\right)^{-1}+\frac{1}{4}\sqrt{\Upsilon}\left(\ln\left(H\right)\right)^{2}\left(\eta\right)\right), \tag{17}$$

$$\varpi_{1,\,3}(x,\,t)=\frac{1}{\Xi}\left(\frac{1}{2}\sqrt{\Upsilon}\left(\ln\left(H\right)\right)^{2}\omega\left(-\frac{1}{2}\frac{\eta}{\upsilon}-\frac{1}{2}\frac{\sqrt{\Upsilon}\left(\tanh\left(\sqrt{\Upsilon}\tau\right)\pm\left(sech\left(\sqrt{\Upsilon}\tau\right)\right)\right)}{\upsilon}\right)^{-1}\right.$$

$$\left.+\frac{1}{4}\sqrt{\Upsilon}\left(\ln\left(H\right)\right)^{2}\left(\eta\right)\right)^{2}, \tag{18}$$

$$V_{1,\,3}(x,\,t)=-\Xi+\frac{1}{2}\sqrt{\Upsilon}\left(\ln\left(H\right)\right)^{2}\omega\left(-\frac{1}{2}\frac{\eta}{\upsilon}-\frac{1}{2}\frac{\sqrt{\Upsilon}\left(\tanh\left(\sqrt{\Upsilon}\tau\right)\pm\left(sech\left(\sqrt{\Upsilon}\tau\right)\right)\right)}{\upsilon}\right)^{-1}$$

$$+\frac{1}{4}\sqrt{\Upsilon}\left(\ln\left(H\right)\right)^{2}\left(\eta\right), \tag{19}$$

$$W_{1,\,3}(x,\,t)=2\Xi^2-2\Xi\left(\frac{1}{2}\sqrt{\Upsilon}\,(\ln(H))^2\,\omega\left(-\frac{1}{2}\frac{\eta}{\upsilon}-\frac{1}{2}\frac{\sqrt{\Upsilon}\left(\tanh\left(\sqrt{\Upsilon}\tau\right)\pm\left(sech\left(\sqrt{\Upsilon}\tau\right)\right)\right)}{\upsilon}\right)^{-1}\right.$$

$$\left.+\frac{1}{4}\sqrt{\Upsilon}\,(\ln(H))^2\,(\eta)\right), \tag{20}$$

$$\varpi_{1,\,4}(x,\,t)=\frac{1}{\Xi}\left(\frac{1}{2}\sqrt{\Upsilon}\,(\ln(H))^2\,\omega\left(-\frac{1}{2}\frac{\eta}{\upsilon}-1/2\,\frac{\sqrt{\Upsilon}\left(\coth\left(\sqrt{\Upsilon}\tau\right)\pm\left(csch\left(\sqrt{\Upsilon}\tau\right)\right)\right)}{\upsilon}\right)^{-1}\right.$$

$$\left.+\frac{1}{4}\sqrt{\Upsilon}\,(\ln(H))^2\,(\eta)\right)^2, \tag{21}$$

$$V_{1,\,4}(x,\,t)=-\Xi+\frac{1}{2}\sqrt{\Upsilon}\,(\ln(H))^2\,\omega\left(-\frac{1}{2}\frac{\eta}{\upsilon}-1/2\,\frac{\sqrt{\Upsilon}\left(\coth\left(\sqrt{\Upsilon}\tau\right)\pm\left(csch\left(\sqrt{\Upsilon}\tau\right)\right)\right)}{\upsilon}\right)^{-1}$$

$$+\frac{1}{4}\sqrt{\Upsilon}\,(\ln(H))^2\,(\eta), \tag{22}$$

$$W_{1,\,4}(x,\,t)=2\Xi^2-2\Xi\left(\frac{1}{2}\sqrt{\Upsilon}\,(\ln(H))^2\,\omega\left(-\frac{1}{2}\frac{\eta}{\upsilon}-1/2\,\frac{\sqrt{\Upsilon}\left(\coth\left(\sqrt{\Upsilon}\tau\right)\pm\left(csch\left(\sqrt{\Upsilon}\tau\right)\right)\right)}{\upsilon}\right)^{-1}\right.$$

$$\left.+\frac{1}{4}\sqrt{\Upsilon}\,(\ln(H))^2\,(\eta)\right), \tag{23}$$

and

$$\varpi_{1,\,5}(x,\,t)=\frac{1}{\Xi}\left(\frac{1}{2}\sqrt{\Upsilon}\,(\ln(H))^2\,\omega\left(-\frac{1}{2}\frac{\eta}{\upsilon}-1/4\,\frac{\sqrt{\Upsilon}\left(\tanh\left(1/4\sqrt{\Upsilon}\tau\right)-\coth\left(1/4\sqrt{\Upsilon}\tau\right)\right)}{\upsilon}\right)^{-1}\right.$$

$$\left.+\frac{1}{4}\sqrt{\Upsilon}\,(\ln(H))^2\,(\eta)\right)^2. \tag{24}$$

$$V_{1,\,5}(x,\,t)=-\Xi+\frac{1}{2}\sqrt{\Upsilon}\,(\ln(H))^2\,\omega\left(-\frac{1}{2}\frac{\eta}{\upsilon}-1/4\,\frac{\sqrt{\Upsilon}\left(\tanh\left(1/4\sqrt{\Upsilon}\tau\right)-\coth\left(1/4\sqrt{\Upsilon}\tau\right)\right)}{\upsilon}\right)^{-1}$$

$$+\frac{1}{4}\sqrt{\Upsilon}\,(\ln(H))^2\,(\eta)\,. \tag{25}$$

$$W_{1,\,5}(x,\,t)=2\Xi^2-2\Xi\left(\frac{1}{2}\sqrt{\Upsilon}\,(\ln(H))^2\,\omega\left(-\frac{1}{2}\frac{\eta}{\upsilon}-1/4\,\frac{\sqrt{\Upsilon}\left(\tanh\left(1/4\sqrt{\Upsilon}\tau\right)-\coth\left(1/4\sqrt{\Upsilon}\tau\right)\right)}{\upsilon}\right)^{-1}\right.$$

$$\left.+\frac{1}{4}\sqrt{\Upsilon}\,(\ln(H))^2\,(\eta)\right). \tag{26}$$

1.2 When $\omega\upsilon>0;\ \eta=0,\ \Upsilon>0$, we have

$$\varpi_{1,\,6}(x,\,t)=\frac{1}{\Xi}\left(\frac{1}{2}\sqrt{\Upsilon}\,(\ln(H))^2\,\omega\frac{1}{\sqrt{\frac{\omega}{\upsilon}}}\left(\tan\left(\sqrt{\omega\upsilon}\tau\right)\right)^{-1}\right)^2, \tag{27}$$

$$V_{1,\,6}(x,\,t)=-\Xi+\frac{1}{2}\sqrt{\Upsilon}\,(\ln(H))^2\,\omega\frac{1}{\sqrt{\frac{\omega}{\upsilon}}}\left(\tan\left(\sqrt{\omega\upsilon}\tau\right)\right)^{-1}, \tag{28}$$

$$W_{1,\,6}(x,\,t)=2\Xi^2-2\Xi\left(\frac{1}{2}\sqrt{\Upsilon}\,(\ln(H))^2\,\omega\frac{1}{\sqrt{\frac{\omega}{\upsilon}}}\left(\tan\left(\sqrt{\omega\upsilon}\tau\right)\right)^{-1}\right), \tag{29}$$

$$\varpi_{1,\,16}(x,\,t)=\frac{1}{\Xi}\left(-\frac{1}{2}\sqrt{\Upsilon}\,(\ln(H))^2\,\omega\frac{1}{\sqrt{\frac{\omega}{\upsilon}}}\left(\cot\left(\sqrt{\omega\upsilon}\tau\right)\right)^{-1}\right)^2, \tag{30}$$

$$V_{1,\,7}(x,\,t)=-\Xi-\frac{1}{2}\sqrt{\Upsilon}\,(\ln(H))^2\,\omega\frac{1}{\sqrt{\frac{\omega}{\upsilon}}}\left(\cot\left(\sqrt{\omega\upsilon}\tau\right)\right)^{-1} \tag{31}$$

$$W_{1,\,7}(x,\,t)=2\Xi^2-2\Xi\left(-\frac{1}{2}\sqrt{\Upsilon}\,(\ln(H))^2\,\omega\frac{1}{\sqrt{\frac{\omega}{\upsilon}}}\left(\cot\left(\sqrt{\omega\upsilon}\tau\right)\right)^{-1}\right), \tag{32}$$

$$\varpi_{1,\,8}(x,\,t)=\frac{1}{\Xi}\left(\frac{1}{2}\sqrt{\Upsilon}\left(\ln\left(H\right)\right)^{2}\omega\frac{1}{\sqrt{\frac{\omega}{\upsilon}}}\left(\tan\left(2\sqrt{\omega\,\upsilon}\tau\right)\pm\left(\sec\left(2\sqrt{\omega\,\upsilon}\tau\right)\right)\right)^{-1}\right)^{2}, \tag{33}$$

$$V_{1,\,8}(x,\,t)=-\Xi+\frac{1}{2}\sqrt{\Upsilon}\left(\ln\left(H\right)\right)^{2}\omega\frac{1}{\sqrt{\frac{\omega}{\upsilon}}}\left(\tan\left(2\sqrt{\omega\,\upsilon}\tau\right)\pm\left(\sec\left(2\sqrt{\omega\,\upsilon}\tau\right)\right)\right)^{-1}, \tag{34}$$

$$W_{1,\,8}(x,\,t)=2\Xi^{2}-2\Xi\left(\frac{1}{2}\sqrt{\Upsilon}\left(\ln\left(H\right)\right)^{2}\omega\frac{1}{\sqrt{\frac{\omega}{\upsilon}}}\left(\tan\left(2\sqrt{\omega\,\upsilon}\tau\right)\pm\left(\sec\left(2\sqrt{\omega\,\upsilon}\tau\right)\right)\right)^{-1}\right), \tag{35}$$

$$\varpi_{1,\,9}(x,\,t)=\frac{1}{\Xi}\left(-\frac{1}{2}\sqrt{\Upsilon}\left(\ln\left(H\right)\right)^{2}\omega\frac{1}{\sqrt{\frac{\omega}{\upsilon}}}\left(\cot\left(2\sqrt{\omega\,\upsilon}\tau\right)\pm\left(\csc\left(2\sqrt{\omega\,\upsilon}\tau\right)\right)\right)^{-1}\right)^{2}, \tag{36}$$

$$V_{1,\,9}(x,\,t)=-\Xi-\frac{1}{2}\sqrt{\Upsilon}\left(\ln\left(H\right)\right)^{2}\omega\frac{1}{\sqrt{\frac{\omega}{\upsilon}}}\left(\cot\left(2\sqrt{\omega\,\upsilon}\tau\right)\pm\left(\csc\left(2\sqrt{\omega\,\upsilon}\tau\right)\right)\right)^{-1}, \tag{37}$$

$$W_{1,\,9}(x,\,t)=2\Xi^{2}-2\Xi\left(-\frac{1}{2}\sqrt{\Upsilon}\left(\ln\left(H\right)\right)^{2}\omega\frac{1}{\sqrt{\frac{\omega}{\upsilon}}}\left(\cot\left(2\sqrt{\omega\,\upsilon}\tau\right)\pm\left(\csc\left(2\sqrt{\omega\,\upsilon}\tau\right)\right)\right)^{-1}\right), \tag{38}$$

and

$$\varpi_{1,\,10}(x,\,t)=\frac{1}{\Xi}\left(\sqrt{\Upsilon}\left(\ln\left(H\right)\right)^{2}\omega\frac{1}{\sqrt{\frac{\omega}{\upsilon}}}\left(\tan\left(\frac{1}{2}\sqrt{\omega\,\upsilon}\tau\right)-\cot\left(1/2\sqrt{\omega\,\upsilon}\tau\right)\right)^{-1}\right)^{2}. \tag{39}$$

$$V_{1,\,10}(x,\,t)=-\Xi+\sqrt{\Upsilon}\left(\ln\left(H\right)\right)^{2}\omega\frac{1}{\sqrt{\frac{\omega}{\upsilon}}}\left(\tan\left(\frac{1}{2}\sqrt{\omega\,\upsilon}\tau\right)-\cot\left(1/2\sqrt{\omega\,\upsilon}\tau\right)\right)^{-1}. \tag{40}$$

$$W_{1,\,10}(x,\,t)=2\Xi^{2}-2\Xi\left(\sqrt{\Upsilon}\left(\ln\left(H\right)\right)^{2}\omega\frac{1}{\sqrt{\frac{\omega}{\upsilon}}}\left(\tan\left(\frac{1}{2}\sqrt{\omega\,\upsilon}\tau\right)-\cot\left(1/2\sqrt{\omega\,\upsilon}\tau\right)\right)^{-1}\right). \tag{41}$$

1.3 When $\eta = 0;\ \omega\upsilon < 0,\ \Upsilon > 0$, we have:

$$\varpi_{1,\,11}(x,\,t) = \frac{1}{V}\left(-\frac{1}{2}\sqrt{\Upsilon}\,(\ln(H))^2\,\omega\frac{1}{\sqrt{-\frac{\omega}{\upsilon}}}\left(\tanh\left(\sqrt{-\omega\,\upsilon}\,\tau\right)\right)^{-1}\right)^2, \tag{42}$$

$$V_{1,\,11}(x,\,t) = -\Xi - \frac{1}{2}\sqrt{\Upsilon}\,(\ln(H))^2\,\omega\frac{1}{\sqrt{-\frac{\omega}{\upsilon}}}\left(\tanh\left(\sqrt{-\omega\,\upsilon}\,\tau\right)\right)^{-1}, \tag{43}$$

$$W_{1,\,11}(x,\,t) = 2\Xi^2 - 2\Xi\left(-\frac{1}{2}\sqrt{\Upsilon}\,(\ln(H))^2\,\omega\frac{1}{\sqrt{-\frac{\omega}{\upsilon}}}\left(\tanh\left(\sqrt{-\omega\,\upsilon}\,\tau\right)\right)^{-1}\right), \tag{44}$$

$$\varpi_{1,\,12}(x,\,t) = \frac{1}{\Xi}\left(-\frac{1}{2}\sqrt{\Upsilon}\,(\ln(H))^2\,\omega\frac{1}{\sqrt{-\frac{\omega}{\upsilon}}}\left(\coth\left(\sqrt{-\omega\,\upsilon}\,\tau\right)\right)^{-1}\right)^2, \tag{45}$$

$$V_{1,\,12}(x,\,t) = -\Xi - \frac{1}{2}\sqrt{\Upsilon}\,(\ln(H))^2\,\omega\frac{1}{\sqrt{-\frac{\omega}{\upsilon}}}\left(\coth\left(\sqrt{-\omega\,\upsilon}\,\tau\right)\right)^{-1}, \tag{46}$$

$$W_{1,\,12}(x,\,t) = 2\Xi^2 - 2\Xi\left(-\frac{1}{2}\sqrt{\Upsilon}\,(\ln(H))^2\,\omega\frac{1}{\sqrt{-\frac{\omega}{\upsilon}}}\left(\coth\left(\sqrt{-\omega\,\upsilon}\,\tau\right)\right)^{-1}\right), \tag{47}$$

$$\varpi_{1,\,13}(x,\,t) = \frac{1}{\Xi}\left(-1/2\sqrt{\Upsilon}\,(\ln(H))^2\,\omega\frac{1}{\sqrt{-\frac{\omega}{\upsilon}}}\left(\tanh\left(2\sqrt{-\omega\,\upsilon}\,\tau\right) \pm \left(isech\left(2\sqrt{-\omega\,\upsilon}\,\tau\right)\right)\right)^{-1}\right)^2, \tag{48}$$

$$V_{1,\,13}(x,\,t) = -\Xi - 1/2\sqrt{\Upsilon}\,(\ln(H))^2\,\omega\frac{1}{\sqrt{-\frac{\omega}{\upsilon}}}\left(\tanh\left(2\sqrt{-\omega\,\upsilon}\,\tau\right) \pm \left(isech\left(2\sqrt{-\omega\,\upsilon}\,\tau\right)\right)\right)^{-1} \tag{49}$$

$$W_{1,\,13}(x,\,t) = 2\Xi^2 - 2\Xi\left(-1/2\sqrt{\Upsilon}\,(\ln(H))^2\,\omega\frac{1}{\sqrt{-\frac{\omega}{\upsilon}}}\left(\tanh\left(2\sqrt{-\omega\,\upsilon}\,\tau\right) \pm \left(isech\left(2\sqrt{-\omega\,\upsilon}\,\tau\right)\right)\right)^{-1}\right), \tag{50}$$

$$\varpi_{1,\ 14}(x,\ t) = \frac{1}{\Xi}\left(-1/2\sqrt{\Upsilon}\left(\ln\left(H\right)\right)^{2}\omega\frac{1}{\sqrt{-\frac{\omega}{\upsilon}}}\left(\coth\left(2\sqrt{-\omega\,\upsilon}\tau\right) \pm \left(csch\left(2\sqrt{-\omega\,\upsilon}\tau\right)\right)\right)^{-1}\right)^{2}, \tag{51}$$

$$V_{1,\ 14}(x,\ t) = -\Xi - 1/2\sqrt{\Upsilon}\left(\ln\left(H\right)\right)^{2}\omega\frac{1}{\sqrt{-\frac{\omega}{\upsilon}}}\left(\coth\left(2\sqrt{-\omega\,\upsilon}\tau\right) \pm \left(csch\left(2\sqrt{-\omega\,\upsilon}\tau\right)\right)\right)^{-1}, \tag{52}$$

$$W_{1,\ 14}(x,\ t) = 2\Xi^{2} - 2\Xi\left(-1/2\sqrt{\Upsilon}\left(\ln\left(H\right)\right)^{2}\omega\frac{1}{\sqrt{-\frac{\omega}{\upsilon}}}\left(\coth\left(2\sqrt{-\omega\,\upsilon}\tau\right) \pm \left(csch\left(2\sqrt{-\omega\,\upsilon}\tau\right)\right)\right)^{-1}\right), \tag{53}$$

and

$$\varpi_{1,\ 15}(x,\ t) = \frac{1}{\Xi}\left(-\sqrt{\Upsilon}\left(\ln\left(H\right)\right)^{2}\omega\frac{1}{\sqrt{-\frac{\omega}{\upsilon}}}\left(\tanh\left(1/2\sqrt{-\omega\,\upsilon}\tau\right) + \coth\left(1/2\sqrt{-\omega\,\upsilon}\tau\right)\right)^{-1}\right). \tag{54}$$

$$V_{1,\ 15}(x,\ t) = -\Xi - \sqrt{\Upsilon}\left(\ln\left(H\right)\right)^{2}\omega\frac{1}{\sqrt{-\frac{\omega}{\upsilon}}}\left(\tanh\left(1/2\sqrt{-\omega\,\upsilon}\tau\right) + \coth\left(1/2\sqrt{-\omega\,\upsilon}\tau\right)\right)^{-1}. \tag{55}$$

$$W_{1,\ 15}(x,\ t) = 2\Xi^{2} - 2\Xi\left(-\sqrt{\Upsilon}\left(\ln\left(H\right)\right)^{2}\omega\frac{1}{\sqrt{-\frac{\omega}{\upsilon}}}\left(\tanh\left(1/2\sqrt{-\omega\,\upsilon}\tau\right) + \coth\left(1/2\sqrt{-\omega\,\upsilon}\tau\right)\right)^{-1}\right). \tag{56}$$

1.4 Suppose $\upsilon = \omega;\ \eta = 0,\ \Upsilon > 0$, we have:

$$\varpi_{1,\ 16}(x,\ t) = \frac{1}{\Xi}\left(1/2\,\frac{\sqrt{\Upsilon}\left(\ln\left(H\right)\right)^{2}\omega}{\tan\left(\omega\,\tau\right)}\right)^{2}, \tag{57}$$

$$V_{1,\ 16}(x,\ t) = -\Xi + 1/2\,\frac{\sqrt{\Upsilon}\left(\ln\left(H\right)\right)^{2}\omega}{\tan\left(\omega\,\tau\right)}, \tag{58}$$

$$W_{1,\ 16}(x,\ t) = 2\lambda^{2} - 2\lambda\left(1/2\,\frac{\sqrt{\Upsilon}\left(\ln\left(H\right)\right)^{2}\omega}{\tan\left(\omega\,\tau\right)}\right), \tag{59}$$

$$\varpi_{1,\,17}(x,\,t)=\frac{1}{\Xi}\left(-1/2\,\frac{\sqrt{\Upsilon}\,(\ln{(H)})^{2}\,\omega}{\cot{(\omega\,\tau)}}\right)^{2},\tag{60}$$

$$V_{1,\,17}(x,\,t)=-\Xi-1/2\,\frac{\sqrt{\Upsilon}\,(\ln{(H)})^{2}\,\omega}{\cot{(\omega\,\tau)}},\tag{61}$$

$$W_{1,\,17}(x,\,t)=2\Xi^{2}-2\Xi\left(-1/2\,\frac{\sqrt{\Upsilon}\,(\ln{(H)})^{2}\,\omega}{\cot{(\omega\,\tau)}}\right),\tag{62}$$

$$\varpi_{1,\,18}(x,\,t)=\frac{1}{\Xi}\left(1/2\,\frac{\sqrt{\Upsilon}\,(\ln{(H)})^{2}\,\omega}{\tan{(2\,\omega\,\tau)}\pm(\sec{(2\,\omega\,\tau)})}\right)^{2},\tag{63}$$

$$V_{1,\,18}(x,\,t)=-\Xi+1/2\,\frac{\sqrt{\Upsilon}\,(\ln{(H)})^{2}\,\omega}{\tan{(2\,\omega\,\tau)}\pm(\sec{(2\,\omega\,\tau)})},\tag{64}$$

$$W_{1,\,18}(x,\,t)=2\Xi^{2}-2\Xi\left(1/2\,\frac{\sqrt{\Upsilon}\,(\ln{(H)})^{2}\,\omega}{\tan{(2\,\omega\,\tau)}\pm(\sec{(2\,\omega\,\tau)})}\right),\tag{65}$$

$$\varpi_{1,\,19}(x,\,t)=\frac{1}{\Xi}\left(1/2\,\frac{\sqrt{\Upsilon}\,(\ln{(H)})^{2}\,\omega}{-\cot{(2\,\omega\,\tau)}\pm(\csc{(2\,\omega\,\tau)})}\right)^{2},\tag{66}$$

$$V_{1,\,19}(x,\,t)=-\Xi+1/2\,\frac{\sqrt{\Upsilon}\,(\ln{(H)})^{2}\,\omega}{-\cot{(2\,\omega\,\tau)}\pm(\csc{(2\,\omega\,\tau)})},\tag{67}$$

$$W_{1,\,19}(x,\,t)=2\Xi^{2}-2\Xi\left(1/2\,\frac{\sqrt{\Upsilon}\,(\ln{(H)})^{2}\,\omega}{-\cot{(2\,\omega\,\tau)}\pm(\csc{(2\,\omega\,\tau)})}\right),\tag{68}$$

and

$$\varpi_{1,\,20}(x,\,t)=\frac{1}{\Xi}\left(1/2\,\frac{\sqrt{\Upsilon}\,(\ln{(H)})^{2}\,\omega}{1/2\,\tan{(1/2\,\omega\,\tau)}-1/2\,\cot{(1/2\,\omega\,\tau)}}\right)^{2}\tag{69}$$

$$V_{1,\,20}(x,\,t)=-\Xi+1/2\,\frac{\sqrt{\Upsilon}\,(\ln{(H)})^{2}\,\omega}{1/2\,\tan{(1/2\,\omega\,\tau)}-1/2\,\cot{(1/2\,\omega\,\tau)}}.\tag{70}$$

$$W_{1,\,20}(x,\,t)=2\Xi^{2}-2\Xi\left(1/2\,\frac{\sqrt{\Upsilon}\,(\ln{(H)})^{2}\,\omega}{1/2\,\tan{(1/2\,\omega\,\tau)}-1/2\,\cot{(1/2\,\omega\,\tau)}}\right).\tag{71}$$

1.5 When $\upsilon = -\omega;\ \eta = 0,\ \Upsilon > 0$, we have:

$$\varpi_{1,\ 21}(x,\ t) = \frac{1}{\Xi}\left(-1/2\,\frac{\sqrt{\Upsilon}\,(\ln(H))^2\,\omega}{\tanh(\omega\,\tau)}\right)^2, \tag{72}$$

$$V_{1,\ 21}(x,\ t) = -\Xi - 1/2\,\frac{\sqrt{\Upsilon}\,(\ln(H))^2\,\omega}{\tanh(\omega\,\tau)}, \tag{73}$$

$$W_{1,\ 21}(x,\ t) = 2\Xi^2 - 2\Xi\left(-1/2\,\frac{\sqrt{\Upsilon}\,(\ln(H))^2\,\omega}{\tanh(\omega\,\tau)}\right), \tag{74}$$

$$\varpi_{1,\ 22}(x,\ t) = \frac{1}{\Xi}\left(-1/2\,\frac{\sqrt{\Upsilon}\,(\ln(H))^2\,\omega}{\coth(\omega\,\tau)}\right)^2, \tag{75}$$

$$V_{1,\ 22}(x,\ t) = -\Xi - 1/2\,\frac{\sqrt{\Upsilon}\,(\ln(H))^2\,\omega}{\coth(\omega\,\tau)}, \tag{76}$$

$$W_{1,\ 22}(x,\ t) = 2\Xi^2 - 2\Xi\left(-1/2\,\frac{\sqrt{\Upsilon}\,(\ln(H))^2\,\omega}{\coth(\omega\,\tau)}\right), \tag{77}$$

$$\varpi_{1,\ 23}(x,\ t) = \frac{1}{\Xi}\left(1/2\,\frac{\sqrt{\Upsilon}\,(\ln(H))^2\,\omega}{-\tanh(2\,\omega\,\tau) \pm (isech(2\,\omega\,\tau))}\right)^2, \tag{78}$$

$$V_{1,\ 23}(x,\ t) = -\Xi + 1/2\,\frac{\sqrt{\Upsilon}\,(\ln(H))^2\,\omega}{-\tanh(2\,\omega\,\tau) \pm (isech(2\,\omega\,\tau))}, \tag{79}$$

$$W_{1,\ 23}(x,\ t) = 2\Xi^2 - 2\Xi\left(1/2\,\frac{\sqrt{\Upsilon}\,(\ln(H))^2\,\omega}{-\tanh(2\,\omega\,\tau) \pm (isech(2\,\omega\,\tau))}\right), \tag{80}$$

$$\varpi_{1,\ 24}(x,\ t) = \frac{1}{\Xi}\left(1/2\,\frac{\sqrt{\Upsilon}\,(\ln(H))^2\,\omega}{-\coth(2\,\omega\,\tau) \pm (csch(2\,\omega\,\tau))}\right)^2, \tag{81}$$

$$V_{1,\ 24}(x,\ t) = -\Xi + 1/2\,\frac{\sqrt{\Upsilon}\,(\ln(H))^2\,\omega}{-\coth(2\,\omega\,\tau) \pm (csch(2\,\omega\,\tau))}, \tag{82}$$

$$W_{1,\ 24}(x,\ t) = 2\Xi^2 - 2\Xi\left(1/2\,\frac{\sqrt{\Upsilon}\,(\ln(H))^2\,\omega}{-\coth(2\,\omega\,\tau) \pm (csch(2\,\omega\,\tau))}\eta\right), \tag{83}$$

and

$$\varpi_{1,\,25}(x,\,t)=\frac{1}{\Xi}\left(1/2\,\frac{\sqrt{\Upsilon}\,(\ln(H))^2\,\omega}{-1/2\,\tanh(1/2\,\omega\,\tau)-1/2\,\coth(1/2\,\omega\,\tau)}\right)^2. \tag{84}$$

$$V_{1,\,25}(x,\,t)=-\Xi+1/2\,\frac{\sqrt{\Upsilon}\,(\ln(H))^2\,\omega}{-1/2\,\tanh(1/2\,\omega\,\tau)-1/2\,\coth(1/2\,\omega\,\tau)}. \tag{85}$$

$$W_{1,\,25}(x,\,t)=2\Xi^2-2\Xi\left(1/2\,\frac{\sqrt{\Upsilon}\,(\ln(H))^2\,\omega}{-1/2\,\tanh(1/2\,\omega\,\tau)-1/2\,\coth(1/2\,\omega\,\tau)}\right). \tag{86}$$

1.6 When $\eta=\varsigma;\ \omega=n\varsigma(n\neq 0);\ \upsilon=0,\ \Upsilon>0$, we have:

$$\varpi_{1,\,26}(x,\,t)=\frac{1}{\Xi}\left(1/2\,\frac{\sqrt{\Upsilon}\,(\ln(H))^2\,\omega}{H^{\tau\varsigma}-n}+1/4\,\sqrt{\Upsilon}\,(\ln(H))^2\,(\eta)\right)^2. \tag{87}$$

$$V_{1,\,26}(x,\,t)=-\Xi+1/2\,\frac{\sqrt{\Upsilon}\,(\ln(H))^2\,\omega}{H^{\tau\varsigma}-n}+1/4\,\sqrt{\Upsilon}\,(\ln(H))^2\,(\eta). \tag{88}$$

$$W_{1,\,26}(x,\,t)=2\Xi^2-2\Xi\left(1/2\,\frac{\sqrt{\Upsilon}\,(\ln(H))^2\,\omega}{H^{\tau\varsigma}-n}+1/4\,\sqrt{\Upsilon}\,(\ln(H))^2\,(\zeta)\right). \tag{89}$$

1.7 When $\eta=0;\ \upsilon=0;\ \omega\neq 0,\ \Upsilon>0$,

$$\varpi_{1,\,27}(x,\,t)=\frac{1}{\Xi}\left(1/2\,\frac{\sqrt{\Upsilon}\,(\ln(H))^2}{\tau\ln(H)}+1/4\,\sqrt{\Upsilon}\,(\ln(H))^2\,(\eta)\right)^2, \tag{90}$$

$$V_{1,\,27}(x,\,t)=-\Xi+1/2\,\frac{\sqrt{\Upsilon}\,(\ln(H))^2}{\tau\ln(H)}+1/4\,\sqrt{\Upsilon}\,(\ln(H))^2\,(\eta), \tag{91}$$

$$W_{1,\,27}(x,\,t)=2\Xi^2-2\Xi\left(1/2\,\frac{\sqrt{\Upsilon}\,(\ln(H))^2}{\tau\ln(H)}+1/4\,\sqrt{\Upsilon}\,(\ln(H))^2\,(\eta)\right), \tag{92}$$

1.8 When $\omega=0,\ \Upsilon>0$, we have:

$$\varpi_{1,\,28}(x,\,t)=\frac{1}{\Xi}\left(1/4\,\sqrt{\Upsilon}\,(\ln(H))^2\,(\eta)\right)^2. \tag{93}$$

$$V_{1,\,28}(x,\,t)=-\Xi+1/4\,\sqrt{\Upsilon}\,(\ln(H))^2\,(\eta). \tag{94}$$

$$W_{1,\,28}(x,\,t) = 2\Xi^2 - 2\Xi\left(1/4\sqrt{\Upsilon}\left(\ln(H)\right)^2(\eta)\right). \tag{95}$$

1.9 When $\eta = \varsigma;\ \upsilon = n\varsigma;\ \alpha = 0,\ \Upsilon > 0$, we have:

$$\varpi_{1,\,29}(x,\,t) = \frac{1}{\Xi}\left(1/2\,\frac{\sqrt{\Upsilon}\left(\ln(H)\right)^2\omega\,(1 - nH^{\tau\varsigma})}{H^{\tau^2}} + 1/4\sqrt{\Upsilon}\left(\ln(H)\right)^2(\eta)\right)^2. \tag{96}$$

$$V_{1,\,29}(x,\,t) = -\Xi + 1/2\,\frac{\sqrt{\Upsilon}\left(\ln(H)\right)^2\omega\,(1 - nH^{\tau\varsigma})}{H^{\tau^2}} + 1/4\sqrt{\Upsilon}\left(\ln(H)\right)^2(\eta). \tag{97}$$

$$W_{1,\,29}(x,\,t) = 2\Xi^2 - 2\Xi\left(1/2\,\frac{\sqrt{\Upsilon}\left(\ln(H)\right)^2\omega\,(1 - nH^{\tau\varsigma})}{H^{\tau\varsigma}} + 1/4\sqrt{\Upsilon}\left(\ln(H)\right)^2(\eta)\right). \tag{98}$$

Now consider case 2, we have:

Household 2.1 When $\Upsilon > 0;\ \upsilon \neq 0$, we have:

$$\varpi_{2,\,1}(x,\,t) = \frac{1}{\Xi}\left(1/4\sqrt{\Upsilon}\left(\ln(H)\right)^2(\eta) + 1/2\sqrt{\Upsilon}\upsilon\,\left(\ln(H)\right)^2\left(-1/2\,\frac{\eta}{\upsilon} - 1/2\,\frac{\sqrt{\Upsilon}\tanh\left(1/2\sqrt{\Upsilon}\tau\right)}{\upsilon}\right)\right)^2, \tag{99}$$

$$V_{2,\,1}(x,\,t) = -\Xi + 1/4\sqrt{\Upsilon}\left(\ln(H)\right)^2(\eta) + 1/2\sqrt{\Upsilon}\upsilon\,\left(\ln(H)\right)^2\left(-1/2\,\frac{\eta}{\upsilon} - 1/2\,\frac{\sqrt{\Upsilon}\tanh\left(1/2\sqrt{\Upsilon}\tau\right)}{\upsilon}\right), \tag{100}$$

$$W_{2,\,1}(x,\,t) = 2\Xi^2 - 2\Xi\left(1/4\sqrt{\Upsilon}\left(\ln(H)\right)^2(\eta) + 1/2\sqrt{\Upsilon}\upsilon\,\left(\ln(H)\right)^2\left(-1/2\,\frac{\eta}{\upsilon} - 1/2\,\frac{\sqrt{\Upsilon}\tanh\left(1/2\sqrt{\Upsilon}\tau\right)}{\upsilon}\right)\right), \tag{101}$$

$$\varpi_{2,\,2}(x,\,t) = \frac{1}{\Xi}\left(1/4\sqrt{\Upsilon}\left(\ln(H)\right)^2(\eta) + 1/2\sqrt{\Upsilon}\upsilon\,\left(\ln(H)\right)^2\left(-1/2\,\frac{\eta}{\upsilon} - 1/2\,\frac{\sqrt{\Upsilon}\coth\left(1/2\sqrt{\Upsilon}\tau\right)}{\upsilon}\right)\right)^2, \tag{102}$$

$$V_{2,\,2}(x,\,t) = -\Xi + 1/4\sqrt{\Upsilon}\left(\ln(H)\right)^2(\eta) + 1/2\sqrt{\Upsilon}\upsilon\,\left(\ln(H)\right)^2\left(-1/2\,\frac{\eta}{\upsilon} - 1/2\,\frac{\sqrt{\Upsilon}\coth\left(1/2\sqrt{\Upsilon}\tau\right)}{\upsilon}\right), \tag{103}$$

$$W_{2,\,2}(x,\,t) = 2\Xi^2 - 2\Xi\left(1/4\sqrt{\Upsilon}\left(\ln(H)\right)^2(\eta) + 1/2\sqrt{\Upsilon}\upsilon\,\left(\ln(H)\right)^2\left(-\frac{1}{2}\frac{\eta}{\upsilon} - 1/2\,\frac{\sqrt{\Upsilon}\coth\left(1/2\sqrt{\Upsilon}\tau\right)}{\upsilon}\right)\right), \tag{104}$$

$$\varpi_{2,\,3}(x,\ t)=\frac{1}{\Xi}\left(1/4\sqrt{\Upsilon}\left(\ln\left(H\right)\right)^{2}\left(\eta\right)+1/2\sqrt{\Upsilon}\upsilon\ \left(\ln\left(H\right)\right)^{2}\right.$$

$$\left.\left(-\frac{1}{2}\frac{\eta}{\upsilon}-1/2\,\frac{\sqrt{\Upsilon}\left(\tanh\left(\sqrt{\Upsilon}\tau\right)\pm\left(sech\left(\sqrt{\Upsilon}\tau\right)\right)\right)}{\upsilon}\right)\right)^{2}, \tag{105}$$

$$V_{2,\,3}(x,\ t)=-\Xi+1/4\sqrt{\Upsilon}\left(\ln\left(H\right)\right)^{2}\left(\eta\right)+\frac{1}{2}\sqrt{\Upsilon}\upsilon\ \left(\ln\left(H\right)\right)^{2}$$

$$\left(-\frac{1}{2}\frac{\eta}{\upsilon}-\frac{1}{2}\frac{\sqrt{\Upsilon}\left(\tanh\left(\sqrt{\Upsilon}\tau\right)\pm\left(sech\left(\sqrt{\Upsilon}\tau\right)\right)\right)}{\upsilon}\right), \tag{106}$$

$$W_{2,\,3}(x,\ t)=2\Xi^{2}-2\Xi\left(\frac{1}{4}\sqrt{\Upsilon}\left(\ln\left(H\right)\right)^{2}\left(\eta\right)+\frac{1}{2}\sqrt{\Upsilon}\upsilon\ \left(\ln\left(H\right)\right)^{2}\right.$$

$$\left.\left(-\frac{1}{2}\frac{\eta}{\upsilon}-\frac{1}{2}\frac{\sqrt{\Upsilon}\left(\tanh\left(\sqrt{\Upsilon}\tau\right)\pm\left(sech\left(\sqrt{\Upsilon}\tau\right)\right)\right)}{\upsilon}\right)\right), \tag{107}$$

$$\varpi_{2,\,4}(x,\ t)=\frac{1}{\Xi}\left(\frac{1}{4}\sqrt{\Upsilon}\left(\ln\left(H\right)\right)^{2}\left(\eta\right)+\frac{1}{2}\sqrt{\Upsilon}\upsilon\ \left(\ln\left(H\right)\right)^{2}\right.$$

$$\left.\left(-\frac{1}{2}\frac{\eta}{\upsilon}-\frac{1}{2}\frac{\sqrt{\Upsilon}\left(\coth\left(\sqrt{\Upsilon}\tau\right)\pm\left(csch\left(\sqrt{\Upsilon}\tau\right)\right)\right)}{\upsilon}\right)\right)^{2}, \tag{108}$$

$$V_{2,\,4}(x,\ t)=-\Xi+\frac{1}{4}\sqrt{\Upsilon}\left(\ln\left(H\right)\right)^{2}\left(\eta\right)+1/2\sqrt{\Upsilon}\upsilon\ \left(\ln\left(H\right)\right)^{2}$$

$$\left(-\frac{1}{2}\frac{\eta}{\upsilon}-\frac{1}{2}\frac{\sqrt{\Upsilon}\left(\coth\left(\sqrt{\Upsilon}\tau\right)\pm\left(csch\left(\sqrt{\Upsilon}\tau\right)\right)\right)}{\upsilon}\right) \tag{109}$$

$$W_{2,\,4}(x,\ t)=2\Xi^{2}-2\Xi\left(1/4\sqrt{\Upsilon}\left(\ln\left(H\right)\right)^{2}\left(\eta\right)+\frac{1}{2}\sqrt{\Upsilon}\upsilon\ \left(\ln\left(H\right)\right)^{2}\right.$$

$$\left.\left(-\frac{1}{2}\frac{\eta}{\upsilon}-\frac{1}{2}\frac{\sqrt{\Upsilon}\left(\coth\left(\sqrt{\Upsilon}\tau\right)\pm\left(csch\left(\sqrt{\Upsilon}\tau\right)\right)\right)}{\upsilon}\right)\right), \tag{110}$$

and

$$\varpi_{2,\,5}(x,\,t)=\frac{1}{\Xi}\left(\frac{1}{4}\sqrt{\Upsilon}\left(\ln\left(H\right)\right)^{2}\left(\eta\right)+\frac{1}{2}\sqrt{\Upsilon}\upsilon\,\left(\ln\left(H\right)\right)^{2}\right.$$

$$\left.\left(-\frac{1}{2}\frac{\eta}{\upsilon}-\frac{1}{4}\frac{\sqrt{\Upsilon}\left(\tanh\left(\frac{1}{4}\sqrt{\Upsilon}\tau\right)-\coth\left(\frac{1}{4}\sqrt{\Upsilon}\tau\right)\right)}{\upsilon}\right)\right)^{2}. \tag{111}$$

$$V_{2,\,5}(x,\,t)=-\Xi+\frac{1}{4}\sqrt{\Upsilon}\left(\ln\left(H\right)\right)^{2}\left(\eta\right)+\frac{1}{2}\sqrt{\Upsilon}\upsilon\,\left(\ln\left(H\right)\right)^{2}$$

$$\left(-\frac{1}{2}\frac{\eta}{\upsilon}-\frac{1}{4}\frac{\sqrt{\Upsilon}\left(\tanh\left(\frac{1}{4}\sqrt{\Upsilon}\tau\right)-\coth\left(\frac{1}{4}\sqrt{\Upsilon}\tau\right)\right)}{\upsilon}\right). \tag{112}$$

$$W_{2,\,5}(x,\,t)=2\Xi^{2}-2\Xi\left(\frac{1}{4}\sqrt{\Upsilon}\left(\ln\left(H\right)\right)^{2}\left(\eta\right)+\frac{1}{2}\sqrt{\Upsilon}\upsilon\,\left(\ln\left(H\right)\right)^{2}\right.$$

$$\left.\left(-\frac{1}{2}\frac{\eta}{\upsilon}-\frac{1}{4}\frac{\sqrt{\Upsilon}\left(\tanh\left(\frac{1}{4}\sqrt{\Upsilon}\tau\right)-\coth\left(\frac{1}{4}\sqrt{\Upsilon}\tau\right)\right)}{\upsilon}\right)\right). \tag{113}$$

Household 2.2 When $\upsilon\omega>0;\ \eta=0,\ \Upsilon>0$, we have:

$$\varpi_{2,\,6}(x,\,t)=\frac{1}{\Xi}\left(\frac{1}{2}\sqrt{\Upsilon}\upsilon\,\left(\ln\left(H\right)\right)^{2}\sqrt{\frac{\omega}{\upsilon}}\tan\left(\sqrt{\omega\,\upsilon}\tau\right)\right)^{2}, \tag{114}$$

$$V_{2,\,6}(x,\,t)=-\Xi+\frac{1}{2}\sqrt{\Upsilon}\upsilon\,\left(\ln\left(H\right)\right)^{2}\sqrt{\frac{\omega}{\upsilon}}\tan\left(\sqrt{\omega\,\upsilon}\tau\right), \tag{115}$$

$$W_{2,\,6}(x,\,t)=2\Xi^{2}-2\Xi\left(\frac{1}{2}\sqrt{\Upsilon}\upsilon\,\left(\ln\left(H\right)\right)^{2}\sqrt{\frac{\omega}{\upsilon}}\tan\left(\sqrt{\omega\,\upsilon}\tau\right)\right), \tag{116}$$

$$\varpi_{2,\,7}(x,\,t)=\frac{1}{\Xi}\left(-\frac{1}{2}\sqrt{\Upsilon}\upsilon\,\left(\ln\left(H\right)\right)^{2}\sqrt{\frac{\omega}{\upsilon}}\cot\left(\sqrt{\omega\,\upsilon}\tau\right)\right)^{2}, \tag{117}$$

$$V_{2,\,7}(x,\,t)=-\Xi-\frac{1}{2}\sqrt{\Upsilon}\upsilon\,\left(\ln\left(H\right)\right)^{2}\sqrt{\frac{\omega}{\upsilon}}\cot\left(\sqrt{\omega\,\upsilon}\tau\right), \tag{118}$$

$$W_{2,\,7}(x,\,t)=2\Xi^{2}-2\Xi\left(-\frac{1}{2}\sqrt{\Upsilon}\upsilon\,\left(\ln\left(H\right)\right)^{2}\sqrt{\frac{\omega}{\upsilon}}\cot\left(\sqrt{\omega\,\upsilon}\tau\right)\right), \tag{119}$$

$$\varpi_{2,\,8}(x,\,t)=\frac{1}{\Xi}\left(\frac{1}{2}\sqrt{\Upsilon}\upsilon\,(\ln(H))^2\sqrt{\frac{\omega}{\upsilon}}\left(\tan\left(2\sqrt{\omega\,\upsilon}\tau\right)\pm\left(\sec\left(2\sqrt{\omega\,\upsilon}\tau\right)\right)\right)\right)^2, \tag{120}$$

$$V_{2,\,8}(x,\,t)=-\Xi+\frac{1}{2}\sqrt{\Upsilon}\upsilon\,(\ln(H))^2\sqrt{\frac{\omega}{\upsilon}}\left(\tan\left(2\sqrt{\omega\,\upsilon}\tau\right)\pm\left(\sec\left(2\sqrt{\omega\,\upsilon}\tau\right)\right)\right), \tag{121}$$

$$W_{2,\,8}(x,\,t)=2\Xi^2-2\Xi\left(\frac{1}{2}\sqrt{\Upsilon}\upsilon\,(\ln(H))^2\sqrt{\frac{\omega}{\upsilon}}\left(\tan\left(2\sqrt{\omega\,\upsilon}\tau\right)\pm\left(\sec\left(2\sqrt{\omega\,\upsilon}\tau\right)\right)\right)\right), \tag{122}$$

$$\varpi_{2,\,9}(x,\,t)=\frac{1}{\Xi}\left(-1/2\sqrt{\Upsilon}\upsilon\,(\ln(H))^2\sqrt{\frac{\omega}{\upsilon}}\left(\cot\left(2\sqrt{\omega\,\upsilon}\tau\right)\pm\left(\csc\left(2\sqrt{\omega\,\upsilon}\tau\right)\right)\right)\right)^2, \tag{123}$$

$$V_{2,\,9}(x,\,t)=-\Xi-1/2\sqrt{\Upsilon}\upsilon\,(\ln(H))^2\sqrt{\frac{\omega}{\upsilon}}\left(\cot\left(2\sqrt{\omega\,\upsilon}\tau\right)\pm\left(\csc\left(2\sqrt{\omega\,\upsilon}\tau\right)\right)\right), \tag{124}$$

$$W_{2,\,9}(x,\,t)=2\Xi^2-2\Xi\left(-1/2\sqrt{\Upsilon}\upsilon\,(\ln(H))^2\sqrt{\frac{\omega}{\upsilon}}\left(\cot\left(2\sqrt{\omega\,\upsilon}\tau\right)\pm\left(\csc\left(2\sqrt{\omega\,\upsilon}\tau\right)\right)\right)\right), \tag{125}$$

and

$$\varpi_{2,\,10}(x,\,t)=\frac{1}{\Xi}\left(1/4\sqrt{\Upsilon}\upsilon\,(\ln(H))^2\sqrt{\frac{\omega}{\upsilon}}\left(\tan\left(1/2\sqrt{\omega\,\upsilon}\tau\right)-\cot\left(1/2\sqrt{\omega\,\upsilon}\tau\right)\right)\right)^2. \tag{126}$$

$$V_{2,\,10}(x,\,t)=-\Xi+1/4\sqrt{\Upsilon}\upsilon\,(\ln(H))^2\sqrt{\frac{\omega}{\upsilon}}\left(\tan\left(\frac{1}{2}\sqrt{\omega\,\upsilon}\tau\right)-\cot\left(1/2\sqrt{\omega\,\upsilon}\tau\right)\right). \tag{127}$$

$$W_{2,\,10}(x,\,t)=2\Xi^2-2\Xi\left(1/4\sqrt{\Upsilon}\upsilon\,(\ln(H))^2\sqrt{\frac{\omega}{\upsilon}}\left(\tan\left(\frac{1}{2}\sqrt{\omega\,\upsilon}\tau\right)-\cot\left(1/2\sqrt{\omega\,\upsilon}\tau\right)\right)\right). \tag{128}$$

Household 2.3 When $\eta=0;\ \upsilon\omega<0,\ \Upsilon>0$, we have:

$$\varpi_{2,\,11}(x,\,t)=\frac{1}{\Xi}\left(-1/2\sqrt{\Upsilon}\upsilon\,(\ln(H))^2\sqrt{-\frac{\omega}{\upsilon}}\tanh\left(\sqrt{-\omega\,\upsilon}\tau\right)\right)^2, \tag{129}$$

$$V_{2,\,11}(x,\,t)=-\Xi-\frac{1}{2}\sqrt{\Upsilon}\upsilon\,(\ln(H))^2\sqrt{-\frac{\omega}{\upsilon}}\tanh\left(\sqrt{-\omega\,\upsilon}\tau\right), \tag{130}$$

$$W_{2,\,11}(x,\,t)=2\Xi^2-2\Xi\left(-\frac{1}{2}\sqrt{\Upsilon}\upsilon\,(\ln(H))^2\sqrt{-\frac{\omega}{\upsilon}}\tanh\left(\sqrt{-\omega\,\upsilon}\tau\right)\right), \tag{131}$$

$$\varpi_{2,\ 12}(x,\ t)=\frac{1}{\Xi}\left(-1/2\sqrt{\Upsilon}\upsilon\ (\ln(H))^2\sqrt{-\frac{\omega}{\upsilon}}\coth\left(\sqrt{-\omega\upsilon}\tau\right)\right)^2, \tag{132}$$

$$V_{2,\ 12}(x,\ t)=-\Xi-\frac{1}{2}\sqrt{\Upsilon}\upsilon\ (\ln(H))^2\sqrt{-\frac{\omega}{\upsilon}}\coth\left(\sqrt{-\omega\upsilon}\tau\right), \tag{133}$$

$$W_{2,\ 17}(x,\ t)=2\Xi^2-2\Xi\left(-\frac{1}{2}\sqrt{\Upsilon}\upsilon\ (\ln(H))^2\sqrt{-\frac{\omega}{\upsilon}}\coth\left(\sqrt{-\omega\upsilon}\tau\right)\right), \tag{134}$$

$$\varpi_{2,\ 12}(x,\ t)=\frac{1}{\Xi}\left(-\frac{1}{2}\sqrt{\Upsilon}\upsilon\ (\ln(H))^2\sqrt{-\frac{\omega}{\upsilon}}\left(\tanh\left(2\sqrt{-\omega\upsilon}\tau\right)\pm\left(isech\left(2\sqrt{-\omega\upsilon}\tau\right)\right)\right)\right)^2, \tag{135}$$

$$V_{2,\ 13}(x,\ t)=-\Xi-\frac{1}{2}\sqrt{\Upsilon}\upsilon\ (\ln(H))^2\sqrt{-\frac{\omega}{\upsilon}}\left(\tanh\left(2\sqrt{-\omega\upsilon}\tau\right)\pm\left(isech\left(2\sqrt{-\omega\upsilon}\tau\right)\right)\right), \tag{136}$$

$$W_{2,\ 13}(x,\ t)=2\Xi^2-2\Xi\left(-1/2\sqrt{\Upsilon}\upsilon\ (\ln(H))^2\sqrt{-\frac{\omega}{\upsilon}}\left(\tanh\left(2\sqrt{-\omega\upsilon}\tau\right)\pm\left(isech\left(2\sqrt{-\omega\upsilon}\tau\right)\right)\right)\right), \tag{137}$$

$$\varpi_{2,\ 14}(x,\ t)=\frac{1}{\Xi}\left(-1/2\sqrt{\Upsilon}\upsilon\ (\ln(H))^2\sqrt{-\frac{\omega}{\upsilon}}\left(\coth\left(2\sqrt{-\omega\upsilon}\tau\right)\pm\left(csch\left(2\sqrt{-\omega\upsilon}\tau\right)\right)\right)\right)^2, \tag{138}$$

$$V_{2,\ 14}(x,\ t)=-\Xi-1/2\sqrt{\Upsilon}\upsilon\ (\ln(H))^2\sqrt{-\frac{\omega}{\upsilon}}\left(\coth\left(2\sqrt{-\omega\upsilon}\tau\right)\pm\left(csch\left(2\sqrt{-\omega\upsilon}\tau\right)\right)\right), \tag{139}$$

$$W_{2,\ 14}(x,\ t)=2\Xi^2-2\Xi\left(-1/2\sqrt{\Upsilon}\upsilon\ (\ln(H))^2\sqrt{-\frac{\omega}{\upsilon}}\left(\coth\left(2\sqrt{-\omega\upsilon}\tau\right)\pm\left(csch\left(2\sqrt{-\omega\upsilon}\tau\right)\right)\right)\right), \tag{140}$$

and

$$\varpi_{2,\ 15}(x,\ t)=\frac{1}{\Xi}\left(-1/4\sqrt{\Upsilon}\upsilon\ (\ln(H))^2\sqrt{-\frac{\omega}{\upsilon}}\left(\tanh\left(1/2\sqrt{-\omega\upsilon}\tau\right)+\coth\left(1/2\sqrt{-\omega\upsilon}\tau\right)\right)\right)^2. \tag{141}$$

$$Vi_{2,\ 15}(x,\ t)=-\Xi-1/4\sqrt{\Upsilon}\upsilon\ (\ln(H))^2\sqrt{-\frac{\omega}{\upsilon}}\left(\tanh\left(1/2\sqrt{-\omega\upsilon}\tau\right)+\coth\left(1/2\sqrt{-\omega\upsilon}\tau\right)\right). \tag{142}$$

$$W_{2,\ 15}(x,\ t)=2\Xi^2-2\Xi\left(-1/4\sqrt{\Upsilon}\upsilon\ (\ln(H))^2\sqrt{-\frac{\omega}{\upsilon}}\left(\tanh\left(1/2\sqrt{-\omega\upsilon}\tau\right)+\coth\left(1/2\sqrt{-\omega\upsilon}\tau\right)\right)\right). \tag{143}$$

Household 2.4 When $\upsilon=\omega;\ \eta=0,\ \Upsilon>0$, we have:

$$\varpi_{2,\ 16}(x,\ t)=\frac{1}{\Xi}\left(1/2\sqrt{\Upsilon}\upsilon\ (\ln(H))^2\tan(\omega\,\tau)\right)^2, \tag{144}$$

$$V_{2,\ 16}(x,\ t)=-\Xi+1/2\sqrt{\Upsilon}\upsilon\ (\ln(H))^2\tan(\omega\,\tau), \tag{145}$$

$$W_{2,\ 16}(x,\ t)=2\Xi^2-2\Xi\left(1/2\sqrt{\Upsilon}\upsilon\ (\ln(H))^2\tan(\omega\,\tau)\right), \tag{146}$$

$$\varpi_{2,\ 17}(x,\ t)=\frac{1}{\Xi}\left(-1/2\sqrt{\Upsilon}\upsilon\ (\ln(H))^2\cot(\omega\,\tau)\right)^2, \tag{147}$$

$$V_{2,\ 17}(x,\ t)=-\Xi-1/2\sqrt{\Upsilon}\upsilon\ (\ln(H))^2\tan(\omega\,\tau), \tag{148}$$

$$W_{2,\ 17}(x,\ t)=2\Xi^2-2\Xi\left(-1/2\sqrt{\Upsilon}\upsilon\ (\ln(H))^2\tan(\omega\,\tau)\right), \tag{149}$$

$$\varpi_{2,\ 18}(x,\ t)=\frac{1}{\Xi}\left(1/2\sqrt{\Upsilon}\upsilon\ (\ln(H))^2(\tan(2\,\omega\,\tau)\pm(\sec(2\,\omega\,\tau)))\right)^2, \tag{150}$$

$$V_{2,\ 18}(x,\ t)=-\Xi+1/2\sqrt{\Upsilon}\upsilon\ (\ln(H))^2(\tan(2\,\omega\,\tau)\pm(\sec(2\,\omega\,\tau))), \tag{151}$$

$$W_{2,\ 18}(x,\ t)=2\Xi^2-2\Xi\left(1/2\sqrt{\Upsilon}\upsilon\ (\ln(H))^2(\tan(2\,\omega\,\tau)\pm(\sec(2\,\omega\,\tau)))\right) \tag{152}$$

$$\varpi_{2,\ 19}(x,\ t)=\frac{1}{\Xi}\left(1/2\sqrt{\Upsilon}\upsilon\ (\ln(H))^2(-\cot(2\,\omega\,\tau)\pm(\csc(2\,\omega\,\tau)))\right)^2, \tag{153}$$

$$V_{2,\ 19}(x,\ t)=-\Xi+1/2\sqrt{\Upsilon}\upsilon\ (\ln(H))^2(-\cot(2\,\omega\,\tau)\pm(\csc(2\,\omega\,\tau))) \tag{154}$$

$$W_{2,\ 19}(x,\ t)=2\Xi^2-2\Xi\left(1/2\sqrt{\Upsilon}\upsilon\ (\ln(H))^2(-\cot(2\,\omega\,\tau)\pm(\csc(2\,\omega\,\tau)))\right), \tag{155}$$

and

$$\varpi_{2,\ 20}(x,\ t)=\frac{1}{\Xi}\left(1/2\sqrt{\Upsilon}\upsilon\ (\ln(H))^2(1/2\tan(1/2\,\omega\,\tau)-1/2\cot(1/2\,\omega\,\tau))\right)^2. \tag{156}$$

$$V_{2,\ 20}(x,\ t)=-\Xi+1/2\sqrt{\Upsilon}\upsilon\ (\ln(H))^2(1/2\tan(1/2\,\omega\,\tau)-1/2\cot(1/2\,\omega\,\tau)). \tag{157}$$

$$W_{2,\ 20}(x,\ t)=2\Xi^2-2\Xi\left(1/2\sqrt{\Upsilon}\upsilon\ (\ln(H))^2(1/2\tan(1/2\,\omega\,\tau)-1/2\cot(1/2\,\omega\,\tau))\right). \tag{158}$$

Household 2.5 When $\upsilon=-\omega;\ \eta=0,\ \Upsilon>0$, we have:

$$\varpi_{2,\,21}(x,\,t)=\frac{1}{\Xi}\left(-1/2\,\sqrt{\Upsilon}\upsilon\,\left(\ln\left(H\right)\right)^{2}\tanh\left(\omega\,\tau\right)\right)^{2}, \tag{159}$$

$$V_{2,\,21}(x,\,t)=-\Xi-1/2\,\sqrt{\Upsilon}\upsilon\,\left(\ln\left(H\right)\right)^{2}\tanh\left(\omega\,\tau\right), \tag{160}$$

$$W_{2,\,21}(x,\,t)=2\Xi^{2}-2\Xi\left(-1/2\,\sqrt{\Pi}\upsilon\,\left(\ln\left(\beta\right)\right)^{2}\tanh\left(\omega\,\tau\right)\right), \tag{161}$$

$$\varpi_{2,\,22}(x,\,t)=\frac{1}{\Xi}\left(-1/2\,\sqrt{\Upsilon}\upsilon\,\left(\ln\left(H\right)\right)^{2}\coth\left(\omega\,\tau\right)\right)^{2}, \tag{162}$$

$$V_{2,\,22}(x,\,t)=-\Xi-1/2\,\sqrt{\Upsilon}\upsilon\,\left(\ln\left(H\right)\right)^{2}\tanh\left(\omega\,\tau\right)\Bigg)^{2}, \tag{163}$$

$$W_{2,\,22}(x,\,t)=2\Xi^{2}-2\Xi\left(-1/2\,\sqrt{\Upsilon}\upsilon\,\left(\ln\left(H\right)\right)^{2}\tanh\left(\omega\,\tau\right)\right)^{2}, \tag{164}$$

$$\varpi_{2,\,23}(x,\,t)=\frac{1}{\Xi}\left(1/2\,\sqrt{\Upsilon}\upsilon\,\left(\ln\left(H\right)\right)^{2}\left(-\tanh\left(2\,\omega\,\tau\right)\pm\left(isech\left(2\,\omega\,\tau\right)\right)\right)\right)^{2}, \tag{165}$$

$$V_{2,\,23}(x,\,t)=-\Xi+1/2\,\sqrt{\Upsilon}\upsilon\,\left(\ln\left(H\right)\right)^{2}\left(-\tanh\left(2\,\omega\,\tau\right)\pm\left(isech\left(2\,\omega\,\tau\right)\right)\right), \tag{166}$$

$$W_{2,\,23}(x,\,t)=2\Xi^{2}-2\Xi\left(1/2\,\sqrt{\Upsilon}\upsilon\,\left(\ln\left(H\right)\right)^{2}\left(-\tanh\left(2\,\omega\,\tau\right)\pm\left(isech\left(2\,\omega\,\tau\right)\right)\right)\right) \tag{167}$$

$$\varpi_{2,\,24}(x,\,t)=\frac{1}{\Xi}\left(1/2\,\sqrt{\Upsilon}\upsilon\,\left(\ln\left(H\right)\right)^{2}\left(-\coth\left(2\,\omega\,\tau\right)\pm\left(csch\left(2\,\omega\,\tau\right)\right)\right)\right)^{2}, \tag{168}$$

$$V_{2,\,24}(x,\,t)=-\Xi+1/2\,\sqrt{\Upsilon}\upsilon\,\left(\ln\left(H\right)\right)^{2}\left(-\coth\left(2\,\omega\,\tau\right)\pm\left(csch\left(2\,\omega\,\tau\right)\right)\right) \tag{169}$$

$$W_{2,\,24}(x,\,t)=2\Xi^{2}-2\Xi\left(1/2\,\sqrt{\Upsilon}\upsilon\,\left(\ln\left(H\right)\right)^{2}\left(-\coth\left(2\,\omega\,\tau\right)\pm\left(csch\left(2\,\omega\,\tau\right)\right)\right)\right), \tag{170}$$

and

$$\varpi_{2,\,25}(x,\,t)=\frac{1}{\Xi}\left(1/2\,\sqrt{\Upsilon}\upsilon\,\left(\ln\left(H\right)\right)^{2}\left(-1/2\,\tanh\left(1/2\,\omega\,\tau\right)-1/2\,\coth\left(1/2\,\omega\,\tau\right)\right)\right)^{2}. \tag{171}$$

$$V_{2,\,25}(x,\,t)=-\Xi+1/2\,\sqrt{\Upsilon}\upsilon\,\left(\ln\left(H\right)\right)^{2}\left(-1/2\,\tanh\left(1/2\,\omega\,\tau\right)-1/2\,\coth\left(1/2\,\omega\,\tau\right)\right). \tag{172}$$

$$W_{2,\,25}(x,\,t)=2\Xi^{2}-2\Xi\left(1/2\,\sqrt{\Upsilon}\upsilon\,\left(\ln\left(H\right)\right)^{2}\left(-1/2\,\tanh\left(1/2\,\omega\,\tau\right)-1/2\,\coth\left(1/2\,\omega\,\tau\right)\right)\right) \tag{173}$$

Household 2.6 When $\eta = \varsigma;\ \omega = n\tau;\ \upsilon = 0,\ \Upsilon > 0$,

$$\varpi_{2,\,26}(x,\,t) = \frac{1}{\Xi}\left(1/4\sqrt{\Upsilon}(\ln(H))^2(\eta) + 1/2\sqrt{\Upsilon}\upsilon\,(\ln(H))^2(H^{\varsigma\tau} - n)\right)^2. \tag{174}$$

$$V_{2,\,26}(x,\,t) = -\Xi + 1/4\sqrt{\Upsilon}(\ln(H))^2(\zeta) + 1/2\sqrt{\Upsilon}\upsilon\,(\ln(H))^2(H^{\varsigma\tau} - n) \tag{175}$$

$$W_{2,\,26}(x,\,t) = 2\Xi^2 - 2\Xi\left(1/4\sqrt{\Upsilon}(\ln(H))^2(\zeta) + 1/2\sqrt{\Upsilon}\upsilon\,(\ln(H))^2(H^{\varsigma\tau} - n)\right) \tag{176}$$

Household 2.7 When $\eta = 0;\ \omega = 0,\ \Upsilon > 0$, we have:

$$\varpi_{2,\,27}(x,\,t) = \frac{-1}{\Xi}\left(-1/2\,\frac{\sqrt{\Upsilon}(\ln(H))^2}{\tau\ln(H)}\right)^2. \tag{177}$$

$$V_{2,\,27}(x,\,t) = -\Xi - 1/2\,\frac{\sqrt{\Upsilon}(\ln(H))^2}{\tau\ln(H)}. \tag{178}$$

$$W_{2,\,27}(x,\,t) = 2\Xi^2 - 2\Xi\left(-1/2\,\frac{\sqrt{\Upsilon}(\ln(H))^2}{\tau\ln(H)}\right). \tag{179}$$

Household 2.8 When $\omega = 0;\ \eta \neq 0;\ \upsilon \neq 0,\ \Upsilon > 0$,

$$\varpi_{2,\,28}(x,\,t) = \frac{-1}{\Xi}\left(1/4\sqrt{\Upsilon}(\ln(H))^2(\eta) - 1/2\,\frac{\sqrt{\Upsilon}(\ln(H))^2\eta}{\cosh(\eta\tau) - \sinh(\eta\tau) + 1}\right)^2, \tag{180}$$

$$V_{2,\,28}(x,\,t) = -\Xi + 1/4\sqrt{\Upsilon}(\ln(H))^2(\eta) - 1/2\,\frac{\sqrt{\Upsilon}(\ln(H))^2\eta}{\cosh(\eta\tau) - \sinh(\eta\tau) + 1}, \tag{181}$$

$$W_{2,\,28}(x,\,t) = 2\Xi^2 - 2\lambda\left(1/4\sqrt{\Upsilon}(\ln(H))^2(\eta) - 1/2\,\frac{\sqrt{\Upsilon}(\ln(H))^2\eta}{\cosh(\eta\tau) - \sinh(\eta\tau) + 1}\right), \tag{182}$$

Household 2.9 When $\eta = \varsigma;\ \upsilon = n\varsigma;\ \omega = 0,\ \Upsilon > 0$, we have:

$$\varpi_{2,\,29}(x,\,t) = \frac{1}{\Xi}\left(1/4\sqrt{\Upsilon}(\ln(H))^2(\eta) + 1/2\,\frac{\sqrt{\Upsilon}\upsilon\,(\ln(H))^2H^{\tau\varsigma}}{1 - nH^{\tau\varsigma}}\right)^2. \tag{183}$$

$$V_{2,\,29}(x,\,t) = -\Xi + 1/4\sqrt{\Upsilon}(\ln(H))^2(\eta) + 1/2\,\frac{\sqrt{\Upsilon}\upsilon\,(\ln(H))^2H^{\tau\varsigma}}{1 - nH^{\tau\varsigma}}. \tag{184}$$

$$W_{2,\,29}(x,\,t) = 2\Xi^2 - 2\Xi\left(1/4\sqrt{\Upsilon}(\ln(H))^2(\zeta) + 1/2\,\frac{\sqrt{\Upsilon}\upsilon\,(\ln(H))^2 H^{\tau\varsigma}}{1 - nH^{\tau\varsigma}}\right). \tag{185}$$

4. Graphics and interpretations

This novel study analyses the Hirota-Satsuma fractional order equation using the Extended Differential Algebraic Method (EDAM), providing new information on the dynamics of multidimensional wave patterns and soliton structures. Our study's outcomes show how EDAM may reliably and effectively identify coherent patterns that differentiate between different stages of nonlinear reaction-diffusion systems. Pattern generation, nonlinear wave propagation, and thermodynamic processes in fractional order systems are only a few of the physical phenomena and temporal development processes that are better understood thanks to the discovered solutions. Short, rapid oscillations that swiftly transition between stable states are a defining feature of the wide range of patterns seen in the ensuing soliton solutions. To comprehend the behaviour of physical processes in Hirota-Satsuma fractional order equation-based systems, such as optical communications and condensed matter physics, it is crucial to identify these soliton waves.

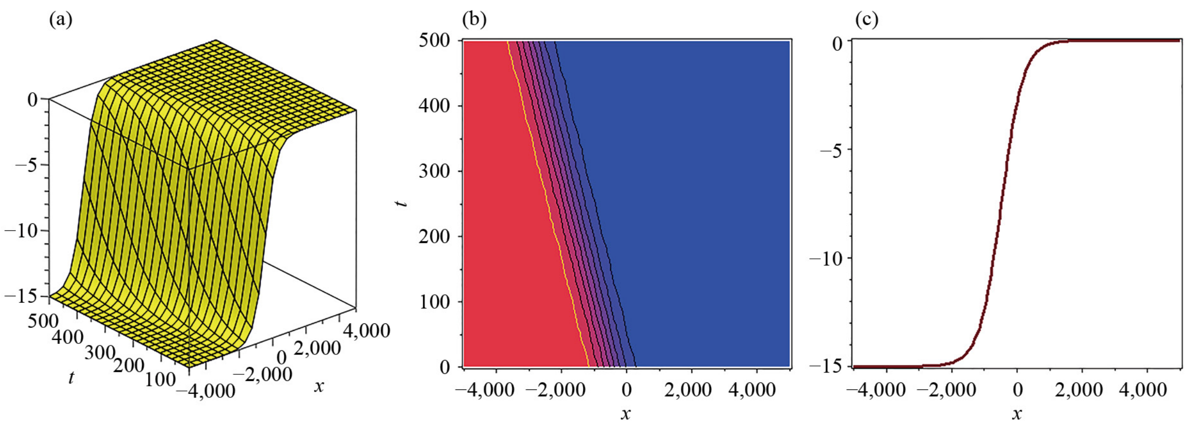

Figure 1. 3-dimensional, contour and 2-dimensional soliton solutions are plotted for Equation (12) with $\eta = 0.4$; $\omega = 1$; $\upsilon = 11$; $H = e$; $\sigma = 0.5$; $\Xi = 0.1$

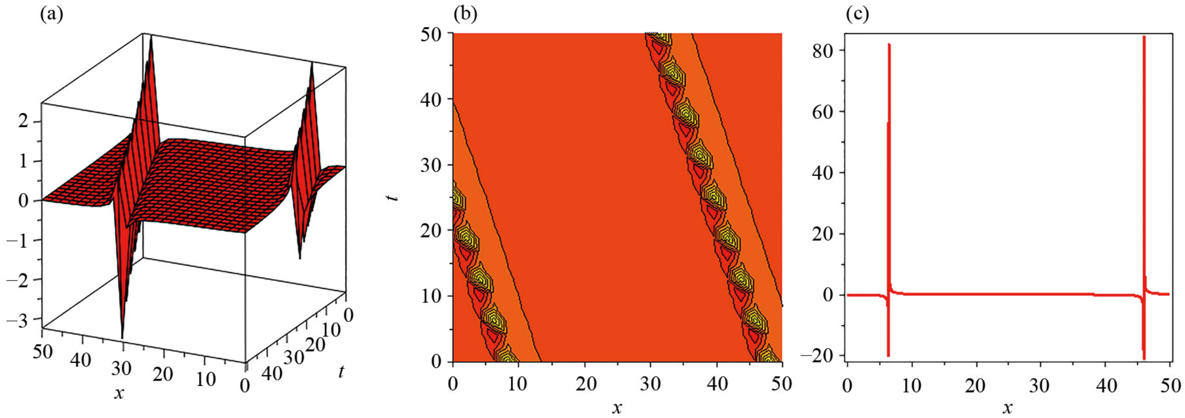

Figure 2. In this figure, 3-dimensional, contour & 2-dimensional soliton solutions are plotted for Eq. (16) with $\eta = 0.14$; $\omega = 11$; $\upsilon = 1.1$; $H = e$; $\sigma = 0.15$; $\Xi = 0.1$

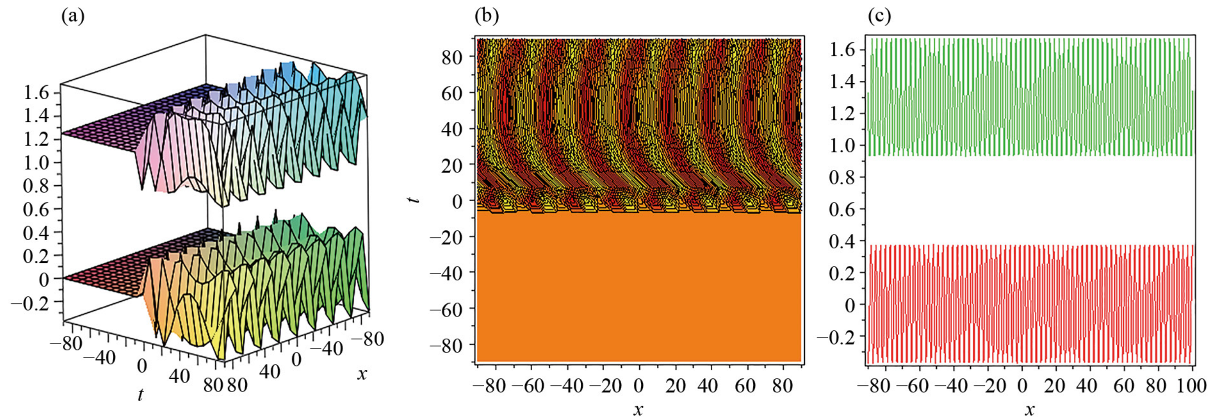

Figure 3. In this figure, 3-dimensional, contour & 2-dimensional soliton solutions are plotted for Eq. (19) with $\eta = 0.4$; $\omega = 0.1$; $\upsilon = 11$; $H = e$; $\sigma = 0.05$; $\Xi = 0.01$

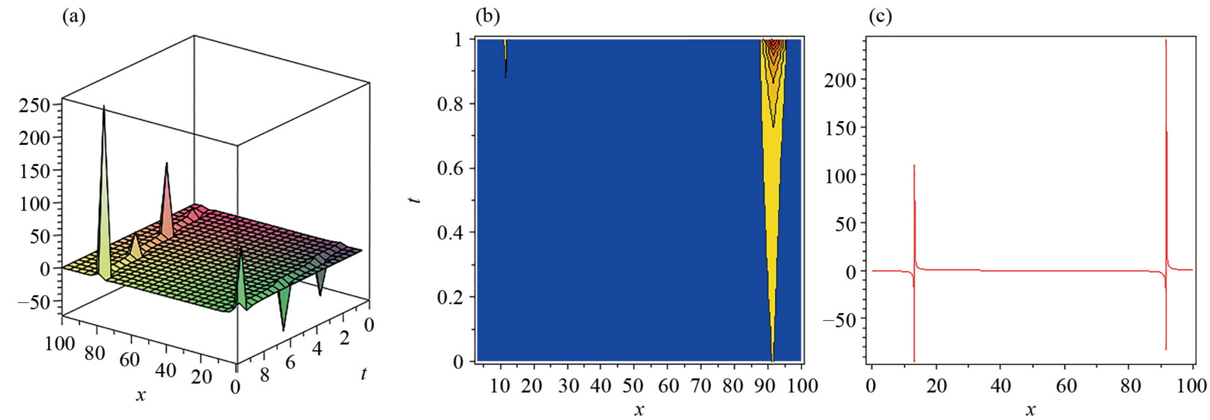

Figure 4. In this figure, 3-dimensional, contour & 2-dimensional soliton solutions are plotted for Eq. (25) with $\eta = 0.4$; $\omega = 1$; $\upsilon = 11$; $H = e$; $\sigma = 0.005$; $\Xi = 0.01$

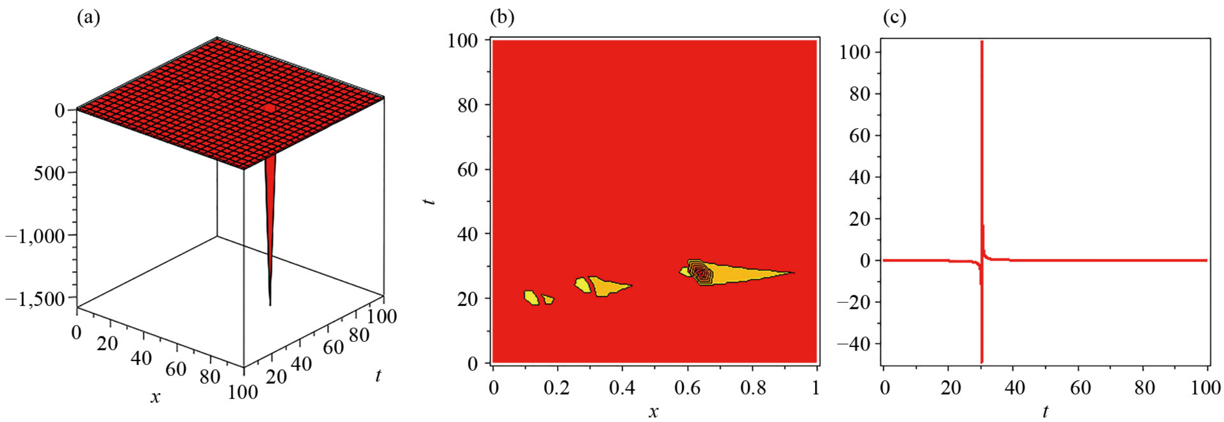

Figure 5. In this figure, 3-dimensional, contour & 2-dimensional soliton solutions are plotted for Eq. (115) with $\eta = 0.04$; $\omega = 0.1$; $\upsilon = 11$; $H = e$; $\sigma = 0.35$; $\Xi = 0.1$

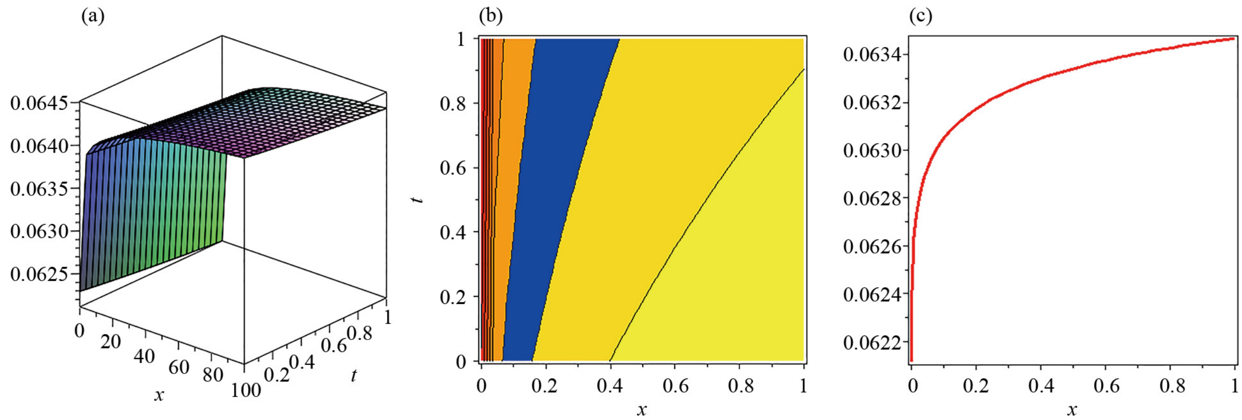

Figure 6. In this figure, 3-dimensional, contour & 2-dimensional soliton solutions are plotted for Eq. (120) with $\eta = 0.4$; $\omega = 1$; $\upsilon = 11$; $H = e$; $\sigma = 0.45$; $\Xi = 0.41$

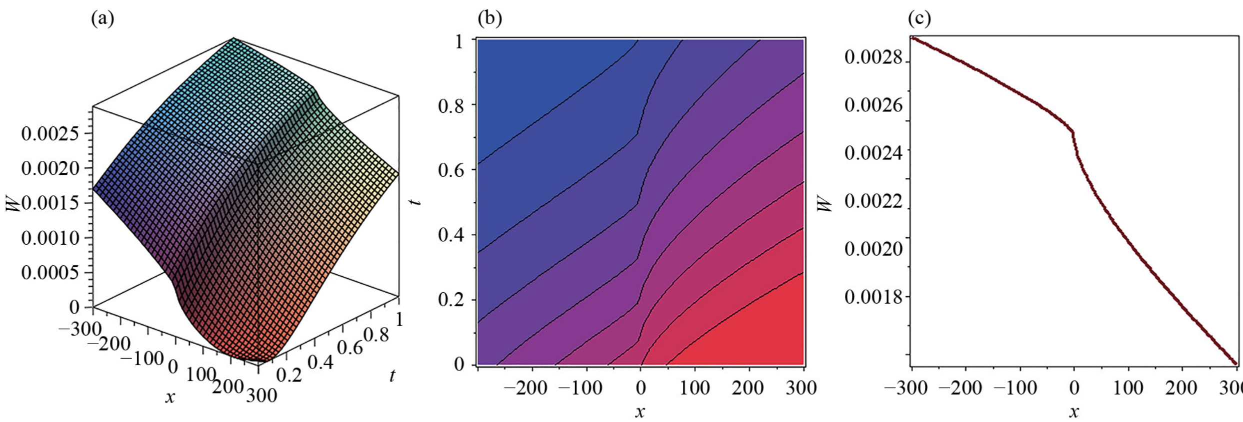

Figure 7. In this figure, 3-dimensional, contour & 2-dimensional soliton solutions are plotted for Eq. (121) with $\eta = 0.4$; $\omega = 0.1$; $\upsilon = 1.1$; $H = e$; $\sigma = 1.5$; $\Xi = 0.1$

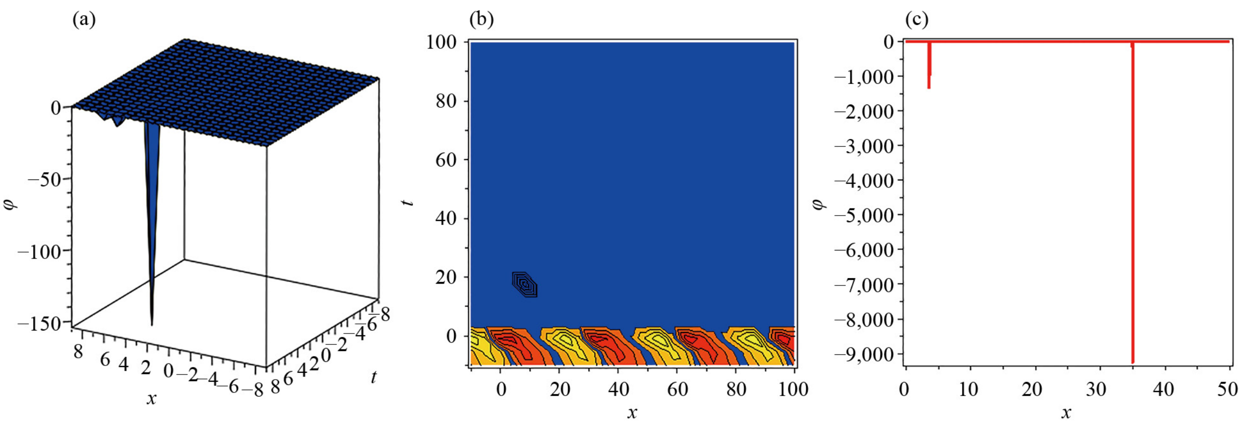

Figure 8. In this figure, 3-dimensional, contour & 2-dimensional soliton solutions are plotted for Eq. (125) with $\eta = 0.14$; $\omega = 0.81$; $\upsilon = 11$; $H = e$; $\sigma = 0.65$; $\Xi = 0.1$

Figures 1-8 illustrate the soliton solutions of the fractional Hirota-Satsuma coupled KdV system for different parameter choices. Figure 1 corresponds to the soliton solution of Equation (12), while Figures 2, 3, and 4 display the solutions associated with Equations (16), (19), and (25), respectively. Moreover, Figure 5 presents the soliton structures obtained from Equation (115), whereas Figures 6 and 7 depict the solutions of Equations (120) and (121). Finally, Figure 8 illustrates the soliton solution corresponding to Equation (125). These figures collectively demonstrate the influence of the fractional parameter on the shape, amplitude, and localization of the obtained soliton solutions.

5. Conclusion

Hirota satsuma coupled KdV equation were successfully investigated using the extended direct algebraic method. The resulting solutions demonstrate the complex interactions and stability of these intricate structures, offering important insights into the dynamic behaviour of solitons in nonlinear systems. The soliton dynamics are further clarified by the graphical representations of the solutions, which provide a deeper comprehension of the physics at play. The findings of this study add to our understanding of fractional calculus and nonlinear dynamics, which may find use in physics, engineering, and optics, among other domains.

Author contributions

Conceptualization, M.V.C., P.O.M., and K.K.A.; Funding acquisition, P.O.M. and M.V.C.; Investigation, M.B., W.W.M., and M.A.Y.; Methodology, M.B., W.W.M., and M.A.Y.; Project administration, P.O.M. and M.V.C.; Software, M.B. and M.A.Y.; Supervision, P.O.M. and M.V.C.; Writing-original draft, M.A.Y., W.W.M. and M.B.; Writing-review & editing, M.V.C., P.O.M., and K.K.A. All authors have read and agreed to the published version of the manuscript.

Conflict of interest

The authors declare no competing financial interest.

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