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Research Article

On the Spectrum of Composition Operators on Harmonic Bloch-Type Spaces

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Abstract: In this paper, we consider the composition operators on harmonic Bloch-type spaces. Then first we compute their spectra on harmonic α-Bloch spaces and harmonic little α-Bloch space and then we characterize isometric composition operators on harmonic α-Bloch-type spaces. Indeed we obtain a relation between the properties of the inducing function φ and the isometricity of the composition operator C_φ.

Keywords: harmonic function, composition operator, spectrum

MSC: 47B33

1. Introduction

This paper is devoted to the study of composition operators on complex-valued harmonic functions in Bloch-type spaces. Composition operators can act on various types of function spaces. In each setting, one of the main goals is to uncover the connection between the properties of the inducing function φ and the operator-theoretic properties of C_φ. Examples of such properties include boundedness, compactness, invertibility, normality, subnormality, isometricity, having closed range, and the Fredholm property, among others. Extensive references for many known results on this topic can be found in [1–7].

In [8], the authors discuss Hardy-type spaces, Bloch-type spaces, and composition operators on complex-valued harmonic functions. In [9–11], we characterized bounded, compact, Fredholm, and essential composition operators on harmonic Bloch function spaces. Also, in [12] the authors have given a complete characterization of the spectrum of composition operators, induced by an automorphism of the open unit disk, acting on a family of Banach spaces of analytic functions that includes the Bloch space and the space of functions of Bounded Mean Oscillation (BMO). Indeed they have shown that for parabolic and hyperbolic automorphisms, the spectrum is the unit circle. For the case of elliptic automorphisms, the spectrum is either the unit circle or a finite cyclic subgroup of the unit circle.

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In this paper, we advance this line of research by determining the spectrum of composition operators on harmonic Bloch-type spaces $HB(\alpha)$. Additionally, we offer a complete characterization of isometric composition operators on these spaces, thereby contributing to a deeper understanding of their structural and spectral properties.

Let φ be an analytic self-map of the open unit disk D. The composition operator C_φ induced by φ is defined on the space of all harmonic functions on D by

$$C_\varphi f = f \circ \varphi.$$

It is straightforward to verify that an operator defined in this manner is linear. Let $f(z) = u(z) + i\upsilon(z)$, where $z = x + iy$, be a continuously differentiable, complex-valued function on the open unit disk D. It is known that the formal derivatives of f are given by

$$f_z = \frac{1}{2}\left(f_x - i f_y\right),$$

$$f_{\bar{z}} = \frac{1}{2}\left(f_x + i f_y\right).$$

A twice continuously differentiable, complex-valued function $f = u + i\upsilon$ on the open unit disk D is called *harmonic* if and only if the real-valued functions u and υ satisfy the Laplace equation:

$$\Delta u = \Delta \upsilon = 0.$$

A straightforward computation reveals that the Laplacian of f can be expressed as

$$\Delta f = 4 f_{z\bar{z}}.$$

Therefore, for functions f with continuous second partial derivatives, f is harmonic if and only if $\Delta f = 0$.

Let f be a complex-valued harmonic function defined on a simply connected domain $D \subset \mathbb{C}$. Such a function f admits a canonical decomposition:

$$f = h + \overline{g},$$

where h and g are analytic functions on D (see [13], p.7).

The purpose of this paper is to study composition operators on spaces of complex-valued harmonic functions. It is well known that analytic functions are preserved under composition, but harmonic functions, in general, are not.

A planar harmonic function f, defined in the domain D and taking complex values, is termed a *harmonic Bloch function* if it satisfies the condition:

$$\beta_f = \sup_{\substack{z,\, w \in D \\ z \neq w}} \frac{\left|f(z) - f(w)\right|}{\rho(z,\, w)} < \infty,$$

where β_f is referred to as the Lipschitz number of f. The metric $\rho(z, w)$ is given by

$$\rho(z, w) = \frac{1}{2}\log\left(\frac{1+\left|\frac{z-w}{1-\bar{z}w}\right|}{1-\left|\frac{z-w}{1-\bar{z}w}\right|}\right)$$

$$= \frac{1}{2}\log\left(\frac{1+\rho(z, w)}{1-\rho(z, w)}\right)$$

$$= \operatorname{artanh}\left|\frac{z-w}{1-\bar{z}w}\right|,$$

and denotes the hyperbolic distance between z and w in D. Here,

$$\rho(z, w) = \left|\varphi_z(w)\right|$$

denote the *pseudo-hyperbolic distance* between z and w in D, where

$$\varphi_z(w) = \frac{z-w}{1-\bar{z}w}$$

is a disk automorphism of D. In this paper, we denote by

$$D(a, r) = \{z \in D : \rho(a, z) < r\}$$

the hyperbolic disk with center a and radius $r > 0$.

Colonna showed in [14] that

$$\beta_f = \sup_{\substack{z,\, w \in D \\ z \neq w}} \frac{|f(z) - f(w)|}{\rho(z, w)} = \sup_{z\in D}\left(1-|z|^2\right)\left[|f_z(z)| + |f_{\bar{z}}(z)|\right].$$

Additionally, the collection of all harmonic Bloch functions, represented as $HB(1)$ or simply HB, constitutes a complex Banach space. The norm for this space is given by:

$$|||f|||_{HB(1)} = |f(0)| + \sup_{z\in D}\left(1-|z|^2\right)\left[|f_z(z)| + |f_{\bar{z}}(z)|\right].$$

For $\alpha \in (0, \infty)$, the *harmonic α-Bloch space*, denoted by $HB(\alpha)$ is defined as the set of all complex-valued harmonic functions f on D that satisfy the condition:

$$\|f\|_{HB(\alpha)} = \sup_{z\in D}(1-|z|^2)^{\alpha}\left[|f_z(z)|+|f_{\bar{z}}(z)|\right] < \infty.$$

In addition, the *harmonic little α-Bloch space*, denoted by $HB_0(\alpha)$, is the subset of $HB(\alpha)$ consisting of all functions that satisfy the additional condition:

$$\lim_{|z|\to 1}(1-|z|^2)^{\alpha}\left[|f_z(z)|+|f_{\bar{z}}(z)|\right] = 0.$$

Clearly, for $\alpha = 1$, we have

$$\|f\|_{HB(\alpha)} = \beta_f.$$

The linear space $HB(\alpha)$ is a Banach space with the norm

$$|||f|||_{HB(\alpha)} = |f(0)| + \|f\|_{HB(\alpha)} = |f(0)| + \sup_{z\in D}(1-|z|^2)^{\alpha}\left[|f_z(z)|+|f_{\bar{z}}(z)|\right].$$

Moreover, $HB_0(\alpha)$ is a closed subspace of $HB(\alpha)$.
Observe that

$$\sup_{z\in D}(1-|z|^2)^{\alpha}\left[|f_z(z)|+|f_{\bar{z}}(z)|\right]$$

is a *pseudo-norm*, which coincides with the harmonic α-Bloch norm on the closed subspace of $HB(\alpha)$ consisting of all functions that vanish at the origin. In general, this pseudo-norm coincides with the quotient norm on $HB(\alpha)/C$, where C denotes the closed subspace of constant functions.

2. The spectrum of C_φ

In this section we compute the spectrum of the composition operator C_φ as a bounded operator on harmonic α-Bloch spaces and *harmonic little α-Bloch space*. Recall from [9] that the composition operator $C_\varphi : HB(\alpha) \to HB(\alpha)$ is bounded if and only if

$$\sup_{z\in D}\frac{(1-|z|^2)^{\alpha}}{(1-|\varphi(z)|^2)^{\alpha}}|\varphi'(z)| < \infty.$$

Similarly, the operator $C_\varphi : HB_0(\alpha) \to HB_0(\alpha)$ is bounded if and only if $\varphi \in B_0(\alpha)$ and

$$\sup_{z\in D}\frac{(1-|z|^2)^{\alpha}}{(1-|\varphi(z)|^2)^{\alpha}}|\varphi'(z)| < \infty.$$

Consider the sequence (z_k), defined as the iteration sequence for $\varphi : D \to D$ with $\varphi(0) = 0$, such that $\varphi(z_k) = z_{k+1}$ for all k.

For $\alpha > 0$, let $HB_m(\alpha) := z^m HB(\alpha)$ denote the subspace of $HB(\alpha)$. The space $HB_m(\alpha)$ can equivalently be described as:

$$HB_m(\alpha) = \{f \in HB(\alpha) : f \text{ has a zero of at least order } m \text{ at zero}\}.$$

In order to estimate the norm of the evaluation map acting on the subspaces $HB_m(\alpha)$, we invoke the following two lemmas from [15].

Lemma 1 Let $\alpha > 1$ and $m \in \mathbb{N}$. Then, there exists a constant $c(\alpha)$ depending only on α such that

$$|f(w)| \leq c(\alpha) \frac{|w|^m}{(1-|w|^2)^{\alpha-1}} \|f\|_{HB(\alpha)}$$

for all $f \in HB_m(\alpha)$ and $w \in D$.

Lemma 2 Let $w \in D$ and $|w| \leq \dfrac{1}{2}$. Then

$$\|\delta_w\|_{HB_m(\alpha)} \leq \|\delta_w\|_{HB(\alpha)} \leq 2^m \|\delta_w\|_{HB_m(\alpha)}.$$

We need the following crucial lemmas due to Cowen and MacCluer.

Lemma 3 ([2], p.293) If φ is not an automorphism and $\varphi(0) = 0$, then for any $0 < r < 1$, there exists a constant $1 \leq M < \infty$ such that if $(z_k)_{k=-K}^{\infty}$ is an iteration sequence with $|z_n| \geq r$ for some non-negative integer n, and $(w_k)_{k=-K}^{n}$ are arbitrary numbers, then there exists $f \in H^\infty$ with

$$\|f\|_\infty \leq M \sup\{|w_k| : -K \leq k \leq n\}.$$

Furthermore, there exists $b < 1$ such that for any iteration sequence $(z_k)_{k=-K}^{\infty}$, we have

$$\frac{|z_{k+1}|}{|z_k|} \leq b \quad \text{whenever } |z_k| \leq \frac{1}{2}.$$

Lemma 4 [16] Let $\alpha > 1$, and suppose that $\varphi(0) = 0$ and $C_\varphi : HB(\alpha) \to HB(\alpha)$ is bounded. Then

$$\{\varphi'(0)^n\}_{n=0}^{\infty} \subset \sigma_{HB(\alpha)}(C_\varphi),$$

and if $\lambda \neq 0$ is an eigenvalue of C_φ, then

$$\lambda \in \{\varphi'(0)^n\}_{n=0}^{\infty}.$$

Here we recall that the essential spectrum of C_φ, as an operator on $HB(\alpha)$, is denoted by $r_{e,\ HB(\alpha)}(C_\varphi)$. In the next theorem, we obtain the spectrum of the composition operator C_φ.

Theorem 1 Let $\alpha > 1$ and suppose that φ, which is not an automorphism, fixes the origin. Assume that $C_\varphi : HB(\alpha) \to HB(\alpha)$ is bounded. Then,

$$\sigma_{HB(\alpha)}(C_\varphi) = \{\lambda \in \mathbb{C} : |\lambda| \leq r_{e,\ HB(\alpha)}(C_\varphi)\} \cup \{\varphi'(0)^n\}_{n=0}^{\infty}.$$

Proof. Whenever $\varphi(0) = 0$, $\{\varphi'(0)^n\}_{n=0}^{\infty}$ is contained in the spectrum by Lemma 4. If $\lambda \in \sigma_{HB(\alpha)}(C_\varphi)$ and $|\lambda| > r_{e,\ HB(\alpha)}(C_\varphi)$, then λ is an eigenvalue of C_φ. This result holds for all bounded operators; see, for example, Proposition 2.2 in [17]. If $\lambda \neq 0$ is an eigenvalue of C_φ, Lemma 4 implies that $\lambda = \varphi'(0)^n$ for some n. Therefore, it suffices to prove that:

$$\{\lambda \in \mathbb{C} : |\lambda| \leq r_{e,\ HB(\alpha)}(C_\varphi)\} \subset \sigma_{HB(\alpha)}(C_\varphi).$$

If $r_{e,\ HB(\alpha)}(C_\varphi) = 0$, then there is nothing to prove because $0 \in \sigma_{HB(\alpha)}(C_\varphi)$ when φ is not an automorphism. Thus, we assume that $\rho = r_{e,\ HB(\alpha)}(C_\varphi) > 0$. Since $\varphi(0) = 0$, we can write $\varphi(z) = z\psi(z)$, where $\psi \in H^\infty$. Consequently, the subspace $HB_m(\alpha)$ is invariant under C_φ, and it has finite codimension in $HB(\alpha)$. Because the spectrum of C_φ is closed, we can choose λ with $0 < |\lambda| < \rho$. By Lemma 7.17 in [2], which is also valid for Banach spaces, we have $\sigma_{HB_m(\alpha)}(C_\varphi) \subset \sigma_{HB(\alpha)}(C_\varphi)$. Thus, it is enough to show that $\lambda \in \sigma_{HB_m(\alpha)}(C_\varphi)$ for some suitable m. Let C_m denote the restriction of C_φ to the invariant closed subspace $HB_m(\alpha)$. We then identify an m such that $(C_m - \lambda I)^*$ is not bounded from below. Proceeding as in the proof of Theorem 7 in [15], let $1 \leq M < \infty$ be the constant in Lemma 3 for $r = \frac{1}{4}$. The iteration sequence $(z_k)_{k=-K}^{\infty}$, where $K > 0$ and $|z_0| \geq \frac{1}{2}$. Note that $(|z_k|)$ is decreasing sequence. Define

$$n := \max\left\{k : |z_k| \geq \frac{1}{4}\right\}. \tag{1}$$

Then $n \geq 0$ and $|z_k| < \frac{1}{4}$ for $k > n$. By Lemma 4, we assume that $\frac{1}{2} < b < 1$ such that $|z_k| \leq b^{k-n}|z_n|$ for $k \geq n$. For fixed $1 < \alpha,\ \alpha' < \infty$, we now choose m large enough so that:

$$\frac{b^m}{|\lambda|} < \frac{9^\alpha}{2M\alpha' 16^\alpha 4^\alpha} < \frac{1}{2}.$$

Given any iteration sequence $(z_k)_{k=-K}^{\infty}$, define the linear functional on $HB_m(\alpha)$ by:

$$L_\lambda(f) = \sum_{k=-K}^{\infty} \lambda^{-k} f(z_k).$$

We observe that L_λ is bounded. This follows because, using (1) and Lemma 1, for arbitrary $f \in HB_m(\alpha)$, we have:

$$|L_\lambda(f)| = \left| \sum_{k=-K}^{\infty} \lambda^{-k} f(z_k) \right|$$

$$\leq C(\alpha)\|f\|_{HB(\alpha)} \left\{ \sum_{k=-K}^{n} |\lambda|^{-k}|z_k|^m(1-|z_k|^2)^{-\alpha+1} + \sum_{k=n+1}^{\infty} |\lambda|^{-k}|z_k|^m(1-|z_k|^2)^{-\alpha+1} \right\}.$$

This is finite since $|z_k| < \frac{1}{4}$ for $k > n$, $|z_k| \leq b^{k-n}|z_n|$ for $k \geq n$, and $\frac{b^m}{|\lambda|} < 1$. Therefore, we get:

$$|L_\lambda(f)| \leq \left(\frac{16}{15}\right)^{\alpha-1} \frac{|z_n|^m}{|\lambda|^n} \sum_{k=n+1}^{\infty} \left(\frac{b^m}{|\lambda|}\right)^{k-n} < \infty.$$

Therefore, L_λ is indeed bounded.

To show that $C_m^* - \lambda I$ is not bounded from below, we need to estimate:

$$\frac{\|(C_m^* - \lambda I)L_\lambda\|_{HB_m(\alpha)}}{\|L_\lambda\|_{HB_m(\alpha)}}.$$

First, notice that:

$$(C_m^* - \lambda I)L_\lambda = -\lambda^{K+1}\delta_{z-k}.$$

Now we find a lower bound for $\|L_\lambda\|_{HB_m(\alpha)}$. For any iteration sequence $(z_k)_{k=-K}^{\infty}$, we know there exists $f \in H^\infty$, with $\|f\|_\infty \leq M$, satisfying the following conditions:

(i) $|f(z_k)| = 1$, for $k = 0$ and $k = n$,

(ii) $\frac{z_k^m f(z_k)}{\lambda^k(1-\bar{z}_0 z_k)^{2\alpha}} \geq 0$, for $k = 0$ and $k = n$,

(iii) $f(z_k) = 0$, for $-K \leq k < n,\ k \neq 0$.

For such f, we have:

$$L_\lambda\left(\frac{z^m f(z)(1-|z_0|^2)^{\alpha+1}}{(1-\bar{z}_0 z)^{2\alpha}}\right) = \sum_{k=-K}^{\infty} \lambda^{-k} z_k^m f(z_k) \frac{(1-|z_0|^2)^{\alpha+1}}{(1-\bar{z}_0 z_k)^{2\alpha}}.$$

Since:

$$\frac{(1-|z_0|^2)^\alpha(1-|z|^2)^\alpha}{(1-\bar{z}_0 z)^{2\alpha}} < 1,$$

the function:

$$g := \frac{(1-|z_0|^2)^{\alpha+1}}{(1-\bar{z_0}z)^{2\alpha}}$$

is in $HB(\alpha)$, with $\|g\|_{HB(\alpha)} < \alpha' < \infty$. Hence, the function $h(z) := z^m f(z)g(z)$, belongs to $HB_m(\alpha)$, and $\|h\|_{HB(\alpha)} \leq \alpha' M$.

Thus, we conclude that:

$$\left|\sum_{k=-K}^{\infty} \lambda^{-k} h(z_k)\right| \geq \frac{|z_0|^m (1-|z_0|^2)}{(1-|z_0|^2)^\alpha}$$

$$+ |\lambda|^{-n}|z_n|^m \frac{(1-|z_0|^2)^{\alpha+1}}{(1-\bar{z_0}z_n)^{2\alpha}} - \left|\sum_{k=n+1}^{\infty} \lambda^{-k} h(z_k)\right|$$

$$:= I + II - III.$$

We now estimate the terms individually. First, we have:

$$II \geq \frac{|z_n|^m (1-|z_0|^2)^{\alpha+1}}{|\lambda|^n 4^\alpha}.$$

Next, for the term III, we have:

$$III \leq \alpha' M \left(\frac{16}{9}\right)^\alpha \frac{|z_n|^m}{|\lambda|^n} \sum_{k=n+1}^{\infty} \left(\frac{b^m}{|\lambda|}\right)^{k-n} (1-|z_0|^2)^{\alpha+1},$$

because $\|f\|_{H^\infty} \leq M$, $|z_k| < \dfrac{1}{4}$, and $|z_k| \leq b^{k-n}|z_n|$ for $k \geq n+1$.

For the sum:

$$\sum_{k=n+1}^{\infty} \left(\frac{b^m}{|\lambda|}\right)^{k-n} < \frac{\dfrac{9^\alpha}{2M\alpha' 16^\alpha 4^\alpha}}{1 - \dfrac{9^\alpha}{2M\alpha' 16^\alpha 4^\alpha}} < \frac{9^\alpha}{M\alpha' 16^\alpha 4^\alpha}.$$

Consequently, we obtain:

$$|L_\lambda(h)| \geq \frac{|z_0|^m}{(1-|z_0|^2)^{\alpha-1}}.$$

This gives us:

$$\|L_\lambda\|_{HB_m(\alpha)} \geq \frac{1}{\alpha' M} \frac{|z_0|^m}{(1-|z_0|^2)^{\alpha-1}} \geq \frac{\|\delta_{z_0}\|_{HB_m(\alpha)}}{\alpha' M C(\alpha)}.$$

Next, we recall that:

$$\rho = r_{e,\ HB(\alpha)}(C_\varphi) = \lim_n \|C_\varphi^n\|_{e,\ HB(\alpha)}^{\frac{1}{n}},$$

where:

$$C_\varphi^n f(w) = C_{\varphi_n} f(w), \quad w \in D,\ f \in HB(\alpha),$$

and $\varphi_n = \varphi \circ \cdots \circ \varphi$ (composed n times).

Hence, $C_{\varphi_n} : HB(\alpha) \to HB(\alpha)$ is a bounded composition operator. Therefore:

$$\|C_{\varphi_n}\|_{e,\ HB(\alpha)} = \lim_{r\to 1} \sup_{|\varphi_n(w)|>r} \frac{|\varphi_n'(w)|(1-|w|^2)^\alpha}{(1-|\varphi_n(w)|^2)^\alpha}. \tag{2}$$

Given $0 < |\lambda| < \rho$, pick μ such that $|\lambda| < \mu < \rho$. Since ρ is the essential spectral radius, there exists n_0 such that for all $n \geq n_0$,

$$\|C_{\varphi_n}\|_{e,\ HB(\alpha)} > \mu^n.$$

Hence, by (2), for each $K > n_0$, we can find a $w \in D$ such that:

$$|\varphi_K'(w)| \left(\frac{1-|w|^2}{1-|\varphi_K(w)|^2}\right)^\alpha > \mu^K > 0, \quad \text{and} \quad |\varphi_K(w)| \geq \frac{1}{2}.$$

Thus, we have:

$$\begin{aligned}
\frac{\|\delta_{\varphi_K(w)}\|_{HB_m(\alpha)}}{\|\delta_{(w)}\|_{HB_m(\alpha)}} &\geq \frac{1}{2^m} \frac{\|\delta_{\varphi_K(w)}\|_{HB(\alpha)}}{\|\delta_{(w)}\|_{HB(\alpha)}} \\
&= \frac{1}{2^m} \left(\frac{1-|w|^2}{1-|\varphi_K(w)|^2}\right)^\alpha \\
&\geq \frac{\mu^K}{2^m|\varphi_K'(w)|}.
\end{aligned}$$

For every $K \leq n_0$, with this choice of w, we can define the iteration sequence $(z_k)_{k=-K}^{\infty}$ by letting $z_k = w$ and $z_{k+1} = \varphi(z_k)$ for $k \geq -K$. Hence, $|z_0| = |\varphi_K(w)| \geq \frac{1}{2}$. Thus, we have:

$$\frac{\|(C_m^* - \lambda I)L_\lambda\|_{HB_m(\alpha)}}{\|L_\lambda\|_{HB_m(\alpha)}} \leq \frac{\alpha' MC(\alpha)}{\|\delta_{z_0}\|_{HB_m(\alpha)}} |\lambda|^{K+1} \|\delta_{z-K}\|_{HB_m(\alpha)}$$

$$\leq \alpha' MC(\alpha)|\lambda| 2^m \left(\frac{|\lambda|}{\mu}\right)^K.$$

Choosing $K \geq n_0$ large enough, it follows that $C_m^* - \lambda I$ is not bounded from below. □

When $C_\varphi : HB_0(\alpha) \to HB_0(\alpha)$ is bounded, we deduce that $C_\varphi = C_\varphi^{**} : HB(\alpha) \to HB(\alpha)$ is also bounded, and we can apply Theorem 1. Furthermore, C_φ^n is bounded on both $HB_0(\alpha)$ and $HB(\alpha)$, and $C_\varphi^n f(w) = C_{\varphi_n} f(w)$. Therefore, C_φ^n is a bounded composition operator on both $HB_0(\alpha)$ and $HB(\alpha)$.

Using Theorem 1 and its proof, we find that the essential norm of C_φ^n on $HB_0(\alpha)$ coincides with the essential norm of C_φ^n on $HB(\alpha)$, provided C_φ is bounded on $HB_0(\alpha)$. Consequently, we conclude that:

$$r_{e,\, HB_0(\alpha)}(C_\varphi) = r_{e,\, HB(\alpha)}(C_\varphi),$$

and we can formulate the following result.

Corollary 1 Let $\alpha > 1$, and suppose that φ, not an automorphism, fixes the origin and that $C_\varphi : HB_0(\alpha) \to HB_0(\alpha)$ is bounded. Then:

$$\sigma_{HB_0(\alpha)}(C_\varphi) = \{\lambda \in \mathbb{C} : |\lambda| \leq r_{e,\, HB_0(\alpha)}(C_\varphi)\} \cup \{\varphi'(0)^n\}_{n=0}^{\infty}.$$

3. When is C_φ an isometry?

In this section, we characterize isometric composition operators on harmonic Bloch spaces. Recall that for $\alpha > 0$, and φ being an analytic self-map of the unit disk D, the function:

$$\tau_{\varphi,\, \alpha}(z) = \frac{(1-|z|^2)^\alpha |\varphi'(z)|}{(1-|\varphi(z)|^2)^\alpha}$$

is defined. We write τ_φ when $\alpha = 1$.

Additionally, we recall the Schwarz-Pick Lemma, which states that if φ is a self-map of D, then:

$$\sup_{z \in D} \frac{(1-|z|^2)|\varphi'(z)|}{1-|\varphi(z)|^2} \leq 1,$$

and equality holds at one point if and only if it holds at every point in D, which happens if and only if φ is a disk automorphism.

Now, we consider the case $\alpha = 1$ and provide an equivalent condition for C_φ to be an isometry.

Theorem 2 Let φ be an analytic self-map of D and let $\alpha = 1$. Then, the composition operator C_φ is an isometry on the harmonic α-Bloch space $HB(\alpha)$ if and only if:

(i) $\varphi(0) = 0$, and

(ii) Either:

(a) φ is a rotation (whenever φ' is bounded or φ is univalent), or

(b) For every $a \in D$, there exists a sequence $\{z_n\} \subset D$ such that $|z_n| \to 1$, $\varphi(z_n) \to a$, and $\tau_\varphi(z_n) \to 1$.

Proof. We first show that if C_φ is an isometry on the harmonic Bloch space $HB(\alpha)$, then necessarily $\varphi(0) = 0$. By the Schwarz-Pick Lemma, we have:

$$\sup_{z\in D} \frac{(1-|z|^2)|\varphi'(z)|}{1-|\varphi(z)|^2} \leq 1.$$

For every $f \in HB$, we obtain:

$$\|f\circ\varphi\|_{HB(\alpha)} = |f(\varphi(0))| + \sup_{z\in D}(1-|z|^2)|\varphi'(z)|\,[|f_z(\varphi(z))| + |f_{\bar{z}}(\varphi(z))|]$$

$$\leq |f(\varphi(0))| + \sup_{z\in D}(1-|\varphi(z)|^2)\,[|f_z(\varphi(z))| + |f_{\bar{z}}(\varphi(z))|]$$

$$= |f(\varphi(0))| - |f(0)| + \|f\|_{HB(\alpha)}.$$

It follows that $|f(\varphi(0))| \geq |f(0)|$ for all $f \in HB$. Setting $\varphi(0) = a$ and choosing $f = \bar{a}\varphi_a + a\bar{\varphi}_a$, where φ_a is the automorphism:

$$\varphi_a(z) = \frac{a-z}{1-\bar{a}z}, \quad z \in D,$$

we get:

$$0 = |(\bar{a}\varphi_a + a\bar{\varphi}_a)(a)| = |f(\varphi(0))| \geq |f(0)| = |(\bar{a}\varphi_a + a\bar{\varphi}_a)(0)| = 2|a|^2,$$

hence $\varphi(0) = 0$.

The identity function $I(z) = z$ belongs to each harmonic Bloch space and has norm one. Thus, since C_φ is an isometry, we have:

$$\|\varphi\|_{HB(\alpha)} = \|C_\varphi I\|_{HB(\alpha)} = \|I\|_{HB(\alpha)} = 1.$$

Next, suppose C_φ is an isometry. As shown above, $\varphi(0) = 0$ and $\|\varphi\|_{HB(\alpha)} = 1$. Hence:

$$\sup_{z\in D} \frac{(1-|z|^2)|\varphi'(z)|}{1-|\varphi(z)|^2} \cdot \left(1-|\varphi(z)|^2\right) = \sup_{z\in D} \tau_\varphi(z) \cdot \left(1-|\varphi(z)|^2\right) = 1.$$

If $\varphi(z) \neq e^{i\theta}z$, the Schwarz-Pick Lemma gives $\tau_\varphi(z) < 1$ for $z \in D$. Thus, there exists a sequence $\{z_n\}$ such that $|z_n| \to 1$, $\varphi(z_n) \to 0$, and $\tau_\varphi(z_n) \to 1$. Therefore:

$$|\varphi'(z)| > \frac{1-|\varphi(z_n)|^2}{2(1-|z_n|^2)} \to \infty,$$

for sufficiently large n, which contradicts the boundedness of φ' on D.

If φ is univalent and $\varphi(0) = 0$, then there exists $\varepsilon > 0$ such that $|\varphi(z_n)| > \varepsilon$ for any sequence $|z_n| \to 1$ with sufficiently large n. If $\varphi(z) \neq e^{i\theta}z$, as observed above, this leads to a contradiction.

Finally, if C_φ is an isometry and φ fixes the origin but is not a rotation, then φ cannot be a disk automorphism. Every disk automorphism has the form $\psi = \lambda\varphi_a$, where λ is a complex number of modulus one and φ_a is the automorphism:

$$\varphi_a(z) = \frac{a-z}{1-\bar{a}z}, \quad z \in D.$$

Choosing $f = \bar{a}\varphi_a + a\bar{\varphi}_a$, we have:

$$\begin{aligned}
\|\bar{a}\varphi_a + a\bar{\varphi}_a\|_{HB(\alpha)} &= |(\bar{a}\varphi_a + a\bar{\varphi}_a)(0)| + \sup_{z\in D} 2|a||\varphi_a'(z)|(1-|z|^2) \\
&= 2|a|^2 + 2|a|\sup_{z\in D}(1-|\varphi_a(z)|^2) \\
&= 2|a|^2 + 2|a|.
\end{aligned}$$

Since

$$\begin{aligned}
\|f\|_{HB(\alpha)} &= \|(\bar{a}\varphi_a + a\bar{\varphi}_a) \circ \varphi\|_{HB(\alpha)} \\
&= |(\bar{a}\varphi_a + a\bar{\varphi}_a)(\varphi(0))| + \sup_{z\in D} 2|a||\varphi'(z)|(1-|z|^2)|\varphi_a'(\varphi(z))| \\
&= 2|a|^2 + 2|a|\sup_{z\in D}\tau_\varphi(z)(1-|\varphi(z)|^2)|\varphi_a'(\varphi(z))| \\
&= 2|a|^2 + 2|a|\sup_{z\in D}\tau_\varphi(z)(1-|\varphi_a(\varphi(z))|^2) \\
&\leq 2|a|^2 + 2|a| \\
&= \|\bar{a}\varphi_a + a\bar{\varphi}_a\|_{HB(\alpha)} \\
&= \|f\|_{HB(\alpha)}.
\end{aligned}$$

It follows that:

$$\sup_{z\in D}\tau_{\varphi}(z)(1-|\varphi_a(\varphi(z))|^2)=1. \tag{3}$$

Recall that both quantities $\tau_{\varphi}(z)$ and $(1-|\varphi_a(\varphi(z))|^2)$ are always bounded above by 1. Note that the supremum in (3) cannot be achieved at any point $z_0\in D$; otherwise, we would have $\tau_{\varphi}(z_0)=1$, which implies that φ is a disk automorphism, a case that has already been excluded.

Thus, there exists at least one sequence $\{z_n\}\subset D$ such that $\lim_{n\to\infty}\tau_{\varphi}(z_n)=1$ and, at the same time, $1-|\varphi_a(\varphi(z_n))|^2\to 1$. That is, $\varphi(z_n)\to a$ as $n\to\infty$.

By the continuity of τ_{φ}, the sequence $\{z_n\}$ cannot accumulate inside D; otherwise, it would again imply that φ is an automorphism. Hence, $|z_n|\to 1$ as $n\to\infty$, which proves the claim.

It is evident that every rotation generates a composition operator that is an isometry on $HB(\alpha)$. Now, we check sufficiency for those self-maps φ that fix the origin but are not rotations. For an arbitrary function in $HB(\alpha)$, the inequality:

$$\|f\circ\varphi\|_{HB(\alpha)}=|f(\varphi(0))|+\sup_{z\in D}(1-|z|^2)|\varphi'(z)|\,[|f_z(\varphi(z))|+|f_{\bar{z}}(\varphi(z))|]$$

$$\leq |f(0)|+\sup_{z\in D}(1-|\varphi(z)|^2)\,[|f_z(\varphi(z))|+|f_{\bar{z}}(\varphi(z))|]=\|f\|_{HB(\alpha)},$$

follows directly from the Schwarz-Pick Lemma.

To verify the reverse inequality, consider two cases:

(i) $\sup_{z\in D}(1-|z|^2)\,[|f_z(z)|+|f_{\bar{z}}(z)|]$ is attained as a maximum at some point $a\in D$,

(ii) $\sup_{z\in D}(1-|z|^2)\,[|f_z(z)|+|f_{\bar{z}}(z)|]=\lim_{n\to\infty}(1-|a_n|^2)\,[|f_z(a_n)|+|f_{\bar{z}}(a_n)|]$, for some sequence $\{a_n\}$ such that $|a_n|\to 1$.

The first case is straightforward; thus, we focus on case (ii). By assumption, for each point a_n, there exists a point z_n such that $\varphi(z_n)\to a_n$. By the continuity of $(1-|z|^2)\,[|f_z(z)|+|f_{\bar{z}}(z)|]$ at each point a_n, for every n, we can find z_n such that:

$$\left|(1-|a_n|^2)\,[|f_z(a_n)|+|f_{\bar{z}}(a_n)|]-(1-|\varphi(z_n)|^2)\,[|f_z(\varphi(z_n))|+|f_{\bar{z}}(\varphi(z_n))|]\right|<\frac{1}{n},$$

and $\tau_{\varphi}(z_n)>1-\frac{1}{n}$. For $\alpha=1$, we get:

$$\|f\circ\varphi\|_{HB(\alpha)}=|f(\varphi(0))|+\sup_{z\in D}\tau_{\varphi}(z)(1-|\varphi(z)|^2)\,[|f_z(\varphi(z))|+|f_{\bar{z}}(\varphi(z))|]$$

$$\geq |f(0)|+\limsup_{n\to\infty}\left(1-\frac{1}{n}\right)(1-|\varphi(z_n)|^2)\,[|f_z(\varphi(z_n))|+|f_{\bar{z}}(\varphi(z_n))|]$$

$$\geq |f(0)|+\limsup_{n\to\infty}\left((1-|a_n|^2)\,[|f_z(a_n)|+|f_{\bar{z}}(a_n)|]-\frac{1}{n}\right)=\|f\|_{HB(\alpha)}.$$

This completes the proof. □

Now, in the next lemma, we find a necessary condition for the composition operator C_φ on the harmonic Bloch spaces $HB(\alpha)$ in the case $\alpha > 0$ to be an isometry.

Lemma 5 If C_φ is an isometry on $HB(\alpha)$ with $\alpha > 0$, then $\varphi(0) = 0$.

Proof. Note first that the function $L(z) = z + \bar{z}$ belongs to each of the harmonic Bloch-type spaces (harmonic α-Bloch spaces) and $\|L\|_{HB(\alpha)} = 2$. Thus, since C_φ is an isometry, we have:

$$
\begin{aligned}
2 &= \|C_\varphi L\|_{HB(\alpha)} \\
&= \|\varphi + \bar{\varphi}\|_{HB(\alpha)} \\
&= \sup_{z\in D} 2|\varphi'(z)|(1-|z|^2)^\alpha.
\end{aligned}
$$

Hence, $\sup_{z\in D} |\varphi'(z)|(1-|z|^2)^\alpha = 1$. Suppose that $\varphi(0) = a \neq 0$. Using the function $f_a = 1 - (\bar{a}z + a\bar{z})$, we see that:

$$
\begin{aligned}
\|C_\varphi f_a\|_{HB(\alpha)} &= \|f_a \circ \varphi\|_{HB(\alpha)} \\
&= |f_a(\varphi(0))| + \sup_{z\in D}(1-|z|^2)^\alpha |\varphi'(z)| \left[|(f_a)_z(\varphi(z))| + |(f_a)_{\bar{z}}(\varphi(z))|\right] \\
&= 1 - 2|a|^2 + 2|a| \sup_{z\in D}(1-|z|^2)^\alpha |\varphi'(z)| \\
&= 1 + 2|a| - 2|a|^2.
\end{aligned}
$$

Since C_φ is an isometry, we also have:

$$
\begin{aligned}
\|f_a\|_{HB(\alpha)} &= |f_a(0)| + \sup_{z\in D}(1-|z|^2)^\alpha \left[|(f_a)_z(z)| + |(f_a)_{\bar{z}}(z)|\right] \\
&= 1 + 2|a| \sup_{z\in D}(1-|z|^2)^\alpha \\
&= 1 + 2|a|.
\end{aligned}
$$

Equating the norms gives:

$$
-2|a|^2 = 0,
$$

which contradicts $a \neq 0$. Therefore, $\varphi(0) = 0$, and $\|\varphi\|_{B(\alpha)} = 1$. The lemma is proved. □

We now show that the only isometric composition operators on harmonic Bloch-type spaces (other than harmonic Bloch spaces) are induced by rotations. In the proof, we use two different ideas for the cases $0<\alpha<1$ and $\alpha>1$, and we divide the theorem correspondingly.

Theorem 3 Let φ be an analytic self-map of D. If $\alpha>0$ and $\alpha\neq 1$, then the composition operator C_φ is an isometry on $HB(\alpha)$ if and only if φ is a rotation.

3.1 *Case* $0<\alpha<1$

For the characterization of isometric composition operators on the spaces $HB(\alpha)$ with $0<\alpha<1$, we use the fact that $B(\alpha)=\mathrm{Lip}_{1-\alpha}$ and that their norms are equivalent. We also utilize the n-th iteration of φ, defined as:

$$\varphi^n=\underbrace{\varphi\circ\varphi\circ\cdots\circ\varphi}_{n\text{ times}}.$$

Theorem 4 Let $0<\alpha<1$ and φ be an analytic self-map of D. Then, the composition operator C_φ is an isometry on $HB(\alpha)$ if and only if φ is a rotation.

Proof. Let $\varphi(z)=\lambda z$ for any $z\in D$, where $|\lambda|=1$, i.e., let φ be a rotation. Then C_φ is an isometry on all harmonic Bloch-type spaces $HB(\alpha)$, since for every $\alpha>0$:

$$\begin{aligned}\|C_\varphi f\|_{HB(\alpha)}&=\|f\circ\varphi\|_{HB(\alpha)}\\&=|f(0)|+\sup_{z\in D}(1-|z|^2)^\alpha|\lambda|\,[|f_z(\lambda z)|+|f_{\bar z}(\lambda z)|]\\&=\|f\|_{HB(\alpha)}.\end{aligned}$$

Now, we prove the converse. Assume that C_φ is an isometry on $HB(\alpha)$. From Lemma 5, we know that $\varphi(0)=0$, and as noted in its proof, $\|\varphi\|_{B(\alpha)}=1$. Since the proof of this theorem is identical to the proof of Theorem 2.2 in [6], we omit the details. □

3.2 *Case* $\alpha>1$

The proof of the characterization of isometric composition operators on spaces $HB(\alpha)$ with $\alpha>1$ relies on the Schwarz-Pick Lemma, which states:

$$\sup_{z\in D}\frac{(1-|z|^2)|\varphi'(z)|}{1-|\varphi(z)|^2}\leq 1,$$

and equality holds at one point if and only if equality holds at every point in D, which happens if and only if φ is a disk automorphism.

Theorem 5 Let $\alpha>1$ and φ be an analytic self-map of D. Then, the composition operator C_φ is an isometry on $HB(\alpha)$ if and only if φ is a rotation.

Proof. As mentioned earlier, we only need to prove that if C_φ is an isometry, then φ must be a rotation. From Lemma 5, we know that $\varphi(0)=0$ and $\|\varphi\|_{B(\alpha)}=1$. Thus, we have:

$$1 = \sup_{z \in D}(1-|z|^2)^{\alpha}|\varphi'(z)| = \sup_{z \in D}(1-|z|^2)^{\alpha-1}\frac{(1-|z|^2)|\varphi'(z)|}{1-|\varphi(z)|^2}(1-|\varphi(z)|^2).$$

Since $\alpha - 1 > 0$, and by the Schwarz-Pick Lemma, all three factors in the last product are ≤ 1. These factors are also continuous functions on D, with $(1-|z|^2)^{\alpha-1} \to 0$ as $|z| \to 1$. Hence, the supremum must be attained at some point in D. By the Schwarz-Pick Lemma, φ must be a disk automorphism. Since $\varphi(0) = 0$, it follows that φ must be a rotation. □

4. Conclusion

We computed the spectrum compute their spectra on harmonic α-Bloch spaces and *harmonic little α-Bloch space* in Theorem 1 and Corollary 1. Also we characterized isometric composition operators on harmonic α-Bloch-type spaces in Theorems 2, 3, 4 and 5. Indeed we obtained some relations between the properties of the inducing function φ and the isometricity of the composition operator C_{φ}.

Finally, many problems still opened may be interested to extend the obtained results in this article for some important systems with composition operators may be applied in future works as in [18, 19].

Conflict of interest

The authors declare no competing financial interest.

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