Contemporary Mathematics
https://ojs.wiserpub.com/index.php/CM/

UNIVERSAL WISER
PUBLISHER

Research Article

Homogenization of a Nonlinear Stochastic Convection-Diffusion Model for Reactive Flows with Noise Boundary

Mogtaba Mohammed

Department of Mathematics, College of Science Al-Zulfi Majmaah University, Al-Majmaah, 11952, Saudi Arabia
E-mail: mogtaba.m@mu.edu.sa

Received: 16 September 2025; **Revised:** 10 October 2025; **Accepted:** 4 November 2025

Abstract: In this paper, we address the transport of a solute through a porous media that involves both convection and diffusion, along with a linear chemical reaction (desorption/adsorption) occurring on the pore surfaces, all of which are influenced by external nonlinear random forces. The mathematical representation of this model is a system of a non-linear stochastic convection-diffusion equation in the saturated fluid phase and a linear stochastic convection-diffusion equation on the surface of porosity coupled with a linear reaction term. We use the method of two-scale convergence with drift and probabilistic compactness results to obtain a homogenized model consisting of a nonlinear stochastic diffusion equation where the concentration of the fluid on the pore surface contributes to the diffusion coefficient of the homogenized model.

***Keywords*:** homogenization, two-scale convergence with drift, stochastic models, nonlinearities, convection-diffusion model, noise boundary

MSC: 60H15, 60H35, 35B27

1. Introduction and problem statement

There are numerous applications for the transport of solute through porous materials, where a mass transfer occurs between the fluid and the pores, in fields that include chemical engineering and soil science [1, 2]. Solute transport in porous media involves convection, diffusion, and interactions between solutes in bulk or on the surface of the skeleton. Convection occurs through the the fluid's velocity field where the solutes dissolve. The majority of experimental data are gathered at a much larger macroscopical scale, although our understanding of flow in porous environments is rooted in fundamental physical laws at the pore level. Since direct simulation of microscopical models usually surpasses current computer capabilities, up-scaling techniques are essential for practical analysis. Several papers have considered homogenization and up-scaling methods to study these types of phenomena; see, for example, [3–7]. In [3] and [4] the authors studied homogenization results for a linear model of fluid flow in a periodic porous medium where convection-diffusion took place in the fluid phase with a chemical reaction at the surface of the pore. The results of [8] were extended to include convection-diffusion in both the phase of fluid and pore surface. The tools used in this paper were the two-scale asymptotic expansion with drift and the two-scale convergence with drift for rigorous justification; see [9] for more details. In all of the above results, the models were mathematically described using systems of linear Partial Differential Equations

DOI: https://doi.org/10.37256/cm.7220268653

(PDEs) with oscillatory parameters in moving domains. However, these models do not account for natural randomness that could affect the transportation of the salute in the porous medium, and for this stochastic PDEs are necessary. For a general account on homogenization of stochastic PDEs, see, [10–14]. The first work that considers homogenization for a stochastic chemical reactive model is [15], in which the author obtains homogenization results for a non-linear stochastic model that describes chemical reactive fluxes in a porous medium. The results in [15] represent the non-linear stochastic counterpart of those in [16]. In [17], the authors investigate the well-posedness combined with numerical simulation for a non-linear stochastic model that describes chemical reactive fluxes. For more results on homogenization of stochastic models, we refer to [18, 19]. In this paper, we assume that a porous medium is filled with a fluid with no compression that cannot pass through the solid portion of the medium. The velocity of this fluid is considered to be given, constant over time, and periodic in space. Our goal is to investigate a single solute transportation within this fluid under the assumption that the solute concentrations are influenced by external nonlinear random fluctuations acting in both the fluid and the skeleton surface phases.

The results of this paper address a highly relevant and complex problem, combining nonlinearity, stochasticity, and multiscale phenomena in a realistic physical context (reactive transport in porous media). It successfully extends the deterministic linear results of [8] and the nonlinear stochastic (but non-convective) results of [15] to a more general and applicable framework. The additional convective terms necessitate the use of the two-scale convergence method with drift to accurately represent the multiscale behavior of the system. The two scale convergence with drift as part of developments of two scale convergence method [20, 21], was first introduced in [22] with more details in [23] as a rigorous justification for the Two scale asymptotic expansions with drift approach. We also mention the related stochastic two scale convergence method, see [24, 25].

The current results are limited in situations where the reaction term has a nonlinear dependence on the solute concentration in both the fluid phase and the pore space, especially when combined with convective transport and stochastic effects. Under such situations, the analytical complexity of the problem is even higher, and it is unclear how the theoretical framework provided here would be extended.

Let D be any open bounded subset of $\mathbb{R}^n$ with smooth boundary ∂D and denote by D_f and D_s the fluid and solid phases, respectively, of the domain D where D_s completely lies inside D. By a scaling of ε we define the periodic porous domain as $D^\varepsilon = \varepsilon D_f$ and $\partial D_\varepsilon = \partial D_s$. We consider transport of a solute in a fluid within the domain D_ε, influenced by nonlinear random forces, the concentration in both phases will evolve according to different dynamics, and there will be interactions at the interface (boundary of the perforations) where the reaction term will play a role, we have

Solute concentration in the Fluid Phase. In the fluid phase D^ε, the solute concentration c_f^ε evolves according to a nonlinear stochastic convection-diffusion equation:

$$dc_f^\varepsilon + \frac{1}{\varepsilon}\mathbf{v}_f^\varepsilon \cdot \nabla c_f^\varepsilon dt - \operatorname{div}(\kappa_f^\varepsilon \nabla c_f^\varepsilon)dt = \beta_f(t, c_f^\varepsilon, \nabla c_f^\varepsilon)dt + \alpha_f^\varepsilon(c_f^\varepsilon)dW_1 \text{ in } D^\varepsilon \times (0, T). \tag{1}$$

Adsorbed solute concentration on the surface. On the surface ∂D^ε, the concentration c_s^ε of the solute evolves according to a stochastic convection-diffusion-reaction equation:

$$dc_s^\varepsilon + \frac{1}{\varepsilon}\mathbf{v}_s^\varepsilon \cdot \nabla_s c_s^\varepsilon dt - \operatorname{div}_s(\kappa_s^\varepsilon \nabla c_s^\varepsilon)dt = \frac{\eta}{\varepsilon^2}\left(c_f^\varepsilon - \frac{c_s^\varepsilon}{\lambda}\right)dt + \alpha_s^\varepsilon dW_2 \text{ on } \partial D^\varepsilon \times (0, T). \tag{2}$$

We apply the following Neumann condition to the concentration of solute c_f^ε

$$-\frac{\kappa_f^\varepsilon}{\varepsilon}\nabla c_f^\varepsilon \cdot \mathbf{n} = \frac{\eta}{\varepsilon^2}\left(c_f^\varepsilon - c_s^\varepsilon/\lambda\right) \text{ on } \partial D^\varepsilon \times (0, T), \tag{3}$$

and the homogeneous Dirichlet boundary condition for the solute concentration c_f^ε on the exterior boundary

$$c_f^\varepsilon = 0 \text{ on } \partial D. \tag{4}$$

The initial concentrations

$$c_f^\varepsilon(x, 0) = c_f^0(x) \text{ in } D^\varepsilon, \quad c_s^\varepsilon(x, 0) = c_s^0(x) \text{ on } \partial D^\varepsilon, \tag{5}$$

where

- $c_f^\varepsilon(x, t)$ and $c_s^\varepsilon(x, t)$ are the concentrations of solute in the fluid and the skeleton's surface phases,
- $\mathbf{v}_f^\varepsilon(x)$ and $\mathbf{v}_s^\varepsilon(x)$ are given vector functions representing the velocity fields of the fluid in the fluid and the skeleton's surface phases,
- $\kappa_f^\varepsilon(x)$ and $\kappa_s^\varepsilon(x)$ are the diffusion coefficient of the fluid concentrations in the fluid and the skeleton's surface phases,
- $\beta_f(t, c_f^\varepsilon, \nabla c_f^\varepsilon)$ is a nonlinear function that represent an external force depends on concentration of the solute c_f^ε and its diffusion ∇c_f^ε,
- $\eta/\varepsilon^2 \left(c_f^\varepsilon - c_s^\varepsilon/\lambda\right)$ is a reaction term coupling the concentrations on the fluid and on the skeleton's surface phases, where $\eta \in (0, 1)$ is the rate of the adsorption in the fluid and $\lambda \in (0, 1)$ is the constant of adsorption equilibrium.
- $\alpha_f^\varepsilon(t, x, c_f^\varepsilon)$ and $\alpha_s^\varepsilon(t, x)$: These functions represent the strength and nature of the random forces in the fluid and the skeleton's surface phases.
- $W_1(t)$ and $W_2(t)$ are independent standard Wiener processes associated with the fluid and the surface phases of the skeleton, respectively.
- The reference cell is given by the unit cube $Y = (0, 1)^n$, the solid part of Y is indicated by S and the fluid part of Y is denoted by F, where S has a smooth boundary, ∂S and $\bar{S} \subset Y$.
- Let D be any open bounded subset of $\mathbb{R}^n$ with smooth boundary ∂D, $D^\varepsilon = \bigcup_{\xi\in\mathbb{Z}^n} \{\varepsilon(F+\xi) \subset D\}$ and $\partial D^\varepsilon = \bigcup_{\xi\in\mathbb{Z}^n} \{\varepsilon(\partial S+\xi)\}$.
- $\nu(y)$ is the outer unit normal to F and $\theta(y) = I_n - \nu(y)\otimes\nu(y)$ is the projection operator on the tangent hyperplane of $\partial F = \partial S$.
- $\nabla_s(\cdot) = \theta(y)\nabla(\cdot)$ and $\operatorname{div}_s \mathbf{u} = \operatorname{div}\theta(y)\mathbf{u}$ for any vector function $\mathbf{u}\colon F \to \mathbb{R}^n$.

1.1 *Assumptions on the data*

A.1. Since the fluid is assumed to be incompressible and do not penetrate the solid part of the porous medium, then

$$\nabla\cdot\mathbf{v}_f(y) = 0 \text{ in } F, \quad \nabla_s\cdot\mathbf{v}_s(y) = 0 \text{ in } \partial S, \quad \text{and } \mathbf{v}_f(y)\cdot\nu(y) = 0 \text{ on } \partial S. \tag{6}$$

A.2. The diffusion coefficients $\kappa_f(y)$ and $\kappa_s(y)$ are periodic and bounded in $L^\infty(F)$ and $L^\infty(\partial S)$, respectively and satisfy the following ellipticity condition

$$\kappa_f(y)\xi\xi \geq c|\xi|^2 \text{ in } F \text{ and } \kappa_s(y)\xi\xi \geq c|\xi|^2 \text{ on } \partial S. \tag{7}$$

A.3. $\beta_f(t, c_f^\varepsilon, \nabla c_f^\varepsilon) = \gamma_f(t, c_f^\varepsilon)\cdot\nabla c_f^\varepsilon$ and $\gamma_f(t, c_f^\varepsilon) = (\gamma_{f_j}(t, c_f^\varepsilon))_{1\leq j\leq n}$ such that $\gamma_j(\cdot, u(\cdot, x))\colon L^2(D^\varepsilon) \to L^2(D^\varepsilon)$ continuous functions in u and measurable for all $t \in (0, T)$. Furthermore, we have

$$\left|(\gamma_f(t,\, u)\cdot \nabla v,\, \phi)_{L^2(D^\varepsilon)}\right| \leq c\|\nabla v\|_{[L^2(D^\varepsilon)]^n}\|\phi\|_{L^2(D^\varepsilon)}, \tag{8}$$

and

$$|\gamma_j(t,\, u_1)-\gamma_j(t,\, u_2)| \leq c|u_1-u_2|, \ \text{ for } j=1,\, 2,\, \cdots n \text{ and } u_1,\, u_2\in\mathbb{R}. \tag{9}$$

A.4. $\alpha_f^\varepsilon(t,\, c_f^\varepsilon) = (\alpha_f^{\varepsilon^j}(t,\, c_f^\varepsilon))_{1\leq j\leq m}$ such that $\alpha_f^\varepsilon(\cdot,\, u(\cdot,\, x))$: $L^2(D^\varepsilon)\to [L^2(D^\varepsilon)]^m$ a continuous function in u and measurable for all $t\in[0,\, T]$. Furthermore, we assume

$$\|\alpha_f^{\varepsilon^j}(t,\, u)\|_{L^2(D^\varepsilon)} \leq c(1+\|u\|_{L^2(D^\varepsilon)}), \tag{10}$$

and

$$\|\alpha_f^{\varepsilon^j}(t,\, u_1)-\alpha_f^{\varepsilon^j}(t,\, u_2)\|_{L^2(D^\varepsilon)} \leq c\|u_1-u_2\|_{L^2(D^\varepsilon)}), \ \text{ for } j=1,\, 2,\, \cdots,\, n, \ \text{ and } u_1,\, u_2\in\mathbb{R}. \tag{11}$$

A.5. Unlike in the case of the fluid phase, the noise on the surface is assumed to be of additive type i.e. $\alpha_s^\varepsilon(t,\, x) = (\alpha_s^{\varepsilon^j}(t,\, x))_{1\leq j\leq m}$ where α_s^ε: $[0,\, T]\times\partial D^\varepsilon\to\mathbb{R}^m$ such that its L^2norm is bounded i.e.

$$\varepsilon\|\alpha_s^{\varepsilon^j}\|_{L^2(\partial D^\varepsilon)} = \varepsilon\sum_{j=1}^m\|\alpha_s^{\varepsilon^j}\|_{L^2(\partial D^\varepsilon)} \leq C. \tag{12}$$

A.6. The initial concentrations are supposed to be $c_f^0\in L^2(\mathbb{R}^n)$ and $c_s^0\in H^1(\mathbb{R}^n)$.

Definition 1 By a weak (in the probabilistic sense) solution to system (1)-(5) we mean the set $(\Omega, \mathbb{P}, \mathbb{F}, c_f^\varepsilon, W_1, c_s^\varepsilon, W_2)$ such that c_f^ε and c_s^ε are random fields on $D^\varepsilon\times(0,\, T)$ and $\partial D^\varepsilon\times(0,\, T)$ respectively, if

(a) $W_1 = (W_1^j)_{1\leq j\leq m}$ and $W_2 = (W_2^j)_{1\leq j\leq m}$ are independent, $m-$dimensional standard Brownian motions and are $\mathscr{F}_t$-adapted, where $(\mathscr{F}_t)_{t\in[0,\, T]}$ is a filtration on $(\Omega,\, \mathbb{P},\, \mathscr{F})$.

(b) The maps $(\omega,\, x,\, t)\in\Omega\times D^\varepsilon\times(0,\, T)\mapsto c_f^\varepsilon(\omega,\, x,\, t)$ and $(\omega,\, x,\, t)\in\Omega\times\partial D^\varepsilon\times(0,\, T)\mapsto c_s^\varepsilon(\omega,\, x,\, t)$ are progressively measurable random fields and

$$c_f^\varepsilon\in C([0,\, T];\, L^2(D^\varepsilon))\cap L^2(\Omega;\, L^2(0,\, T;\, W^{1,\,2}(D^\varepsilon))),$$

and

$$c_s^\varepsilon\in C([0,\, T];\, L^2(\partial D^\varepsilon))\cap L^2(\Omega;\, L^2(0,\, T;\, W^{1,\,2}(\partial D^\varepsilon))).$$

(c) The couple $(c_f^\varepsilon,\, c_s^\varepsilon)$ satisfies,

$$\int_{D^\varepsilon} c_f^\varepsilon(t)\varphi^\varepsilon dx + \int_0^t \int_{D^\varepsilon} \left[\frac{1}{\varepsilon}\mathbf{v}^\varepsilon \cdot \nabla c_f^\varepsilon \varphi^\varepsilon + \kappa_f^\varepsilon \nabla c_f^\varepsilon \cdot \nabla \varphi^\varepsilon\right] dxd\tau$$

$$+\frac{\varepsilon}{\lambda}\left[\int_{\partial D^\varepsilon} c_s^\varepsilon(t)\phi^\varepsilon d\sigma^\varepsilon(x) + \int_0^t \int_{\partial D^\varepsilon} \left[\frac{1}{\varepsilon}\mathbf{v}_s^\varepsilon \cdot \nabla_s c_s^\varepsilon \phi^\varepsilon + \kappa_s^\varepsilon \nabla c_s^\varepsilon \cdot \nabla_s \phi^\varepsilon\right] d\sigma^\varepsilon(x)d\tau\right]$$

$$+\frac{\eta}{\varepsilon}\int_0^t \int_{\partial D^\varepsilon} \left(c_f^\varepsilon - \frac{c_s^\varepsilon}{\lambda}\right)\left(\varphi^\varepsilon - \frac{\phi^\varepsilon}{\lambda}\right) d\sigma^\varepsilon(x)d\tau = \int_{D^\varepsilon} c_f^0 \varphi^\varepsilon dx + \frac{\varepsilon}{\lambda}\int_{\partial D^\varepsilon} c_s^0 \varphi^\varepsilon dx$$

$$+\int_0^t \int_{D^\varepsilon} \beta_f(c_f^\varepsilon, \nabla c_f^\varepsilon)\varphi^\varepsilon dxd\tau + \frac{\varepsilon}{\lambda}\int_0^t \int_{\partial D^\varepsilon} \phi^\varepsilon \alpha_s^\varepsilon d\sigma^\varepsilon(x)dW_2(\tau)$$

$$+\int_0^t \int_{D^\varepsilon} \varphi^\varepsilon \alpha_f^\varepsilon(c_f^\varepsilon) d\sigma^\varepsilon(x)dW_1(\tau), \tag{13}$$

for almost all $\omega \in \Omega$, all $t \in [0, T]$, and all $(\varphi^\varepsilon, \phi^\varepsilon) \in W^{1,2}(D^\varepsilon) \times W^{1,2}(\partial D^\varepsilon)$, where $d\sigma^\varepsilon(x)$ is the surface measure on ∂D^ε.

Definition 2 Let $(\Omega, \mathbb{P}, \mathscr{F}, c_f^\varepsilon, W_1, c_s^\varepsilon, W_2)$ and $(\Omega, \mathbb{P}, \mathscr{F}, b_f^\varepsilon, W_1, b_s^\varepsilon, W_2)$ be two solutions of (1)-(5) by means of Definition 1, having the same initial data (c_f^0, c_s^0). Thus the system, (1)-(5) has a unique a path-wise solution if

$$(c_f^\varepsilon, c_s^\varepsilon)(x, t) = (b_f^\varepsilon, b_s^\varepsilon)(x, t) \text{ a.e. in } (D^\varepsilon \times (0, T)) \times (\partial D^\varepsilon \times (0, T)),\ \mathbb{P} - a.s..$$

For the proof of Theorems 1 and 2 below, we refer to [17].

Theorem 1 Let assumptions A.1.-A.6. holds for all $\varepsilon > 0$ and all $T > 0$. Then, $(\Omega, \mathbb{P}, \mathscr{F}, c_f^\varepsilon, W_1, c_s^\varepsilon, W_2)$ is a weak probabilistic solution for (1)-(5) by means of Definition 1.

Theorem 2 The system (1)-(5) possesses a pathwise unique solution.

The paper is organized as follows: In Section 2, we derive energy estimates for the solutions of system (1)-(5), enabling us to gain some crucial two-scale convergence with drift to the system's solution. However, due to the system's nonlinearities, some compactness is required to produce strong convergences; for this, we additionally acquire a finite difference estimate on the solution of equation (1). In Section 3, we use the estimates derived in Section 2 to prove tightness of a probability measure generated by the solution of our system (1)-(5). This leads to the application of Prokhorov's and Skorokhod's Theorems, which transform our system into a wider probability space with $\mathbb{P} - a.s.$ strong convergence for the solution of the new system. In Section 4, we state certain definitions and convergence results on the two scale convergence with drift, which will be used in the last Section 5, where we achieve the key results of the study, that is, the homogenization result.

2. Energy and finite difference estimates

Theorem 3 Under Assumptions A.1.-A.6., there exists a constant $C > 0$ that does not depend on ε such that the solutions of system (1)-(5) satisfy the following estimates:

$$\mathbb{E}\sup_t \|c_f^\varepsilon\|^2_{L^2(D^\varepsilon)} + \varepsilon\mathbb{E}\sup_t \|c_s^\varepsilon\|^2_{L^2(\partial D^\varepsilon)} \le C, \tag{14}$$

$$\mathbb{E}\int_0^T \|\nabla c_f^\varepsilon(t)\|^2_{L^2(D^\varepsilon)}dt + \varepsilon\mathbb{E}\int_0^T \|\nabla c_s^\varepsilon(t)\|^2_{L^2(\partial D^\varepsilon)}dt \le C, \tag{15}$$

$$\varepsilon\mathbb{E}\int_0^T \left\|\varepsilon^{-1}\left(c_f^\varepsilon - \frac{c_s^\varepsilon}{\lambda}\right)\right\|^2_{L^2(\partial D^\varepsilon)} dt \le C. \tag{16}$$

Proof. Let us first note that from Green formula, see, for example, [26, Theorem 3.33, p.51] and the boundary condition (3), we have

$$\begin{aligned} -(\operatorname{div}(\kappa_f^\varepsilon \nabla c_f^\varepsilon),\, c_f^\varepsilon)_{L^2(D^\varepsilon)} =& (\kappa_f^\varepsilon \nabla c_f^\varepsilon,\, \nabla c_f^\varepsilon)_{L^2(D^\varepsilon)} - (\kappa_f^\varepsilon \nabla c_f^\varepsilon \cdot \mathbf{n},\, c_f^\varepsilon)_{L^2(\partial D^\varepsilon)} \\ =& (\kappa_f^\varepsilon \nabla c_f^\varepsilon,\, \nabla c_f^\varepsilon)_{L^2(D^\varepsilon)} + \frac{\varepsilon\eta}{\varepsilon^2}(c_f^\varepsilon - \frac{c_s^\varepsilon}{\lambda},\, c_f^\varepsilon)_{L^2(\partial D^\varepsilon)} \end{aligned} \tag{17}$$

We now apply Ito's formula to the functions $\phi(t,\, c_f^\varepsilon) = \|c_f^\varepsilon\|^2_{L^2(D^\varepsilon)}$ and $\varphi(t,\, c_s^\varepsilon) = \|c_s^\varepsilon\|^2_{L^2(\partial D^\varepsilon)}$ and use (17), we have

$$\begin{aligned} &\frac{1}{2}d\|c_f^\varepsilon(t)\|^2_{L^2(D^\varepsilon)} + (\kappa_f^\varepsilon \nabla c_f^\varepsilon,\, \nabla c_f^\varepsilon)_{L^2(D^\varepsilon)}dt + \frac{\varepsilon\eta}{\varepsilon^2}(c_f^\varepsilon - \frac{c_s^\varepsilon}{\lambda},\, c_f^\varepsilon)_{L^2(\partial D^\varepsilon)}dt \\ =& (\beta_f(c_f^\varepsilon,\, \nabla c_f^\varepsilon),\, c_f^\varepsilon)_{L^2(D^\varepsilon)}dt + \frac{1}{2}\|\alpha_f^\varepsilon(c_f^\varepsilon)\|^2_{L^2(D^\varepsilon)}dt + (\alpha_f^\varepsilon(c_f^\varepsilon)dW_1,\, c_f^\varepsilon)_{L^2(D^\varepsilon)}, \end{aligned} \tag{18}$$

and

$$\begin{aligned} &\frac{\varepsilon}{2\lambda}d\|c_s^\varepsilon(t)\|^2_{L^2(\partial D^\varepsilon)} + \frac{\varepsilon}{\lambda}(\kappa_s^\varepsilon \nabla c_s^\varepsilon,\, \nabla c_s^\varepsilon)_{L^2(\partial D^\varepsilon)}dt - \frac{\varepsilon\eta}{\varepsilon^2}(c_f^\varepsilon - \frac{c_s^\varepsilon}{\lambda},\, c_s^\varepsilon/\lambda)_{L^2(\partial D^\varepsilon)}dt \\ =& \frac{\varepsilon}{2\lambda}\|\alpha_s^\varepsilon\|^2_{L^2(\partial D^\varepsilon)}dt + \frac{\varepsilon}{\lambda}(\alpha_s^\varepsilon dW_2,\, c_s^\varepsilon)_{L^2(\partial D^\varepsilon)}, \end{aligned} \tag{19}$$

where the convection terms vanish by assumption A.1. Adding (18) to (19), integrating from 0 to t, taking the sup for all $t \in [0,\, T]$ followed by the expectation and using assumptions A.2., we have

$$\begin{aligned} &\mathbb{E}\sup_t \|c_f^\varepsilon(t)\|^2_{L^2(D^\varepsilon)} + \varepsilon\mathbb{E}\sup_t \|c_s^\varepsilon(t)\|^2_{L^2(\partial D^\varepsilon)} + \mathbb{E}\int_0^T \|\nabla c_f^\varepsilon(t)\|^2_{L^2(D^\varepsilon)}dt \\ &+ \varepsilon\mathbb{E}\int_0^T \|\nabla c_s^\varepsilon(t)\|^2_{L^2(\partial D^\varepsilon)}dt + \varepsilon\mathbb{E}\int_0^T \left\|\varepsilon^{-1}\left(c_f^\varepsilon - \frac{c_s^\varepsilon}{\lambda}\right)\right\|^2_{L^2(\partial D^\varepsilon)} dt \end{aligned}$$

$$\leq C\Bigg[\|c_f^0\|^2_{L^2(D^\varepsilon)}+\|c_s^0\|^2_{L^2(\partial D^\varepsilon)}+\mathbb{E}\int_0^T|(\beta_f(c_f^\varepsilon,\nabla c_f^\varepsilon),c_f^\varepsilon)_{L^2(D^\varepsilon)}|dt$$

$$+\mathbb{E}\int_0^T\|\alpha_f^\varepsilon(c_f^\varepsilon)(t)\|^2_{L^2(D^\varepsilon)}dt+\varepsilon\int_0^T\|\alpha_s^\varepsilon(t)\|^2_{L^2(\partial D^\varepsilon)}dt$$

$$+\mathbb{E}\sup_t\left|\int_0^t(\alpha_f^\varepsilon(c_f^\varepsilon)dW_1,c_f^\varepsilon)_{L^2(D^\varepsilon)}\right|+\varepsilon\mathbb{E}\sup_t\left|\int_0^t(\alpha_s^\varepsilon dW_2,c_s^\varepsilon)_{L^2(\partial D^\varepsilon)}\right|\Bigg]. \tag{20}$$

To estimate the right-hand side of (20), we begin with the third term therein. We use Young's inequality and condition (8) for small enough ρ, we have

$$\begin{aligned}\mathbb{E}\int_0^T|(\beta_f(c_f^\varepsilon,\nabla c_f^\varepsilon),c_f^\varepsilon)_{L^2(D^\varepsilon)}|dt=&\mathbb{E}\int_0^T|(\gamma_f(c_f^\varepsilon)\cdot\nabla c_f^\varepsilon,c_f^\varepsilon)_{L^2(D^\varepsilon)}|dt\\ \leq& c\mathbb{E}\int_0^T\|\nabla c_f^\varepsilon\|_{[L^2(D^\varepsilon)]^n}\|c_f^\varepsilon\|_{L^2(D^\varepsilon)}dt\\ \leq&\rho\sup_t\mathbb{E}\|c_f^\varepsilon(t)\|^2_{L^2(D^\varepsilon)}+c_\rho\mathbb{E}\int_0^T\|\nabla c_f^\varepsilon(t)\|^2_{[L^2(D^\varepsilon)]^n}dt.\end{aligned} \tag{21}$$

It is easy to see from the linear growth (10) the following

$$\mathbb{E}\int_0^T\|\alpha_f^\varepsilon(c_f^\varepsilon)(t)\|^2_{L^2(D^\varepsilon)}dt\leq c\left(T+\mathbb{E}\int_0^T\|c_f^\varepsilon)(t)\|^2_{L^2(D^\varepsilon)}dt\right). \tag{22}$$

From Burkhlder-Davis-Gundy, Cauchy-Schowartz and Young inequalities for large enough ρ, we obtain the following

$$\mathbb{E}\sup_t\left|\int_0^t(\alpha_f^\varepsilon(c_f^\varepsilon)dW_1(t),c_f^\varepsilon(t))_{L^2(D^\varepsilon)}\right|$$

$$\leq c_p\mathbb{E}\left[\int_0^T\left(\alpha_f^\varepsilon(c_f^\varepsilon(t)),c_f^\varepsilon(t)\right)^2_{L^2(D^\varepsilon)}dt\right]^{\frac{1}{2}}$$

$$\leq c_p\mathbb{E}\left[\int_0^T\|\alpha_f^\varepsilon(c_f^\varepsilon(t))\|^2_{L^2(D^\varepsilon)}\|c_f^\varepsilon(t)\|^2_{L^2(D^\varepsilon)}dt\right]^{\frac{1}{2}}$$

$$\leq c_p\mathbb{E}\sup_t\|c_f^\varepsilon(t)\|_{L^2(D^\varepsilon)}\left[\int_0^T\|\alpha_f^\varepsilon(c_f^\varepsilon(t))\|^2_{L^2(D^\varepsilon)}dt\right]^{\frac{1}{2}}$$

$$\leq \frac{c_p}{\rho} \mathbb{E} \sup_t \|c_f^\varepsilon(t)\|_{L^2(D^\varepsilon)}^2 + c_p \rho \mathbb{E} \int_0^T \|\alpha_f^\varepsilon(c_f^\varepsilon(t))\|_{L^2(D^\varepsilon)}^2 dt. \tag{23}$$

Similarly, we obtain

$$\varepsilon \mathbb{E} \sup_t \left| \int_0^t (\alpha_s^\varepsilon dW_2,\, c_s^\varepsilon)_{L^2(\partial D^\varepsilon)} \right|$$

$$\leq \frac{\varepsilon c_p}{\rho} \mathbb{E} \sup_t \|c_s^\varepsilon(t)\|_{L^2(\partial D^\varepsilon)}^2 + c_p \rho \varepsilon \mathbb{E} \int_0^T \|\alpha_s^\varepsilon(t)\|_{L^2(\partial D^\varepsilon)}^2 dt. \tag{24}$$

From the inequalities (21)-(24), condition (12) and Assumption A.6., we get

$$\mathbb{E} \sup_t \|c_f^\varepsilon(t)\|_{L^2(D^\varepsilon)}^2 + \varepsilon \mathbb{E} \sup_t \|c_s^\varepsilon(t)\|_{L^2(\partial D^\varepsilon)}^2 + \mathbb{E} \int_0^T \|\nabla c_f^\varepsilon(t)\|_{L^2(D^\varepsilon)}^2 dt$$

$$+ \varepsilon \mathbb{E} \int_0^T \|\nabla c_s^\varepsilon(t)\|_{L^2(\partial D^\varepsilon)}^2 dt + \varepsilon \mathbb{E} \int_0^T \left\| \varepsilon^{-1} \left(c_f^\varepsilon - \frac{c_s^\varepsilon}{\lambda} \right) \right\|_{L^2(\partial D^\varepsilon)}^2 dt$$

$$\leq c_1 + c_2 \left[\mathbb{E} \int_0^T \|c_f^\varepsilon\|_{L^2(D^\varepsilon)} dt + \mathbb{E} \int_0^T \|c_s^\varepsilon\|_{L^2(\partial D^\varepsilon)} dt \right]. \tag{25}$$

The proof is completed by Gronwall's inequality. □

These estimates allow us to obtain weak limits only for subsequences of the solution sequence to our problem. To achieve strong convergence, certain compactness conditions are necessary, which are typically obtained by estimating the time derivative of the solution. However, this is only possible when working with deterministic PDEs. In the case of stochastic processes, the following theorem offers bounds for the finite difference for c_f^ε.

Theorem 4 Under Assumptions A.1-A.6., and for $\tau > 0$ (in a similar way, we can deal with $\tau < 0$) such that $t + \tau \in [0,\, T]$, we have a positive c independent of ε, where

$$\mathbb{E} \sup_{\tau \in [0,\, \theta]} \int_0^{T-\tau} \|c_f^\varepsilon(t+\tau) - c_f^\varepsilon(t)\|_{H^{-1}(D^\varepsilon)} dt \leq c\theta. \tag{26}$$

Proof. From equation (1), we write

$$c_f^\varepsilon(t+\tau) - c_f^\varepsilon(t) = \frac{-1}{\varepsilon} \int_t^{t+\tau} \mathbf{v}_f^\varepsilon \cdot \nabla c_f^\varepsilon ds + \int_t^{t+\tau} \operatorname{div}(\kappa_f^\varepsilon \nabla c_f^\varepsilon) ds$$

$$+ \int_t^{t+\tau} \beta_f(c_f^\varepsilon,\, \nabla c_f^\varepsilon) ds + \int_t^{t+\tau} \alpha_f^\varepsilon(c_f^\varepsilon) dW_1(s), \tag{27}$$

where $0 \le t \le T-\theta$. We can write for all $\psi \in H_0^1(D^\varepsilon)$ and $\|\psi\|_{H_0^1(D^\varepsilon)} = 1$

$$\begin{aligned}
\left\| \frac{-1}{\varepsilon} \int_t^{t+\tau} \mathbf{v}_f^\varepsilon \cdot \nabla c_f^\varepsilon ds \right\|_{H^{-1}(D^\varepsilon)} &= \left\| \int_t^{t+\tau} \frac{1}{\varepsilon} c_f^\varepsilon \nabla \cdot \mathbf{v}_f^\varepsilon - \nabla \cdot (\mathbf{v}_f^\varepsilon c_f^\varepsilon) ds \right\|_{H^{-1}(D^\varepsilon)} \\
&\le \sup_\psi \int_t^{t+\tau} \int_{D^\varepsilon} \left[\underbrace{\frac{1}{\varepsilon} c_f^\varepsilon \nabla \cdot \mathbf{v}_f^\varepsilon \psi}_{=0} - \nabla \cdot (\mathbf{v}_f^\varepsilon c_f^\varepsilon) \psi \right] ds dx \\
&\le \sup_\psi \left| \int_t^{t+\tau} \int_{D^\varepsilon} \mathbf{v}_f^\varepsilon c_f^\varepsilon \nabla \psi ds dx \right| \\
&\le \|\mathbf{v}_f^\varepsilon\|_{L^\infty(D^\varepsilon)} \|\nabla \psi\|_{L^2(D^\varepsilon)} \int_t^{t+\tau} \|c_f^\varepsilon\|_{L^2(D^\varepsilon)} ds \\
&\le c\tau^{\frac{1}{2}} \left\{ \int_t^{t+\tau} \|c_f^\varepsilon\|_{L^2(D^\varepsilon)}^2 ds \right\}^{\frac{1}{2}}. \qquad (28)
\end{aligned}$$

Thus, by estimate (14), we have

$$\mathbb{E} \sup_{\tau \in [0,\, \theta]} \int_0^{T-\tau} \left\| \frac{-1}{\varepsilon} \int_t^{t+\tau} \mathbf{v}_f^\varepsilon \cdot \nabla c_f^\varepsilon ds \right\|_{H^{-1}(D^\varepsilon)}^2 dt \le c\mathbb{E} \sup_{\tau \in [0,\, \theta]} \tau^{\frac{1}{2}} \int_0^{T-\tau} \left\| \int_0^T \|c_f^\varepsilon\|_{L^2(D^\varepsilon)}^2 dt \right\|_{H^{-1}(D^\varepsilon)} dt \le c\theta \qquad (29)$$

From the boundary condition (3) and assumption A.2., we have

$$\begin{aligned}
\left\| \int_t^{t+\tau} \operatorname{div}(\kappa_f^\varepsilon \nabla c_f^\varepsilon) ds \right\|_{H^{-1}(D^\varepsilon)} &\le \sup_\psi \int_t^{t+\tau} \int_{Q^f} \operatorname{div}(\kappa_f^\varepsilon \nabla c_f^\varepsilon) \psi dx ds \\
&= \sup_\psi \int_t^{t+\tau} \int_{D^\varepsilon} \kappa_f^\varepsilon \nabla c_f^\varepsilon(s) \cdot \nabla \phi dx ds \\
&\quad + \sup_\psi \frac{\eta}{\varepsilon} \int_t^{t+\tau} \int_{\partial D^\varepsilon} \left(c_f^\varepsilon(s) - \frac{c_f^\varepsilon(s)}{\lambda} \right) \psi d\sigma_x ds \\
&\le \|\kappa_f^\varepsilon\|_{L^\infty(D^\varepsilon)} \|\nabla \psi\|_{L^2(D^\varepsilon)} \int_t^{t+\tau} \|\nabla c_f^\varepsilon(s)\|_{L^2(D^\varepsilon)} ds \\
&\quad + \eta \|\psi\|_{L^2(D^e\varepsilon)} \int_t^{t+\tau} \left\| \varepsilon^{-1} \left(c_f^\varepsilon(s) - \frac{c_s^\varepsilon(s)}{\lambda} \right) \right\|_{L^2(\partial D^\varepsilon)} ds. \qquad (30)
\end{aligned}$$

Thus, by estimates (15) and (16), we have

$$
\begin{aligned}
&\mathbb{E}\sup_{0\le\tau\le\theta}\int_0^{T-\theta}\left\|\int_t^{t+\tau}\operatorname{div}(\kappa_f^{\varepsilon}\nabla c_f^{\varepsilon})ds\right\|^2_{H^{-1}(D^{\varepsilon})}\Big\}dt\\
&\le c\tau\mathbb{E}\sup_{0\le\tau\le\theta}\int_0^{T-\theta}\Big\{\int_0^T\|\nabla c_f^{\varepsilon}(s)\|^2_{L^2(D^{\varepsilon})}ds\Big\}dt\\
&\quad+c\tau\varepsilon\mathbb{E}\sup_{0\le\tau\le\theta}\int_0^{T-\theta}\Big\{\int_0^T\left\|\varepsilon^{-1}\left(c_f^{\varepsilon}(s)-\frac{c_s^{\varepsilon}(s)}{\lambda}\right)\right\|^2_{L^2(\partial D^{\varepsilon})}ds\Big\}dt\\
&\le c\theta.
\end{aligned}
\tag{31}
$$

From assumption A.3. and estimate (15), we get

$$
\begin{aligned}
&\mathbb{E}\sup_{0\le\tau\le\theta}\int_0^{T-\theta}\left\{\left\|\int_t^{t+\tau}\beta_f(c_f^{\varepsilon},\nabla c_f^{\varepsilon})ds\right\|^2_{W^{-1,2}(D^{\varepsilon})}\right\}dt\\
&\le c\mathbb{E}\sup_{0\le\tau\le\theta}\tau\int_0^{T-\theta}\Big\{\int_0^T\|\nabla c_f^{\varepsilon}(s)\|^2_{L^2(D^{\varepsilon})}ds\Big\}dt\;\le\;c\theta.
\end{aligned}
\tag{32}
$$

As for the stochastic integral, we use the fact that $L^2(D^{\varepsilon})$ is compactly embedded in $H^{-1}(D^{\varepsilon})$, Itô’s isometry, Fubini’s theorem, and Cauchy-Schwartz’s inequality to obtain

$$
\begin{aligned}
&\mathbb{E}\sup_{0\le\tau\le\theta}\int_0^{T-\theta}\Big\{\left\|\int_t^{t+\tau}\alpha_f^{\varepsilon}(c_f^{\varepsilon})dW_1(s)\right\|^2_{H^{-1}(D^{\varepsilon})}\Big\}dt\\
&\le\mathbb{E}\sup_{0\le\tau\le\theta}\left\|\int_t^{t+\tau}\alpha_f^{\varepsilon}(c_f^{\varepsilon})dW_1(s)\right\|^2_{L^2(D^{\varepsilon})}dt\\
&\le\mathbb{E}\int_0^{T-\theta}\sup_{0\le\tau\le\theta}\left\|\int_t^{t+\tau}\alpha_f^{\varepsilon}(c_f^{\varepsilon})dW_1(s)\right\|^2_{L^2(\varepsilon)}dt\\
&\le\int_0^{T-\theta}\left(\int_{D^{\varepsilon}}\mathbb{E}\sup_{0\le\tau\le\theta}\left(\int_t^{t+\tau}\alpha_f^{\varepsilon}(c_f^{\varepsilon})dW_1(s)\right)^2dx\right)dt\\
&\le\int_0^{T-\theta}\left(\mathbb{E}\sup_{0\le\tau\le\theta}\int_t^{t+\tau}\|\alpha_f^{\varepsilon}(c_f^{\varepsilon})\|^2_{L^2(D^{\varepsilon})}ds\right)dt.
\end{aligned}
\tag{33}
$$

Thus, by assumption A.4. and estimate (14), we have

$$\mathbb{E}\sup_{0\leq\tau\leq\theta}\int_0^{T-\theta}\left\{\left\|\int_t^{t+\tau}\alpha_f^{\varepsilon}(c_f^{\varepsilon})dW_1(s)\right\|_{H^{-1}(D^{\varepsilon})}^2\right\}dtt$$

$$\leq c\mathbb{E}\sup_{0\leq\tau\leq\theta}\tau\int_0^{T-\theta}\left\{\int_0^T\left(1+\|c_f^{\varepsilon}(t)\|_{L^2(D^{\varepsilon})}\right)^2dt\right\}dt\leq c\theta. \tag{34}$$

Combining the estimates in (29)-(34) with (27), we obtain (26). □

3. Tightness of probability distribution and compactness

In this section we aim to establish some type of strong convergence for the solution c_f^{ε} of equation (1). For this, we define the following extension by zero

$$\tilde{c_f^{\varepsilon}}=\begin{cases}c_f^{\varepsilon} & \text{in } D^{\varepsilon},\\ \\ 0 & \text{in } D\backslash D^{\varepsilon},\end{cases} \tag{35}$$

and define the set

$$\mathbb{Z}=\left\{\varphi^{\varepsilon}\in L^2(\Omega;L^2(0,T;L^2(D)))\cap L^2(\Omega;L^2(0,T;W_0^{1,2}(D)));\ \mathbb{E}\int_0^{T-s}\|\varphi(t+s)-\varphi(t)\|_{W^{-1,2}(D)}dt\leq Cs\right\}.$$

It is well known that $\mathbb{Z}$ is a compact subset of $L^2(\Omega;L^2(0,T;L^2(D)))$, see, for example, [10, 12, 27]. Also, $\Pi_{\varepsilon}:=\Pi_{\varepsilon}(\tilde{c_f^{\varepsilon}},W_1)$ the probability distribution (law) of $(\tilde{c_f^{\varepsilon}},W_1)$ is tight in $(\mathbb{K},\mathscr{B}(\mathbb{K}))$ where $\mathbb{K}=L^2(\Omega;L^2(0,T;L^2(D)))\times C([0,T];\mathbb{R}^n)$ and $\mathscr{B}(\mathbb{K})$ its $\sigma-$algebra of Borel sets. From this tightness and Prokhorov's compactness result [28] we are able to construct a subsequence Π_{ε_k} of Π_{ε} that weakly converges to a probability distribution Π in $\mathbb{K}$. Following this, by Skorokhod's representation theorem [28], see also [12] for a detailed implementation, we construct a new probability space $(\hat{\Omega},\hat{\mathbb{P}},\hat{\mathscr{F}})$ and two sets of random variables $(\tilde{c_f^{\varepsilon_k}},W_1^{\varepsilon_k})$ (with probability law Π_{ε_k}) and $(c_f,\hat{W}_1)$ (with probability law Π) such that

$$(\tilde{c_f^{\varepsilon_k}},W_1^{\varepsilon_k})\to(c_f,\hat{W}_1)\text{ in }\mathscr{B}(\mathbb{K})\text{ as }\varepsilon_k\to 0. \tag{36}$$

To avoid confusion and make notation easier to follow, we write the new processes on the new probability space $(\hat{\Omega},\hat{\mathbb{P}},\hat{\mathscr{F}})$ without hat. In the following theorem, we show that $(c_f^{\varepsilon_k},c_s^{\varepsilon_k},W_1^{\varepsilon_k},W_2)$ satisfies the stochastic model (1)-(5). This allows us to establish that the sequence of solutions possesses improved properties; in particular, it exhibits strong convergence in a probabilistic sense. Our proof follows the approach outlined in [11, p.352-356].

Theorem 5 Suppose that $\hat{\mathscr{F}}_t:=\sigma(c_f^{\varepsilon_k}(\tau),c_s^{\varepsilon_k}(\tau),W_1^{\varepsilon_k}(\tau),W_2(\tau))_{\tau\in[0,T]}$ is a filtration on $(\hat{\Omega},\hat{\mathbb{P}},\hat{\mathscr{F}})$, then

- $W_1^{\varepsilon_k}$ and W_2 are independent, $m-$dimensional standard Wiener processes and are $\mathscr{F}_t$-adapted.
- The couple $(c_f^{\varepsilon_k},c_s^{\varepsilon_k})$ satisfies,

$$\int_{D^{\varepsilon_k}} dc_f^{\varepsilon_k} \varphi^{\varepsilon_k} dx + \int_{D^{\varepsilon_k}} \left[\frac{1}{\varepsilon_k} \mathbf{v}^{\varepsilon_k} \cdot \nabla c_f^{\varepsilon_k} \varphi^{\varepsilon_k} + \kappa_f^{\varepsilon_k} \nabla c_f^{\varepsilon_k} \cdot \nabla \varphi^{\varepsilon_k} \right] dxdt$$

$$+ \frac{\varepsilon_k}{\lambda} \left[\int_{\partial D^{\varepsilon_k}} dc_s^{\varepsilon_k} \phi^{\varepsilon_k} d\sigma^{\varepsilon_k}(x) + \int_{\partial D^{\varepsilon_k}} \left[\frac{1}{\varepsilon_k} \mathbf{v}_s^{\varepsilon_k} \cdot \nabla_s c_s^{\varepsilon_k} \phi^{\varepsilon_k} + \kappa_s^{\varepsilon_k} \nabla c_s^{\varepsilon_k} \cdot \nabla_s \phi^{\varepsilon_k} \right] d\sigma^{\varepsilon_k}(x) dt \right]$$

$$+ \frac{\eta}{\varepsilon_k} \int_{\partial D^{\varepsilon_k}} \left(c_f^{\varepsilon_k} - \frac{c_s^{\varepsilon_k}}{\lambda} \right) \left(\varphi^{\varepsilon_k} - \frac{\phi^{\varepsilon_k}}{\lambda} \right) d\sigma^{\varepsilon_k}(x) dt = \int_{D^{\varepsilon_k}} \beta_f(c_f^{\varepsilon_k}, \nabla c_f^{\varepsilon_k}) \varphi^{\varepsilon_k} dxdt$$

$$+ \frac{\varepsilon_k}{\lambda} \int_{\partial D^{\varepsilon_k}} \phi^{\varepsilon_k} \alpha_s^{\varepsilon_k} d\sigma^{\varepsilon_k}(x) dW_2(t) + \int_{D^{\varepsilon_k}} \varphi^{\varepsilon_k} \alpha_f^{\varepsilon_k}(c_f^{\varepsilon_k}) dx dW_1^{\varepsilon_k}(t), \tag{37}$$

for all $t \in [0, T]$ and all $(\varphi^{\varepsilon_k}, \phi^{\varepsilon_k}) \in W^{1,2}(D^{\varepsilon_k}) \times W^{1,2}(\partial D^{\varepsilon_k})$, where $d\sigma^{\varepsilon_k}(x)$ is the surface measure on $\partial D^{\varepsilon_k}$.

4. Two scale convergence with drift

Here, we introduce the concept and some important results on two-scale convergence with drift in perforated domains and on their surfaces; see [9, 23, 29, 30] for more details and proofs.

Definition 3 If u^{ε} is a sequence in $L^2((0, T) \times \mathbb{R}^n)$ and $\mathbf{v}^* \in \mathbb{R}^n$ is a constant, we say that v^{ε} is two-scale converge with drift $\mathbf{v}^*$ to $v(t, x, y)$ in $L^2((0, T) \times \mathbb{R}^n \times Y)$ (denoted by $v^{\varepsilon} \rightharpoonup v$ 2-s with drift) if we have for all $\phi(t, x, y) \in \mathscr{D}((0, T) \times \mathbb{R}^n; C_{\text{per}}^{\infty})$ the following:

$$\int_0^T \int_{\mathbb{R}^n} u^{\varepsilon}(t, x) \phi\left(t, x - \frac{\mathbf{v}^*}{\varepsilon} t, \frac{x}{\varepsilon}\right) dxdt \to \int_0^T \int_{\mathbb{R}^n} \int_Y u(t, x, y) \phi(t, x, y) dydxdt. \tag{38}$$

Theorem 6 [9, 23] Assume that $v^{\varepsilon} \in L^2((0, T) \times \mathbb{R}^n)$ is a bounded sequence, i.e., $\|v^{\varepsilon}\|_{L^2((0,T)\times\mathbb{R}^n)} \leq C$ for some $C > 0$ and let $\mathbf{v}^* \in \mathbb{R}^n$ be a constant. Subsequently, there is a subsequence still indicated by v^{ε} with a function $v(t, x, y) \in L^2((0, T) \times \mathbb{R}^n; L^2_{\text{per}}(Y))$ where

$$v^{\varepsilon} \rightharpoonup v \text{ 2-s with drift.} \tag{39}$$

Theorem 7 [9, 23] Let $v^{\varepsilon} \in L^2((0, T); H^1(\mathbb{R}^n))$ be a uniformly bounded sequence, and let $\mathbf{v}^* \in \mathbb{R}^n$ be a constant. Subsequently, there is a subsequence still indicated by v^{ε} with functions $(v(t, x), \hat{v}(t, x, y)) \in L^2((0, T); H^1(\mathbb{R}^n)) \times L^2((0, T) \times \mathbb{R}^n; H^1_{\text{per}}(Y))$ where

$$v^{\varepsilon} \rightharpoonup v \text{ 2-s with drift,} \tag{40}$$

and

$$\nabla v^{\varepsilon} \rightharpoonup \nabla_x v + \nabla_y \hat{v} \text{ 2-s with drift.} \tag{41}$$

Theorem 8 [9, 23] Assume that $v^{\varepsilon} \in L^2((0, T) \times \mathbb{R}^n)$ which is two scale converge with drift to v in $L^2((0, T) \times \mathbb{R}^n; L^2_{\text{per}}(Y))$ and

$$\lim_{\varepsilon \to 0} \|v^{\varepsilon}\|_{L^2((0, T)\times\mathbb{R}^n)} \geq \|v\|_{L^2((0, T)\times\mathbb{R}^n\times Y)}. \tag{42}$$

Furthermore, if

$$\lim_{\varepsilon \to 0} \|v^{\varepsilon}\|_{L^2((0, T)\times\mathbb{R}^n)} = \|v\|_{L^2((0, T)\times\mathbb{R}^n\times Y)}. \tag{43}$$

Thus, $v^{\varepsilon} \to v$ 2-s with drift strongly (if v is smooth enough), i.e.,

$$\lim_{\varepsilon \to 0} \int_0^T \int_{\mathbb{R}^n} \left| v^{\varepsilon}(t, x) - v\left(t, x - \frac{\mathbf{v}^*}{\varepsilon} t, \frac{x}{\varepsilon}\right) \right|^2 dxdt = 0. \tag{44}$$

Definition 4 Assume that $v^{\varepsilon} \in L^2((0, T) \times \partial D^{\varepsilon})$ and $\mathbf{v}^* \in \mathbb{R}^n$ a constant, we say that v^{ε} is two scale converge with drift $\mathbf{v}^*$ to $v(t, x, y)$ in $L^2((0, T) \times \mathbb{R}^n \times \partial S)$ (denoted by $v^{\varepsilon} \rightharpoonup v$ 2-s with drift) if for all $\varphi(t, x, y) \in \mathscr{D}((0, T) \times \mathbb{R}^n; C^{\infty}_{\text{per}})$, we have

$$\varepsilon \int_0^T \int_{\partial D^{\varepsilon}} v^{\varepsilon}(t, x) \varphi\left(t, x - \frac{\mathbf{v}^*}{\varepsilon} t, \frac{x}{\varepsilon}\right) dxdt \to \int_0^T \int_{\mathbb{R}^n} \int_{\partial S} v(t, x, y) \varphi(t, x, y) dydxdt. \tag{45}$$

Theorem 9 [9] Assume that $v^{\varepsilon} \in L^2((0, T) \times \partial D^{\varepsilon}))$ is a bounded sequence, i.e. $\varepsilon \|v^{\varepsilon}\|_{L^2((0, T)\times\partial D^{\varepsilon})} \leq C$ for some $C > 0$ and let $\mathbf{v}^*$ be a constant vector in $\mathbb{R}^n$. Subsequently, there is a subsequence still indicated by v^{ε} with a function $v(t, x, y) \in L^2((0, T) \times \mathbb{R}^n \times \partial S)$ where

$$v^{\varepsilon} \rightharpoonup v \text{ 2-s with drift.} \tag{46}$$

Theorem 10 [9] Let $\mathbf{v}^* \in \mathbb{R}^n$ be a constant and $v^{\varepsilon} \in L^2((0, T); H^1(\mathbb{R}^n))$ such that

$$\varepsilon \|v^{\varepsilon}\|_{L^2((0, T)\times\partial D^{\varepsilon})} + \varepsilon \|\nabla v^{\varepsilon}\|_{L^2((0, T)\times\partial D^{\varepsilon})} \leq C.$$

Thus, there is a subsequence still indicated by v^{ε} with functions $(v(t, x), \hat{v}(t, x, y)) \in L^2((0, T); H^1(\mathbb{R}^n)) \times L^2((0, T) \times \mathbb{R}^n; H^1_{\text{per}}(\partial S))$ where

$$v^{\varepsilon} \rightharpoonup v \text{ 2-s with drift,} \tag{47}$$

and

$$\nabla_s v^\varepsilon \rightharpoonup \theta(y)\nabla_x v + \nabla_{s_y}\hat{v} \text{ 2-s with drift.} \tag{48}$$

5. Homogenization results

Here, we present the main results of the paper that is homogenization of the system (37). For simplicity, we use ε instead of ε_k.

Theorem 11 Let c_f^ε and c_s^ε be the solution of the stochastic model (1)-(5). Then there exist functions $c_f \in L^2(\Omega;\, L^2(0,\, T);\, H^1(D))$, $\hat{c_f} \in L^2(\Omega;\, L^2((0,\, T)\times D);\, H^1_{\text{per}}(F))$, and $\hat{c_s} \in L^2(\Omega;\, L^2((0,\, T)\times D);\, H^1_{\text{per}}(S))$ such that the following two-scale convergences with drift $\mathbf{v}^*$ hold with probability almost surely.

$$\begin{cases} c_f^\varepsilon \rightharpoonup c_f(t,\, x) \text{ 2-s with drift,} \\ c_s^\varepsilon \rightharpoonup \lambda c_f(t,\, x) \text{ 2-s with drift,} \\ \nabla c_f^\varepsilon \rightharpoonup \nabla_x c_f(t,\, x) + \nabla_y \hat{c_f}(t,\, x,\, y) \text{ 2-s with drift,} \\ \nabla c_s^\varepsilon \rightharpoonup \lambda\theta(y)\nabla_x c_f(t,\, x) + \nabla_{s_y}\hat{c_s}(t,\, x,\, y) \text{ 2-s with drift,} \\ \frac{1}{\varepsilon}\left(c_f^\varepsilon - \frac{c_s^\varepsilon}{\lambda}\right) \rightharpoonup \hat{c_f}(t,\, x,\, y) - \frac{\hat{c_s}}{\lambda}(t,\, x,\, y) \text{ 2-s with drift,} \end{cases} \tag{49}$$

Proof. The proof is done as in [9, Theorem 3.8.2.] □

Theorem 12 The limit c_f obtained in Theorem 11 satisfies the following homogenized model

$$\lambda_n dc_f - \operatorname{div}(\kappa^0\nabla_x c_f)dt = \tilde{\beta}(c_f,\, \nabla c_f)dt + \alpha_f(c_f)dW_1 + \alpha_s dW_2, \tag{50}$$

with the initial condition

$$\lambda_n c_f(0,\, x) = |F|c_f^0(x) + |\partial S|c_s^0(x), \tag{51}$$

where $\lambda_n = |F| + \lambda|\partial S|_{n-1}$, and

$$\kappa^0_{i,\, j} = \int_F \kappa(y)\,(\nabla_y\omega_j + e_j)\cdot(\nabla_y\omega_j + e_j)dy + \eta\int_{\partial S}\left(\omega_j + \frac{\xi_j}{\lambda}\right)\left(\omega_j + \frac{\xi_j}{\lambda}\right)d\sigma_y$$
$$+\frac{1}{\lambda}\int_{\partial S}\kappa_s(y)(\lambda e_j + \nabla_{s_y})\cdot(\lambda e_j + \nabla_{s_y})d\sigma_y, \tag{52}$$

such that $\omega = (\omega_j)_{1\le j\le n}$, $\xi = (\xi_j)_{1\le j\le n}$ satisfy

$$\hat{c_f}(t, x, y) = \omega(y)\cdot\nabla_x c_f(t, x) \text{ and } \hat{c_s}(t, x, y) = \xi(y)\cdot\nabla_x c_f(t, x), \tag{53}$$

$$\tilde{\beta}(c_f, \nabla c_f) = \gamma(t, c_f)\cdot\nabla_x c_f(t, x)\sum_{j=1}^{n}\int_F(\nabla_y\omega_j + e_j)dy, \tag{54}$$

and

$$\begin{cases} \mathbf{v}(y)\cdot\nabla_y\omega_j - \mathrm{div}_y(\kappa(\nabla_y\omega_j + e_j)) = (\mathbf{v}^* - \mathbf{v})e_j \text{ in } F, \\ \\ \mathbf{v}(y)\cdot\nabla_{s_y}\xi_j - \mathrm{div}_{s_y}(\kappa_s(\nabla_{s_y}\xi_j + \lambda e_j)) = \lambda(\mathbf{v}^* - \mathbf{v}_s)e_j + \eta(\omega_j - \frac{1}{\lambda}\xi_j) \text{ in } \partial S, \\ \\ -\kappa(\nabla_y\omega_j + e_j)\cdot\nu = \eta(\omega_j - \frac{1}{\lambda}\xi_j) \text{ on } \partial S, \\ \\ \omega_j \text{ and } \xi_j \text{ are } Y- \text{ periodic.} \end{cases} \tag{55}$$

Proof. Let us replace the test functions in the weak formulation (37) by

$$\varphi^\varepsilon = \varphi\left(t, x - \frac{\mathbf{v}^*}{\varepsilon}t\right) + \varepsilon\hat{\varphi}\left(t, x - \frac{\mathbf{v}^*}{\varepsilon}t, \frac{x}{\varepsilon}\right), \tag{56}$$

$$\phi^\varepsilon = \lambda\varphi\left(t, x - \frac{\mathbf{v}^*}{\varepsilon}t\right) + \varepsilon\hat{\phi}\left(t, x - \frac{\mathbf{v}^*}{\varepsilon}t, \frac{x}{\varepsilon}\right), \tag{57}$$

where φ, $\hat{\varphi}$ and $\hat{\phi}$ are smooth with compact support functions and $\varphi(T, x) = \hat{\varphi}(T, x, y) = \hat{\phi}(T, x, y) = 0$. The presence of the equilibrium constant of adsorption λ in the test function in (57) aligns with the last convergence convergence in (49). It also helps to cancel the reaction term when considering $\hat{\varphi} = \hat{\phi} = 0$. We consider as a first case that $\varphi = 0$ in (37) and integrating by parts, we get

$$\int_0^T\int_{D^\varepsilon} c_f^\varepsilon \mathrm{v}^*\nabla_x\hat{\varphi}\left(t, x - \frac{\mathbf{v}^*}{\varepsilon}t, \frac{x}{\varepsilon}\right)dxdt + \int_0^T\int_{D^\varepsilon}\mathbf{v}^\varepsilon\cdot\nabla c_f^\varepsilon\hat{\varphi}\left(t, x - \frac{\mathbf{v}^*}{\varepsilon}t, \frac{x}{\varepsilon}\right)dxdt$$

$$+\int_0^T\int_{D^\varepsilon}\kappa_f^\varepsilon\nabla c_f^\varepsilon\cdot\nabla_y\hat{\varphi}\left(t, x - \frac{\mathbf{v}^*}{\varepsilon}t, \frac{x}{\varepsilon}\right)dxdt + \frac{\varepsilon}{\lambda}\int_0^T\int_{\partial D^\varepsilon} c_s^\varepsilon \mathrm{v}^*\nabla_s\hat{\phi}\left(t, x - \frac{\mathbf{v}^*}{\varepsilon}t, \frac{x}{\varepsilon}\right)d\sigma_x dt$$

$$+\frac{\varepsilon}{\lambda}\int_0^T\int_{\partial D^\varepsilon}\mathbf{v}_s^\varepsilon\cdot\nabla c_s^\varepsilon\hat{\phi}\left(t, x - \frac{\mathbf{v}^*}{\varepsilon}t, \frac{x}{\varepsilon}\right)d\sigma_x dt + \frac{\varepsilon}{\lambda}\int_0^T\int_{\partial D^\varepsilon}\kappa_s^\varepsilon\nabla_s c_s^\varepsilon\cdot\nabla_y\hat{\phi}\left(t, x - \frac{\mathbf{v}^*}{\varepsilon}t, \frac{x}{\varepsilon}\right)d\sigma_x dt$$

$$+\varepsilon\int_0^T\int_{\partial D^\varepsilon}\frac{\eta}{\varepsilon}\left(c_f^\varepsilon - \frac{c_s^\varepsilon}{\lambda}\right)\left(\hat{\varphi} - \frac{\hat{\phi}}{\lambda}\right)d\sigma_x dt = \varepsilon\int_0^T\int_{D^\varepsilon}\beta_f(c_f^\varepsilon, \nabla c_f^\varepsilon)\hat{\varphi}dxdt$$

$$+\int_0^T\int_{D^\varepsilon}\varepsilon\hat{\varphi}\alpha_f^\varepsilon(c_f^\varepsilon)dxdW_1^\varepsilon(t)+\frac{\varepsilon}{\lambda}\int_0^T\int_{\partial D^\varepsilon}\varepsilon\hat{\phi}\alpha_s^\varepsilon d\sigma_x dW_2(t). \tag{58}$$

We now use the convergences obtained in Theorem 11 and the assumptions A.3.-A.5. to make a passage to the limit in the above equation. From the first convergence in (49), we get

$$\int_0^T\int_{D^\varepsilon}c_f^\varepsilon \mathrm{v}^*\nabla_x\hat{\varphi}\left(t,\, x-\frac{\mathbf{v}^*}{\varepsilon}t,\, \frac{x}{\varepsilon}\right)dxdt\to\int_0^T\int_{\mathbb{R}^n\times F}c_f(t,\, x)\cdot \mathrm{v}^*\nabla\hat{\varphi}\,(t,\, x,\, y)\,dxdydt$$

$$=-\int_0^T\int_{\mathbb{R}^n\times F}\nabla c_f(t,\, x)\cdot \mathrm{v}^*\hat{\varphi}\,(t,\, x,\, y)\,dxdydt. \tag{59}$$

By the third convergence in (49), we get

$$\int_0^T\int_{D^\varepsilon}\mathbf{v}^\varepsilon\cdot\nabla c_f^\varepsilon\hat{\varphi}\left(t,\, x-\frac{\mathbf{v}^*}{\varepsilon}t,\, \frac{x}{\varepsilon}\right)dxdt$$

$$\to\int_0^T\int_{\mathbb{R}^n\times F}\mathbf{v}(y)\cdot\left(\nabla_x c_f(t,\, x)+\nabla_y\hat{c_f}(t,\, x,\, y)\right)\hat{\varphi}\,(t,\, x,\, y)\,dxydt. \tag{60}$$

Using the third convergence in (49) and integrating by parts, we get

$$\int_0^T\int_{D^\varepsilon}\kappa_f^\varepsilon\nabla c_f^\varepsilon\cdot\nabla_y\hat{\varphi}\left(t,\, x-\frac{\mathbf{v}^*}{\varepsilon}t,\, \frac{x}{\varepsilon}\right)dxdt$$

$$\to\int_0^T\int_{\mathbb{R}^n\times F}\kappa(y)\left(\nabla_x c_f(t,\, x)+\nabla_y\hat{c_f}(t,\, x,\, y)\right)\nabla\hat{\varphi}\,(t,\, x,\, y)\,dxdydt$$

$$=-\int_0^T\int_{\mathbb{R}^n\times F}\operatorname{div}\left(\kappa(y)\left(\nabla_x c_f(t,\, x)+\nabla_y\hat{c_f}(t,\, x,\, y)\right)\right)\hat{\varphi}\,(t,\, x,\, y)\,dxdydt. \tag{61}$$

By the second convergence in (49) and by integrating parts, we have

$$\frac{\varepsilon}{\lambda}\int_0^T\int_{\partial D^\varepsilon}c_s^\varepsilon \mathrm{v}^*\nabla_s\hat{\phi}\left(t,\, x-\frac{\mathbf{v}^*}{\varepsilon}t,\, \frac{x}{\varepsilon}\right)d\sigma_x dt$$

$$\frac{1}{\lambda}\int_0^T\int_{\mathbb{R}^n\times\partial S}\lambda c_f(t,\, x)\cdot \mathrm{v}^*\nabla\hat{\phi}\,(t,\, x,\, y)\,dxdydt$$

$$=-\int_0^T\int_{\mathbb{R}^n\times\partial S}\nabla c_f(t,\, x)\cdot \mathrm{v}^*\hat{\phi}\,(t,\, x,\, y)\,dxdydt. \tag{62}$$

From the fourth convergence in (49), we have

$$\frac{\varepsilon}{\lambda}\int_0^T\int_{\partial D^\varepsilon}\mathbf{v}_s^\varepsilon\cdot\nabla c_s^\varepsilon\hat{\phi}\left(t,\, x-\frac{\mathbf{v}^*}{\varepsilon}t,\, \frac{x}{\varepsilon}\right)d\sigma_x dt$$

$$\to\frac{1}{\lambda}\int_0^T\int_{\mathbb{R}^n\times\partial S}\mathbf{v}_s(y)\cdot\left(\lambda\theta\nabla_x c_f(t,\, x)+\nabla_y\hat{c_s}(t,\, x,\, y)\right)\hat{\phi}\,(t,\, x,\, y)\,dx\sigma_y dt. \tag{63}$$

The fourth convergence in (49) and integration by parts give:

$$\frac{\varepsilon}{\lambda}\int_0^T\int_{\partial D^\varepsilon}\kappa_s^\varepsilon\nabla_s c_s^\varepsilon\cdot\nabla_y\hat{\phi}\left(t,\, x-\frac{\mathbf{v}^*}{\varepsilon}t,\, \frac{x}{\varepsilon}\right)d\sigma_x dt$$

$$\to-\frac{1}{\lambda}\int_0^T\int_{\mathbb{R}^n\times\partial S}\operatorname{div}_y\left(\kappa_s(y)\left(\lambda\nabla_x c_f(t,\, x)+\nabla_y\hat{c_s}(t,\, x,\, y)\right)\right)\hat{\phi}\,(t,\, x,\, y)\,dx\sigma_y dt. \tag{64}$$

The last convergence in (49) leads to:

$$\varepsilon\int_0^T\int_{\partial D^\varepsilon}\frac{\eta}{\varepsilon}\left(c_f^\varepsilon-\frac{c_s^\varepsilon}{\lambda}\right)\left(\hat{\varphi}-\frac{\hat{\phi}}{\lambda}\right)d\sigma_x dt$$

$$\to\int_0^T\int_{\mathbb{R}^n\times\partial S}\eta\left(\hat{c_f}-\frac{\hat{c}_s}{\lambda}\right)\left(\hat{\varphi}-\frac{\hat{\phi}}{\lambda}\right)dxd\sigma_y dt. \tag{65}$$

From assumption (8) and estimate (15), we have

$$\varepsilon\mathbb{E}\int_0^T\int_{D^\varepsilon}\beta_f(c_f^\varepsilon,\, \nabla c_f^\varepsilon)\hat{\varphi}dxdt\leq\varepsilon\mathbb{E}\int_0^T\left|\int_{D^\varepsilon}\gamma_f(c_f^\varepsilon)\cdot\nabla c_f^\varepsilon\hat{\varphi}dx\right|dt$$

$$\leq\varepsilon\mathbb{E}\int_0^T\|\nabla c_f^\varepsilon\|_{L^2(D^\varepsilon)}\|\nabla\hat{\varphi}\|_{L^2(D^\varepsilon)}dt\leq\varepsilon c\to 0. \tag{66}$$

We use assumption A.4. and Burkholder-Davis-Gundy’s inequality, we get

$$\varepsilon\mathbb{E}\sup_t\left|\int_0^T\int_{D^\varepsilon}\alpha_f^\varepsilon(c_f^\varepsilon)\hat{\varphi}dxdW_1^\varepsilon(t)\right|\leq\varepsilon\mathbb{E}\left\{\int_0^T\left\{\int_{D^\varepsilon}\alpha_f^\varepsilon(c_f^\varepsilon)\hat{\varphi}dx\right\}^2dt\right\}^{\frac{1}{2}}$$

$$\leq c\varepsilon\mathbb{E}\left\{\int_0^T\|\alpha_f^\varepsilon(c_f^\varepsilon)\|^2_{[L^2(D^\varepsilon)]^n}\|\hat{\varphi}\|^2_{L^2(D^\varepsilon)}dt\right\}^{\frac{1}{2}}$$

$$\varepsilon c\to 0 \text{ when } \varepsilon\to 0. \tag{67}$$

Same as in (67), we have

$$\frac{\varepsilon}{\lambda}\mathbb{E}\sup_{t}\left|\int_0^T\int_{\partial D^\varepsilon}\varepsilon\hat{\phi}\alpha_s^\varepsilon d\sigma_x dW_2(t)\right| \to 0 \text{ when } \varepsilon \to 0. \tag{68}$$

From (59)-(68), we have

$$\begin{aligned}
&-\int_0^T\int_{\mathbb{R}^n\times F}\nabla c_f(t,\,x)\cdot\mathrm{v}^*\hat{\varphi}\,(t,\,x,\,y)\,dxdydt\\
&+\int_0^T\int_{\mathbb{R}^n\times F}\mathbf{v}(y)\cdot\left(\nabla_x c_f(t,\,x)+\nabla_y\hat{c_f}(t,\,x,\,y)\right)\hat{\varphi}\,(t,\,x,\,y)\,dxydt\\
&-\int_0^T\int_{\mathbb{R}^n\times F}\operatorname{div}\left(\kappa(y)\left(\nabla_x c_f(t,\,x)+\nabla_y\hat{c_f}(t,\,x,\,y)\right)\right)\hat{\varphi}\,(t,\,x,\,y)\,dxdydt\\
&-\int_0^T\int_{\mathbb{R}^n\times\partial S}\nabla c_f(t,\,x)\cdot\mathrm{v}^*\hat{\phi}\,(t,\,x,\,y)\,dxdydt\\
&+\frac{1}{\lambda}\int_0^T\int_{\mathbb{R}^n\times\partial S}\mathbf{v}_s(y)\cdot\left(\lambda\theta\nabla_x c_f(t,\,x)+\nabla_y\hat{c_s}(t,\,x,\,y)\right)\hat{\phi}\,(t,\,x,\,y)\,dx\sigma_y dt\\
&-\frac{1}{\lambda}\int_0^T\int_{\mathbb{R}^n\times\partial S}\operatorname{div}_y\left(\kappa_s(y)\left(\lambda\nabla_x c_f(t,\,x)+\nabla_y\hat{c_s}(t,\,x,\,y)\right)\right)\hat{\phi}\,(t,\,x,\,y)\,dx\sigma_y dt\\
&+\int_0^T\int_{\mathbb{R}^n\times\partial S}\eta\left(\hat{c_f}-\frac{\hat{c_s}}{\lambda}\right)\left(\hat{\varphi}-\frac{\hat{\phi}}{\lambda}\right)dxd\sigma_y dt=0.
\end{aligned}\tag{69}$$

From the linearity of the integral equation (69) one can write $\hat{c_f}(t,\,x,\,y)=\omega(y)\cdot\nabla_x c_f(t,\,x)$ and $\hat{c_s}(t,\,x,\,y)=\xi(y)\cdot\nabla_x c_f(t,\,x),$ and then realize that (69) is nothing but the variational formulation of (55). We now take $\hat{\varphi}=\hat{\phi}=0$ in (37) and integrate by part, we have

$$\begin{aligned}
&-\int_0^T\int_{D^\varepsilon}c_f^\varepsilon\frac{\partial\varphi}{\partial t}\left(t,\,x-\frac{\mathbf{v}^*}{\varepsilon}t\right)dxdt+\frac{1}{\varepsilon}\int_0^T\int_{D^\varepsilon}c_f^\varepsilon\mathbf{v}^*\cdot\nabla\varphi\left(t,\,x-\frac{\mathbf{v}^*}{\varepsilon}t\right)dxdt\\
&+\int_0^T\int_{D^\varepsilon}\kappa_f^\varepsilon\nabla c_f^\varepsilon\cdot\nabla\varphi\left(t,\,x-\frac{\mathbf{v}^*}{\varepsilon}t\right)dxdt-\frac{1}{\varepsilon}\int_0^T\int_{D^\varepsilon}c_f^\varepsilon\mathbf{v}^\varepsilon\cdot\nabla\varphi\left(t,\,x-\frac{\mathbf{v}^*}{\varepsilon}t\right)dxdt\\
&-\varepsilon\int_0^T\int_{\partial D^\varepsilon}c_s^\varepsilon\frac{\partial\varphi}{\partial t}\left(t,\,x-\frac{\mathbf{v}^*}{\varepsilon}t\right)d\sigma_x dt+\int_0^T\int_{\partial D^\varepsilon}c_s^\varepsilon\mathbf{v}^*\cdot\nabla\varphi\left(t,\,x-\frac{\mathbf{v}^*}{\varepsilon}t\right)d\sigma_x dt\\
&+\varepsilon\int_0^T\int_{\partial D^\varepsilon}\kappa_s^\varepsilon\nabla c_s^\varepsilon\cdot\nabla\varphi\left(t,\,x-\frac{\mathbf{v}^*}{\varepsilon}t\right)d\sigma_x dt-\int_0^T\int_{\partial D^\varepsilon}c_s^\varepsilon\mathbf{v}^\varepsilon\cdot\nabla\varphi\left(t,\,x-\frac{\mathbf{v}^*}{\varepsilon}t\right)d\sigma_x dt
\end{aligned}$$

$$-\int_{D^\varepsilon} c_f^0 \varphi(0, x) dx - \varepsilon \int_{\partial D^\varepsilon} c_s^0 \varphi(0, x) d\sigma_x = \int_0^T \int_{D^\varepsilon} \beta_f(c_f^\varepsilon, \nabla c_f^\varepsilon) \varphi dx dt$$

$$+ \varepsilon \int_0^T \int_{\partial D^\varepsilon} \alpha_s^\varepsilon \varphi d\sigma_x dW_2(t) + \int_0^T \int_{D^\varepsilon} \alpha_f^\varepsilon (c_f^\varepsilon) \varphi dx dW_1^\varepsilon(t). \tag{70}$$

Before passing to the limit, we combine the terms of order ε^{-1} as follows:

$$\frac{1}{\varepsilon} \int_0^T \int_{D^\varepsilon} c_f^\varepsilon \mathbf{v}^* \cdot \nabla \varphi \left(t, x - \frac{\mathbf{v}^*}{\varepsilon} t\right) dx dt - \frac{1}{\varepsilon} \int_0^T \int_{D^\varepsilon} c_f^\varepsilon \mathbf{v}^\varepsilon \cdot \nabla \varphi \left(t, x - \frac{\mathbf{v}^*}{\varepsilon} t\right) dx dt$$

$$+ \int_0^T \int_{\partial D^\varepsilon} c_s^\varepsilon \mathbf{v}^* \cdot \nabla \varphi \left(t, x - \frac{\mathbf{v}^*}{\varepsilon} t\right) d\sigma_x dt - \int_0^T \int_{\partial D^\varepsilon} c_s^\varepsilon \mathbf{v}^\varepsilon \cdot \nabla \varphi \left(t, x - \frac{\mathbf{v}^*}{\varepsilon} t\right) d\sigma_x dt$$

$$= \frac{1}{\varepsilon} \int_0^T \int_{D^\varepsilon} c_f^\varepsilon (\mathbf{v}^* - \mathbf{v}^\varepsilon) \cdot \nabla \varphi \left(t, x - \frac{\mathbf{v}^*}{\varepsilon} t\right) dx dt$$

$$+ \lambda \int_0^T \int_{\partial D^\varepsilon} c_f^\varepsilon (\mathbf{v}^* - \mathbf{v}^\varepsilon) \cdot \nabla \varphi \left(t, x - \frac{\mathbf{v}^*}{\varepsilon} t\right) d\sigma_x dt$$

$$+ \varepsilon \lambda \int_0^T \int_{\partial D^\varepsilon} \frac{1}{\varepsilon} \left(c_f^\varepsilon - \frac{c_s^\varepsilon}{\lambda}\right) (\mathbf{v}^* - \mathbf{v}^\varepsilon) \cdot \nabla \varphi \left(t, x - \frac{\mathbf{v}^*}{\varepsilon} t\right) d\sigma_x dt$$

$$= \sum_{j=1}^n \int_0^T \int_{D^\varepsilon} \nabla z_j^\varepsilon \cdot \nabla \left(\partial_{x_j} \varphi \left(t, x - \frac{\mathbf{v}^*}{\varepsilon} t\right) c_f^\varepsilon\right) dx dt$$

$$+ \varepsilon \lambda \int_0^T \int_{\partial D^\varepsilon} \frac{1}{\varepsilon} \left(c_f^\varepsilon - \frac{c_s^\varepsilon}{\lambda}\right) (\mathbf{v}^* - \mathbf{v}^\varepsilon) \cdot \nabla \varphi \left(t, x - \frac{\mathbf{v}^*}{\varepsilon} t\right) d\sigma_x dt, \tag{71}$$

where $z^\varepsilon(x) = z\left(\frac{z}{\varepsilon}\right)$ is the unique solution of the following auxiliary problem, see [9, p.91-92]

$$\begin{cases} \Delta z_j(y) = \mathbf{v}_j^f(y) - \mathbf{v}_j^* \text{ in } F, \\ \\ \nabla z_j \cdot \nu = \lambda(\mathbf{v}_j^* - \mathbf{v}_j^s(y)) \text{ on } \partial S, \\ \\ z_j \text{ is } Y-\text{periodic}. \end{cases} \tag{72}$$

From equations (70) and (71), we have

$$
\begin{aligned}
&-\int_0^T\int_{D^\varepsilon} c_f^\varepsilon \frac{\partial\varphi}{\partial t}\left(t, x-\frac{\mathbf{v}^*}{\varepsilon}t\right)dxdt + \int_0^T\int_{D^\varepsilon}\kappa_f^\varepsilon\nabla c_f^\varepsilon\cdot\nabla\varphi\left(t, x-\frac{\mathbf{v}^*}{\varepsilon}t\right)dxdt \\
&-\varepsilon\int_0^T\int_{\partial D^\varepsilon} c_s^\varepsilon \frac{\partial\varphi}{\partial t}\left(t, x-\frac{\mathbf{v}^*}{\varepsilon}t\right)d\sigma_x dt + \varepsilon\int_0^T\int_{\partial D^\varepsilon}\kappa_s^\varepsilon\nabla c_s^\varepsilon\cdot\nabla\varphi\left(t, x-\frac{\mathbf{v}^*}{\varepsilon}t\right)d\sigma_x dt \\
&-\int_{D^\varepsilon} c_f^0\varphi(0, x)dx - \varepsilon\int_{\partial D^\varepsilon} c_s^0\varphi(0, x)d\sigma_x + \sum_{j=1}^n\int_0^T\int_{D^\varepsilon}\nabla z_j^\varepsilon\cdot\nabla\left(\partial_{x_j}\varphi\left(t, x-\frac{\mathbf{v}^*}{\varepsilon}t\right)c_f^\varepsilon\right)dxdt \\
&+\varepsilon\lambda\int_0^T\int_{\partial D^\varepsilon}\frac{1}{\varepsilon}\left(c_f^\varepsilon-\frac{c_s^\varepsilon}{\lambda}\right)(\mathbf{v}^*-\mathbf{v}^\varepsilon)\cdot\nabla\varphi\left(t, x-\frac{\mathbf{v}^*}{\varepsilon}t\right)d\sigma_x dt \qquad (73) \\
&=\int_0^T\int_{D^\varepsilon}\beta_f(c_f^\varepsilon, \nabla c_f^\varepsilon)\varphi dxdt + \int_0^T\int_{D^\varepsilon}\alpha_f^\varepsilon(c_f^\varepsilon)\varphi dxdW_1^\varepsilon(t) + \varepsilon\int_0^T\int_{\partial D^\varepsilon}\alpha_s^\varepsilon\varphi d\sigma_x dW_2(t).
\end{aligned}
$$

Let us pass to the limit (73) using the convergences in Theorems 7 and 11 and the strong convergence (36). From the first convergence in (49), we have

$$
\begin{aligned}
-\int_0^T\int_{D^\varepsilon} c_f^\varepsilon \frac{\partial\varphi}{\partial t}\left(t, x-\frac{\mathbf{v}^*}{\varepsilon}t\right)dxdt &\to -\int_0^T\int_{\mathbb{R}^n}\int_F c_f(t, x)\frac{\partial\varphi}{\partial t}(t, x)\,dydxdt \\
&=|F|\int_0^T\int_{\mathbb{R}^n}\frac{\partial c_f}{\partial t}(t, x)\varphi(t, x)\,dxdt. \qquad (74)
\end{aligned}
$$

Using the third convergence in (49), we get

$$
\begin{aligned}
&\int_0^T\int_{D^\varepsilon}\kappa_f^\varepsilon\nabla c_f^\varepsilon\cdot\nabla\varphi\left(t, x-\frac{\mathbf{v}^*}{\varepsilon}t\right)dxdt \\
&\to\int_0^T\int_{\mathbb{R}^n\times F}\kappa_f(y)\left(\nabla c_f(t, x)+\nabla_y\tilde{c_f}(t, x, y)\right)\cdot\nabla_x\varphi dxdydt. \qquad (75)
\end{aligned}
$$

The second convergence in (49) gives the following:

$$
\begin{aligned}
-\varepsilon\int_0^T\int_{\partial D^\varepsilon} c_s^\varepsilon \frac{\partial\varphi}{\partial t}\left(t, x-\frac{\mathbf{v}^*}{\varepsilon}t\right)d\sigma_x dt &\to -\int_0^T\int_{\mathbb{R}^n}\int_{\partial S}\lambda c_f(t, x)\frac{\partial\varphi}{\partial t}(t, x)\,d\sigma_y dxdt \\
&=\lambda|\partial S|\int_0^T\int_{\mathbb{R}^n}\frac{\partial c_f}{\partial t}(t, x)\varphi(t, x)\,dxdt. \qquad (76)
\end{aligned}
$$

We use the fourth convergence in (49) to get

$$\varepsilon\int_0^T\int_{\partial D^\varepsilon}\kappa_s^\varepsilon\nabla c_s^\varepsilon\cdot\nabla\varphi\left(t,x-\frac{\mathbf{v}^*}{\varepsilon}t\right)d\sigma_x dt$$

$$\to\int_0^T\int_{\mathbb{R}^n\times\partial S}\kappa_s(y)\left(\lambda\theta(y)\nabla_x c_f(t,x)+\nabla_{s_y}\hat{c_s}(t,x,y)\right)\cdot\nabla_x\varphi dxd\sigma_y dt. \tag{77}$$

It is also easy to see that the initial terms satisfy the following:

$$-\int_{D^\varepsilon}c_f^0\varphi(0,x)dx-\varepsilon\int_{\partial D^\varepsilon}c_s^0\varphi(0,x)d\sigma_x$$

$$\to-|F|\int_{\mathbb{R}^n}c_f^0\varphi(0,x)dx-|\partial S|\int_{\mathbb{R}^n}c_s^0\varphi(0,x)dx. \tag{78}$$

By (72), we have

$$\sum_{j=1}^n\int_0^T\int_{D^\varepsilon}\nabla z_j^\varepsilon\cdot\nabla\left(\partial_{x_j}\varphi\left(t,x-\frac{\mathbf{v}^*}{\varepsilon}t\right)c_f^\varepsilon\right)dxdt$$

$$\to\sum_{j=1}^n\int_0^T\int_{\mathbb{R}^n\times F}\nabla z_j(y)\cdot\nabla\left(\partial_{x_j}\varphi c_f\right)dxdydt. \tag{79}$$

From the last convergence in (49), we get

$$\varepsilon\lambda\int_0^T\int_{\partial D^\varepsilon}\frac{1}{\varepsilon}\left(c_f^\varepsilon-\frac{c_s^\varepsilon}{\lambda}\right)(\mathbf{v}^*-\mathbf{v}^\varepsilon)\cdot\nabla\varphi\left(t,x-\frac{\mathbf{v}^*}{\varepsilon}t\right)d\sigma_x dt$$

$$\to\lambda\int_0^T\int_{\mathbb{R}^n\times\partial S}\left(\hat{c_f}(t,x,y)-\frac{\hat{c_s}}{\lambda}(t,x,y)\right)(\mathbf{v}^*-\mathbf{v}(y))\cdot\nabla\varphi dxd\sigma_y dt. \tag{80}$$

We use A.3. to write the first term on the right side of (73) as:

$$\int_0^T\int_{D^\varepsilon}\beta_f(c_f^\varepsilon,\nabla c_f^\varepsilon)\varphi dxdt=\int_0^T\int_{D^\varepsilon}\gamma_f(c_f^\varepsilon)\cdot\nabla c_f^\varepsilon\varphi dxdt$$

$$=\int_0^T\int_{D^\varepsilon}[\gamma_f(c_f^\varepsilon)-\gamma_f(c_f)]\cdot\nabla c_f^\varepsilon\varphi dxdt+\int_0^T\int_{D^\varepsilon}\gamma_f(c_f)\cdot\nabla c_f^\varepsilon\varphi dxdt. \tag{81}$$

By estimate (15), condition (9) and convergence (36), we have

$$\mathbb{E}\int_0^T\int_{D^\varepsilon}[\gamma_f(c_f^\varepsilon)-\gamma_f(c_f)]\cdot\nabla c_f^\varepsilon\varphi dxdt \le C\mathbb{E}\int_0^T \|\gamma_f(c_f^\varepsilon)-\gamma_f(c_f)\|_{L^2(D^\varepsilon}\|\nabla c_f^\varepsilon\|_{[L^2(D^\varepsilon]^n}dt$$

$$C_1\mathbb{E}\int_0^T \|c_f^\varepsilon-c_f\|_{L^2(D^\varepsilon}dt \to 0 \text{ as } \varepsilon\to 0. \tag{82}$$

Thus,

$$\lim_{\varepsilon\to 0}\int_0^T\int_{D^\varepsilon}\beta_f(c_f^\varepsilon,\,\nabla c_f^\varepsilon)\varphi dxdt = \lim_{\varepsilon\to 0}\int_0^T\int_{D^\varepsilon}\gamma_f(c_f)\cdot\nabla c_f^\varepsilon\varphi dxdt$$

$$=\int_0^T\int_{\mathbb{R}^n\times F}\gamma_f(c_f)\cdot[\nabla c_f(t,\,x)+\nabla_y\tilde{c_f}(t,\,x,\,y)]\varphi dxdt. \tag{83}$$

For the stochastic integrals, we write

$$\mathbb{E}\int_0^T\int_{D^\varepsilon}\alpha_f^\varepsilon(c_f^\varepsilon)\varphi dxdW_1^\varepsilon(t) = \mathbb{E}\int_0^T\int_{D^\varepsilon}\alpha_f^\varepsilon(c_f^\varepsilon)\varphi dxd(W_1^\varepsilon(t)-\tilde{W}_1(t))$$

$$+\mathbb{E}\int_0^T\int_{D^\varepsilon}\alpha_f^\varepsilon(c_f^\varepsilon)\varphi dxd\tilde{W}_1(t). \tag{84}$$

Given the unbounded variation of $W_1^\varepsilon(t)-\tilde{W}_1(t)$, special attention is required to ensure the convergence of the first term on the right-hand side of (84) to fully utilize the $\mathbb{P}-$almost sure uniform convergence (36). For this, we introduce the notion of regularization for the function $\alpha_f^\varepsilon(t,\,x,\,c_f^\varepsilon)$ as follows:

$$\alpha_\delta^\varepsilon(t)=\frac{1}{\delta}\int_0^T\rho\left(-\frac{t-\tau}{\delta}\right)\alpha_f^\varepsilon(\tau,\,x,\,c_f^\varepsilon(\tau))d\tau,$$

where ρ is a classical mollifier and $\delta>0$. By construction $\alpha_\delta^\varepsilon$ is differentiable in time and for all $\varepsilon,\,\delta>0$, we have

$$\mathbb{E}\int_0^T\|\alpha_\delta^\varepsilon(t)\|_{L^2(D^\varepsilon)}^2dt\le\mathbb{E}\int_0^T\|\alpha_f^\varepsilon(t,\,x,\,c_f^\varepsilon(t))\|_{L^2(D^\varepsilon)}^2dt, \tag{85}$$

and

$$\alpha_\delta^\varepsilon(t)\to\alpha_f^\varepsilon(t,\,x,\,c_f^\varepsilon(t)) \text{ in } L^2(\Omega;\,L^2((0,\,T);\,L^2(D^\varepsilon))) \text{ as } \delta\to 0. \tag{86}$$

Let's write (84)'s first term on the right side as follows:

$$\mathbb{E}\int_0^T\int_{D^\varepsilon}\alpha_f^\varepsilon(c_f^\varepsilon)\varphi dxd(W_1^\varepsilon(t)-\tilde{W}_1(t)) =\mathbb{E}\int_0^T\int_{D^\varepsilon}\alpha_\delta^\varepsilon(t)\varphi dxd(W_1^\varepsilon(t)-\tilde{W}_1(t))$$

$$+\mathbb{E}\int_0^T\int_{D^\varepsilon}[\alpha_f^\varepsilon(t,x,c_f^\varepsilon(t))-\alpha_\delta^\varepsilon(t)]\varphi dxd(W_1^\varepsilon(t)-\tilde{W}_1(t)). \tag{87}$$

From Burkholder-Davis-Gundy's inequality and convergence (86), we get the following:

$$\begin{aligned}
&\mathbb{E}\int_0^T\int_{D^\varepsilon}[\alpha_f^\varepsilon(t,x,c_f^\varepsilon(t))-\alpha_\delta^\varepsilon(t)]\varphi dxd(W_1^\varepsilon(t)-\tilde{W}_1(t))\\
\leq&\mathbb{E}\sup_t\left|\int_0^T\int_{D^\varepsilon}[\alpha_f^\varepsilon(t,x,c_f^\varepsilon(t))-\alpha_\delta^\varepsilon(t)]\varphi dxd(W_1^\varepsilon(t)-\tilde{W}_1(t))\right|\\
\leq&\varepsilon\mathbb{E}\left\{\int_0^T\left\{\int_{D^\varepsilon}[\alpha_f^\varepsilon(t,x,c_f^\varepsilon(t))-\alpha_\delta^\varepsilon(t)]\varphi dx\right\}^2dt\right\}^{\frac{1}{2}}\\
\leq&c\mathbb{E}\left\{\int_0^T\|\alpha_f^\varepsilon(t,x,c_f^\varepsilon(t))-\alpha_\delta^\varepsilon(t)\|^2_{[L^2(D^\varepsilon)]^n}\|\varphi\|^2_{L^2(D^\varepsilon)}dt\right\}^{\frac{1}{2}}\\
\to&0 \text{ as } \delta\to 0.
\end{aligned}\tag{88}$$

Since $\alpha_\delta^\varepsilon$ is differentiable in time, the first term on the right side of (87) is addressed as follows:

$$\begin{aligned}
&\mathbb{E}\int_0^T\int_{D^\varepsilon}\alpha_\delta^\varepsilon(t)\varphi dxd(W_1^\varepsilon(t)-\tilde{W}_1(t))\\
=&\mathbb{E}\int_0^T\int_{D^\varepsilon}(W_1^\varepsilon(t)-\tilde{W}_1(t))\frac{\partial}{\partial t}[\alpha_\delta^\varepsilon(t)\varphi]dxdt+\mathbb{E}\int_{D^\varepsilon}(W_1^\varepsilon(T)-\tilde{W}_1(T))\alpha_\delta^\varepsilon(T)\varphi(T,x)dx\\
\leq&\mathbb{E}\sup_t|W_1^\varepsilon(t)-\tilde{W}_1(t)|\left\{\int_0^T\int_{D^\varepsilon}\left|\frac{\partial}{\partial t}[\alpha_\delta^\varepsilon(t)\varphi]\right|dxdt+\int_{D^\varepsilon}|\alpha_\delta^\varepsilon(T)\varphi(T,x)|dx\right\}\\
\leq&c\mathbb{E}\sup_t|W_1^\varepsilon(t)-\tilde{W}_1(t)|\to 0 \text{ as } \varepsilon\to 0.
\end{aligned}\tag{89}$$

From (88), (89) and (87), we have

$$\lim_{\varepsilon\to 0}\mathbb{E}\int_0^T\int_{D^\varepsilon}\alpha_f^\varepsilon(c_f^\varepsilon)\varphi dxdW_1^\varepsilon(t)=\lim_{\varepsilon\to 0}\mathbb{E}\int_0^T\int_{D^\varepsilon}\alpha_f^\varepsilon(c_f^\varepsilon)\varphi dxd\tilde{W}_1(t). \tag{90}$$

From the convergence theorem for stochastic integrals by Rozovskii [31, Theorem 4, p.63] and the condition (9) on $\alpha_f^\varepsilon(c_f^\varepsilon)$, we pass to the limit in (90) to obtain the following.

$$\lim_{\varepsilon\to 0}\mathbb{E}\int_0^T\int_{D^\varepsilon}\alpha_f^\varepsilon(c_f^\varepsilon)\varphi dxdW_1^\varepsilon(t)=\mathbb{E}\int_0^T\int_{\mathbb{R}^n}\alpha_f(c_f)\varphi dxd\tilde{W}_1(t). \tag{91}$$

In a similar way, we have

$$\lim_{\varepsilon\to 0}\mathbb{E}\varepsilon\int_0^T\int_{\partial D^\varepsilon}\alpha_s^\varepsilon\varphi d\sigma_x dW_2(t)=\mathbb{E}\int_0^T\int_{\mathbb{R}^n}\alpha_s\varphi dxdW_2(t). \tag{92}$$

All the above obtained convergences lead to the following weak formulation.

$$\begin{aligned}
&|F|\int_0^T\int_{\mathbb{R}^n}dc_f\varphi dt+\int_0^T\int_{\mathbb{R}^n\times F}\kappa_f(y)\left(\nabla c_f(t,x)+\nabla_y\tilde{c_f}(t,x,y)\right)\cdot\nabla_x\varphi dxdydt\\
&\lambda|\partial S|\int_0^T\int_{\mathbb{R}^n}dc_f\varphi dxdt+\int_0^T\int_{\mathbb{R}^n\times\partial S}\kappa_s(y)\left(\lambda\theta(y)\nabla_x c_f(t,x)+\nabla_{s_y}\hat{c_s}(t,x,y)\right)\cdot\nabla_x\varphi dxd\sigma_y dt\\
&-|F|\int_{\mathbb{R}^n}c_f^0\varphi(0,x)dx-|\partial S|\int_{\mathbb{R}^n}c_s^0\varphi(0,x)dx+\sum_{j=1}^n\int_0^T\int_{\mathbb{R}^n\times F}\nabla z_j(y)\cdot\nabla\left(\partial_{x_j}\varphi c_f\right)dxdydt\\
&+\lambda\int_0^T\int_{\mathbb{R}^n\times\partial S}\left(\hat{c_f}(t,x,y)-\frac{\hat{c_s}}{\lambda}(t,x,y)\right)(\mathbf{v}^*-\mathbf{v}(y))\cdot\nabla\varphi dxd\sigma_y dt \qquad (93)\\
=&\int_0^T\int_{\mathbb{R}^n}\tilde{\beta}_f(c_f,\nabla c_f)\varphi dxdt+\int_0^T\int_{\mathbb{R}^n}\alpha_f(c_f)\varphi dxd\tilde{W}_1(t)\\
&+\int_0^T\int_{\mathbb{R}^n}\alpha_s\varphi dxdW_2(t).
\end{aligned}$$

We use the expressions in (53) to obtain

$$\begin{aligned}
&(|F|+\lambda|\partial S|)\int_0^T\int_{\mathbb{R}^n}dc_f\varphi dxdt+\int_0^T\int_{\mathbb{R}^n\times F}\sum_{i,\,j=1}^n\kappa_{f_{i,\,j}}(y)\frac{\partial c_f}{\partial x_j}\frac{\partial\varphi}{\partial x_i}dxdydt\\
&+\int_0^T\int_{\mathbb{R}^n\times F}\sum_{i,\,j=1}^n\sum_{k=1}^n\kappa_{f_{i,\,j}}(y)\frac{\partial\omega_j}{\partial y_k}\frac{\partial c_f}{\partial x_j}\frac{\partial\varphi}{\partial x_i}dxdydt\\
&+\lambda\int_0^T\int_{\mathbb{R}^n\times\partial S}\sum_{i,\,j=1}^n\sum_{k=1}^n\kappa_{s_{i,\,j}}(y)\theta_{j,\,k}(y)\frac{\partial c_f}{\partial x_j}\frac{\partial\varphi}{\partial x_i}dxd\sigma_y dt
\end{aligned}$$

$$+\int_0^T\int_{\mathbb{R}^n\times\partial S}\sum_{i,\,j=1}^n\sum_{k=1}^n\kappa_{s_{i,\,j}}(y)\frac{\partial_s\xi_j}{\partial y_k}\frac{\partial c_f}{\partial x_j}\frac{\partial\varphi}{\partial x_i}dxd\sigma_ydt$$

$$+\lambda\int_0^T\int_{\mathbb{R}^n\times\partial S}\left(\omega_j-\frac{\xi_j}{\lambda}\right)(\mathbf{v}_{s_i}-\mathbf{v}_i^*)\frac{\partial c_f}{\partial x_j}\frac{\partial\varphi}{\partial x_i}dxd\sigma_ydt$$

$$+\int_0^T\int_{\mathbb{R}^n\times F}\sum_{i,\,j=1}^n\sum_{k=1}^n\frac{\partial z_j}{\partial y_k}\frac{\partial\omega_j}{\partial y_k}\frac{\partial c_f}{\partial x_j}\frac{\partial\varphi}{\partial x_i}dxdydt$$

$$-\int_{\mathbb{R}^n}(|F|c_f^0(x)+|\partial S|c_s^0(x))\varphi(0,\,x)dx$$

$$=\int_0^T\int_{\mathbb{R}^n}\tilde{\beta}_f(c_f,\,\nabla c_f)\varphi dxdt+\int_0^T\int_{\mathbb{R}^n}\alpha_f(c_f)\varphi dxd\tilde{W}_1(t$$

$$+\int_0^T\int_{\mathbb{R}^n}\alpha_s\varphi dxdW_2(t). \tag{94}$$

This equation is basically the variational formulation of the homogenized model (50). With the effective coefficients given by (52). The existence of a weak probabilistic model of the homogenized model is standard, and this, combined with pathwise uniqueness, shows that the convergences took place on the entire sequence rather than on a subsequence. This completes the proof of the main result. □

Acknowledgment

The author expresses his deepest gratitude to Anonymous reviewers for the careful reading of the paper with their insightful comments and suggestions that improved the paper.The author also extends the appreciation to the Deanship of Postgraduate Studies and Scientific Research at Majmaah University for funding this research work through the project number R-2025-2115.

Conflict of interest

The author declares no competing financial interest.

References

[1] Bear J, Bachmat Y. *Introduction to Modeling of Transport Phenomena in Porous Media*. Springer Science & Business Media; 2012.

[2] Greenkorn RA. *Flow Phenomena in Porous Media: Fundamentals and Applications in Petroleum, Water and Food Production*. CRC Press; 1983.

[3] Allaire A, Brizzi R, Mikelić A, Piatnitski A. Two-scale expansion with drift approach to the taylor dispersion for reactive transport through porous media. *Chemical Engineering Science*. 2010; 65(7): 2292-2300. Available from: https://doi.org/10.1016/j.ces.2009.09.010.

[4] Allaire G, Mikelić A, Piatnitski A. Homogenization approach to the dispersion theory for reactive transport through porous media. *SIAM Journal on Mathematical Analysis*. 2010; 42(1): 125-144. Available from: https://doi.org/10.1137/090754935.

[5] Van Duijn CJ, Pop IS. Crystal dissolution and precipitation in porous media: pore scale analysis. *Journal of Pure and Applied Mathematics*. 2004; 577: 171-211. Available from: https://doi.org/10.1515/crll.2004.2004.577.171.

[6] Hornung U. *Homogenization and Porous Media*. Springer Science & Business Media; 1997.

[7] Mauri R. Dispersion, convection, and reaction in porous media. *Physics of Fluids A: Fluid Dynamics*. 1991; 3(5): 743-756. Available from: https://doi.org/10.1063/1.858007.

[8] Allaire G, Hutridurga H. Homogenization of reactive flows in porous media and competition between bulk and surface diffusion. *IMA Journal of Applied Mathematics*. 2012; 77(6): 788-815. Available from: https://doi.org/10.1093/imamat/hxs049.

[9] Ramaiah HH. *Homogenization of complex flows in porous media and applications*. PhD Thesis. Palaiseau: Ecole Polytechnique; 2013.

[10] Sango M. Asymptotic behavior of a stochastic evolution problem in a varying domain. *Stochastic Analysis and Applications*. 2002; 20(6): 1331-1358.

[11] Sango M. Homogenization of stochastic semilinear parabolic equations with non-lipschitz forcings in domains with fine grained boundaries. *Communications in Mathematical Sciences*. 2013; 12(2): 345-382. Available from: https://doi.org/10.4310/CMS.2014.V12.N2.A7.

[12] Mohammed M, Sango M. Homogenization of linear hyperbolic stochastic partial differential equation with rapidly oscillating coefficients: The two scale convergence method. *Asymptotic Analysis*. 2015; 91(3-4): 341-371. Available from: https://doi.org/10.3233/ASY-141269.

[13] Mohammed M, Sango M. Homogenization of nonlinear hyperbolic stochastic partial differential equations with nonlinear damping and forcing. *Networks and Heterogeneous Media*. 2019; 14(2): 341-369. Available from: https://doi.org/10.3934/nhm.2019014.

[14] Mohammed M, Khan WA. Homogenization and correctors for stochastic hyperbolic equations in domains with periodically distributed holes. *Journal of Multiscale Modelling*. 2021; 12(3): 2150008. Available from: https://doi.org/10.1142/S1756973721500086.

[15] Mohammed M. Homogenization of a nonlinear stochastic model with nonlinear random forces for chemical reactive flows in porous media. *Discrete and Continuous Dynamical Systems-Series B*. 2023; 28(8): 1-25. Available from: https://doi.org/10.3934/dcdsb.2023032.

[16] Conca C, Díaz JI, Timofte C. Effective chemical processes in porous media. *Mathematical Models and Methods in Applied Sciences*. 2003; 13(10): 1437-1462. Available from: https://doi.org/10.1142/S0218202503002982.

[17] Eddahbi M, Mohammed M, El-Otmany H. Well-posedness of a nonlinear stochastic model for a chemical reaction in porous media and applications. *AIMS Mathematics*. 2024; 9(9): 24860-24886. Available from: https://doi.org/10.3934/math.20241211.

[18] Bessaih H, Mohammed M, Tayel IM. Homogenization and corrector results for a stochastic coupled thermoelastic model. *Discrete and Continuous Dynamical Systems*. 2025; 45(7): 2349-2380. Available from: https://doi.org/10.3934/dcds.2024168.

[19] Emereuwa C, Mohammed M. Homogenization of a stochastic model of a single phase flow in partially fissured media. *Asymptotic Analysis*. 2022; 129(3-4): 413-450. Available from: https://doi.org/10.3233/ASY-211735.

[20] Allaire G. Homogenization and two-scale convergence. *SIAM Journal on Mathematical Analysis*. 1992; 23(6): 1482-1518. Available from: https://doi.org/10.1137/0523084.

[21] Nguetseng G. A general convergence result for a functional related to the theory of homogenization. *SIAM Journal on Mathematical Analysis*. 1989; 20(3): 608-623. Available from: https://doi.org/10.1137/0520043.

[22] Marušić-Paloka E, Piatnitski AL. Homogenization of a nonlinear convection-diffusion equation with rapidly oscillating coefficients and strong convection. *Journal of the London Mathematical Society*. 2005; 72(2): 391-409. Available from: https://doi.org/10.1112/S0024610705006824.

[23] Allaire G. Periodic homogenization and effective mass theorems for the schrödinger equation. In: Abdallah NB, Frosali G. (eds.) *Quantum Transport. Lecture Notes in Mathematics*. Springer, Berlin, Heidelberg; 2008. p.1-44.

[24] Bourgeat A, Mikelic A, Wright S. Stochastic two-scale convergence in the mean and applications. *Journal of Pure and Applied Mathematics*. 1994; 456(1): 19-51. Available from: https://doi.org/10.1515/crll.1994.456.19.

[25] Sango M, Woukeng JL. Stochastic two-scale convergence of an integral functional. *Asymptotic Analysis*. 2011; 73(1-2): 97-123. Available from: https://doi.org/10.3233/ASY-2011-1038.

[26] Cioranescu D, Donato P. *An Introduction to Homogenization*. Oxford University Press; 1999.

[27] Bensoussan A. Some existence results for stochastic partial differential equations. *Pitman Research Notes in Mathematics Series*. 1992; 268: 37-53.

[28] Billingsley P. *Convergence of Probability Measures*. John Wiley & Sons; 2013.

[29] Allaire G, Hutridurga H. Homogenization of reactive flows in periodic porous media. In: *CFM 2011–20ème Congrès Français de Mécanique [CFM 2011–20th French Congress on Mechanics]*; Besançon, France. 2011.

[30] Allaire G, Murat F. Homogenization of the neumann problem with nonisolated holes. *Asymptotic Analysis*. 1993; 7(2): 81-95. Available from: https://doi.org/10.3233/ASY-1993-7201.

[31] Rozovskii BL. *Stochastic Evolution Systems: Linear Theory and Applications to Non-Linear Filtering*. Springer Science & Business Media; 2012.