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Research Article

The CN_s Tensor Product of Graphs: Structure, Spectra and Application

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Abstract: In this study, we present a product graph called CN_s tensor product based on the concept of CN_s vertices—those with identical neighborhoods. The structure of this product graph is closely tied to the CN sets—induced by equivalence among CN_s vertices, of the factor graphs, giving rise to multipartite components which are the complements of Rook's graphs and isolated vertices. Also, the spectral analysis reveals that the adjacency matrix of the product graph is the kronecker product of the CN_s matrices of the factor graphs, leading to an integral spectrum. Moreover, we examine the Laplacian and Signless Laplacian spectrum of the product graph. This framework also offers insight into biological applications such as homology modeling, where proteins are represented as graphs. The automorphism group of the CN_s tensor product reflects the symmetries inherited from its factor graphs, and can be written in terms of the symmetric groups corresponding to the sizes of CN sets with more than one element. We also analyze the independence number, matching number, and chromatic number of the product graph, showing that they are influenced by the structure of the multipartite components and the distribution of isolated vertices.

Keywords: CN set, CN_s matrix, CN_s graph, CN_s tensor product

MSC: 05C76, 05C50, 05C90, 05C25

1. Introduction

Spectral graph theory is a mathematical field that brings linear algebra and graph theory together. It focuses on studying graphs through the eigenvalues of associated matrices. Commonly studied graph matrices [1] are the adjacency matrix A, the Laplacian $L = D - A$, and the signless Laplacian $Q = D + A$, where D is a diagonal matrix of vertex degrees. Several additional matrices have been introduced, and their applications explored in various studies as referenced in [2–6]. These matrices encapsulate fundamental, structural and connectivity information about the graph and are extensively used in spectral analysis.

Over time, various graph metrics have been introduced to capture increasingly refined structural properties of graphs. One important such metric is the *graph energy* [7], defined as the sum of the absolute values of the eigenvalues of adjacency matrix of the graph. This measure offers insights into structural symmetry and overall stability of the graph.

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An important development in this area was introduced in [8], where the *common neighborhood matrix* and the corresponding *common neighborhood energy* were defined and analyzed for strongly regular and other graph classes. In that study, the authors also defined the *common neighborhood graph* (or *congraph*) of a graph G, denoted as $\text{con}(G)$, where two vertices are adjacent if and only if they share at least one common neighbor in G. This graph preserves the vertex set of G but modifies the adjacency based on neighborhood overlap.

Subsequent works, such as in [9], extended these ideas to various graph operations, including Cartesian products, compositions, corona products, and even splice and link graphs. Another significant study [10] highlighted the relevance of neighborhood structures in network reliability by connecting neighborhood-based parameters to stability measures such as toughness and neighbor-integrity.

In this paper, we focus on a particular class of graph called *CN_s graph* [11], wherein vertices are grouped according to identical neighborhoods. The vertices with same neighborhood are called CN_s vertices, which exhibit maximal structural symmetry, and is captured using the *CN_s matrix*—a binary $n \times n$ matrix where the entry at position (i, j) is 1 if the vertices v_i and v_j share identical neighborhoods, and 0 otherwise [11]. This matrix not only reflects local symmetries but also facilitates efficient encoding of neighborhood equivalence in complex networks.

Using this foundation, we propose a graph product, called the *CN_s tensor product*, denoted by $G_1 Ⓣ G_2$, inspired from the ordinary tensor product of the graphs G_1 and G_2. This operation focuses on combining identical neighborhood structures of two graphs G_1 and G_2, producing a graph whose adjacency matrix is the Kronecker product of the CN_s matrices of the factor graphs. This product reveals how identical neighbor relationships between vertices interact across graphs and allows a rich spectral characterization.

Graph product operations have served as a foundational tool for constructing complex network from simpler components. Graph products have been applied to parallel computing, molecular networks, and interconnection systems. Recent research has highlighted the potential of graph product methods in Network-on-Chip (NoC) design, emphasizing their ability to achieve scalability, modularity, and efficient communication layouts [12, 13] .

Also, matching [14] and independence [15] are two central concepts in graph theory, each providing key insights into the structure and function of complex networks. A matching in a graph is a set of pairwise non-adjacent edges, and a perfect matching is one in which every vertex is incident to exactly one edge from the matching. On the other hand, the independence number of a graph, denoted by $\alpha(G)$, represents the maximum number of vertices that are mutually non-adjacent.

Our main contributions are summarized as follows:

• Formal introduction of the CN_s tensor product.

• Spectral analysis of the resulting product graphs, showing that they are integral.

• Evaluation of graph energy to quantify structural insights of the CN_s tensor product.

• Biological application in homology modeling of proteins, where conserved motifs are captured by CN_s sets, and the CN_s tensor product helps to identify functional similarities.

• Investigation of classical graph invariants for the CN_s tensor product, including chromatic number, independence number, matching number.

• The automorphism group of the CN_s tensor product.

Our work complements prior studies such as in [16], which surveys the spectral properties of classical graph products. However, the CN_s tensor product introduces a structurally enriched product model tailored to neighborhood equivalence, which offers novel insights into the topological and spectral behavior of graphs.

Overall, this study contributes a new lens through which structural symmetry, neighborhood-based partitioning, and graph product operations can be analyzed in both mathematical and applied settings, particularly those involving complex networks and biological systems.

2. Preliminaries

A vertex u is a neighbor [17] of a vertex v in a graph G if there is an edge connecting u and v. The vertices having identical neighbors play a crucial role in identifying common structures within a graph, allowing us to analyze connectivity patterns and graph behaviors. These vertices can be grouped together to form subgraphs that exhibit similar properties, facilitating the study of larger graph characteristics, such as clustering and community detection. Moreover, the presence of such vertices can influence the overall graph energy, as they contribute to the symmetry and stability of the structure of the graph.

Before delving further, let's clarify some basic terminologies necessary for the following discussions:

Definition 1 [17] Let G be a graph with vertex set V and edge set E. The neighborhood, $N(v)$ of a vertex $v \in V$ is the set of all neighbors of v. i.e., $N(v) = \{u: \{u, v\} \in E\}$.

Definition 2 [18] If $\lambda_1 < \lambda_2 < \cdots < \lambda_k$ are distinct eigenvalues of a graph G with respective multiplicities $p_1, p_2, \ldots, p_k$, then the spectrum of G, $S(G) = \{\lambda_1^{p_1}, \lambda_2^{p_2}, \ldots, \lambda_k^{p_k}\}$.

Definition 3 [19] The nullity $\eta(G)$ of a graph G is the multiplicity of the eigenvalue zero in $S(G)$.

Definition 4 [20] If a graph has nonzero nullity, then it is called a singular graph otherwise it is called nonsingular graph.

Definition 5 [11] The set of all eigenvalues of the CN_s matrix of a graph G is called CN_s spectrum of G.

Remark 1 The relation of identical neighborhoods partitions the vertex set of a graph into disjoint subsets, each referred to as a CN set.

Definition 6 [11] Let $G = (V, E)$ be a graph and $v \in V$ be a vertex of G. Then CN set corresponding to v is denoted by $CN_G(v)$ or simply $CN(v)$ and is defined as

$$CN_G(v) = \{u \in V: N(v) = N(u)\}.$$

Definition 7 [11] A vertex whose CN set is a singleton is called DN vertex. A CN set containing two elements is referred to as a Binary CN set.

Definition 8 [21] Let A be an $m \times n$ matrix and B be a $p \times q$ matrix. The Kronecker product, denoted by $A \otimes B$, is an $(mp) \times (nq)$ matrix, defined as;

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \ldots & a_{1n}B \\ a_{21}B & a_{22}B & \ldots & a_{2n}B \\ & \ldots\ldots\ldots\ldots & & \\ a_{m1}B & a_{m2}B & \ldots & a_{mn}B \end{bmatrix}$$

where each entry a_{ij} of A is multiplied by the entire matrix B.

Theorem 1 [21] The spectrum of the Kronecker product of two matrices A and B, consists of the pairwise product of the eigenvalues of A and B.

Definition 9 [22] A bipartite graph G is one whose vertex set can be partitioned into two subsets X and Y, such that each edge has one end in X and one end in Y. Such a partition (X, Y) is called the bipartition of the graph G, and the sets X and Y are called the partite sets of the graph. A bipartite graph with two partite sets of sizes r_1 and r_2 is represented by B_{r_1, r_2}.

Multipartite graphs are a generalization of bipartite graphs. In an n-partite graph, the vertex set is divided into n disjoint sets, and no two vertices within the same set are adjacent. The multipartite graph with partite sets of sizes $r_1, r_2, \ldots, r_m$ is denoted as $B_{r_1, r_2, \ldots, r_m}$.

A regular graph is one in which every vertex has the same number of edges incident on it. Hence, every vertices of a regular graph has the same degree, and a k-regular graph means every vertex has degree k. A regular bipartite graph is a bipartite graph that is also regular.

Definition 10 [17] Let G be a graph. Two vertices u and v of G are said to be connected if there is a u-v path in G. The relation connected is an equivalence relation on the vertex set V. Let $C_1, C_2, \ldots, C_k$ be the equivalence classes. The sub-graphs generated by these equivalence classes are called the components of G.

Theorem 2 [11] If the graph G has d number of DN vertices and k number of CN sets containing more than one element, with sizes $r_1, r_2, \ldots, r_k$ (where each $r_i \neq 1$, $i = 1, 2, \ldots, k$), then the CN_s spectrum of G consists of the following eigenvalues:

$$0^d, (-1)^{n-d-k}, (r_1 - 1)^1, (r_2 - 1)^1, \ldots, (r_k - 1)^1. \tag{1}$$

3. CN_s tensor product of two graphs

Now, we can define the CN_s tensor product of two graphs using the CN_s vertices of the factor graphs. This operation takes two graphs and systematically integrates the CN_s vertices of both graphs to form a combined structure. This process highlights the shared neighborhood patterns between two graphs, producing a new graph whose adjacency structure is closely related to the CN_s relationships between the vertices of G_1 and G_2. Hence, this product can serve as a powerful tool for studying complex networks, offering a new perspective on how CN_s vertices influence graph dynamics and structure.

Definition 11 Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. Then, CN_s tensor product of G_1 and G_2 is denoted by $G_1 \textcircled{T} G_2$, and is the graph with vertex set $V_1 \times V_2$. Two vertices (u_1, v_1) and (u_2, v_2) in $V_1 \times V_2$ are adjacent in $G_1 \textcircled{T} G_2$, if and only if u_1 and u_2 are CN_s vertices in G_1, and v_1 and v_2 are CN_s vertices in G_2.

Theorem 3 Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two CN_s graphs. Then every edge of the CN_s tensor product is an edge of the complement of the standard tensor product.

Proof. Let (u, v) and (u', v') be two distinct vertices of $V_1 \times V_2$. Suppose $((u, v), (u', v')) \in E(G_1 \textcircled{T} G_2)$. By the definition of the CN_s tensor product,

1. u and u' belong to the same CN set of G_1, and
2. v and v' belong to the same CN set of G_2, with $u \neq u'$, $v \neq v'$.

But the vertices in the same CN set are pairwise non-adjacent, and hence $uu' \notin E_1$ and $vv' \notin E_2$. Therefore $((u, v), (u', v'))$ is an edge of the complement graph $(G_1 \times G_2)^c$.

$$E\left(G_1 \textcircled{T} G_2\right) \subseteq E\left((G_1 \times G_2)^c\right). \tag{2}$$

□

Remark 2 Figure 1 illustrates the CN_s tensor product of S_5 and C_4, while Figure 2 shows their standard tensor product. It is evident from the figures that one forms a subset of the complement of the other.

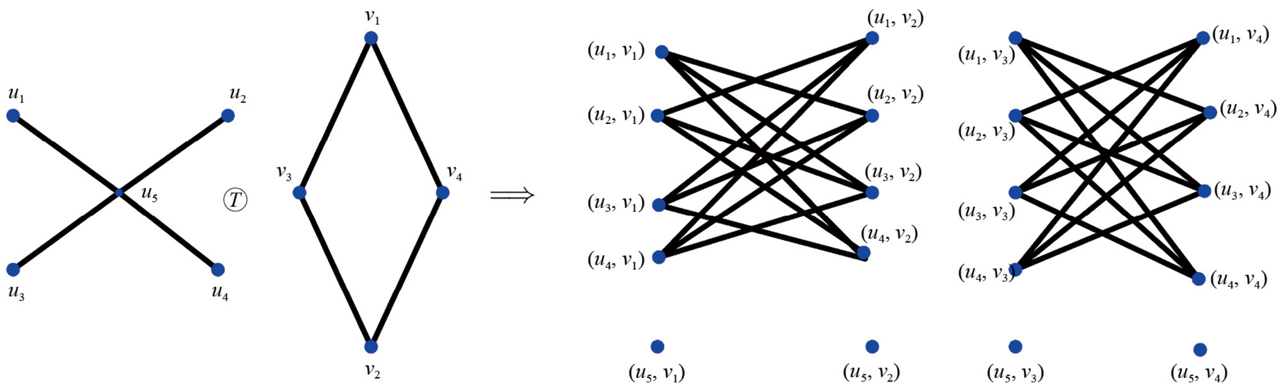

Figure 1. Illustration of the CN_s tensor product of graphs

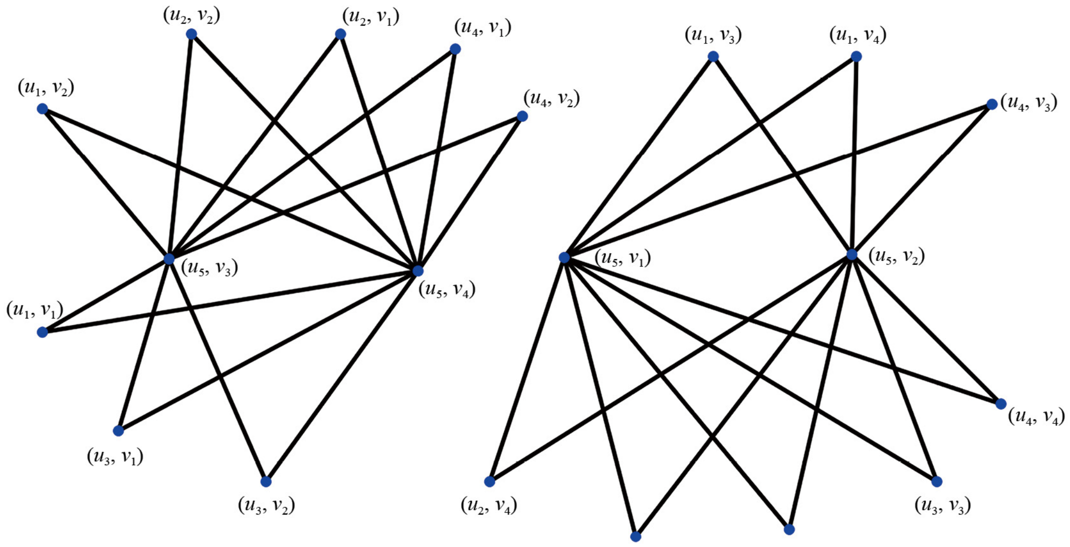

Figure 2. Standard tensor product of S_5 and C_4

3.1 *Properties of CN_s tensor product*

The CN_s tensor product of two graphs is a graph operation that provides a unique way to analyse and understand graph structures based on CN_s vertices. In this section, we will examine some fundamental properties of CN_s tensor product as follows.

- If $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$ are two graphs of order n_1, n_2, respectively, then order of $G_1 Ⓣ G_2$ is $n_1 n_2$.
- If G_1, G_2 are two graphs, then $G_1 Ⓣ G_2 \simeq G_2 Ⓣ G_1$.
- If G_1, G_2, G_3 are any three graphs, then $(G_1 Ⓣ G_2) Ⓣ G_3 = G_1 Ⓣ (G_2 Ⓣ G_3)$.
- If either G_1 or G_2 are DN_s graph with n_1, n_2 vertices, respectively, then $G_1 Ⓣ G_2$ is a null graph with $n_1 n_2$ vertices.
- If G_1 is a complete graph, it contains no CN_s vertices and hence $G_1 Ⓣ G_2$ is a null graph, regardless of the structure of G_2.

4. Structural properties of the CN_s tensor product

Let G_1 and G_2 be two CN_s graphs. For each pair of CN sets with more than one element in G_1 and G_2, the CN_s tensor product will always have regular components. In this section, we determined the order and size of the CN_s tensor product, along with the number of isolated vertices and the structure of its connected components. These results provide a clear understanding of how the CN set partitioning of the factor graphs influences the overall connectivity and regularity of the CN_s product graph.

Proposition 1 If (u, v) is a vertex of $G_1 \textcircled{T} G_2$, then neighbors of (u, v) in $G_1 \textcircled{T} G_2$ is

$$N_{G_1 \textcircled{T} G_2}((u, v)) = (CN_{G_1}(u) - \{u\}) \times (CN_{G_2}(v) - \{v\}). \tag{3}$$

Proof. By the definition of $G_1 \textcircled{T} G_2$, two vertices (u_1, v_1) and (u_2, v_2) are adjacent if and only if $u_1 \neq u_2$ and both belong to same CN set in G_1, and $v_1 \neq v_2$ and both belong to same CN set in G_2.
Hence, $N_{G_1 \textcircled{T} G_2}((u, v)) = (CN_{G_1}(u) - \{u\}) \times (CN_{G_2}(v) - \{v\})$. □

Corollary 1 If G_1 and G_2 are two CN_s graphs, then $G_1 \textcircled{T} G_2$ is a DN_s graph.

Proof. Let (u, v) be an arbitrary vertex in $G_1 \textcircled{T} G_2$.

Then $N_{G_1 \textcircled{T} G_2}((u, v)) = (CN_{G_1}(u) - \{u\}) \times (CN_{G_2}(v) - \{v\})$ and no other vertices in $G_1 \textcircled{T} G_2$ have this set as neighborhood.

Hence $G_1 \textcircled{T} G_2$ is a DN_s graph. □

Theorem 4 If G_1 is a CN_s graph with a binary CN set and G_2 has CN sets of sizes $r_1, r_2, \ldots, r_k$, where $r_i \geq 2$: $i = 1, 2, \ldots, k$, then $G_1 \textcircled{T} G_2$ has k bipartite components with $2r_i$: $i = 1, 2, \ldots, k$ elements.

Proof. Let the binary CN set of G_1 be $\{u_1, u_2\}$. Let $C = \{v_1, v_2, \ldots, v_p\}$: $p > 1$, be one of the CN sets of G_2. Then for each $i = 1, 2, \ldots, p$, the vertices (u_1, v_i) of $G_1 \textcircled{T} G_2$ are adjacent to (u_2, v_j): $j \neq i$, $j = 1, 2, \ldots, p$. By definition of CN_s tensor product, no other vertices are adjacent with these vertices. Hence, corresponding to $\{u_1, u_2\} \times \{v_1, v_2, \ldots, v_p\}$ we get a bipartite component in $G_1 \textcircled{T} G_2$ with $2p$ elements. There are k number of CN sets for G_2 so that there will be k number of bipartite components. □

Corollary 2 If a graph G_1 has k_1 number of binary CN sets and G_2 has k_2 number of binary CN sets, then the graph resulting from their CN_s tensor product $G_1 \textcircled{T} G_2$ has $k_1 k_2$ bipartite components.

Lemma 1 If G_1 has a CN set of r_1 elements and G_2 has a CN set of r_2 elements with $r_1, r_2 \geq 2$ then $G_1 \textcircled{T} G_2$ has a multipartite regular component $B_{r_1, r_1, \ldots, r_1 (r_2 \, times)}$ with each vertex has degree $(r_1 - 1)(r_2 - 1)$.

Proof. Let $\{u_1, u_2, \ldots, u_{r_1}\}$ be a CN set of G_1 and $\{v_1, v_2, \ldots, v_{r_2}\}$ be a CN set of G_2, where $r_1, r_2 \geq 2$. Then, the vertices (u_{k_1}, v_{k_2}): $1 \leq k_1 \leq r_1, 1 \leq k_2 \leq r_2$ are adjacent to the vertices (u_{p_1}, v_{p_2}); $1 \leq p_1 \leq r_1, p_1 \neq k_1, 1 \leq p_2 \leq r_2, p_2 \neq k_2$ in $G_1 \textcircled{T} G_2$. Therefore, the vertices have degree $(r_1 - 1)(r_2 - 1)$. Hence, the corresponding multipartite component with $r_1 r_2$ vertices are in r_2 partitions of r_1 elements or equivalently in r_1 partitions of r_2 elements. □

Corollary 3 Number of edges in $G_1 \textcircled{T} G_2$ is

$$\frac{1}{2} \sum_{\mathscr{C}_j \in \mathscr{C}(G_2)} \sum_{\mathscr{C}_i \in \mathscr{C}(G_1)} (|\mathscr{C}_i|^2 - |\mathscr{C}_i|) \times (|\mathscr{C}_j|^2 - |\mathscr{C}_j|), \tag{4}$$

where $\mathscr{C}(G_1)$, $\mathscr{C}(G_2)$ are the collection of all CN sets of G_1 and G_2, respectively, with more than one element and $|\mathscr{C}_i|$ is the number of vertices in $\mathscr{C}_i$.

Proof. We have the CN sets $\mathscr{C}_1$ and $\mathscr{C}_2$ of G_1 and G_2, respectively, determines regular components with $|\mathscr{C}_1| \times |\mathscr{C}_2|$ vertices.

By Lemma 1, each vertices have degree $(|\mathscr{C}_1|-1)(|\mathscr{C}_2|-1)$.

Hence, the number of edges is

$$\frac{1}{2}\sum_{\mathscr{C}_j\in\mathscr{C}(G_2)}\sum_{C_i\in\mathscr{C}(G_1)}|\mathscr{C}_i|\times|\mathscr{C}_j|\times(|\mathscr{C}_i|-1)\times(|\mathscr{C}_j|-1)=\frac{1}{2}\sum_{\mathscr{C}_j\in\mathscr{C}(G_2)}\sum_{\mathscr{C}_i\in\mathscr{C}(G_1)}(|\mathscr{C}_i|^2-|\mathscr{C}_i|)\times(|\mathscr{C}_j|^2-|\mathscr{C}_j|). \quad (5)$$

□

Theorem 5 If G_1 is a CN_s graph with n_1 vertices, of which d_1 are DN vertices, and G_2 is a CN_s graph with n_2 vertices, of which d_2 are DN vertices, then $G_1 Ⓣ G_2$ has $d_1n_2+d_2n_1-d_1d_2$ isolated vertices.

Proof. For each vertex $u\in V(G_1)$ and each DN vertex v in G_2, the pair (u, v) is isolated in $G_1 Ⓣ G_2$. Similarly, for each $v\in V(G_2)$ and DN vertex u in G_1, the pair (u, v) is also isolated in $G_1 Ⓣ G_2$. However, when both u and v are DN vertices in G_1 and G_2, respectively, the vertex (u, v) is counted twice in the two previous cases. To correct this double counting, we subtract the number of such overlapping pairs. Therefore, the total number of isolated vertices in $G_1 Ⓣ G_2$ is $d_1n_2+d_2n_1-d_1d_2$. □

Corollary 4 Let G_1 and G_2 be two CN_s graphs with n_1 and n_2 vertices, respectively. Suppose G_1 has d_1 number of DN vertices and k_1 number of CN sets with more than one element, and G_2 has d_2 number of DN vertices and k_2 number of CN sets with more than one element. Then number of components in the CN_s tensor product $G_1 Ⓣ G_2$ is $k_1k_2+n_1d_2+n_2d_1-d_1d_2$.

Proof. Let $\mathscr{C}_1, \mathscr{C}_2, \ldots, \mathscr{C}_{k_1}$ be the CN sets with more than one element in G_1, and let $\mathscr{C}'_1, \mathscr{C}'_2, \ldots, \mathscr{C}'_{k_2}$ be the CN sets in G_2. Then, by Lemma 1, for each pair $(\mathscr{C}_i, \mathscr{C}'_j)$ with $1\le i\le k_1$ and $1\le j\le k_2$, the Cartesian product $\mathscr{C}_i\times\mathscr{C}'_j$ forms a multipartite component in the product graph $G_1 Ⓣ G_2$. The total number of such multipartite components is k_1k_2. According to Theorem 5, the number of isolated vertices in $G_1 Ⓣ G_2$ is $d_1n_2+d_2n_1-d_1d_2$. Therefore, the total number of components in the CN_s tensor product $G_1 Ⓣ G_2$ is $k_1k_2+d_1n_2+d_2n_1-d_1d_2$. □

4.1 *Rook's graph and CN_s tensor product*

In graph theory, the Rook's graph, denoted $R(m, n)$, is a well known combinatorial structure that can be defined as the line graph of the complete bipartite graph $K_{m,n}$. Equivalently, it can be interpreted as the graph whose vertices correspond to the squares of an $m\times n$ chessboard, where two squares are adjacent if a rook can legally move between them in one step, i.e., they lie in the same row or the same column. The CN_s tensor product produces complex graph components, with highly regular multipartite structures which can be identified as the complement of the Rook's graph.

Lemma 2 If G_1 contains a CN set $\mathscr{C}_1$ of size r_1 and G_2 contains a CN set $\mathscr{C}_2$ of size r_2, where $r_1, r_2\ge 2$ then the component induced by $\mathscr{C}_1\times\mathscr{C}_2$ in $G_1 Ⓣ G_2$ is isomorphic to the complement of Rook's Graph with r_1r_2 elements.

Proof. Let $\mathscr{C}_1=\{u_1, u_2, \ldots, u_{r_1}\}$ and $C_2=\{v_1, v_2, \ldots, v_{r_2}\}$ be CN sets of G_1 and G_2, respectively, with $r_1, r_2\ge 2$. In the CN_s tensor product graph $G_1 Ⓣ G_2$, the subgraph induced by $\mathscr{C}_1\times\mathscr{C}_2$ has r_1r_2 vertices, and two vertices (u_i, v_j) and (u_k, v_l) are adjacent if and only if $i\ne k$, $j\ne l$ and u_i, u_k and v_j, v_l are CN_s vertices in G_1 and G_2, respectively. Such pair of vertices corresponds to non-adjacent vertices in Rook's graph $R(r_1, r_2)$ and therefore adjacent in its complement. Hence, the subgraph induced by $\mathscr{C}_1\times\mathscr{C}_2$ in $G_1 Ⓣ G_2$ is isomorphic to the complement of Rook's Graph $\overline{R(r_1, r_2)}$. □

Corollary 5 If G_1 has a CN set $\mathscr{C}_1$ with r_1 elements and G_2 has a CN set $\mathscr{C}_2$ with r_2 elements and $r_1, r_2\ge 2$, then the component induced by $\mathscr{C}_1\times\mathscr{C}_2$ in $G_1 Ⓣ G_2$ has $2\binom{r_1}{2}\binom{r_2}{2}$ edges.

Proof. Let $\mathscr{C}_1$ and $\mathscr{C}_2$ be arbitrary CN sets of G_1 and G_2, respectively, where, $\mathscr{C}_1=\{u_1, u_2, \ldots, u_{r_1}\}$ and $\mathscr{C}_2=\{v_1, v_2, \ldots, v_{r_2}\}$ with $r_1, r_2\ge 2$. By Lemma 2, total number of vertices in the subgraph induced by $\mathscr{C}_1\times\mathscr{C}_2$ in $G_1 Ⓣ G_2$ is r_1r_2. Also, for each u_i, there are $\binom{r_2}{2}$ vertices are non-adjacent to vertices with u_i as first component. Similarly, for each v_j, there are $\binom{r_1}{2}$ vertices are non-adjacent to vertices with v_j as second component. Hence, there are $r_1\binom{r_2}{2}+r_2\binom{r_1}{2}$ non-adjacent vertices. Thus,

$$\text{Number of edges} = \binom{r_1 r_2}{2} - \left[r_1\binom{r_2}{2}+r_2\binom{r_1}{2}\right]$$

$$= \frac{r_1^2 r_2^2 - r_1 r_2 - r_1 r_2(r_2-1) - r_2 r_1(r_1-1)}{2}$$

$$= \frac{r_1^2 r_2^2 - r_1 r_2 - r_1 r_2^2 + r_1 r_2 - r_1^2 r_2 + r_1 r_2}{2}$$

$$= \frac{r_1 r_2(r_1 r_2 - r_2 - r_1 + 1)}{2}$$

$$= \frac{r_1 r_2(r_1-1)(r_2-1)}{2}$$

$$= 2\binom{r_1}{2}\binom{r_2}{2}.$$

□

5. Main results in CN_s tensor product

Now, we can derive the adjacency, Laplacian and signless Laplacian spectrum of CN_s Tensor product of graphs. These spectra play a crucial role in understanding the structural and spectral characteristics of the resulting product graph. In particular, the adjacency spectrum reveals regularity patterns, while the Laplacian and signless Laplacian spectra provide insights into the flow, clustering, and balance of the network.

5.1 *Adjacency spectrum of the CN_s tensor product*

In this section, we evaluated the adjacency matrix of CN_s tensor product and hence calculated the adjacency spectrum and identified that the spectrum of the graph obtained by CN_s tensor product of two graphs contain only integer values. Hence, CN_s tensor product graph is always an integral graph.

Theorem 6 If G_1 and G_2 are two CN_s graphs then the adjacency matrix of $G_1 Ⓣ G_2$ is given by $CN_s(G_1)\otimes CN_s(G_2)$, the Kronecker product of $CN_s(G_1)$ and $CN_s(G_2)$.

Proof. Let $\{u_1, u_2, \ldots, u_{n_1}\}$ be the vertices of G_1 and $\{v_1, v_2, \ldots, v_{n_2}\}$ be the vertices of G_2. The adjacency matrix of $G_1 Ⓣ G_2$ can be constructed using block matrices as shown in Table 1.

By definition of CN_s tensor product, each block on the diagonal is a zero matrix, while the non-diagonal blocks are nonzero in the position where 1 occurs in the matrix $CN_s(G_1)$. The entries in those blocks are determined by the CN_s vertices of G_2. Hence, the adjacency matrix of $G_1 Ⓣ G_2$ is given by $CN_s(G_1)\otimes CN_s(G_2)$. □

Table 1. Block structure of $A(G_1 \textcircled{T} G_2)$

	$(u_1, v_1), ..., (u_1, v_{n_2})$	$(u_2, v_1), ..., (u_2, v_{n_2})$	$\ldots\ldots$	$(u_{n_1}, v_1), \ldots, (u_{n_1}, v_{n_2})$
(u_1, v_1) $\vdots$ (u_1, v_{n_2})	B_{11}	B_{12}	$\ldots$	B_{1n_2}
(u_2, v_1) $\vdots$ (u_2, v_{n_2})	B_{21}	B_{22}	$\ldots$	B_{2n_2}
$\vdots$ $\vdots$	$\vdots$ $\vdots$	$\vdots$ $\vdots$	$\vdots$ $\vdots$	$\vdots$ $\vdots$
(u_{n_1}, v_1) $\vdots$ (u_{n_1}, v_{n_2})	$B_{n_1 1}$	$B_{n_1 2}$	$\ldots$	$B_{n_1 n_2}$

Corollary 6 The eigenvalues of $G_1 \textcircled{T} G_2$ are the pairwise product of the eigenvalues of $CN_s(G_1)$ and $CN_s(G_2)$.

Proof. By Theorem 6, the adjacency matrix of $G_1 \textcircled{T} G_2$ is the Kronecker product of $CN_s(G_1)$ and $CN_s(G_2)$. Hence, the eigenvalues of $G_1 \textcircled{T} G_2$ is the pairwise product of eigenvalues of $CN_s(G_1)$ and $CN_s(G_2)$. $\square$

Corollary 7 Let G_1 and G_2 be two CN_s graphs having n_1 and n_2 number of vertices, respectively, of which d_1 and d_2 number of vertices are DN vertices. Also, G_1 has p_1 number of CN sets with more than one elements of order $r_1, r_2, \ldots, r_{p_1}$, and G_2 has p_2 number of CN sets with more than one elements of order $t_1, t_2, \ldots, t_{p_2}$ then

(i) the nullity of $G_1 \textcircled{T} G_2$ is $d_1 n_2 + d_2 n_1 - d_1 d_2$.

(ii) the distinct eigenvalues of $G_1 \textcircled{T} G_2$ are

$$0,\ 1,\ -(r_1-1),\ -(r_2-1),\ \ldots,\ -(r_{p_1}-1),\ -(t_1-1),\ -(t_2-1),\ \ldots,\ -(t_{p_2}-1)$$

$$(r_1-1)(t_1-1),\ (r_1-1)(t_2-1),\ \ldots,\ (r_1-1)(t_{p_2}-1)$$

$$(r_2-1)(t_1-1),\ (r_2-1)(t_2-1),\ \ldots,\ (r_2-1)(t_{p_2}-1)$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$(r_1-1)(t_{p_2}-1),\ (r_2-1)(t_{p_2}-1)\ldots,\ (r_{p_1}-1)(t_{p_2}-1).$$

Proof. By Theorem 2, the eigenvalues of $CN_s(G_1)$ are 0^{d_1}, $(-1)^{n_1-d_1-p_1}$, $r_1-1, r_2-1, \ldots, r_{p_1}-1$.

And the eigenvalues of $CN_s(G_2)$ are 0^{d_2}, $(-1)^{n_2-d_2-p_2}$, $t_1-1, t_2-1, \ldots, t_{p_2}-1$.

By Corollary 6, the eigenvalues of $G_1 \textcircled{T} G_2$ are the pairwise product of the eigenvalues of $CN_s(G_1)$ and $CN_s(G_2)$, so that the spectrum of $G_1 \textcircled{T} G_2$ is

$$\{0^{d_1n_2+d_2n_1-d_1d_2}, 1^{(n_1-d_1-p_1)(n_2-d_2-p_2)}, (1-r_1)^{n_2-d_2-p_2}, (1-r_2)^{n_2-d_2-p_2}, \ldots, (1-r_{p_1})^{n_2-d_2-p_2},$$

$$(1-t_1)^{n_1-d_1-p_1}, (1-t_2)^{n_1-d_1-p_1}, \ldots, (1-t_{p_2})^{n_1-d_1-p_1}, (r_1-1)(t_1-1), (r_1-1)(t_2-1), \ldots,$$

$$(r_1-1)(t_{p_2}-1), (r_2-1)(t_1-1), (r_2-1)(t_2-1), \ldots, (r_2-1)(t_{p_2}-1), (r_{p_1}-1)(t_1-1),$$

$$(r_{p_1}-1)(t_2-1), \ldots, (r_{p_1}-1)(t_{p_2}-1)\}.$$

□

Corollary 8 If G_1 and G_2 have no DN vertices, then $G_1 Ⓣ G_2$ is nonsingular.

Proof. The eigenvalue 0 of $G_1 Ⓣ G_2$, does not exist for a graph with no DN vertices.
Hence $G_1 Ⓣ G_2$ is nonsingular. □

Corollary 9 The CN_s tensor product graph of two CN_s graphs is always an integral graph.

Proof. Let G_1 and G_2 be two CN_s graph. By Corollary 7, the eigenvalues for $G_1 Ⓣ G_2$ are all integers. So $G_1 Ⓣ G_2$ is an integral graph. □

Theorem 7 Let G_1 and G_2 be two CN_s graphs of order n_1 and n_2, respectively. Suppose that d_1 and d_2 number of vertices in G_1 and G_2 are DN vertices, and that G_1 and G_2 have p_1 and p_2 number of CN sets, respectively, each with more than one element. Then energy of $G_1 Ⓣ G_2$ is $4(n_1-d_1-p_1)(n_2-d_2-p_2)$.

Proof. Let G_1 has CN sets of order $r_1, r_2, \ldots, r_{p_1}$, and G_2 has CN sets of order $t_1, t_2, \ldots, t_{p_2}$, where each $r_i, t_j \geq 2$: $1 \leq i \leq p_1$, $1 \leq j \leq p_2$.
Then, $n_1 = d_1 + r_1 + r_2 + \cdots + r_{p_1}$ and $n_2 = d_2 + t_1 + t_2 + \cdots + t_{p_2}$.
As we know energy of a graph is sum of all its eigenvalues and by Corollary 7,

$$\begin{aligned} Energy\ of\ G_1 Ⓣ G_2 = &(n_1-d_1-p_1)(n_2-d_2-p_2) \\ &+(n_2-d_2-p_2)\big(r_1+r_2+\cdots+r_{p_1}-p_1\big) \\ &+(n_1-d_1-p_1)\big(t_1+t_2+\cdots+t_{p_2}-p_2\big) \\ &+\big(r_1+r_2+\cdots+r_{p_1}-p_1\big)\big(t_1+t_2+\cdots+t_{p_2}-p_2\big) \\ =& 4(n_1-d_1-p_1)(n_2-d_2-p_2) \end{aligned}$$

. □

Corollary 10 Let G_1 and G_2 be two CN_s graphs. Then energy of $G_1 Ⓣ G_2$ is always a multiple of 4.

Example: From Figure 1, $G_1 = S_5$ has two CN sets $\{u_1, u_2, u_3, u_4\}$, $\{u_5\}$ and $G_2 = C_4$ has two CN sets $\{v_1, v_2\}, \{v_3, v_4\}$. Also $d_1 = 1, p_1 = 1, p_2 = 2, n_1 = 5, n_2 = 4$ so that the spectrum of $G_1 Ⓣ G_2$ is $\{0^4, 1^6, (-3)^2, (-1)^6, 3^2\}$ and energy of $G_1 Ⓣ G_2$ is 24.

5.2 *Laplacian spectrum of the CN_s tensor product*

The Laplacian matrix [23] captures information about graph connectivity and is defined as $L(G) = D(G) - A(G)$, where $D(G)$ is the degree matrix and $A(G)$ is the adjacency matrix of the graph G. The Laplacian eigenvalues are non-negative real numbers that provide insights into many graph properties such as connectivity, clustering, and synchronization dynamics. In the context of the CN_s tensor product, analyzing the Laplacian spectrum reveals how local neighborhood symmetries combine to form global structural patterns.

In this section, we explore how the Laplacian eigenvalues of the CN_s tensor product are influenced by the structures of its component graphs. These eigenvalues reflect the interaction patterns between CN_s vertices and DN vertices in the factor graphs, revealing how graph symmetries and connectivity propagate through the CN_s tensor product structure.

Lemma 3 [7] If G and H are graphs with Laplacian eigenvalues $\{\lambda_i\}$ and $\{\mu_j\}$, respectively, then the Laplacian eigenvalues of their Cartesian product $G \square H$ are given by:

$$\{\lambda_i + \mu_j \mid \lambda_i \in \mathrm{Spec}(L(G)),\ \mu_j \in \mathrm{Spec}(L(H))\}. \tag{6}$$

Lemma 4 [24] Let A be a real symmetric matrix. Then any two eigenvectors v_1 and v_2 corresponding to distinct eigenvalues $\lambda_1 \neq \lambda_2$ are orthogonal.

Theorem 8 Let G_1 and G_2 be two CN_s graph with n_1 and n_2 number of vertices, respectively. Suppose that G_1 has d_1 number of DN vertices and p_1 number of CN sets with sizes $r_1, r_2, \ldots, r_{p_1}$, where each $r_i > 1$: $i = 1, 2, 3, \ldots, p_1$. And, let G_2 contains d_2 number of DN vertices and p_2 number of CN sets with sizes $s_1, s_2, \ldots, s_{p_2}$, where each $s_i > 1$: $i = 1, 2, 3, \ldots, p_2$. Then $G_1 Ⓣ G_2$ has an integral Laplacian spectrum.

Proof. The Cartesian product of DN vertices of G_1 and G_2, as well as the product of a DN vertex of G_1 with a CN set having more than one element of G_2 and the product of a CN set having more than one element of G_1 with a DN vertex of G_2, each forms an edgeless component in $G_1 Ⓣ G_2$. As these components are edgeless, their Laplacian spectra consist solely of the eigenvalue 0.

Now, consider the components corresponding to the Cartesian product of CN sets $\mathscr{C}_i$: $1 \leq i \leq p_1$ of G_1 with the CN sets $\mathscr{C}'_j$: $1 \leq j \leq p_2$ of G_2, where each CN set has more than one element. By Lemma 2, each such product $\mathscr{C}_i \times \mathscr{C}'_j$ induces a subgraph in $G_1 Ⓣ G_2$ that is isomorphic to the complement of the Rook's graph $R(r_i, s_j)$, where $r_i = |\mathscr{C}_i|$ and $s_j = |\mathscr{C}'_j|$.

Clearly 0 is in the Laplacian spectrum of $\overline{R(r_i, s_j)}$.

Hence, $L(\overline{R(r_i, s_j)})x = 0 \Rightarrow D(\overline{R(r_i, s_j)})x = A(\overline{R(r_i, s_j)}))x$ which is possible only if $x = 1$, the all one vector. Hence, 0 is in the Laplacian spectrum of $\overline{R(r_i, s_j)})$ with multiplicity 1 and the corresponding eigenvector is the all one vector.

Now,

$$\begin{aligned} L(\overline{R(r_i, s_j)}) &= D(\overline{R(r_i, s_j)}) - A(\overline{R(r_i, s_j)}) \\ &= D(K_{r_is_j}) - D(R(r_i, s_j)) - \big(A(K_{r_is_j}) - A(R(r_i, s_j))\big) \\ &= (r_is_j - 1)I_{r_is_j} - D(R(r_i, s_j)) - \big(J_{r_is_j} - I_{r_is_j}\big) + A(R(r_i, s_j)\big) \\ &= r_is_jI_{r_is_j} - J_{r_is_j} - L(R(r_i, s_j)) \end{aligned}$$

If $\lambda \neq 0$ is an eigenvalue of $L(R(r_i, s_j))$ with eigenvector x', then by Lemma 4, x' is orthogonal to x. Since $J = xx^T$ we get, $J_{r_is_j}x' = (xx^T)x' = x(x^Tx') = 0$.

Then,

$$(L(\overline{R(r_i, s_j)}) - (r_is_j - \lambda)I_{r_is_j})x' = (r_is_jI_{r_is_j} - J_{r_is_j} - L(R(r_i, s_j)) - (r_is_j - \lambda)I_{r_is_j})x' = (r_is_j - \lambda)x'$$

Therefore, if $\lambda \neq 0$ is an eigenvalue of $L(R(r_i, s_j))$, then $r_is_j - \lambda$ is an eigenvalue of $L(\overline{R(r_i, s_j)})$. Next, we can find the nonzero Laplacian eigenvalues of $R(r_i, s_j)$

The Rook's graph $R(r_i, s_j)$ is defined as the Cartesian product of two complete graphs:

$$R(r_i, s_j) = K_{r_i} \square K_{s_j}.$$

The Laplacian spectrum of a complete graph K_p is well known to be

$$\{0^{(1)},\ p^{(p-1)}\}.$$

Applying Lemma 3, which describes the Laplacian spectrum of Cartesian products of graphs, the Laplacian eigenvalues of the Rook's graph $R(r_i, s_j)$ are given by all possible sums of the eigenvalues of K_{r_i} and K_{s_j}. Hence, the Laplacian spectrum of $R(r_i, s_j)$ is:

- $0+0=0$, with multiplicity 1;
- $0+s_j=s_j$, with multiplicity r_i-1;
- $r_i+0=r_i$, with multiplicity s_j-1;
- r_i+s_j, with multiplicity $(r_i-1)(s_j-1)$.

Consequently, the Laplacian spectrum of the complement graph $\overline{R(r_i, s_j)}$, is given by:

- 0, with multiplicity 1;
- $r_is_j - s_j = s_j(r_i-1)$, with multiplicity r_i-1;
- $r_is_j - r_i = r_i(s_j-1)$, with multiplicity s_j-1;
- $r_is_j - r_i - s_j$, with multiplicity $(r_i-1)(s_j-1)$.

Therefore, each component $\mathscr{C}_i \times \mathscr{C}'_j$ contributes an integral Laplacian spectrum to the overall Laplacian spectrum of $G_1 Ⓣ G_2$.

As each component of $G_1 Ⓣ G_2$ is either an edgeless one, contributing only 0 as a Laplacian eigenvalue, or isomorphic to a complement of Rook's graph, which has an integral Laplacian spectrum, we conclude that all eigenvalues of the Laplacian matrix of $G_1 Ⓣ G_2$ are integers. Hence, the Laplacian spectrum of $G_1 Ⓣ G_2$ is entirely integral. □

Example: The CN_s tensor product graph constructed in the Figure 1 has Laplacian spectrum $\{0^6, 6^6, 4^2, 2^6.\}$

5.3 *Signless Laplacian spectrum of the CN_s tensor product*

In this section, we analyze the signless Laplacian spectrum of CN_s tensor product which are shaped by the sizes of the CN sets of G_1 and G_2. This spectral perspective reveals the integral nature of the CN_s tensor product and provides new insights into its structural properties, including the presence of symmetric subgraphs and their spectral invariants.

Theorem 9 If G_1 and G_2 are two CN_s graphs, then $G_1 Ⓣ G_2$ has an integral signless Laplacian spectrum.

Proof. Let G_1 and G_2 be two CN_s graph with n_1 and n_2 number of vertices, respectively. Suppose that G_1 has d_1 number of DN vertices and p_1 number of CN sets denoted $\mathscr{C}_i$: $1 \leq i \leq p_1$, with sizes $r_1, r_2, \ldots, r_{p_1}$ where each $r_i > 1$: $i = 1, 2, 3, \ldots, p_1$. Similarly assume G_2 contains d_2 number of DN vertices and p_2 number of CN sets $\mathscr{C}'_j$ for $1 \leq j \leq p_2$, with sizes $s_1, s_2, \ldots, s_{p_2}$ where $s_i > 1$: $i = 1, 2, 3, \ldots, p_2$. By Lemma 2, each product $\mathscr{C}_i \times \mathscr{C}'_j$ induces a subgraph in $G_1 Ⓣ G_2$ that is isomorphic to the complement of the Rook's graph $R(r_i, s_j)$.

The signless Laplacian matrix of $\overline{R(r_i, s_j)}$ is given by:

$$Q(\overline{R(r_i,\, s_j)}) = D(\overline{R(r_i,\, s_j)}) + A(\overline{R(r_i,\, s_j)}) = (r_i-1)(s_j-1)I + A(\overline{R(r_i,\, s_j)})$$

Also, $R(r_i,\ s_j)$ is the Cartesian product of the complete graphs K_{r_i} and K_{s_j}. The adjacency spectrum of K_{r_i} is $\{(-1)^{r_i-1},\ (r_i-1)^1\}$ and that of K_{s_j} is $\{(-1)^{s_j-1},\ (s_j-1)^1\}$. From Lemma 3, the eigenvalues of $R(r_i,\ s_j)$ are $\{(-2)^{(r_i-1)(s_j-1)},\ (r_i-2)^{s_j-1},\ (s_j-2)^{r_i-1},\ (r_i+s_j-2)^1\}$.

Claim: All one vector is the eigenvector corresponding to the largest eigenvalue $\lambda = r_i+s_j-2$.

Let x be the all one vector. The degree of each vertex in $R(r_i,\, s_j)$ is (r_i+s_j-2) so that $A(R(r_i,\, s_j))x = (r_i+s_j-2)x$. Hence, the eigenvector corresponding to (r_i+s_j-2) is x.

Now consider, $A(\overline{R(r_i,\, s_j)}) = J_{r_i\times s_j} - I_{r_i\times s_j} - A(R(r_i,\, s_j))$ and

$$\begin{aligned} A(\overline{R(r_i,\, s_j)})x &= \left(J_{r_i\times s_j} - I_{r_i\times s_j} - A(R(r_i,\, s_j))\right)x \\ &= J_{r_i\times s_j}x - I_{r_i\times s_j}x - A(R(r_i,\, s_j))x \\ &= r_i s_j x - x - (r_i+s_j-2)x \\ &= (r_i s_j - r_i - s_j + 1)x \end{aligned}$$

Hence, $r_is_j - r_i - s_j + 1$ is an eigenvalue of $\overline{R(r_i,\, s_j)}$ with all-one vector as corresponding eigenvector.

Now, assume $\lambda \neq r_i+s_j-2$ with x' as the corresponding eigenvector. Then, by Lemma 4, $x'x^T = 0$ and $J.x' = (x.x^T).x' = x.(x^T.x') = x.0 = 0$. If x' is an eigenvector corresponding to λ then $A(R(r_i,\, s_j))x' = \lambda x'$, and

$$\begin{aligned} A(\overline{R(r_i,\, s_j)})x' &= \left(J_{r_i\times s_j} - I_{r_i\times s_j} - A(R(r_i,\, s_j))\right)x' \\ &= J_{r_i\times s_j}x' - I_{r_i\times s_j}x' - A(R(r_i,\, s_j))x' \\ &= -x' - \lambda x' \\ &= (-1-\lambda)x' \end{aligned}$$

Corresponding to the eigenvalue $\lambda \neq r_i+s_j-2$ of $R(r_i,\, s_j)$, the eigenvalue corresponding to its complement is $-1-\lambda$ with eigenvector x' which is orthogonal to all one vector. Hence, the eigenvalues of $\overline{R(r_i,\, s_j)}$ are given as in Table 2.

Table 2. Eigenvalues of $\overline{R(r_i,\, r_j)}$

Eigenvalues of $R(r_i,\, s_j)$	Multiplicity	Eigenvalues of $\overline{R(r_i,\, s_j)}$
-2	$(r_i-1)(s_j-1)$	1
r_i-2	s_j-1	$1-r_i$
s_j-2	r_i-1	$1-s_j$
r_i+s_j-2	1	$r_is_j-r_i-s_j+1$

If X is an eigenvector of $\overline{R(r_i, s_j)}$ corresponding to the eigenvalue λ, then

$$Q(\overline{R(r_i, s_j)})X = (r_i - 1)(s_j - 1)IX + A(\overline{R(r_i, s_j)})X$$

$$= (r_i - 1)(s_j - 1)X + \lambda X$$

$$= ((r_i - 1)(s_j - 1) + \lambda)X$$

Hence, X is also the signless Lapalacian eigenvector of $\overline{R(r_i, s_j)}$ with $(r_i - 1)(s_j - 1) + \lambda$ as corresponding eigenvalue. So the signless Laplacian eigenvalues of $\overline{R(r_i, s_j)}$ are given in Table 3.

Table 3. Signless Laplacian eigenvalues of $\overline{R(r_i, r_j)}$

Eigenvalues of $\overline{R(r_i, s_j)}$	Multiplicity	Signless laplacian eigenvalues of $\overline{R(r_i, s_j)}$
1	$(r_i - 1)(s_j - 1)$	$(r_i - 1)(s_j - 1) + 1$
$1 - r_i$	$s_j - 1$	$(r_i - 1)(s_j - 1) + 1 - r_i = (r_i - 1)(s_j - 2)$
$1 - s_j$	$r_i - 1$	$(r_i - 1)(s_j - 1) + 1 - s_j = (r_i - 2)(s_j - 1)$
$r_i s_j - r_i - s_j + 1$	1	$(r_i - 1)(s_j - 1) + r_i s_j - r_i - s_j + 1 = 2(r_i s_j - r_i - s_j + 1)$

Since each component of $G_1 ⓣ G_2$ is either an edgeless one, contributing only 0 as a Signless Laplacian eigenvalue, or isomorphic to a complement of Rook's graph, which has an integral Signless Laplacian spectrum, we conclude that the Signless Laplacian spectrum of $G_1 ⓣ G_2$ is entirely integral. □

Corollary 11 The signless Laplacian eigenvectors of $G_1 ⓣ G_2$ are either all one vector or balanced.

Proof. Each component of $G_1 ⓣ G_2$ is either an isolated vertex or isomorphic to the complement of a Rook's graph. For the complement of the Rook's graph, the eigenvector corresponding to the largest eigenvalue is all-one vector, while all other eigenvectors are orthogonal to it, forming balanced vectors. □

Example: The CN_s tensor product graph constructed in the Figure 1 has signless Laplacian spectrum $\{0^6, 4^{12}, 10^2\}$.

5.4 *Comparison of CN_s tensor product with other graph products*

Classical graph products combine graphs based on adjacency of vertices only—they ignore the internal structural symmetry of the factor graphs. However, in many real-world graph models, set of vertices share identical neighborhood patterns. The CN_s Tensor product is designed precisely to capture the identical neighborhood relations of the vertices during the product operation and it reduces redundant vertices. In the standard Tensor product, disconnection occurs due to the bipartite nature of both graphs, whereas in the CN_s Tensor product, it results from the partitioning of vertices into CN_s sets. A comparative summary of properties of the CN_s Tensor product graph with classical graph products are presented in Table 4.

Table 4. Comparative properties: CN_s tensor vs classical graph products

Property	CN_s Tensor $G_1 \textcircled{T} G_2$	Tensor $G_1 \times G_2$	Cartesian $G_1 \square G_2$/Strong $G_1 \boxtimes G_2$/Lexicographic $G_1[G_2]$
Adjacency rule	Adjacency determined by CN sets. $(u, v) \sim (u', v')$ if and only if u, u' lie in same CN set of G_1 and v, v' lie in same CN set of G_2.	$(u, v) \sim (u', v')$ if and only if $uu' \in E(G_1)$ and $vv' \in E(G_2)$.	Cartesian: edge in one factor and equality in other. Strong: union of Cartesian and Tensor adjacencies. Lexicographic: $u \neq u'$ gives all edges between copies of G_2.
Adjacency matrix form	$CN_s(G_1) \otimes CN_s(G_2)$	$A(G_1) \otimes A(G_2)$	Cartesian: $A(G_1) \otimes I + I \otimes A(G_2)$. Strong: sum of Cartesian and tensor forms. Lexicographic: $A(G_1) \otimes J + I \otimes A(G_2)$ with J all-ones between blocks.
Spectrum	eigenvalues are pairwise products of CN_s spectrum of factor graphs	Eigenvalues are pairwise products of eigenvalues of factor graphs $(\lambda_i \mu_j)$	Cartesian/strong/lexicographic have spectrum formed from sums $(\lambda_i + \mu_j)$/sum of sum and products $(\lambda_i + \mu_j + \lambda_i \mu_j)$/scaled sums of factor spectra $(n_2 \lambda_i + \mu_j)$.
Multiplicity of eigenvalues	multiplicities determined by CN-set sizes r_i, t_j	Multiplicities follow multiplicative structure of adjacency spectrum of the factor graphs	Multiplicities derive from factor multiplicities with additive/combinational rules.
Regularity	Not necessarily regular; degrees inside a CN-block follow $(r_i - 1)(t_j - 1)$ pattern for block $\mathscr{C}_i \times \mathscr{C}'_j$. Regularity may occur per block if CN sets have uniform size.	If G_1, G_2 are r, s-regular then product is rs-regular	If G_1, G_2 are r, s-regular then Cartesian: product is $r+s$-regular/strong: product is $r+s+rs$-regular/lexicographic: product is $n_2 r + s$-regular.
Number of components	determined by CN set partitioning of the factor graphs	The tensor product is connected if and only if at least one of the graphs is not bipartite	Cartesian/strong: connected if and only if both factors connected. Lexicographic: connected if and only if G_1 connected.
Typical use	compresses repeated local motifs, highlights conserved modules so that useful in Protein-Protein Interaction (PPI), homology modeling	useful for Kronecker-style network models, where two systems interact jointly, such as multi-layer or coupled networks.	Cartesian/strong/lexicographic: model layered/interleaved interactions; lexicographic good for hierarchical networks.

Theorem 10 The standard Tensor product of two graphs is a CN_s graph if and only if each of the factor graphs are CN_s.

Proof. For the standard Tensor product of $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ we have, for every $(u, v) \in V_1 \times V_2$, $N_{G_1 \times G_2}(u, v) = N_{G_1}(u) \times N_{G_2}(v)$.

- First suppose the Tensor product, $G_1 \times G_2$ is a CN_s graph.

Then, $G_1 \times G_2$ contains at least two distinct vertices $(u_1, v_1) \neq (u_2, v_2)$ with identical neighborhoods. Hence,

$$N_{G_1}(u_1) \times N_{G_2}(v_1) = N_{G_1}(u_2) \times N_{G_2}(v_2).$$

$$\Rightarrow N_{G_1}(u_1) = N_{G_1}(u_2) \quad \text{and} \quad N_{G_2}(v_1) = N_{G_2}(v_2).$$

Since $(u_1, v_1) \neq (u_2, v_2)$, we have $u_1 \neq u_2$ and $v_1 \neq v_2$, so both G_1 and G_2 are CN graphs.

- Conversely, suppose that both G_1 and G_2 are CN_s graphs. Then, there exist distinct vertices $u_1, u_2 \in V_1$, such that $N_{G_1}(u_1) = N_{G_1}(u_2)$, and distinct vertices $v_1, v_2 \in V_2$ such that $N_{G_2}(v_1) = N_{G_2}(v_2)$. Then

$$N_{G_1 \times G_2}(u_1, v_1) = N_{G_1}(u_1) \times N_{G_2}(v_1) = N_{G_1}(u_2) \times N_{G_2}(v_2) = N_{G_1 \times G_2}(u_2, v_2).$$

Therefore, (u_1, v_1) and (u_2, v_2) are distinct vertices having identical neighborhoods. Hence $G_1 \times G_2$ is a CN_s graph. □

6. CN_s tensor product for homology modeling of protein structures

In homology modeling, the primary objective is to predict the three-dimensional structure of a target protein by aligning it with a structurally known template protein. Both proteins can be modeled as graphs—denoted G_1 (target) and G_2 (template)—where:

• Vertices represent amino acid residues, and

• Edges represent significant intramolecular interactions.

To capture shared topological or structural features between two proteins, we employ the CN_s tensor product of graphs. This construction yields a new graph in which each component reflects structurally conserved motifs or aligned regions between the target and the template.

Let the number of CN sets in G_1 and G_2 be p_1 and p_2, respectively. The resulting CN_s Tensor product, comprises $p_1 p_2$ multipartite components. Each component corresponds to a pairwise combination of CN sets from the two proteins and represents a conserved structural motif, thereby offering insights into potential functional similarities.

In contrast, the presence of isolated vertices in the product graph reflects regions of structural dissimilarity. These isolated vertices represent parts of the target or template that do not share common structure with the other. Such dissimilarities may indicate evolutionary divergence, adaptation, or functional specialization unique to the target protein.

Thus, the CN_s Tensor product framework serves not only to highlight conserved structures but also to pinpoint functionally or evolutionarily significant deviations.

Algorithm: Construction of the CN_s Tensor Product for Protein Homology Modeling

Input: Two protein graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, each annotated with CN sets (with more than one element) and DN vertices.

1. **Identification of CN Sets:** Partition the vertex sets of G_1 and G_2 into CN sets:

$$CN_1 = \{\mathscr{C}_1, \mathscr{C}_2, \ldots, \mathscr{C}_{p_1}\}, \quad CN_2 = \{\mathscr{C}'_1, \mathscr{C}'_2, \ldots, \mathscr{C}'_{p_2}\}$$

2. **Vertex Set Construction:** Define the vertex set of the CN_s Tensor product graph as the Cartesian product:

$$V(G_1 \circledT G_2) = V_1 \times V_2$$

3. **Edge Construction Rules:** Add an edge between vertices (u, v) and (u', v') in $G_1 \circledT G_2$ if and only if:

• $u \neq u'$ and $v \neq v'$,

• $u, u' \in \mathscr{C}_i$ for some i, and

• $v, v' \in \mathscr{C}'_j$ for some j.

4. **Component Construction:** For each i, j the vertices in $\mathscr{C}_i \times \mathscr{C}'_j$ induce a multipartite subgraph isomorphic to the complement of a Rook's graph $R(|\mathscr{C}_i|, |\mathscr{C}'_j|)$ in $G_1 \circledT G_2$. This results in $p_1 p_2$ connected multipartite components, each corresponding to a structurally conserved motif in the given two proteins.

Vertices (u, v) appear as isolated vertices in the product graph if u or v is a DN vertex.

5. **Structural Analysis:**

• Total number of multipartite components: $p_1 \cdot p_2$.

• Total number of isolated vertices can be found using Theorem 5.

• Compute Laplacian eigenvalue and hence the algebraic connectivity (second smallest Laplacian eigenvalue).

Biological Interpretation:

• Multipartite components signify structurally conserved regions between the target and template proteins.

• Isolated vertices denote non-aligned residues, potentially indicative of unique or divergent structural elements.

• Component with high algebraic connectivity signifies the most highly conserved motif or well-aligned region between the target and the template protein.

Other components, arranged in decreasing order of their algebraic connectivity, indicate progressively weaker structural correspondence—reflecting partial similarity between the compared protein regions.

Algorithmic Framework for Protein Homology Modeling Using the CN_s Tensor Product

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Algorithm 1 Construction of the CN_s Tensor Product for Protein Homology Modeling

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Require: Two protein graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, each with CN sets and DN vertices

Ensure: $G = G_1 \textcircled{T} G_2$, the CN_s tensor product graph

Identify CN partitions:

$$CN(G_1) = \{\mathscr{C}_1, \mathscr{C}_2, \ldots, \mathscr{C}_{p_1}\}, \quad CN(G_2) = \{\mathscr{C}'_1, \mathscr{C}'_2, \ldots, \mathscr{C}'_{p_2}\}.$$

Construct the vertex set:

$$V(G) = V_1 \times V_2.$$

for each $\mathscr{C}_i \in CN(G_1)$ **do**

  for each $\mathscr{C}'_j \in CN(G_2)$ **do**

    Form subgraph H_{ij} induced by $\mathscr{C}_i \times \mathscr{C}'_j$

    Add edges between (u, v) and (u', v') if $u, u' \in \mathscr{C}_i$, $v, v' \in \mathscr{C}'_j$, $u \neq u'$, and $v \neq v'$

    Compute Laplacian eigenvalues of H_{ij}:

$$\lambda_L \in \{0,\ r_i s_j - s_j,\ r_i s_j - r_i,\ r_i s_j - r_i - s_j\}$$

    Determine algebraic connectivity (second smallest λ_L) as $r_i s_j - r_i - s_j$ to quantify structural similarity

  end for

end for

Construct the final product graph:

$$G = \bigcup_{i=1}^{p_1} \bigcup_{j=1}^{p_2} H_{ij}.$$

Add isolated vertices corresponding to DN vertices of G_1 or G_2

return G

Complexity Analysis. Let $n_1 = |V_1|$, $n_2 = |V_2|$, and p_1, p_2 denote the numbers of CN sets in G_1 and G_2, respectively.

• Vertex set generation: $O(n_1 n_2)$.

• Component construction: $O(p_1 p_2)$.

• Memory usage: $O(n_1 n_2)$.

Numerical Illustration. Consider two protein graphs G_1 and G_2 say Lysozyme C from Chicken (PDB: 1LYZ, UniProt: P00698) and Human Lysozyme C (PDB: 1REX, UniProt: P61626) with the the *CN* partitions as given in Table 5.

Table 5. *CN* partitions

CN Set in G_1	Size	*CN* Set in G_2	Size	Component in $G_1 Ⓣ G_2$
$\mathscr{C}_1$	3	$\mathscr{C}_1'$	2	Bipartite (6 vertices)
$\mathscr{C}_2$	2	$\mathscr{C}_2'$	3	Bipartite (6 vertices)
DN vertices	2	1		Isolated

Each $\mathscr{C}_i \times \mathscr{C}_j'$ induces a subgraph isomorphic to the complement of a Rook's graph $\overline{R(|\mathscr{C}_i|, |\mathscr{C}_j'|)}$. The Laplacian eigenvalues of each component reflect its internal connectivity and hence the level of structural conservation as given in Table 6.

Table 6. Structural analysis

Component	Laplacian Eigenvalues	Laplacian Energy	Biological Interpretation
$\mathscr{C}_1 \times \mathscr{C}_1'$	$\{0, 1, 3, 4\}$	12	Moderately conserved motif
$\mathscr{C}_2 \times \mathscr{C}_2'$	$\{0, 1, 3, 4\}$	13	Moderately conserved region
Isolated DN vertices	$\{0\}$	0	Divergent residues
$\mathscr{C}_1 \times \mathscr{C}_2'$	$\{0, 3, 6\}$	36	Highly conserved motif
$\mathscr{C}_2 \times \mathscr{C}_1'$	$\{0, 2\}$	10	Moderately conserved motif

The computational complexity of constructing the CN_s matrix for a graph primarily depends on the process of identifying vertices with identical neighborhoods. In the context of protein graphs, which are typically sparse, the optimized approach ensures scalability. Moreover, for large-scale biological networks, parallel neighborhood hashing and sparse matrix representations can further improve efficiency, making the CN_s matrix feasible for analyzing high-resolution protein homology models.

7. Symmetry, independence and matching in CN_s tensor product graphs

7.1 *Automorphism group of* CN_s *tensor product*

In graph theory, automorphisms reflect the structural symmetries of a graph—mapping vertices to vertices such that adjacency is preserved. When graphs model complex systems, like molecular or protein structures, their automorphism groups reveal redundancies, conserved motifs, and functionally equivalent configurations.

The CN_s Tensor product of two graphs is defined to capture shared local connectivity, emphasizing regions of functional or structural similarity. When both factor graphs are CN_s-structured, their CN_s Tensor product results in a union of multipartite regular components, whose symmetries are dictated by the *CN* sets of the input graphs.

Understanding the automorphism group of the CN_s Tensor product graph is crucial: it allows researchers to detect structurally invariant regions between two biological networks, such as protein-protein interaction graphs or secondary structure motifs in homologous proteins. These automorphisms correspond to symmetries in protein folding patterns, conserved contact maps, or evolutionarily preserved interfaces.

In this section, we explore how automorphisms arise in the CN_s Tensor product of two CN_s graphs, how these symmetries can be classified, and what they imply for biological data interpretation in homology modeling and comparative protein structure analysis.

Theorem 11 Let G_1 and G_2 be two CN_s graphs having n_1 and n_2 vertices, respectively, of which k_1 and k_2 vertices are DN vertices. Also, G_1 and G_2 have p_1 and p_2 number of CN sets, respectively, with more than one element, then the automorphism group of $G_1 Ⓣ G_2$ is:

$$\text{Aut}(G_1 Ⓣ G_2) \cong \prod_{i=1}^{p_1}\prod_{j=1}^{p_2}\left(S_{r_i^{(1)}} \times S_{r_j^{(2)}}\right),$$

where S_n denotes the symmetric group on n elements, and $r_i^{(1)}$, $r_j^{(2)}$ are orders of CN sets with more than one element of G_1 and G_2, respectively.

Proof. By Lemma 2, if G_1 has a CN set $\mathscr{C}_i$ with r_i elements and G_2 has a CN set $\mathscr{C}_j$ with r_j elements and $r_i, r_j \geq 2$, then the component induced by $\mathscr{C}_i \times \mathscr{C}_j$ in $G_1 Ⓣ G_2$ is isomorphic to the complement of Rook's Graph $\overline{R(r_i, r_j)}$. Since automorphisms preserve both adjacency and non-adjacency the automorphism group of the complement of a Rook's graph is same as the automorphism group of the original Rook's graph. In Rook's graph with $r_i r_j$ elements two vertices (u_1, v_1) and (u_2, v_2) are adjacent if and only if either $u_1 = u_2$ or $v_1 = v_2$. Any permutation $\sigma \in S_{r_i}$ induces an automorphism of the graph: $(u_k, v_l) \to (u_{\sigma(k)}, v_l)$. Similarly, any permutation $\mu \in S_{r_j}$ induces an automorphism of the graph: $(u_k, v_l) \to (u_k, v_{\mu(l)})$. And the automorphism group of $\mathscr{C}_i \times \mathscr{C}_j$ is $S_{r_i} \times S_{r_j}$. Hence,

$$\text{Aut}(G_1 Ⓣ G_2) \cong \prod_{i=1}^{p_1}\prod_{j=1}^{p_2}\left(S_{r_i^{(1)}} \times S_{r_j^{(2)}}\right).$$

□

7.2 *Independence and matching number of CN_s tensor product*

Here, we study both matching and independence within the CN_s Tensor product. We show that under certain conditions—specifically, when the product of the sizes of CN sets is even-a perfect matching exists across the components of the CN_s Tensor product. This has implications for pairing and resource allocation tasks in network design and biology. In the study of graph products, computing the independence number becomes a nontrivial task, as it heavily depends on how adjacency relations evolve through the specific product operation applied.

The CN_s Tensor product of two graphs yields multipartite components determined by the CN sets of the factor graphs. The independence number and matching number of the CN_s Tensor product graph is then derived from the multipartite components, as established in the following theorems.

Theorem 12 Let G_1 and G_2 be two CN_s graphs with CN sets $\mathscr{C}_1, \mathscr{C}_2, \ldots, \mathscr{C}_{r_1}$ in G_1 and $\mathscr{C}'_1, \mathscr{C}'_2, \ldots, \mathscr{C}'_{r_2}$ in G_2. The independence number of the CN_s Tensor product of these two graphs is given by $\sum\limits_{i=1}^{r_1}\sum\limits_{j=1}^{r_2} max(|\mathscr{C}_i|, |\mathscr{C}'_j|)$, where $|\mathscr{C}_i|$ denote the number of vertices in each CN set $\mathscr{C}_i$.

Proof. Let $\mathscr{C}_i = \{u_1, u_2, \ldots, u_{p_1}\}$ and $\mathscr{C}'_j = \{v_1, v_2, \ldots, v_{p_2}\}$ be two arbitrary CN sets of G_1 and G_2, respectively, with $|\mathscr{C}_i| = p_1$ and $|\mathscr{C}'_j| = p_2$. Then, by Lemma 1, the Cartesian product $\mathscr{C}_i \times \mathscr{C}'_j$ forms a multipartite regular component in $G_1 Ⓣ G_2$.

Within this component, the nonadjacent vertex pairs are of the form (u_i, v_j), (u_i, v_k) or (u_i, v_j), $(u_l, v_j)$where $i, l = 1, 2, \ldots, p_1$ and $j, k = 1, 2, \ldots, p_2$. This structure implies that the maximum number of mutually nonadjacent vertices in such a component is $max(p_1, p_2)$.

Therefore, the independence number of the CN_s Tensor product is given by:

$$\alpha(G_1 Ⓣ G_2) = \sum_{i=1}^{r_1} \sum_{j=1}^{r_2} max(|\mathscr{C}_i|, |\mathscr{C}'_j|).$$

□

Theorem 13 Let G_1 and G_2 be two CN_s graphs with n_1 and n_2 vertices, respectively. If G_1 has k_1 number of CN sets, each with more than one vertex, having sizes $r_1, r_2, \ldots, r_{k_1}$, and G_2 has k_2 number of CN sets with sizes $t_1, t_2, \ldots, t_{k_2}$: $t_i > 1$, $1 \le i \le k_2$. Then, the matching number of $G_1 Ⓣ G_2$ is denoted by $\mu(G_1 Ⓣ G_2)$ and is given by $\sum_{i=1}^{k_1} \sum_{j=1}^{k_2} \left\lfloor \frac{r_i t_j}{2} \right\rfloor$.

Proof. Let $\mathscr{C}_1, \mathscr{C}_2, \ldots, \mathscr{C}_{k_1}$ be the CN sets of G_1 with sizes $r_1, r_2, \ldots, r_{k_1}$, and let $\mathscr{S}_1, \mathscr{S}_2, \ldots, \mathscr{S}_{k_2}$ be the CN sets of G_2 with sizes $t_1, t_2, \ldots, t_{k_2}$. Then, by Lemma 2, each component corresponding to $\mathscr{C}_i \times \mathscr{S}_j$ for $1 \le i \le k_1$, $1 \le j \le k_2$ is isomorphic to the complement of Rook's Graph on $r_i t_j$ vertices. In this component, two vertices are adjacent if and only if they do not lie in the same row or the same column. Since each edge in a matching connects two distinct vertices and no two edges share a vertex, the maximum number of matched vertices is bounded above by $\left\lfloor \frac{r_i t_j}{2} \right\rfloor$. And the isolated vertices-those formed from DN vertices do not contribute to any matching and are hence excluded. Therefore, $\mu(G_1 Ⓣ G_2) = \sum_{i=1}^{k_1} \sum_{j=1}^{k_2} \left\lfloor \frac{r_i t_j}{2} \right\rfloor$. □

Corollary 12 Let G_1 and G_2 be two CN_s graphs with no DN vertices. Suppose G_1 has k_1 number of CN sets, each containing more than one vertex, with respective sizes $r_1, r_2, \ldots, r_{k_1}$, and G_2 has k_2 number of CN sets with sizes $t_1, t_2, \ldots, t_{k_2}$: $t_i > 1$, $1 \le i \le k_2$. If the product $r_i t_j$ is even for all $1 \le i \le k_1$ and $1 \le j \le k_2$, then the graph $G_1 Ⓣ G_2$ admits a perfect matching.

Proof. Given $r_i t_j$ is even for all $1 \le i \le k_1$ and $1 \le j \le k_2$, it follows from Theorem 13, that the matching in each component contains exactly $\frac{r_i t_j}{2}$ edges with no shared vertices, thus forming a perfect matching. Moreover, since every vertex in each component is matched, the matching is perfect across the entire graph. □

7.3 *Chromatic number of CN_s tensor product*

The chromatic number [25] of a graph, denoted by $\chi(G)$, is one of the most fundamental and widely studied concept in graph theory. It represents the minimum number of colors needed to color the vertices of a graph G such that no two adjacent vertices share the same color. This concept plays a crucial role in both theoretical studies and practical applications, including scheduling, register allocation in compilers, frequency assignment in wireless networks, and map colouring problems.

In the context of graph products and particularly specialized products such as the CN_s Tensor product, understanding how colouring properties translate under such operations becomes a rich and challenging problem. The chromatic number of the resulting product graph is often governed by the structure of its multipartite components, which are determined by the CN sets of the factor graphs.

Theorem 14 If G_1 and G_2 are two CN_s graphs where the largest CN set of G_1 and G_2 have sizes r_1 and r_2, respectively, then the chromatic number of $G_1 Ⓣ G_2$ is $min(r_1, r_2)$.

Proof. Let the largest CN sets of the graphs G_1 and G_2 be denoted by C_1 and C_2, with cardinalities r_1 and r_2, respectively. Then, $C_1 \times C_2$ gives rise to the largest multipartite component in $G_1 Ⓣ G_2$, consisting of $min(r_1, r_2)$ partite sets. In this component, vertices within each partite sets are mutually non-adjacent, while each vertex is adjacent to all vertices in the other partite sets, except for those in its own. Hence, this component is a regular multipartite graph. Therefore, to assign distinct colors to adjacent vertices within this component, at least $\min(r_1, r_2)$ colors are required. To ensure that adjacent vertices are assigned distinct colors, at least $\min(r_1, r_2)$ colors are needed for proper colouring of this component. Furthermore, all other components in $G_1 \circledT G_2$ involve smaller CN sets and hence require strictly fewer colors. Therefore, the chromatic number of the CN_s Tensor product is determined by the largest multipartite component, and is:

$$\chi(G_1 \circledT G_2) = \min(r_1, r_2)$$

□

Remark. The CN_s Tensor product opens new directions for studying graph products that emphasize structural equivalence rather than mere adjacency. Future work can extend this framework to directed graphs, weighted interactions, and dynamic networks. Additionally, exploring the interplay between structure and other graph invariants could yield further insights into symmetry, modularity, and complexity in real-world networks.

8. Conclusion

In this paper, we introduced and explored a graph operation called the *CN_s Tensor product*, built upon the concept of CN_s vertices. By exploiting the CN_s matrices of input graphs, we showed that the adjacency matrix of the resulting product graph is given by their Kronecker product. This structure reveals that the CN_s Tensor product graph is integral, with both its adjacency, laplacian as well as the signless Laplacian spectra reflecting deep symmetries and structural regularities. Our spectral analysis demonstrated how the eigenvalues of the CN_s Tensor product are determined by those of the factor graphs, providing insights into graph energy. We also discussed combinatorial parameters such as the automorphism group, independence number, matching number and chromatic number of the product graph, which are closely tied to its multipartite and isolated components. Furthermore, we connected this framework to practical applications in structural bioinformatics, particularly homology modeling of proteins. The CN_s Tensor product captures conserved motifs as multipartite components and identifies divergent regions as isolated vertices, thus serving as a useful tool for comparing structural similarities between biological macromolecules.

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Confilict of interest

The authors declare no competing financial interest.

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