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New Approach of Generalized α-Nonexpansive Mappings via HK-Iteration Process with Applications to Fractional Differential Equation and Boundary Value Problems

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Abstract: This paper introduces the HK-Iteration process and investigates its application to generalized α-nonexpansive mappings in Banach spaces. The proposed scheme not only unifies but also extends several classical iterative processes, thereby offering a broader framework within fixed point theory. Convergence analysis is carried out in detail: weak convergence is established by employing Opial's property, while strong convergence is obtained under the assumptions of uniform convexity together with condition (I). To demonstrate the practical utility of the process, we further apply the HK-Iteration to a boundary value problem associated with a fractional differential equation involving the Caputo derivative, where Greens function is used to reformulate the problem in operator form. A numerical example is provided, indicating that the HK-Iteration achieves faster convergence and improved accuracy compared with existing iterative schemes, thereby underscoring both the originality and the practical significance of the method.

Keywords: generalized α-nonexpansive mapping, uniformly convex Banach space, weak convergence, strong convergence

MSC: 47H10, 46B45, 46B50, 34K06

1. Introduction and background

Throughout this paper, $\mathbb{N}$ and $\mathbb{R}$ denote the sets of natural and real numbers, respectively. Let K be a subset of a Banach space B and let $\mathscr{T}: K \to K$ be a mapping. The mapping $\mathscr{T}$ is said to be *nonexpansive* if

$$\|\mathscr{T}t - \mathscr{T}y\| \leq \|t - y\|, \quad \forall t, y \in K.$$

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An element $q \in K$ is called a fixed point of $\mathscr{T}$ if $q = \mathscr{T}q$, and the set of all such points is denoted by $F_{\mathscr{T}}$. If $F_{\mathscr{T}} \neq \emptyset$ and

$$\|\mathscr{T}t - \mathscr{T}q\| \leq \|t - q\|, \quad \forall t \in K, q \in F_{\mathscr{T}},$$

then $\mathscr{T}$ is called *quasi-nonexpansive*. A foundational contribution in this area was first made by Browder [1], while similar results were obtained independently by Gohde [2] and Kirk [3]. They established that if K is a nonempty, closed, bounded, and convex subset of a uniformly convex Banach space, then every nonexpansive mapping $\mathscr{T} : K \to K$ possesses at least one fixed point.

Since then, the theory has been significantly broadened. In 2008, Suzuki [4] introduced the *condition (C)*, providing an important extension of nonexpansive mappings. A mapping $\mathscr{T} : K \to K$ satisfies condition (C) if

$$\frac{1}{2}\|t - \mathscr{T}t\| \leq \|t - y\| \implies \|\mathscr{T}t - \mathscr{T}y\| \leq \|t - y\|, \quad \forall t, y \in K.$$

This condition enabled a new class of iterative results beyond the classical nonexpansive setting. Later, Aoyama and Kohsaka [5] proposed the notion of α*-nonexpansive mappings*, defined by the existence of $\alpha \in [0,1)$ such that

$$\|\mathscr{T}t - \mathscr{T}y\|^2 \leq \alpha\|\mathscr{T}t - y\|^2 + \alpha\|t - \mathscr{T}y\|^2 + (1-2\alpha)\|t - y\|^2, \quad \forall t, y \in K.$$

As pointed out by Ariza-Ruiz et al. [6], this definition is trivial for $\alpha < 0$, while every nonexpansive mapping is 0-nonexpansive. Moreover, Suzuki [4] observed that such mappings need not be continuous, and this fact was further emphasized by Pant and Shukla [7].

Further progress was made by Pant and Shukla [7], who introduced the broader class of *generalized* α*-nonexpansive mappings*. This framework subsumes Suzukis generalized nonexpansive mappings and is defined as follows: for some $\alpha \in [0,1)$,

$$\frac{1}{2}\|t - \mathscr{T}t\| \leq \|t - y\| \Rightarrow \|\mathscr{T}t - \mathscr{T}y\| \leq \alpha\|\mathscr{T}t - y\| + \alpha\|\mathscr{T}y - t\| + (1-2\alpha)\|t - y\|.$$

Alongside these theoretical developments, a parallel line of research has focused on the construction of iterative algorithms to approximate fixed points of mappings. The Banach contraction principle guarantees convergence of the Picard iteration $t_{n+1} = \mathscr{T}t_n$ for contractive mappings; however, this approach generally fails in the nonexpansive case. To overcome this difficulty, a wide variety of iterative processes have been proposed. The Mann [8] and Ishikawa iteration [9] provided early alternatives, followed later by the Noor [10] and the Agarwal et al. [11]. More recently, Thakur et al. [12] presented a three-step process. Comparative analyses, such as those by Rhoades [13], have shown that no single scheme is universally optimal, since performance often depends on the nature of the underlying function.

In response to the limitations of earlier approaches, scholars have consistently worked toward designing more efficient iterative methods. Over the years, a wide range of refinements to the classical schemes have been presented, each contributing to the advancement of fixed point theory and its applications. Among the most prominent are the following processes:

Mann [8] introduced the one-step iteration scheme:

$$\begin{cases} t_1 \in K, \\ t_{i+1} = (1-\alpha_i)t_i + \alpha_i \mathscr{T}t_i, \end{cases} \tag{1}$$

where $\{\alpha_i\} \subset (0,1)$ and $i \in \mathbb{N}$. This classical method laid the foundation for the development of subsequent iterative schemes in fixed point theory.

Ishikawa [9] proposed a two-step iteration defined as:

$$\begin{cases} t_1 \in K, \\ y_i = (1-\beta_i)t_i + \beta_i \mathscr{T} t_i, \\ t_{i+1} = (1-\alpha_i)t_i + \alpha_i \mathscr{T} y_i, \end{cases} \tag{2}$$

with $\{\alpha_i\}, \{\beta_i\} \subset (0,1)$. This process provided a refinement of the Mann iteration by incorporating an intermediate step.

Agarwal et al. [11] introduced another two-step iterative scheme:

$$\begin{cases} t_1 \in K, \\ y_i = (1-\beta_i)t_i + \beta_i \mathscr{T} t_i, \\ t_{i+1} = (1-\alpha_i)\mathscr{T} t_i + \alpha_i \mathscr{T} y_i, \end{cases} \tag{3}$$

where $\{\alpha_i\}, \{\beta_i\} \subset (0,1)$. This iteration has been widely used as a basis for extending fixed point approximation methods.

Thakur [12] suggested the following three-step iteration scheme:

$$\begin{cases} t_1 \in K, \\ z_i = (1-\beta_i)t_i + \beta_i \mathscr{T} t_i, \\ y_i = \mathscr{T}\big((1-\alpha_i)t_i + \alpha_i z_i\big), \\ t_{i+1} = \mathscr{T} y_i, \end{cases} \tag{4}$$

with $\{\alpha_i\}, \{\beta_i\} \subset (0,1)$. This scheme is particularly effective in improving convergence speed for nonlinear operators.

Mann [14] developed the following iterative process:

$$\begin{cases} t_1 \in K, \\ z_i = (1-\beta_i)t_i + \beta_i \mathscr{T} t_i, \\ y_i = \mathscr{T} z_i, \\ t_{i+1} = \mathscr{T} y_i, \end{cases} \tag{5}$$

where $\{\beta_i\} \subset (0,1)$. This modification further simplified the structure while retaining convergence properties.

Abbas and Nazir [15] proposed a generalized three-step iteration scheme:

$$\begin{cases} t_1 \in K, \\ w_i = (1-\alpha_i)t_i + \alpha_i \mathscr{T} t_i, \\ y_i = (1-\beta_i)\mathscr{T} t_i + \beta_i \mathscr{T} w_i, \\ t_{i+1} = (1-\gamma_i)\mathscr{T} y_i + \gamma_i \mathscr{T} y_i, \end{cases} \tag{6}$$

with $\{\alpha_i\}, \{\beta_i\}, \{\gamma_i\} \subset (0,1)$. This process unified different iterative ideas under a flexible framework.

Eke and Akewe [16] introduced the Picard-Noor iteration:

$$\begin{cases} t_1 \in K, \\ w_i = (1-\alpha_i)t_i + \alpha_i \mathscr{T} t_i, \\ y_i = (1-\beta_i)t_i + \beta_i w_i, \\ z_i = (1-\gamma_i)t_i + \gamma_i \mathscr{T} y_i, \\ t_{i+1} = \mathscr{T} z_i, \end{cases} \tag{7}$$

where $\{\alpha_i\}, \{\beta_i\}, \{\gamma_i\} \subset (0,1)$. This iteration scheme generalized several known processes and enhanced stability in fixed-point computations. More recent investigations have continued to refine and expand hybrid Picard-type schemes. For instance, Ali and Jubair [17] studied enriched nonexpansive mappings in geodesic spaces, while Ozger et al. [18] analyzed Fredholm integral equations through fixed point methods. These works highlight the current trend of extending iterative processes to broader classes of operators and integral formulations, reinforcing the relevance of new iterative frameworks.

Most recently, Ullah and Ullah [19] introduced the Picard-P iteration, which demonstrated superior accuracy and faster convergence compared to many existing schemes. Alongside the Picard-P scheme, Nawaz et al. [20] recently examined the convergence of the Picard SP iteration for generalized α-nonexpansive mappings, reporting improved numerical performance. Similarly, several works have applied iterative methods to fractional problems by constructing Greens functions for Caputo-type boundary value problems [21, 22]. These contributions underline the strong interplay between iterative fixed point techniques and fractional calculus.

Taken together, these recent developments [23–29] illustrate that fixed point theory continues to evolve rapidly, with significant progress in both theoretical generalizations and computational applications. Motivated by this active research landscape, we now introduce a new hybrid process, the HK-Iteration:

$$\begin{cases} u_i = (1-\zeta_i)w_i + \zeta_i \mathscr{T} w_i, \\ v_i = (1-\kappa_i)\mathscr{T} w_i + \kappa_i \mathscr{T}^2 u_i, \\ w_{i+1} = \mathscr{T} v_i, \end{cases} \tag{8}$$

with control sequences $\{\zeta_i\}, \{\kappa_i\} \subset [0,1]$. This novel iteration unifies several existing schemes into a single framework and ensures faster convergence with improved computational efficiency.

2. Motivation and methodological framework

The study of fixed points for nonexpansive-type mappings has long required the development of new strategies, since direct applications of the Banach contraction principle are not sufficient in this setting. Although a variety of iterative schemes have been introduced, the central challenge remains twofold: establishing convergence for generalized classes of mappings and designing algorithms that are simultaneously robust in theory and efficient in computation.

Generalized αnon-expansive mappings introduced as a broad extension of Suzuki's generalized nonexpansive mappings illustrate these challenges clearly. Such mappings do not necessarily preserve continuity, and classical techniques alone cannot guarantee convergence. This necessitates the use of deeper structural properties of Banach spaces, including uniform convexity, Opials property, and asymptotic centers. These tools are fundamental in ensuring weak or strong convergence of iterative processes in this generalized setting.

From a computational standpoint, classical and modern schemes alike (Mann, Ishikawa, Noor, and even the more recent Picard-P iteration) exhibit limitations in convergence speed and numerical precision when applied to generalized α-nonexpansive mappings. These shortcomings motivate the introduction of the *HK-Iteration process*, which integrates the strengths of multi-step constructions into a unified framework. As will be demonstrated, the HK-Iteration not only subsumes several existing schemes but also achieves faster convergence and improved numerical stability in practice. At the same time, its performance depends on the choice of control sequences, which highlights the importance of analyzing both the advantages and potential sensitivities of the method.

To rigorously establish these properties, we recall in the next section a number of foundational concepts that form the backbone of the analysis. These include asymptotic centers, the geometry of uniformly convex Banach spaces, Opials property, and condition (I), together with auxiliary results on generalized α-nonexpansive mappings. This framework provides the analytical tools necessary for proving convergence of the HK-Iteration and for situating it within the broader context of fixed point theory.

3. Preliminaries

Let B be a Banach space and $K \subset B$ a nonempty, closed and convex subset. For a bounded sequence $\{t_n\} \subset K$ and a point $t \in B$, the *asymptotic radius* of $\{t_n\}$ relative to K is defined by

$$r(K, \{t_n\}) := \inf\{\limsup_{n\to\infty} \|t - t_n\| : t \in K\}.$$

The *asymptotic centre* of $\{t_n\}$ relative to K is

$$A(K, \{t_n\}) := \{t \in K : \limsup_{n\to\infty} \|t - t_n\| = r(K, \{t_n\})\}.$$

It is well known (see e.g. [1, 3]) that in a uniformly convex Banach space the asymptotic centre $A(K, \{t_n\})$ contains a unique point. Moreover, if K is weakly compact and convex then, $A(K, \{t_n\})$ is nonempty and convex.

Definition 1 (Uniform Convexity [30]) A Banach space B is *uniformly convex* if for each $\varepsilon \in (0,2]$ there exists $\delta > 0$ such that whenever $x, y \in B$ satisfy $\|x\| \leq 1$, $\|y\| \leq 1$ and $\|x - y\| > \varepsilon$, it follows that

$$\left\| \frac{x+y}{2} \right\| \leq 1 - \delta.$$

Definition 2 (Opials Property [31]) A Banach space B satisfies *Opial's property* if for every sequence $\{t_n\}$ converging weakly to $t \in B$ and for all $y \in B \setminus \{t\}$ one has

$$\limsup_{n\to\infty} \|t_n - t\| < \limsup_{n\to\infty} \|t_n - y\|.$$

Definition 3 (Condition (I) [4]) Let K be a nonempty subset of a Banach space B. A mapping $T : K \to K$ satisfies *condition (I)* if there exists a function $\psi : [0,\infty) \to [0,\infty)$ with $\psi(0) = 0$ and $\psi(r) > 0$ for all $r > 0$ such that

$$\|t - Tt\| \geq \psi\big(\operatorname{dist}(t, F_T)\big) \quad \text{for all } t \in K,$$

where F_T denotes the fixed point set of T and $\operatorname{dist}(t, F_T)$ is the distance from t to F_T.

Proposition 1 ([7, 6]) Let K be a nonempty subset of a Banach space B and $T : K \to K$ a mapping.

(i) Every Suzuki generalized nonexpansive mapping is generalized α-nonexpansive.

(ii) If T is generalized α-nonexpansive and has a fixed point, then T is quasi-nonexpansive.

(iii) For such a mapping T, the set F_T is closed. Moreover, if B is strictly convex and K is convex, then F_T is convex.

(iv) If T is generalized α-nonexpansive, then for all $t, y \in K$,

$$\|t - Ty\| \leq \frac{3+\alpha}{1-\alpha} \|t - Tt\| + \|t - y\|.$$

If B has Opial's property, T is generalized α-nonexpansive, $\{t_n\}$ converges weakly to w, and $\|Tt_n - t_n\| \to 0$, then $w \in F_T$.

Lemma 1 ([32]) Let B be a uniformly convex Banach space and let $\{\alpha_n\}$ be a sequence satisfying $0 < p \leq \alpha_n \leq q < 1$ for all $n \in \mathbb{N}$. If $\{t_n\}$ and $\{s_n\}$ are sequences in B such that $\limsup_{n\to\infty} \|t_n\| \leq c$, $\limsup_{n\to\infty} \|s_n\| \leq c$, and

$$\lim_{n\to\infty} \|\alpha_n t_n + (1-\alpha_n)s_n\| = c$$

for some $c \geq 0$, then $\lim_{n\to\infty} \|t_n - s_n\| = 0$.

4. Convergence theorems in uniformly convex Banach spaces

We first establish a preliminary result that will serve as a cornerstone for proving the subsequent main theorems.

Lemma 2 Let K be a nonempty closed convex subset of a Banach space B, and let $\mathscr{T} : K \to K$ be a generalized α-nonexpansive mapping with $F_{\mathscr{T}} \neq \emptyset$. Suppose that $\{w_i\}$ is the sequence generated by the HK-Iteration (8). Then, for each $q \in F_{\mathscr{T}}$, the limit

$$\lim_{i\to\infty} \|w_i - q\|$$

exists.

Proof. Fix an arbitrary $q \in F_{\mathscr{T}}$. By Proposition 1(ii), every generalized α-nonexpansive mapping that admits a fixed point is quasi-nonexpansive. Hence, for any $t \in K$ and $q \in F_{\mathscr{T}}$, we have $\|\mathscr{T}t - q\| \leq \|t - q\|$.

Applying this to the iterative scheme, we first consider u_i:

$$\begin{aligned}\|u_i - q\| &= \big\|(1-\zeta_i)w_i + \zeta_i \mathscr{T}w_i - q\big\| \\ &\leq (1-\zeta_i)\|w_i - q\| + \zeta_i\|\mathscr{T}w_i - q\| \\ &\leq (1-\zeta_i)\|w_i - q\| + \zeta_i\|w_i - q\| \\ &= \|w_i - q\|.\end{aligned}$$

Next, for v_i, we obtain

$$\begin{aligned}\|v_i - q\| &= \big\|(1-\kappa_i)\mathscr{T}w_i + \kappa_i \mathscr{T}^2 u_i - q\big\| \\ &\leq (1-\kappa_i)\|\mathscr{T}w_i - q\| + \kappa_i\|\mathscr{T}^2 u_i - q\| \\ &\leq (1-\kappa_i)\|w_i - q\| + \kappa_i\|u_i - q\|,\end{aligned}$$

where the last inequality again uses the quasi-nonexpansiveness of $\mathscr{T}$.

Finally, the update step gives

$$\|w_{i+1} - q\| = \|\mathscr{T}v_i - q\|$$

$$\le \|v_i - q\|$$

$$\le (1-\kappa_i)\|w_i - q\| + \kappa_i\|u_i - q\|$$

$$\le (1-\kappa_i)\|w_i - q\| + \kappa_i\|w_i - q\|$$

$$= \|w_i - q\|.$$

Thus, the sequence $\{\|w_i - q\|\}$ is monotone nonincreasing and bounded below by zero. Consequently, it converges, and the limit $\lim_{i\to\infty}\|w_i - q\|$ exists for every $q \in F_{\mathscr{T}}$. □

Definition 4 [32] A sequence $\{w_i\}$ in a Banach space $\mathscr{U}$ is said to be *asymptotically regular* if

$$\lim_{i\to\infty}\|w_{i+1} - w_i\| = 0.$$

We now verify that the sequence $\{w_i\}$ generated by the HK-Iteration (8) is asymptotically regular. From the nonexpansiveness of $\mathscr{T}$, we have

$$\|w_{i+1} - w_i\| = \|\mathscr{T}v_i - \mathscr{T}v_{i-1}\| \le \|v_i - v_{i-1}\|.$$

Since $\{v_i\}$ and $\{w_i\}$ are bounded, the sequence $\{\|w_{i+1} - w_i\|\}$ is nonincreasing and bounded below by 0. Therefore, $\lim_{i\to\infty}\|w_{i+1} - w_i\|$ exists. Moreover, by standard arguments for nonexpansive mappings in uniformly convex Banach spaces with nonempty fixed-point sets, this limit equals 0. Hence, $\{w_i\}$ is asymptotically regular.

Theorem 1 Let K be a nonempty closed convex subset of a uniformly convex Banach space B, and let $\mathscr{T}: K \to K$ be a generalized α-nonexpansive mapping. Suppose the sequence $\{w_i\}$ is generated by the HK-Iteration (8) with control sequences $\{\zeta_i\}, \{\kappa_i\} \subset (0,1)$ satisfying

$$0 < p \le \zeta_i \le q < 1 \quad \text{and} \quad 0 < r \le \kappa_i \le s < 1 \qquad (\forall i \in \mathbb{N}).$$

Then $F_{\mathscr{T}} \ne \emptyset$ if and only if $\{w_i\}$ is bounded and

$$\lim_{i\to\infty}\|\mathscr{T}w_i - w_i\| = 0.$$

Proof. ($\Rightarrow$) Assume that $F_{\mathscr{T}} \ne \emptyset$ and fix $q \in F_{\mathscr{T}}$. By Lemma 2, the limit

$$c := \lim_{i\to\infty}\|w_i - q\|$$

exists, and hence the sequence $\{w_i\}$ is bounded.

From Proposition 1(ii) (quasi-nonexpansiveness) together with the HK-update rules, we obtain

$$\|u_i - q\| = \big\|(1-\zeta_i)w_i + \zeta_i\mathscr{T}w_i - q\big\|$$

$$\le (1-\zeta_i)\|w_i - q\| + \zeta_i\|\mathscr{T}w_i - q\|$$

$$\leq (1-\zeta_i)\|w_i - q\| + \zeta_i \|w_i - q\| = \|w_i - q\|,$$

and

$$\|v_i - q\| = \left\|(1-\kappa_i)\mathscr{T} w_i + \kappa_i \mathscr{T}^2 u_i - q\right\|$$

$$\leq (1-\kappa_i)\|\mathscr{T} w_i - q\| + \kappa_i \|\mathscr{T}^2 u_i - q\|$$

$$\leq (1-\kappa_i)\|w_i - q\| + \kappa_i \|u_i - q\|.$$

Therefore,

$$\|w_{i+1} - q\| = \|\mathscr{T} v_i - q\|$$

$$\leq \|v_i - q\|$$

$$\leq (1-\kappa_i)\|w_i - q\| + \kappa_i \|u_i - q\|.$$

Rearranging yields

$$\frac{\|w_{i+1} - q\| - \|w_i - q\|}{\kappa_i} \leq \|u_i - q\| - \|w_i - q\|,$$

so that $\|w_{i+1} - q\| \leq \|u_i - q\|$ for all i. Taking limits, we obtain

$$c \leq \liminf_{i\to\infty} \|u_i - q\|.$$

On the other hand, from the earlier estimate, $\limsup_{i\to\infty} \|u_i - q\| \leq c$. Thus,

$$\lim_{i\to\infty} \|u_i - q\| = c.$$

Now, define $t_i := \mathscr{T} w_i - q$ and $s_i := w_i - q$. We have

$$u_i - q = (1-\zeta_i)s_i + \zeta_i t_i,$$

and from Proposition 1(ii),

$$\limsup_{i\to\infty} \|t_i\| \leq \limsup_{i\to\infty} \|s_i\| = c.$$

Since $\|u_i - q\| \to c$ and $0 < p \leq \zeta_i \leq q < 1$, Lemma 1 applies, yielding

$$\lim_{i\to\infty} \|t_i - s_i\| = \lim_{i\to\infty} \|\mathscr{T} w_i - w_i\| = 0.$$

Hence, the sequence $\{w_i\}$ is both bounded and asymptotically regular.

($\Leftarrow$) Conversely, suppose that $\{w_i\}$ is bounded and $\|\mathscr{T}w_i - w_i\| \to 0$. Let $q \in A(K, \{w_i\})$ denote the (unique) asymptotic centre of $\{w_i\}$ in K (uniqueness follows from uniform convexity).

By Proposition 1(iv), we have

$$r(\mathscr{T}q, \{w_i\}) = \limsup_{i\to\infty} \|w_i - \mathscr{T}q\|$$

$$\leq \frac{3+\alpha}{1-\alpha} \limsup_{i\to\infty} \|\mathscr{T}w_i - w_i\| + \limsup_{i\to\infty} \|w_i - q\|$$

$$= r(q, \{w_i\}).$$

Therefore, $\mathscr{T}q \in A(K, \{w_i\})$. By uniqueness of the asymptotic centre, it follows that $\mathscr{T}q = q$. Thus $q \in F_{\mathscr{T}}$, and hence $F_{\mathscr{T}} \neq \emptyset$. □

Theorem 2 Let K be a nonempty closed convex subset of a uniformly convex Banach space B satisfying Opial's property, and let $\mathscr{T} : K \to K$ be a generalized α-nonexpansive mapping with $F_{\mathscr{T}} \neq \emptyset$. Consider the sequence $\{w_i\}$ generated by the HK-Iteration (8):

$$u_i = (1-\zeta_i)w_i + \zeta_i \mathscr{T}w_i, \qquad v_i = (1-\kappa_i)\mathscr{T}w_i + \kappa_i \mathscr{T}^2 u_i, \qquad w_{i+1} = \mathscr{T}v_i,$$

where $\{\zeta_i\}, \{\kappa_i\} \subset [0,1]$. Then the sequence $\{w_i\}$ converges weakly to a fixed point of $\mathscr{T}$.

Proof. From Theorem 1, we know that $\{w_i\}$ is bounded and that

$$\|\mathscr{T}w_i - w_i\| \longrightarrow 0 \qquad (i \to \infty).$$

Since B is uniformly convex, it follows that B is reflexive. Hence, there exists a subsequence $\{w_{i_k}\}$ and some $w_1 \in K$ such that

$$w_{i_k} \rightharpoonup w_1.$$

Because B possesses Opial's property and $\mathscr{T}$ is generalized α-nonexpansive, Proposition 1(v) ensures that $w_1 \in F_{\mathscr{T}}$.

We now claim that the entire sequence $\{w_i\}$ converges weakly to w_1. Suppose, for contradiction, that this is not the case. Then there exists another subsequence $\{w_{j_\ell}\}$ converging weakly to some $w_2 \in K$ with $w_2 \neq w_1$. By the same reasoning as above, Proposition 1(v) implies that $w_2 \in F_{\mathscr{T}}$.

Next, by Lemma 2, we know that the limit

$$\lim_{i\to\infty} \|w_i - q\|$$

exists for every $q \in F_{\mathscr{T}}$. Applying this fact together with Opial's property, we obtain

$$\lim_{i\to\infty} \|w_i - w_1\| = \lim_{k\to\infty} \|w_{i_k} - w_1\| < \lim_{k\to\infty} \|w_{i_k} - w_2\| = \lim_{i\to\infty} \|w_i - w_2\|$$

$$= \lim_{\ell\to\infty} \|w_{j_\ell} - w_2\| < \lim_{\ell\to\infty} \|w_{j_\ell} - w_1\| = \lim_{i\to\infty} \|w_i - w_1\|,$$

which is a contradiction. Therefore, the assumption was false, and the entire sequence $\{w_i\}$ converges weakly to $w_1 \in F_{\mathscr{T}}$. $\square$

Theorem 3 Let K be a nonempty closed convex subset of a uniformly convex Banach space B, and let $\mathscr{T} : K \to K$ be a generalized α-nonexpansive mapping with $F_{\mathscr{T}} \neq \emptyset$. Suppose that $\{w_i\}$ is the sequence generated by the HK-Iteration (8). If

$$\liminf_{i\to\infty} \operatorname{dist}(w_i, F_{\mathscr{T}}) = 0,$$

then $\{w_i\}$ converges strongly to a point in $F_{\mathscr{T}}$.

Proof. By Lemma 2 (applied to the HK-Iteration), for each $q \in F_{\mathscr{T}}$ the limit $\lim_{i\to\infty} \|w_i - q\|$ exists. Consequently, the scalar sequence $\{\operatorname{dist}(w_i, F_{\mathscr{T}})\}$ admits a limit as $i \to \infty$. Together with the hypothesis $\liminf_{i\to\infty} \operatorname{dist}(w_i, F_{\mathscr{T}}) = 0$, this yields

$$\lim_{i\to\infty} \operatorname{dist}(w_i, F_{\mathscr{T}}) = 0.$$

Thus there exists a subsequence $\{w_{i_k}\}$ and a sequence $\{y_k\} \subset F_{\mathscr{T}}$ such that

$$\|w_{i_k} - y_k\| \le \tfrac{1}{2^k}, \qquad k \in \mathbb{N}.$$

From the proof of Lemma 2, for every fixed $q \in F_{\mathscr{T}}$ the sequence $\{\|w_i - q\|\}$ is nonincreasing in i. Hence for each k,

$$\|w_{i_{k+1}} - y_k\| \le \|w_{i_k} - y_k\| \le \tfrac{1}{2^k}.$$

Consequently, for $k \ge 1$,

$$\|y_{k+1} - y_k\| \le \|y_{k+1} - w_{i_{k+1}}\| + \|w_{i_{k+1}} - y_k\|$$

$$\le \tfrac{1}{2^{k+1}} + \tfrac{1}{2^k} \le \tfrac{1}{2^{k-1}} \longrightarrow 0 \quad (k \to \infty).$$

Thus $\{y_k\}$ is Cauchy in $F_{\mathscr{T}}$, and hence converges to some $q \in F_{\mathscr{T}}$ (by Proposition 1(iii), closedness of $F_{\mathscr{T}}$).

Finally, for this q the limit $\lim_{i\to\infty} \|w_i - q\|$ exists (Lemma 2). Since $\operatorname{dist}(w_i, F_{\mathscr{T}}) \to 0$, it follows that

$$\lim_{i\to\infty} \|w_i - q\| = 0.$$

Therefore, $\{w_i\}$ converges strongly to $q \in F_{\mathscr{T}}$. $\square$

Theorem 4 Let K be a nonempty closed convex subset of a uniformly convex Banach space B and let $\mathscr{T} : K \to K$ be a generalized α-nonexpansive mapping with $F_{\mathscr{T}} \neq \emptyset$. If $\mathscr{T}$ satisfies Definition 3, then the sequence $\{w_i\}$ generated by the HK-Iteration (8),

$$u_i = (1-\zeta_i)w_i + \zeta_i \mathscr{T} w_i, \qquad v_i = (1-\kappa_i)\mathscr{T} w_i + \kappa_i \mathscr{T}^2 u_i, \qquad w_{i+1} = \mathscr{T} v_i,$$

converges strongly to a point in $F_{\mathscr{T}}$.

Proof. By Theorem 1, if $F_{\mathscr{T}} \neq \emptyset$ then the HK-sequence $\{w_i\}$ is bounded and satisfies

$$\lim_{i\to\infty} \|\mathscr{T} w_i - w_i\| = 0.$$

In particular,

$$\liminf_{i\to\infty} \|\mathscr{T} w_i - w_i\| = 0. \tag{9}$$

Since $\mathscr{T}$ satisfies Definition 3, there exists a function $\psi : [0,\infty) \to [0,\infty)$ with $\psi(0) = 0$ and $\psi(r) > 0$ for all $r > 0$ such that

$$\|\mathscr{T} t - t\| \geq \psi\big(\operatorname{dist}(t, F_{\mathscr{T}})\big), \qquad \forall t \in K.$$

Applying this to w_i and using (9) yields

$$0 = \liminf_{i\to\infty} \|\mathscr{T} w_i - w_i\| \ \geq \ \liminf_{i\to\infty} \psi\big(\operatorname{dist}(w_i, F_{\mathscr{T}})\big).$$

Because ψ is nondecreasing with $\psi(r) > 0$ whenever $r > 0$, it follows that

$$\liminf_{i\to\infty} \operatorname{dist}(w_i, F_{\mathscr{T}}) = 0.$$

Thus all assumptions of Theorem 3 are fulfilled. Hence, by Theorem 3, the sequence $\{w_i\}$ converges strongly to a point $q \in F_{\mathscr{T}}$. This completes the proof. □

Remark 1 In the proofs of Theorems 1-4, the assumption of uniform convexity of the Banach space is mainly used to ensure reflexivity and the uniqueness of asymptotic centers. Reflexivity alone does not generally imply these properties. Consequently, the results may fail in merely reflexive (but not uniformly convex) Banach spaces. For example, the space $L^1[0,1]$ is reflexive but not uniformly convex, and the uniqueness of asymptotic centers fails there. It is also worth noting that the notion of *asymptotic regularity* that is, a sequence $\{x_n\}$ satisfying, $\|x_{n+1} - x_n\| \to 0$ plays a key role in the convergence analysis of iterative schemes and has been extensively studied in the literature (see, e.g., [33]). Hence, the assumption of uniform convexity cannot be relaxed in this framework.

5. Illustrative example of generalized α-nonexpansive mapping

Example 1 Let $K = [-2, 9] \subset \mathbb{R}$ with the usual absolute value norm $\|\cdot\| = |\cdot|$. Define $\mathscr{T} : K \to K$ by

$$\mathscr{T} t = \begin{cases} \dfrac{t+4}{5}, & t \in [-2, 2), \\ \dfrac{t+12}{3}, & t \in [2, 9]. \end{cases}$$

Then $\mathscr{T}$ fails to satisfy Suzuki's condition (C) (hence is not Suzuki generalized-nonexpansive), but it is a generalized α-nonexpansive mapping for $\alpha = \frac{1}{3}$. Moreover, $\mathscr{T}$ has fixed points at $t = 1$ and $t = 6$.

Proof. Failure of condition (C). Take $t = 1$ and $y = 2$. Then $t \in [-2, 2)$ and $y \in [2, 9]$, so

$$\mathscr{T} t = \frac{1+4}{5} = 1, \qquad \mathscr{T} y = \frac{2+12}{3} = \frac{14}{3}.$$

Hence

$$\frac{1}{2}|t-\mathscr{T}t| = \frac{1}{2}|1-1| = 0 < |t-y| = 1,$$

while

$$|\mathscr{T}t-\mathscr{T}y| = \left|1-\frac{14}{3}\right| = \frac{11}{3} > 1 = |t-y|.$$

Thus the implication in condition (C) fails; $\mathscr{T}$ is not Suzuki generalized-nonexpansive.

Generalized α-nonexpansiveness for $\alpha = \frac{1}{3}$. We show that for all $t, y \in K$,

$$\frac{1}{2}|t-\mathscr{T}t| \leq |t-y| \quad \Longrightarrow \quad |\mathscr{T}t-\mathscr{T}y| \leq \tfrac{1}{3}|\mathscr{T}t-y| + \tfrac{1}{3}|t-\mathscr{T}y| + \tfrac{1}{3}|t-y|. \tag{$*$}$$

Consider four cases.

Case I: $t, y \in [-2, 2)$. Then $\mathscr{T}t = \dfrac{t+4}{5}$, $\mathscr{T}y = \dfrac{y+4}{5}$, and

$$|\mathscr{T}t-\mathscr{T}y| = \frac{1}{5}|t-y| \leq \tfrac{1}{3}|t-y| \leq \tfrac{1}{3}|\mathscr{T}t-y| + \tfrac{1}{3}|t-\mathscr{T}y| + \tfrac{1}{3}|t-y|.$$

Case II: $t, y \in [2, 9]$. Then $\mathscr{T}t = \dfrac{t+12}{3}$, $\mathscr{T}y = \dfrac{y+12}{3}$, and

$$|\mathscr{T}t-\mathscr{T}y| = \frac{1}{3}|t-y| \leq \tfrac{1}{3}|\mathscr{T}t-y| + \tfrac{1}{3}|t-\mathscr{T}y| + \tfrac{1}{3}|t-y|.$$

Case III: $t \in [2, 9]$, $y \in [-2, 2)$. Here

$$\mathscr{T}t = \frac{t+12}{3}, \qquad \mathscr{T}y = \frac{y+4}{5},$$

so

$$|\mathscr{T}t-\mathscr{T}y| = \left|\frac{t+12}{3} - \frac{y+4}{5}\right|$$

$$= \frac{1}{15}\left|5t-3y+48\right|.$$

By the triangle inequality and its reverse, $|a-b| \leq |a-c| + |c-b|$ and $\big||a-c|-|b-c|\big| \leq |a-b|$, we obtain

$$\tfrac{1}{3}|\mathscr{T}t-y| + \tfrac{1}{3}|t-\mathscr{T}y| \;\geq\; \tfrac{1}{3}\big||\mathscr{T}t-y|-|t-\mathscr{T}y|\big|.$$

Substituting $\mathscr{T}t = \frac{t+12}{3}$ and $\mathscr{T}y = \frac{y+4}{5}$, a straightforward algebraic comparison yields

$$\tfrac{1}{3}\big||\mathscr{T}t-y|-|t-\mathscr{T}y|\big| + \tfrac{1}{3}|t-y| \;\geq\; \frac{1}{15}|5t-3y+48|,$$

which gives $(*)$ in this case.

Case IV: $t \in [-2,2)$, $y \in [2,9]$. This is symmetric to Case III; writing

$$|\mathscr{T}t - \mathscr{T}y| = \left|\frac{t+4}{5} - \frac{y+12}{3}\right|$$

$$= \frac{1}{15}|3t - 5y - 48|,$$

the same triangle-inequality argument produces the desired bound.

Combining the four cases proves $(*)$ for all $t, y \in K$. Hence $\mathscr{T}$ is generalized $\frac{1}{3}$-nonexpansive on K, while the explicit counterexample above shows it does not satisfy Suzukis condition (C). Finally, solving $t = \dfrac{t+4}{5}$ and $t = \dfrac{t+12}{3}$ gives the fixed points $t = 1$ (on $[-2,2)$) and $t = 6$ (on $[2,9]$). □

Table 1 and Figure 1 shows tabular and graphical comparison of HK iteration with Ishikawa [9], Noor [10], Agarwal [11] process respectively; where $\alpha_n = 0.33$, $\beta_n = 0.66$, $\gamma_n = 0.25$ and $x_1 = 5.9$.

Table 1. Numerical results produced by HK, Ishikawa, Noor and Agarwal iterative schemes for $\mathscr{T}$ of the Example 1

n	HK	Ishikawa	Noor	Agarwal
1	5.9	5.9	5.9	5.9
2	5.748004444	5.372973333	5.349482222	5.458395555
4	5.979515561	5.873173406	5.863492458	5.905375686
5	5.998334843	5.974347208	5.971354651	5.983468081
6	5.999864641	5.994811295	5.993988933	5.997111690
7	5.999988997	5.998950498	5.998738611	5.999495380
8	5.999999106	5.999787721	5.999735305	5.999911837
9	5.999999927	5.999957063	5.999944455'	5.999984596
10	5.999999994	5.999991315	5.999988344	5.999997309
11	6.000000000	5.999998243	5.999997554	5.999999529
12	6.000000000	5.999999645	5.999999487	5.999999917
13	6.000000000	5.999999928	5.999999892	5.999999986
14	6.000000000	5.999999985	5.999999977	5.999999997
15	6.000000000	5.999999997	5.999999995	6.000000000
16	6.000000000	5.999999999	5.999999999	6.000000000
17	6.000000000	6.000000000	6.000000000	6.000000000
18	6.000000000	6.000000000	6.000000000	6.000000000
19	6.000000000	6.000000000	6.000000000	6.000000000
20	6.000000000	6.000000000	6.000000000	6.000000000

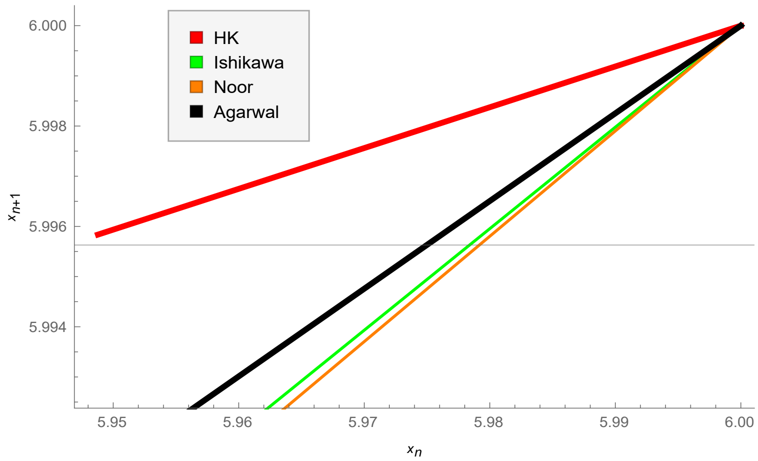

Figure 1. 3D Graphical analysis of iteration schemes towards the fixed point of T in Example 1

6. Application to a fractional differential equation

Fractional Differential Equations (FDEs) have recently attracted significant attention due to their extensive applications in various scientific and engineering domains, such as electromagnetic theory, fluid dynamics, electrical circuits, and probability theory. Many problems involving FDEs are analytically intractable, which motivates the use of iterative schemes to approximate their solutions.

In recent works (see, for instance, [34–36]), researchers have investigated such problems for nonexpansive operators, which are continuous. In contrast, here we deal with generalized α-nonexpansive mappings that may be discontinuous and propose using the HK-iteration process to approximate solutions.

Consider the following boundary value problem:

$$\begin{cases} D^{\xi} h(u) + \Upsilon(u, h(u)) = 0, & 0 \leq u \leq 1, \\ h(0) = h(1) = 0, & 1 < \xi < 2, \end{cases} \tag{10}$$

where D^{ξ} denotes the Caputo fractional derivative of order ξ and $\Upsilon : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a given function.

Let $B = C[0,1]$, the Banach space of continuous real-valued functions on $[0,1]$ endowed with the maximum norm. The associated Green's function for (10) is given explicitly, enabling us to reformulate the problem in operator form and apply HK-Iteration.

$$G(u,v) = \begin{cases} \dfrac{u(1-v)^{\xi-1} - (u-v)^{\xi-1}}{\Gamma(\xi)}, & 0 \leq v \leq u \leq 1, \\ \dfrac{u(1-v)^{\xi-1}}{\Gamma(\xi)}, & 0 \leq u \leq v \leq 1. \end{cases}$$

For clarity and completeness, the boundary value problem (10) can equivalently be expressed through the Green's function representation:

$$h(u) = \int_0^1 G(u,v)\,\Upsilon(v, h(v))\,dv,$$

where $G(u,v)$ is the Green's function associated with the homogeneous Caputo operator satisfying $D^{\xi}h(u)=0$ and the boundary conditions $h(0)=h(1)=0$. This formulation explicitly connects the fractional differential equation to its integral counterpart and provides the correct functional framework for applying the HK-iteration process.

Theorem 5 Let $B=C[0,1]$ and define an operator $H:B\to B$ by

$$(Hh)(u)=\int_0^1 G(u,v)\Upsilon(v,h(v))\,dv, \quad \forall h\in B.$$

If

$$|\Upsilon(v,h(v))-\Upsilon(v,g(v))|\leq\frac{1}{2}\left(|h(v)-H(g(v))|+|g(v)-H(h(v))|\right),$$

then the HK-iteration scheme

$$\begin{cases} u_n=(1-\zeta_n)q_n+\zeta_n H(q_n), \\ v_n=(1-\kappa_n)H(q_n)+\kappa_n H(u_n), \\ q_{n+1}=H(v_n), \end{cases} \tag{11}$$

associated with H converges to a solution S of (10), provided that $\liminf_{n\to\infty}\operatorname{dist}(q_n,S)=0$.

Proof. A function $h\in B$ solves (10) if and only if it satisfies

$$h(u)=\int_0^1 G(u,v)\Upsilon(v,h(v))\,dv.$$

For any $h,g\in B$ and $0\leq u\leq 1$, we have

$$\|H(h(u))-H(g(u))\|\leq\int_0^1 G(u,v)\,|\Upsilon(v,h(v))-\Upsilon(v,g(v))|\,dv$$

$$\leq\frac{1}{2}\|h-H(g)\|+\frac{1}{2}\|g-H(h)\|.$$

Thus,

$$\|H(h)-H(g)\|\leq\frac{1}{2}\left(\|h-H(g)\|+\|g-H(h)\|\right),$$

which shows that H is a Suzuki generalized α-nonexpansive mapping. By Theorem 4, the sequence $\{q_n\}$ generated by (11) converges strongly to a fixed point of H, which corresponds to a solution of (10). □

6.1 *Solving a Caputo-type fractional boundary value problem*

We now apply the HK-Iteration to a standard Caputo-type nonlinear FDE. This demonstrates how the method extends beyond abstract fixed point settings and can be effectively utilized for fractional boundary value problems.

Consider:

$$\begin{cases} {}^cD^{\alpha}u(t)+f(t,u(t))=0, \quad 1<\alpha<2,\ t\in[0,1], \\ u(0)=u(1)=0, \end{cases} \tag{12}$$

where ${}^cD^\alpha$ denotes the Caputo fractional derivative of order α, and $f : [0,1]\times\mathbb{R}\to\mathbb{R}$ is continuous [37–39].

Problem (12) can be rewritten as the integral equation

$$u(t) = \int_0^1 G(s,t) f(s,u(s))\, ds, \tag{13}$$

where

$$G(s,t) = \begin{cases} \dfrac{s(1-t)^{\alpha-1} - (s-t)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \le t \le s \le 1, \\ \dfrac{s(1-t)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \le s \le t \le 1. \end{cases}$$

Define $\mathscr{T} : B \to B$ by

$$(\mathscr{T}u)(t) = \int_0^1 G(s,t) f(s,u(s))\, ds, \tag{14}$$

and apply the HK-iteration (11) with H replaced by $\mathscr{T}$.

Theorem 6 If f satisfies the Lipschitz condition

$$|f(t,x) - f(t,y)| \le L|x-y|, \quad \forall x,y \in \mathbb{R},\ t \in [0,1], \tag{15}$$

for some $0 < L < 1$, then (12) has a unique solution in B and the sequence $\{q_n\}$ from (11) converges to it.

Proof. From (14) and (15), $\mathscr{T}$ is a contraction on B. By the Banach fixed point theorem, $\mathscr{T}$ has a unique fixed point u^*, and Theorem 4, ensures that the HK-iteration converges to u^*. □

Consider

$$\begin{cases} {}^cD^{1.8} y(t) + t^3 = 0, & t \in [0,1], \\ y(0) = y(1) = 0. \end{cases} \tag{16}$$

Here $f(t,y) = t^3$ satisfies (15) with $L = 0$. Using $q_0(t) = t^2(1-t)$ with parameter choices $\zeta_n = 0.7$, $\kappa_n = 0.8$, the HK-iteration (11) is shown to converge rapidly, thereby illustrating both its stability and numerical efficiency compared with classical schemes.

7. Conclusion

In this paper, we introduced and investigated the HK-Iteration process as a new iterative scheme for approximating fixed points of generalized α-nonexpansive mappings in uniformly convex Banach spaces. The scheme unifies and extends several existing iterative processes into a broader framework, thus providing a versatile tool in fixed point theory.

We established weak convergence of the sequence generated by the HK-Iteration using Opial's property and proved strong convergence under additional suitable conditions such as condition (I). These results not only strengthen but also generalize earlier findings in the literature. Furthermore, a numerical example was presented to demonstrate the practical efficiency of the HK-Iteration, confirming that it achieves faster convergence and improved accuracy compared to classical iterative methods. At the same time, the performance of the scheme can be influenced by the choice of control sequences, which calls for a deeper analysis of stability and sensitivity in higher-dimensional settings.

Overall, the HK-Iteration enriches the existing family of iterative schemes, offering both theoretical and computational advantages. This work opens further directions for research, including benchmarking against modern approaches (e.g., spectral or machine learning-based solvers), applications to nonlinear operator equations, and possible extensions to more general metric or hyperbolic spaces.

Confilict of interest

The authors declare no competing financial interest.

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