Contemporary Mathematics
https://ojs.wiserpub.com/index.php/CM/

Research Article

Some Application on Subclass of Meromorphic Functions Associated with q-Analogue Multiplier Operator

Ekram E. Ali[1,2*], Rabha M. El-ashwah[3], Abeer M. Albalahi[1]

[1] Department of Mathematics, College of Science, University of Ha'il, Ha'il, 2440, Saudi Arabia
[2] Department of Mathematics and Computer Science, Faculty of Science, Port Said University, Port Said, 42521, Egypt
[3] Department of Mathematics, Faculty of Science, Damietta University, New Damietta, 34517, Egypt
E-mail: e.ahmad@uoh.edu.sa

Received: 22 September 2025; **Revised:** 20 November 2025; **Accepted:** 1 December 2025

Abstract: In this paper, we introduce a new linear operator $\mathfrak{D}_{\mathrm{q}}^{r}(\lambda, \ell)\mathfrak{f}(\zeta)$ and employ it to define a q-analogue of a differential operator acting on a newly developed subclass of meromorphic functions in the punctured unit disk. The subclass, denoted $\Sigma_{\mathrm{q}}^{*, r}(\Theta, \lambda, \ell)$, is examined comprehensively, and several of its key analytic and geometric properties are established. Specifically, we derive coefficient conditions for functions in this class and study convolution properties, closure results, convex combination criteria, and the interaction of the class with the q-Bernardi integral operator. Moreover, we address and resolve the neighborhoods problem associated with this subclass. These findings offer new insights into meromorphic function theory within the framework of q-calculus operators and underscore the potential for further development of operator generated function classes in geometric function theory.

Keywords: analytic function, meromorphic functions, q-difference operator, q-analogue multiplier operator

MSC: 30C45, 30C80

1. Introduction

Let Σ denote the class of meromorphic functions of the form

$$\mathfrak{f}(\xi) = \frac{1}{\zeta} + \sum_{\nu=1}^{\infty} a_{\nu}\zeta^{\nu} \qquad (\zeta \in \mathbb{U}^*), \tag{1}$$

which are analytic in the punctured unit disc $\mathbb{U}^* = \{\zeta : \zeta \in \mathbb{C} \text{ and } 0 < |\zeta| < 1\}$. Let $\mathfrak{f}$ and $\mathfrak{l}$ be analytic in $\mathbb{U}$. We say that $\mathfrak{f}$ is subordinate to $\mathfrak{l}$, denoted $\mathfrak{f}(\xi) \prec \mathfrak{l}(\xi)$, if there exists an analytic function ω, with $\omega(0) = 0$ and $|\omega(\xi)| < 1$, for all $\xi \in \mathbb{U}$, such that $\mathfrak{f}(\xi) = \mathfrak{l}(\omega(\xi))$, $\xi \in \mathbb{U}$. If the function $\mathfrak{l}$ is univalent in $\mathbb{U}$, $\mathfrak{f}(\xi) \prec \mathfrak{l}(\xi)$ is given as (see [1–3]):

$$\mathfrak{f}(0) = \mathfrak{l}(0) \text{ and } \mathfrak{f}(\mathbb{U}) \subset \mathfrak{l}(\mathbb{U}).$$

DOI: https://doi.org/10.37256/cm.7120268695

For $\mathfrak{f}$ given by (1) and $\hbar$ of the form

$$\hbar(\zeta)=\frac{1}{\zeta}+\sum_{\nu=1}^{\infty}b_{\nu}\zeta^{\nu},\ \ \zeta\in\mathbb{U}^{*},$$

the well-known *convolution product* is

$$(\mathfrak{f}*\hbar)(\zeta):=(\hbar*\mathfrak{f})(\zeta)=\frac{1}{\zeta}+\sum_{\nu=1}^{\infty}a_{\nu}b_{\nu}\zeta^{\nu}=:(\hbar*\mathfrak{f})(\zeta).$$

A function $f\in\Sigma$ is said to be in the subclass $\Sigma^{*}(\alpha)$ of meromorphic starlike functions of order α if it satisfies the inequality

$$Re\left\{-\frac{\zeta\mathfrak{f}'(\zeta)}{\mathfrak{f}(\zeta)}\right\}>\alpha\ \ (0\leq\alpha<1,\ \zeta\in\mathbb{U}). \tag{2}$$

As usual, let $\Sigma^{*}(0)=\Sigma^{*}$.

$\mathfrak{q}$-difference equations are an important aspect of mathematical analysis, particularly in the field known as Geometric Function Theory (GFT). Quantum calculus is frequently used in mathematical disciplines because its numerous possible applications in basic hypergeometric functions [4], orthogonal polynomials [5], combinatorics [6], and number theory [7]. Several fundamental ideas in $\mathfrak{q}$-calculus and [8, 9] demonstrate how it is integrated into mathematical ideas. Srivastava's 1989 book chapter [10] offered the appropriate foundation for integrating the concepts of $\mathfrak{q}$-calculus into GFT. The study of GFT is fascinating and relevant since basic (or quantum or $\mathfrak{q}$-) derivatives have several applications in diverse areas of mathematics (see [11] and [12]). Aral and Gupta [13] and Elhaddad et al. [14] have examined a few $\mathfrak{q}$-operator applications. Numerous $\mathfrak{q}$-derivative problems are also included in [15–21].

Definition 1 [12, 13] The *Jackson derivative* of a function $\mathfrak{f}$ is defined by

$$(\eth_{\mathfrak{q}})\mathfrak{f}(\zeta):=\eth_{\mathfrak{q}}\mathfrak{f}(\zeta)=\frac{\mathfrak{f}(\mathfrak{q}\zeta)-\mathfrak{f}(\zeta)}{(\mathfrak{q}-1)\zeta},\ \ \mathfrak{q}\in(0,1),\ \ \zeta\neq 0.$$

If the function $\mathfrak{f}$ has the form (1) it follows that

$$(\eth_{\mathfrak{q}})\mathfrak{f}(\zeta)=\eth_{\mathfrak{q}}\left(\zeta+\sum_{\nu=2}^{\infty}a_{\nu}\zeta^{\nu}\right)=1+\sum_{\nu=2}^{\infty}[\nu]_{\mathfrak{q}}a_{\nu}\zeta^{\nu-1}, \tag{3}$$

where

$$[\nu]_{\mathfrak{q}}:=\frac{1-\mathfrak{q}^{\nu}}{1-\mathfrak{q}}=1+\sum_{\ell=1}^{\nu-1}\mathfrak{q}^{\ell},\ \ [0]_{\mathfrak{q}}:=0, \tag{4}$$

and

$$\lim_{\mathfrak{q}\to 1^-}[\nu]_{\mathfrak{q}} = \nu.$$

The q-difference operator is subject to the following basic laws.

$$\mathfrak{d}_{\mathfrak{q}}\left(c\mathfrak{f}(\zeta) \pm d\mathfrak{l}(\zeta)\right) = c\mathfrak{d}_{\mathfrak{q}}\mathfrak{f}(\zeta) \pm d\mathfrak{d}_{\mathfrak{q}}\mathfrak{l}(\zeta),$$

$$\mathfrak{d}_{\mathfrak{q}}\left(\mathfrak{f}(\zeta)\mathfrak{l}(\zeta)\right) = \mathfrak{f}(\mathfrak{q}\zeta)\mathfrak{d}_{\mathfrak{q}}\left(\mathfrak{l}(\zeta)\right) + \mathfrak{l}(\zeta)\mathfrak{d}_{\mathfrak{q}}\left(\mathfrak{f}(\zeta)\right),$$

$$\mathfrak{d}_{\mathfrak{q}}\left(\frac{\mathfrak{f}(\zeta)}{\mathfrak{l}(\zeta)}\right) = \frac{\mathfrak{d}_{\mathfrak{q}}\left(\mathfrak{f}(\zeta)\right)\mathfrak{l}(\zeta) - \mathfrak{f}(\zeta)\mathfrak{d}_{\mathfrak{q}}\left(\mathfrak{l}(\zeta)\right)}{\mathfrak{l}(\mathfrak{q}\zeta)\mathfrak{l}(\zeta)}, \ \mathfrak{l}(\mathfrak{q}\zeta)\mathfrak{l}(\zeta) \neq 0,$$

$$\mathfrak{d}_{\mathfrak{q}}\left(\log\mathfrak{f}(\zeta)\right) = \frac{\ln\mathfrak{q}}{\mathfrak{q}-1}\frac{\mathfrak{d}_{\mathfrak{q}}\left(\mathfrak{f}(\zeta)\right)}{\mathfrak{f}(\zeta)},$$

where $\mathfrak{f}$, $\mathfrak{l} \in \Sigma$, and c and d are real or complex constants.

As long ago as 1910, Jackson [13] introduced the $\mathfrak{q}$-integral defined by

$$\int_0^{\zeta}\mathfrak{f}(\tau)d_{\mathfrak{q}}\tau = (1-\mathfrak{q})\zeta\sum_{k=0}^{\infty}\mathfrak{q}^k\mathfrak{f}(\mathfrak{q}^k\zeta), \tag{5}$$

provided that the series involved in (5) is convergent. In particular, for a power function $\mathfrak{f}(\zeta) = \zeta^k$, we note that

$$\int_0^{\zeta}\mathfrak{f}(\tau)d_{\mathfrak{q}}\tau = \int_0^{\zeta}\tau^k d_{\mathfrak{q}}\tau = \frac{1}{[k+1]_{\mathfrak{q}}}\zeta^{k+1} \qquad (k \neq -1),$$

and

$$\lim_{\mathfrak{q}\to 1-}\int_0^{\zeta}\mathfrak{f}(\tau)d_{\mathfrak{q}}\tau = \lim_{\mathfrak{q}\to 1-}\frac{1}{[k+1]_{\mathfrak{q}}}\zeta^{k+1} = \frac{1}{k+1}\zeta^{k+1} = \int_0^{\zeta}\mathfrak{f}(\tau)d\tau,$$

where $\int_0^{\zeta}\mathfrak{f}(\tau)d\tau$ denotes the ordinary integral.

When $\mathfrak{f}$ of the form (1), it is clear that

$$\mathfrak{d}_{\mathfrak{q}}[\mathfrak{f}(\zeta)] = -\frac{1}{\mathfrak{q}\zeta^2} + \sum_{\nu=0}^{\infty}[\nu]_{\mathfrak{q}}a_{\nu}\zeta^{\nu-1}.$$

The $\mathfrak{q}$-analogue of the multiplier operator is a significant concept in the field of mathematics, particularly in the study of special functions and combinatorial analysis. In simple terms, a $\mathfrak{q}$-analogue is a generalization of a mathematical concept that incorporates a parameter "q" which often takes values between 0 and 1. This concept allows mathematicians to explore connections between various areas of mathematics, such as number theory, representation theory, and combinatorics, by extending traditional ideas into a new framework. The multiplier operator, in various contexts, helps in transforming functions and sequences, and understanding its q-analogue opens up new avenues for research and application.

Now, we defined a new operator $\mathfrak{D}_{\mathfrak{q}}^{r}(\lambda, \ell)\mathfrak{f}(\zeta)\colon \Sigma \to \Sigma$ for $\mathfrak{f}(\zeta) \in \Sigma$, $r \in \mathbb{N}_0 = \mathbb{N} \cup \{0\} = \{1, 2, 3, ...\}$, $\ell, \lambda \geq 0$, $0 < \mathfrak{q} < 1$ as followes:

$$\mathfrak{I}_{\ell, \lambda}^{0, \mathfrak{q}}\mathfrak{f}(\zeta) = \mathfrak{I}_{\ell, \lambda}^{\mathfrak{q}}\mathfrak{f}(\zeta) = \mathfrak{f}(\zeta),$$

$$\mathfrak{I}_{\ell, \lambda}^{1, \mathfrak{q}}\mathfrak{f}(\zeta) = (1-\lambda)\,\mathfrak{f}(\zeta) + \frac{\lambda}{[\ell]_{\mathfrak{q}}\zeta^{\ell}}\partial_{\mathfrak{q}}\left(\zeta^{\ell+1}\mathfrak{f}(\zeta)\right),$$

$$\mathfrak{I}_{\ell, \lambda}^{1, \mathfrak{q}}\mathfrak{f}(\zeta) = \frac{1}{\zeta} + \sum_{\nu=1}^{\infty}\left(\frac{[\ell]_{\mathfrak{q}} + \lambda([\nu+\ell+1]_{\mathfrak{q}} - [\ell]_{\mathfrak{q}})}{[\ell]_{\mathfrak{q}}}\right)a_{\nu}\zeta^{\nu},$$

$$\vdots$$

$$\mathfrak{I}_{\ell, \lambda}^{r, \mathfrak{q}}\mathfrak{f}(\zeta) = (1-\lambda)\,\mathfrak{I}_{\ell, \lambda}^{r-1, \mathfrak{q}}\mathfrak{f}(\zeta) + \frac{\lambda}{[\ell]_{\mathfrak{q}}\zeta^{\ell}}\partial_{\mathfrak{q}}\left(\zeta^{\ell+1}\mathfrak{I}_{\ell, \lambda}^{r-1, \mathfrak{q}}\mathfrak{f}(\zeta)\right), \quad r \geq 1,$$

$$\mathfrak{I}_{\ell, \lambda}^{r, \mathfrak{q}}\mathfrak{f}(\zeta) = \frac{1}{\zeta} + \sum_{\nu=1}^{\infty}\left(\frac{[\ell]_{\mathfrak{q}} + \lambda([\nu+\ell+1]_{\mathfrak{q}} - [\ell]_{\mathfrak{q}})}{[\ell]_{\mathfrak{q}}}\right)^{r}a_{\nu}\zeta^{\nu},$$

$$(r \in \mathbb{N}_0,\ \ell, \lambda \geq 0,\ 0 < \mathfrak{q} < 1). \tag{6}$$

We define a new function $\mathfrak{D}_{\mathfrak{q}}^{r}(\lambda, \ell)(\zeta)$ in terms of the Hadamard product (or convolution) by:

$$\mathfrak{D}_{\mathfrak{q}}^{r}(\lambda, \ell)(\zeta) * \mathfrak{I}_{\mathfrak{q}, \lambda, \ell}^{r}(\zeta) = \frac{1}{\zeta(1-\zeta)}.$$

Then,

$$\mathfrak{D}_{\mathfrak{q}}^{r}(\lambda, \ell)(\zeta) = \frac{1}{\zeta} + \sum_{\nu=1}^{\infty}\left(\frac{[\ell]_{\mathfrak{q}}}{[\ell]_{\mathfrak{q}} + \lambda([\nu+\ell+1]_{\mathfrak{q}} - [\ell]_{\mathfrak{q}})}\right)^{r}\zeta^{\nu}.$$

Motivated essentially by the q-analogue Cătas operator, we defined

$$\mathfrak{D}^r_{\mathfrak{q}}(\lambda,\,\ell)\mathfrak{f}(\zeta)=\frac{1}{\zeta}+\sum_{\nu=1}^{\infty}\left(\frac{[\ell]_{\mathfrak{q}}}{[\ell]_{\mathfrak{q}}+\lambda([\nu+\ell+1]_{\mathfrak{q}}-[\ell]_{\mathfrak{q}})}\right)^r a_\nu\zeta^\nu, \tag{7}$$

$r\in\mathbb{N}_0,\,\ell,\,\lambda\geq 0,\,0<\mathfrak{q}<1.$

From (7) we get

$$\lambda\mathfrak{q}^{\ell+1}\zeta\mathfrak{d}_{\mathfrak{q}}(\mathfrak{D}^r_{\mathfrak{q}}(\lambda,\,\ell)\mathfrak{f}(\zeta))=[\ell]_{\mathfrak{q}}\mathfrak{D}^{r+1}_{\mathfrak{q}}(\lambda,\,\ell)\mathfrak{f}(\zeta)-\left(\lambda\mathfrak{q}^{\ell}+[\ell]_{\mathfrak{q}}\right)\mathfrak{D}^r_{\mathfrak{q}}(\lambda,\,\ell)\mathfrak{f}(\zeta),\ (\lambda>0).$$

We note that:
If we set $\mathfrak{q}\to 1^-$ we get $I^m_p(\lambda,\,\ell)\mathfrak{f}(\zeta)$ it was investigated by Bulboaca et al. [22] and El-Ashwah [23], (with $p=1$).
We also observe that:
(i) $\mathfrak{D}^r_{\mathfrak{q}}(1,\,\ell)\mathfrak{f}(\zeta)=\mathfrak{D}^r_{\mathfrak{q}}(\ell)\mathfrak{f}(\zeta)$

$$\mathfrak{f}(\zeta)\in\Sigma\colon\ \mathfrak{D}^r_{\mathfrak{q}}(\ell)\mathfrak{f}(\zeta)=\frac{1}{\zeta}+\sum_{\nu=2}^{\infty}\left(\frac{[\ell]_{\mathfrak{q}}}{[\nu+\ell+1]_{\mathfrak{q}}}\right)^r a_\nu\zeta^\nu,$$

$$(r\in\mathbb{N}_0,\ \ 0<\mathfrak{q}<1,\,\zeta\in\mathbb{U}^*);$$

(ii) $\mathfrak{D}^r_{\mathfrak{q}}(1,\,1)\mathfrak{f}(\zeta)=\mathfrak{D}^r_{\mathfrak{q}}\mathfrak{f}(\zeta)$

$$\mathfrak{f}(\zeta)\in\Sigma\colon\ \mathfrak{D}^r_{\mathfrak{q}}\mathfrak{f}(\zeta)=\frac{1}{\zeta}+\sum_{\nu=2}^{\infty}\left(\frac{1}{[\nu+2]_{\mathfrak{q}}}\right)^r a_\nu\zeta^\nu,$$

$$(r\in\mathbb{N}_0,\,\ell>0,\,0<\mathfrak{q}<1,\,\zeta\in\mathbb{U}^*);$$

(iii) $\mathfrak{D}^r_{\mathfrak{q}}(\lambda,\,1)\mathfrak{f}(\zeta)=\mathfrak{D}^r_{\mathfrak{q}}(\lambda)\mathfrak{f}(\zeta)$

$$\mathfrak{f}(\zeta)\in\Sigma\colon\ \mathfrak{D}^r_{\mathfrak{q}}(\lambda)\mathfrak{f}(\zeta)=\frac{1}{\zeta}+\sum_{\nu=2}^{\infty}\left(\frac{1}{1+\lambda([\nu+2]_{\mathfrak{q}}-1)}\right)^r a_\nu\zeta^\nu,$$

$$(r\in\mathbb{N}_0,\,\lambda>0,\,0<\mathfrak{q}<1,\,\zeta\in\mathbb{U}^*).$$

Now, we define a subclass $\Sigma^{*;\,r}_{\mathfrak{q}}(\Theta,\,\lambda,\,\ell)$ of meromorphic analytic functions on $\mathbb{U}^*$ as follows:

Definition 2 For $0\leq\Theta<1$ and $\ell,\,\lambda\geq 0$. $\mathfrak{f}\in\Sigma$ as in (1) is said to be in the class $\Sigma^{*;\,r}_{\mathfrak{q}}(\Theta,\,\lambda,\,\ell)$ of meromorphic starlike function of order Θ, if it satisfies

$$Re\left\{\frac{-q\zeta\partial_q(\mathfrak{D}_q^r(\lambda,\ell)\mathfrak{f}(\zeta))}{\mathfrak{D}_q^r(\lambda,\ell)\mathfrak{f}(\zeta)}\right\}\geq\Theta,\quad \zeta\in\mathbb{U}^*. \tag{8}$$

We note that

(i) For $r=0$ the class $\Sigma_q^{*,r}(\Theta,\lambda,\ell)$ reduce to the class $\Sigma_q^*(\Theta)$ the class of meromorphic starlike function of order Θ,

(ii) For $q\to 1^-$ and $r=\Theta=0$ the class $\Sigma_q^{*,r}(\Theta,\lambda,\ell)$ reduce to the class Σ^* the class of meromorphic starlike function (see [24]).

Let Φ be the class of analytic functions and univalent convex functions in $\mathbb{U}$, with $\varphi(0)=1,\ \varphi(\zeta)=\frac{1+(1-2\Theta)\zeta}{1-\zeta}$, and $\Re\varphi(\zeta)>\Theta$ in $\mathbb{U}$.

Using the subordination principle we can defined the class $\Sigma_q^{*,r}(\Theta,\lambda,\ell)$ as following

$$\frac{-q\zeta\partial_q(\mathfrak{D}_q^r(\lambda,\ell)\mathfrak{f}(\zeta))}{\mathfrak{D}_q^r(\lambda,\ell)\mathfrak{f}(\zeta)}\prec\varphi(\zeta), \tag{9}$$

where ∂_q is the q-difference operator.

To illustrate our conclusions, the following lemma is necessary:

Lemma 1 [25] Let υ and ε be complex numbers with $\gamma\neq 0$ and let $\varphi(\zeta)$ be analytic in $\mathbb{U}$ with $\varphi(0)=1$ and $Re\{\upsilon\varphi(\zeta)+\varepsilon\}>0$. If $\omega(\zeta)=1+\omega_1\zeta+\omega_2\zeta^2+....$ is analytic in $\mathbb{U}$, then

$$\omega(\zeta)+\frac{\zeta\partial_q\omega(\zeta)}{\upsilon\omega(\zeta)+\varepsilon}\prec\varphi(\zeta),$$

then $\omega(\zeta)\prec\varphi(\zeta)$.

In this paper, we construct, coefficient conditions and show some properties for function $\mathfrak{f}$ belonging to this subclass. These include the convolution conditions, closure theorem, convex combinations, the q-Bernardi integral operator and the neighbourhoods problem.

2. Some properties of the class $\Sigma_q^{*,r}(\Theta,\lambda,\ell)$

Unless otherwise mentioned we shall assume throughout the paper that $0\leq\Theta<1,\ \lambda>0,\ \ell\geq 0$ and $0<q<1$.

Our first result represent a sufficient condition for $\mathfrak{f}\in\Sigma_q^{*,r}(\Theta,\lambda,\ell)$.

Theorem 1 If $\mathfrak{f}\in\Sigma$ of the form (1). Then $\mathfrak{f}\in\Sigma_q^{*,r}(\Theta,\lambda,\ell)$ iff

$$\sum_{\nu=1}^{\infty}(q[\nu]_q+\Theta)\left(\frac{[\ell]_q}{[\ell]_q+\lambda([\nu+\ell+1]_q-[\ell]_q)}\right)^r|a_\nu|\leq 1-\Theta. \tag{10}$$

Proof. Consider $\mathfrak{f}\in\Sigma_q^{*,r}(\Theta,\lambda,\ell)$ then by the definition (8), we have

$$Re\left\{\frac{-q\zeta\partial_q\mathfrak{D}_q^r(\lambda,\ell)\mathfrak{f}(\zeta))}{\mathfrak{D}_q^r(\lambda,\ell)\mathfrak{f}(\zeta)}\right\}\geq\Theta$$

which is equivalent to

$$Re\left\{\frac{1-\sum_{\nu=1}^{\infty}\mathfrak{q}[\nu]_{\mathfrak{q}}\left(\frac{[\ell]_{\mathfrak{q}}}{[\ell]_{\mathfrak{q}}+\lambda([\nu+\ell+1]_{\mathfrak{q}}-[\ell]_{\mathfrak{q}})}\right)^{r}a_{\nu}\zeta^{\nu+1}}{1+\sum_{\nu=1}^{\infty}\left(\frac{[\ell]_{\mathfrak{q}}}{[\ell]_{\mathfrak{q}}+\lambda([\nu+\ell+1]_{\mathfrak{q}}-[\ell]_{\mathfrak{q}})}\right)^{r}a_{\nu}\zeta^{\nu+1}}\right\}\geq\Theta$$

choosing a value of $\zeta = r \in (0, 1)$ on the real axis such that $r \to 1^-$, we have

$$1-\sum_{\nu=1}^{\infty}\mathfrak{q}[\nu]_{\mathfrak{q}}\left(\frac{[\ell]_{\mathfrak{q}}}{[\ell]_{\mathfrak{q}}+\lambda([\nu+\ell+1]_{\mathfrak{q}}-[\ell]_{\mathfrak{q}})}\right)^{r}|a_{\nu}|$$

$$\geq\Theta+\Theta\sum_{\nu=1}^{\infty}\left(\frac{[\ell]_{\mathfrak{q}}}{[\ell]_{\mathfrak{q}}+\lambda([\nu+\ell+1]_{\mathfrak{q}}-[\ell]_{\mathfrak{q}})}\right)^{r}|a_{\nu}|,$$

on simplification we get,

$$\sum_{\nu=1}^{\infty}(\mathfrak{q}[\nu]_{\mathfrak{q}}+\Theta)\left(\frac{[\ell]_{\mathfrak{q}}}{[\ell]_{\mathfrak{q}}+\lambda([\nu+\ell+1]_{\mathfrak{q}}-[\ell]_{\mathfrak{q}})}\right)^{r}|a_{\nu}|\leq 1-\Theta.$$

Conversely, suppose that $\mathfrak{f}\in\Sigma$ of the form (1) and the inequality (10) holds for all $\zeta\in\mathbb{U}^{*}$ we will prove that $\mathfrak{f}\in\Sigma_{\mathfrak{q}}^{*,r}(\Theta,\lambda,\ell)$, according to definition of the class $\Sigma_{\mathfrak{q}}^{*,r}(\Theta,\lambda,\ell)$, and for $0\leq\Theta<1$, we want to show

$$Re\left\{\frac{-\mathfrak{q}\zeta\mathfrak{d}_{\mathfrak{q}}\mathfrak{D}_{\mathfrak{q}}^{r}(\lambda,\ell)\mathfrak{f}(\zeta))}{\mathfrak{D}_{\mathfrak{q}}^{r}(\lambda,\ell)\mathfrak{f}(\zeta)}\right\}\geq\Theta.$$

By using $Re(\omega)\geq\Theta$ if $|1-\Theta+\omega|\geq|1+\Theta-\omega|$, it suffices to show $|\psi(\zeta)|-|\Theta(\zeta)|\geq 0$, where

$$\psi(\zeta)=-\mathfrak{q}\zeta\mathfrak{d}_{\mathfrak{q}}\mathfrak{D}_{\mathfrak{q}}^{r}(\lambda,\ell)\mathfrak{f}(\zeta))+(1-\Theta)\mathfrak{D}_{\mathfrak{q}}^{r}(\lambda,\ell)\mathfrak{f}(\zeta)$$

and

$$\Theta(\zeta)=\mathfrak{q}\zeta\mathfrak{d}_{\mathfrak{q}}\mathfrak{D}_{\mathfrak{q}}^{r}(\lambda,\ell)\mathfrak{f}(\zeta))+(1+\Theta)\mathfrak{D}_{\mathfrak{q}}^{r}(\lambda,\ell)\mathfrak{f}(\zeta).$$

Now

$$|\psi(\zeta)|-|\Theta(\zeta)|=\left|\frac{2-\Theta}{\zeta}-\sum_{\nu=1}^{\infty}(\mathfrak{q}[\nu]_{\mathfrak{q}}-1+\Theta)\left(\frac{[\ell]_{\mathfrak{q}}}{[\ell]_{\mathfrak{q}}+\lambda([\nu+\ell+1]_{\mathfrak{q}}-[\ell]_{\mathfrak{q}})}\right)^{r}a_{\nu}\zeta^{\nu}\right|-$$

$$\left|\frac{\Theta}{\zeta}-\sum_{\nu=1}^{\infty}(\mathfrak{q}[\nu]_{\mathfrak{q}}+1+\Theta)\left(\frac{[\ell]_{\mathfrak{q}}}{[\ell]_{\mathfrak{q}}+\lambda([\nu+\ell+1]_{\mathfrak{q}}-[\ell]_{\mathfrak{q}})}\right)^{r}a_{\nu}\zeta^{\nu-1}\right|,$$

$$\geq\left|\frac{2-2\Theta}{\zeta}-\sum_{\nu=1}^{\infty}2(\mathfrak{q}[\nu]_{\mathfrak{q}}+\Theta)\left(\frac{[\ell]_{\mathfrak{q}}}{[\ell]_{\mathfrak{q}}+\lambda([\nu+\ell+1]_{\mathfrak{q}}-[\ell]_{\mathfrak{q}})}\right)^{r}a_{\nu}\zeta^{\nu}\right|,$$

$$\geq\frac{2(1-\Theta)}{|\zeta|}(1-\sum_{\nu=1}^{\infty}\frac{(\mathfrak{q}[\nu]_{\mathfrak{q}}+\Theta)}{1-\Theta}\left(\frac{[\ell]_{\mathfrak{q}}}{[\ell]_{\mathfrak{q}}+\lambda([\nu+\ell+1]_{\mathfrak{q}}-[\ell]_{\mathfrak{q}})}\right)^{r}|a_{\nu}|\,|\zeta^{\nu+1}|,$$

$$\geq 2(1-\Theta)(1-\sum_{\nu=1}^{\infty}\frac{(\mathfrak{q}[\nu]_{\mathfrak{q}}+\Theta)}{1-\Theta}\left(\frac{[\ell]_{\mathfrak{q}}}{[\ell]_{\mathfrak{q}}+\lambda([\nu+\ell+1]_{\mathfrak{q}}-[\ell]_{\mathfrak{q}})}\right)^{r}|a_{\nu}|.$$

By condition (10) this completes the proof. □

Corollary 1 If $\mathfrak{f}\in\Sigma_{\mathfrak{q}}^{*,\,r}(\Theta,\,\lambda,\,\ell),$ then

$$|a_{\nu}|\leq\frac{(1-\Theta)}{(\mathfrak{q}[\nu]_{\mathfrak{q}}+\Theta)\left(\frac{[\ell]_{\mathfrak{q}}}{[\ell]_{\mathfrak{q}}+\lambda([\nu+\ell+1]_{\mathfrak{q}}-[\ell]_{\mathfrak{q}})}\right)^{r}},\quad(\nu\geq 1).$$

The result is sharp for the function

$$\mathfrak{f}(\zeta)=\frac{1}{\zeta}+\frac{(1-\Theta)}{(\mathfrak{q}[\nu]_{\mathfrak{q}}+\Theta)\left(\frac{[\ell]_{\mathfrak{q}}}{[\ell]_{\mathfrak{q}}+\lambda([\nu+\ell+1]_{\mathfrak{q}}-[\ell]_{\mathfrak{q}})}\right)^{r}}\zeta^{\nu},\quad(\nu\geq 1).$$

In the next theorem, we show the main result of closure of such functions belonging to the class $\Sigma_{\mathfrak{q}}^{*,\,r}(\Theta,\,\lambda,\,\ell)$.

Theorem 2 Let $\mathfrak{f}_i(\zeta)$ be in the class $\Sigma_{\mathfrak{q}}^{*,\,r}(\Theta,\,\lambda,\,\ell)$, for every $i=1,\,2,\,...,\,\beta,$ where

$$\mathfrak{f}_i(\zeta)=\frac{1}{\zeta}+\sum_{\nu=1}^{\infty}a_{\nu,\,i}\zeta^{\nu},\quad(a_{\nu,\,i}\geq 0),$$

then the function

$$\xi(\zeta)=\frac{1}{\zeta}+\sum_{\nu=1}^{\infty}b_{\nu}\zeta^{\nu},\quad(b_{\nu}\geq 0),$$

is also in the class $\Sigma_{\mathfrak{q}}^{*,\,r}(\Theta,\,\lambda,\,\ell),$ where $b_{\nu}=\frac{1}{\beta}\sum_{i=1}^{\beta}a_{\nu,\,i}.$

Proof. To show that $\mathfrak{f}_i(\zeta)\in\Sigma_{\mathfrak{q}}^{*,\,r}(\Theta,\,\lambda,\,\ell)$, it is enough to show that condition (10) holds, such that

$$\sum_{\nu=1}^{\infty}(\mathfrak{q}[\nu]_{\mathfrak{q}}+\Theta)\left(\frac{[\ell]_{\mathfrak{q}}}{[\ell]_{\mathfrak{q}}+\lambda([\nu+\ell+1]_{\mathfrak{q}}-[\ell]_{\mathfrak{q}})}\right)^{r}|b_{\nu}|$$

$$=\sum_{\nu=1}^{\infty}(\mathfrak{q}[\nu]_{\mathfrak{q}}+\Theta)\left(\frac{[\ell]_{\mathfrak{q}}}{[\ell]_{\mathfrak{q}}+\lambda([\nu+\ell+1]_{\mathfrak{q}}-[\ell]_{\mathfrak{q}})}\right)^{r}\frac{1}{\beta}\sum_{i=1}^{\beta}|a_{\nu,i}|,$$

$$=\frac{1}{\beta}\sum_{i=1}^{\beta}\sum_{\nu=1}^{\infty}(\mathfrak{q}[\nu]_{\mathfrak{q}}+\Theta)\left(\frac{[\ell]_{\mathfrak{q}}}{[\ell]_{\mathfrak{q}}+\lambda([\nu+\ell+1]_{\mathfrak{q}}-[\ell]_{\mathfrak{q}})}\right)^{r}|a_{\nu,i}|,$$

as $\mathfrak{f}_i(\zeta)\in\Sigma_{\mathfrak{q}}^{*,r}(\Theta,\lambda,\ell)$ for all $i=1,2,...,\beta,$ then it satisfies condition (10), therefore

$$\sum_{\nu=1}^{\infty}(\mathfrak{q}[\nu]_{\mathfrak{q}}+\Theta)\left(\frac{[\ell]_{\mathfrak{q}}}{[\ell]_{\mathfrak{q}}+\lambda([\nu+\ell+1]_{\mathfrak{q}}-[\ell]_{\mathfrak{q}})}\right)^{r}|b_{\nu}|$$

$$\leq\frac{1}{\beta}\sum_{i=1}^{\beta}(1-\Theta)\leq 1-\Theta,$$

therefore, we have $\mathfrak{f}_i(\zeta)\in\Sigma_{\mathfrak{q}}^{*,r}(\Theta,\lambda,\ell)$ and this completes the proof. □

In next theorem, we proved the class $\Sigma_{\mathfrak{q}}^{*,r}(\Theta,\lambda,\ell)$ is closed under convex combination.

Theorem 3 If $\mathfrak{f}\in\Sigma$ of the form (1). Then $\mathfrak{f}\in\Sigma_{\mathfrak{q}}^{*,r}(\Theta,\lambda,\ell)$ iff

$$\mathfrak{f}(\zeta)=\sum_{\nu=0}^{\infty}\vartheta_{\nu}\mathfrak{f}_{\nu}(\zeta), \tag{11}$$

where

$$\mathfrak{f}_0(\zeta)=\frac{1}{\zeta},$$

$$\mathfrak{f}_{\nu}(\zeta)=\frac{1}{\zeta}+\left(\frac{(1-\Theta)}{(\mathfrak{q}[\nu]_{\mathfrak{q}}+\Theta)\left(\frac{[\ell]_{\mathfrak{q}}}{[\ell]_{\mathfrak{q}}+\lambda([\nu+\ell+1]_{\mathfrak{q}}-[\ell]_{\mathfrak{q}})}\right)^{r}}\right)\zeta^{\nu},\quad(\nu=1,2,...), \tag{12}$$

where $0\leq\vartheta_{\nu}\leq 1$ and $\sum_{\nu=0}^{\infty}\vartheta_{\nu}=1$.

Proof. Let

$$\mathfrak{f}(\zeta)=\sum_{\nu=0}^{\infty}\vartheta_\nu\mathfrak{f}_\nu(\zeta)$$

$$=\vartheta_0\mathfrak{f}_0+\sum_{\nu=1}^{\infty}\vartheta_\nu\mathfrak{f}_\nu$$

$$=\frac{\vartheta_0}{\zeta}+\sum_{\nu=1}^{\infty}\vartheta_\nu\left[\frac{1}{\zeta}+\left(\frac{(1-\Theta)}{(\mathfrak{q}[\nu]_\mathfrak{q}+\Theta)\left(\frac{[\ell]_\mathfrak{q}}{[\ell]_\mathfrak{q}+\lambda([\nu+\ell+1]_\mathfrak{q}-[\ell]_\mathfrak{q})}\right)^r}\right)\zeta^\nu\right],$$

by applying (10), we get

$$\sum_{\nu=1}^{\infty}(\mathfrak{q}[\nu]_\mathfrak{q}+\Theta)\left(\frac{[\ell]_\mathfrak{q}}{[\ell]_\mathfrak{q}+\lambda([\nu+\ell+1]_\mathfrak{q}-[\ell]_\mathfrak{q})}\right)^r\times\left(\frac{(1-\Theta)}{(\mathfrak{q}[\nu]_\mathfrak{q}+\Theta)\left(\frac{[\ell]_\mathfrak{q}}{[\ell]_\mathfrak{q}+\lambda([\nu+\ell+1]_\mathfrak{q}-[\ell]_\mathfrak{q})}\right)^r}\vartheta_\nu\right)$$

$$(1-\Theta)\sum_{\nu=1}^{\infty}\vartheta_\nu=(1-\Theta)(1-\vartheta_0)\leq 1-\Theta,$$

then $\mathfrak{f}\in\Sigma_\mathfrak{q}^{*,r}(\Theta,\lambda,\ell)$.

Conversely, assume that $\mathfrak{f}\in\Sigma_\mathfrak{q}^{*,r}(\Theta,\lambda,\ell)$. Put

$$\vartheta_\nu=\frac{(\mathfrak{q}[\nu]_\mathfrak{q}+\Theta)\left(\frac{[\ell]_\mathfrak{q}}{[\ell]_\mathfrak{q}+\lambda([\nu+\ell+1]_\mathfrak{q}-[\ell]_\mathfrak{q})}\right)^r}{(1-\Theta)}a_\nu,\quad 0\leq\vartheta_\nu\leq 1,$$

$$\vartheta_0=1-\sum_{\nu=1}^{\infty}\vartheta_\nu.$$

Then, $\mathfrak{f}$ can be expressed as

$$\mathfrak{f}(\zeta)=\frac{1}{\zeta}+\sum_{\nu=1}^{\infty}a_\nu\zeta^\nu=\frac{1}{\zeta}+\sum_{\nu=1}^{\infty}\frac{[(1-\Theta)}{(\mathfrak{q}[\nu]_\mathfrak{q}+\Theta)\left(\frac{[\ell]_\mathfrak{q}}{[\ell]_\mathfrak{q}+\lambda([\nu+\ell+1]_\mathfrak{q}-[\ell]_\mathfrak{q})}\right)^r}\vartheta_\nu\zeta^\nu$$

$$=\frac{\vartheta_0}{\zeta}+\sum_{\nu=1}^{\infty}\frac{1}{\zeta}+\left(\frac{(1-\Theta)}{(\mathfrak{q}[\nu]_\mathfrak{q}+\Theta)\left(\frac{[\ell]_\mathfrak{q}}{[\ell]_\mathfrak{q}+\lambda([\nu+\ell+1]_\mathfrak{q}-[\ell]_\mathfrak{q})}\right)^r}\zeta^\nu\right)\vartheta_\nu$$

$$=\sum_{\nu=0}^{\infty}\vartheta_\nu\mathfrak{f}_\nu.$$

And this completes the proof. □

The convolution conditions is obtained in the next theorem.

Theorem 4 If $\mathfrak{f} \in \Sigma$ of the form (1) and $\mathfrak{g}(\zeta) = \frac{1}{\zeta} + \sum\limits_{\nu=1}^{\infty} c_\nu \zeta^\nu$ be in the class $\Sigma_{\mathfrak{q}}^{*,r}(\Theta, \lambda, \ell)$. Then $(\mathfrak{f} * \mathfrak{g}) \in \Sigma_{\mathfrak{q}}^{*,r}(\sigma, \lambda, \ell)$, where

$$\sigma = 1 - \frac{(1-\Theta)^2 (\mathfrak{q}[\nu]_{\mathfrak{q}} + 1)}{(1-\Theta)^2 + (\mathfrak{q}[\nu]_{\mathfrak{q}} + \Theta)^2 \left(\frac{[\ell]_{\mathfrak{q}}}{[\ell]_{\mathfrak{q}} + \lambda([\nu+\ell+1]_{\mathfrak{q}} - [\ell]_{\mathfrak{q}})} \right)^r}$$

Proof. Since $\mathfrak{f}, \mathfrak{g} \in \Sigma_{\mathfrak{q}}^{*,r}(\Theta, \lambda, \ell)$, then

$$\sum_{\nu=1}^{\infty} \frac{(\mathfrak{q}[\nu]_{\mathfrak{q}} + \Theta)}{(1-\Theta)} \left(\frac{[\ell]_{\mathfrak{q}}}{[\ell]_{\mathfrak{q}} + \lambda([\nu+\ell+1]_{\mathfrak{q}} - [\ell]_{\mathfrak{q}})} \right)^r |a_\nu| \leq 1,$$

and

$$\sum_{\nu=1}^{\infty} \frac{(\mathfrak{q}[\nu]_{\mathfrak{q}} + \Theta)}{(1-\Theta)} \left(\frac{[\ell]_{\mathfrak{q}}}{[\ell]_{\mathfrak{q}} + \lambda([\nu+\ell+1]_{\mathfrak{q}} - [\ell]_{\mathfrak{q}})} \right)^r |a_\nu| \leq 1.$$

We have to find the largest σ such that

$$\sum_{\nu=1}^{\infty} \frac{(\mathfrak{q}[\nu]_{\mathfrak{q}} + \sigma)}{(1-\sigma)} \left(\frac{[\ell]_{\mathfrak{q}}}{[\ell]_{\mathfrak{q}} + \lambda([\nu+\ell+1]_{\mathfrak{q}} - [\ell]_{\mathfrak{q}})} \right)^r |a_\nu| |c_\nu| \leq 1, \tag{13}$$

by using Cauchy-Schwartz inequality, we have

$$\sum_{\nu=1}^{\infty} \frac{(\mathfrak{q}[\nu]_{\mathfrak{q}} + \Theta)}{(1-\Theta)} \left(\frac{[\ell]_{\mathfrak{q}}}{[\ell]_{\mathfrak{q}} + \lambda([\nu+\ell+1]_{\mathfrak{q}} - [\ell]_{\mathfrak{q}})} \right)^r \sqrt{|a_\nu| |c_\nu|} \leq 1, \tag{14}$$

it is enough to show that

$$\sum_{\nu=1}^{\infty} \frac{(\mathfrak{q}[\nu]_{\mathfrak{q}} + \sigma)}{(1-\sigma)} \left(\frac{[\ell]_{\mathfrak{q}}}{[\ell]_{\mathfrak{q}} + \lambda([\nu+\ell+1]_{\mathfrak{q}} - [\ell]_{\mathfrak{q}})} \right)^r |a_\nu| |c_\nu|$$

$$\leq \sum_{\nu=1}^{\infty} \frac{(\mathfrak{q}[\nu]_{\mathfrak{q}} + \Theta)}{(1-\Theta)} \left(\frac{[\ell]_{\mathfrak{q}}}{[\ell]_{\mathfrak{q}} + \lambda([\nu+\ell+1]_{\mathfrak{q}} - [\ell]_{\mathfrak{q}})} \right)^r \sqrt{|a_\nu| |c_\nu|}.$$

This is equivalent to

$$\sqrt{|a_\nu||c_\nu|} \leq \frac{(\mathfrak{q}[\nu]_\mathfrak{q}+\Theta)(1-\sigma)}{(1-\Theta)(\mathfrak{q}[\nu]_\mathfrak{q}+\sigma)}.$$

From (14) we have

$$\sqrt{|a_\nu||c_\nu|} \leq \frac{(1-\Theta)}{(\mathfrak{q}[\nu]_\mathfrak{q}+\Theta)\left(\frac{[\ell]_\mathfrak{q}}{[\ell]_\mathfrak{q}+\lambda([\nu+\ell+1]_\mathfrak{q}-[\ell]_\mathfrak{q})}\right)^r}.$$

Thus it is enough to show that

$$\frac{(1-\Theta)}{(\mathfrak{q}[\nu]_\mathfrak{q}+\Theta)\left(\frac{[\ell]_\mathfrak{q}}{[\ell]_\mathfrak{q}+\lambda([\nu+\ell+1]_\mathfrak{q}-[\ell]_\mathfrak{q})}\right)^r} \leq \frac{(\mathfrak{q}[\nu]_\mathfrak{q}+\Theta)(1-\sigma)}{(1-\Theta)(\mathfrak{q}[\nu]_\mathfrak{q}+\sigma)},$$

$$(1-\Theta)(1-\Theta)(\mathfrak{q}[\nu]_\mathfrak{q}+\sigma) \leq (\mathfrak{q}[\nu]_\mathfrak{q}+\Theta)(1-\sigma)(\mathfrak{q}[\nu]_\mathfrak{q}+\Theta)\left(\frac{[\ell]_\mathfrak{q}}{[\ell]_\mathfrak{q}+\lambda([\nu+\ell+1]_\mathfrak{q}-[\ell]_\mathfrak{q})}\right)^r$$

$$\sigma\left[(1-\Theta)^2+(\mathfrak{q}[\nu]_\mathfrak{q}+\Theta)^2\left(\frac{[\ell]_\mathfrak{q}}{[\ell]_\mathfrak{q}+\lambda([\nu+\ell+1]_\mathfrak{q}-[\ell]_\mathfrak{q})}\right)^r\right]$$

$$\leq (\mathfrak{q}[\nu]_\mathfrak{q}+\Theta)^2\left(\frac{[\ell]_\mathfrak{q}}{[\ell]_\mathfrak{q}+\lambda([\nu+\ell+1]_\mathfrak{q}-[\ell]_\mathfrak{q})}\right)^r-(1-\Theta)^2\mathfrak{q}[\nu]_\mathfrak{q}$$

then

$$\sigma \leq \frac{(\mathfrak{q}[\nu]_\mathfrak{q}+\Theta)^2\left(\frac{[\ell]_\mathfrak{q}}{[\ell]_\mathfrak{q}+\lambda([\nu+\ell+1]_\mathfrak{q}-[\ell]_\mathfrak{q})}\right)^r-(1-\Theta)^2\mathfrak{q}[\nu]_\mathfrak{q}}{(1-\Theta)^2+(\mathfrak{q}[\nu]_\mathfrak{q}+\Theta)^2\left(\frac{[\ell]_\mathfrak{q}}{[\ell]_\mathfrak{q}+\lambda([\nu+\ell+1]_\mathfrak{q}-[\ell]_\mathfrak{q})}\right)^r}$$

$$\leq 1-\frac{(1-\Theta)^2(\mathfrak{q}[\nu]_\mathfrak{q}+1)}{(1-\Theta)^2+(\mathfrak{q}[\nu]_\mathfrak{q}+\Theta)^2\left(\frac{[\ell]_\mathfrak{q}}{[\ell]_\mathfrak{q}+\lambda([\nu+\ell+1]_\mathfrak{q}-[\ell]_\mathfrak{q})}\right)^r}.$$

□

Theorem 5 If $\mathfrak{f} \in \Sigma$ of the form (1) and $\mathfrak{g}(\zeta) = \frac{1}{\zeta} + \sum_{\nu=1}^{\infty} c_\nu \zeta^\nu$ be in the class $\Sigma_\mathfrak{q}^{*,r}(\Theta, \lambda, \ell)$. Then

$$\mathfrak{h}(\zeta) = \frac{1}{\zeta} + \sum_{\nu=1}^{\infty}(|a_\nu|^2+|c_\nu|^2)\zeta^\nu \in \Sigma_\mathfrak{q}^{*,r}(\rho, \lambda, \ell),$$

where

$$\rho = 1 - \frac{2(1-\Theta)^2(\mathfrak{q}+1)!}{2(1-\Theta)^2 + (\mathfrak{q}+\Theta)^2\left(\frac{[\ell]_\mathfrak{q}}{[\ell]_\mathfrak{q}+\lambda([\ell+2]_\mathfrak{q}-[\ell]_\mathfrak{q})}\right)^r}.$$

Proof. We want to find the largest ρ such that

$$\sum_{\nu=1}^{\infty} \frac{(\mathfrak{q}[\nu]_\mathfrak{q}+\rho)}{(1-\rho)} \left(\frac{[\ell]_\mathfrak{q}}{[\ell]_\mathfrak{q}+\lambda([\nu+\ell+1]_\mathfrak{q}-[\ell]_\mathfrak{q})}\right)^r (|a_\nu|^2+|c_\nu|^2) \leq 1. \tag{15}$$

Since $\mathfrak{f}, \mathfrak{g} \in \Sigma_\mathfrak{q}^{*,r}(\Theta, \lambda, \ell)$, then from Theorem 1

$$\sum_{\nu=1}^{\infty} \left[\frac{(\mathfrak{q}[\nu]_\mathfrak{q}+\Theta)}{(1-\Theta)} \left(\frac{[\ell]_\mathfrak{q}}{[\ell]_\mathfrak{q}+\lambda([\nu+\ell+1]_\mathfrak{q}-[\ell]_\mathfrak{q})}\right)^r\right]^2 |a_\nu|^2$$

$$\leq \left[\sum_{\nu=1}^{\infty} \frac{(\mathfrak{q}[\nu]_\mathfrak{q}+\Theta)}{(1-\Theta)} \left(\frac{[\ell]_\mathfrak{q}}{[\ell]_\mathfrak{q}+\lambda([\nu+\ell+1]_\mathfrak{q}-[\ell]_\mathfrak{q})}\right)^r |a_\nu|\right]^2 \leq 1 \tag{16}$$

and

$$\sum_{\nu=1}^{\infty} \left[\frac{(\mathfrak{q}[\nu]_\mathfrak{q}+\Theta)}{(1-\Theta)} \left(\frac{[\ell]_\mathfrak{q}}{[\ell]_\mathfrak{q}+\lambda([\nu+\ell+1]_\mathfrak{q}-[\ell]_\mathfrak{q})}\right)^r\right]^2 |c_\nu|^2 \tag{17}$$

$$\leq \left[\sum_{\nu=1}^{\infty} \frac{(\mathfrak{q}[\nu]_\mathfrak{q}+\Theta)}{(1-\Theta)} \left(\frac{[\ell]_\mathfrak{q}}{[\ell]_\mathfrak{q}+\lambda([\nu+\ell+1]_\mathfrak{q}-[\ell]_\mathfrak{q})}\right)^r |c_\nu|\right]^2 \leq 1.$$

Summing (16) and (17), we get

$$\sum_{\nu=1}^{\infty} \frac{1}{2} \left[\frac{(\mathfrak{q}[\nu]_\mathfrak{q}+\Theta)}{(1-\Theta)} \left(\frac{[\ell]_\mathfrak{q}}{[\ell]_\mathfrak{q}+\lambda([\nu+\ell+1]_\mathfrak{q}-[\ell]_\mathfrak{q})}\right)^r\right]^2 (|a_\nu|^2+|c_\nu|^2) \leq 1.$$

But, $\mathfrak{h}(\zeta) \in \Sigma_\mathfrak{q}^{*,r}(\Theta, \lambda, \ell)$ iff

$$\sum_{\nu=1}^{\infty} \left[\frac{(\mathfrak{q}[\nu]_\mathfrak{q}+\rho)}{(1-\rho)} \left(\frac{[\ell]_\mathfrak{q}}{[\ell]_\mathfrak{q}+\lambda([\nu+\ell+1]_\mathfrak{q}-[\ell]_\mathfrak{q})}\right)^r\right] (|a_\nu|^2+|c_\nu|^2) \leq 1.$$

This true if

$$\left[\frac{(\mathfrak{q}[\nu]_{\mathfrak{q}}+\rho)}{(1-\rho)}\left(\frac{[\ell]_{\mathfrak{q}}}{[\ell]_{\mathfrak{q}}+\lambda([\nu+\ell+1]_{\mathfrak{q}}-[\ell]_{\mathfrak{q}})}\right)^{r}\right]$$

$$\leq\frac{1}{2}\left[\frac{(\mathfrak{q}[\nu]_{\mathfrak{q}}+\Theta)}{(1-\Theta)}\left(\frac{[\ell]_{\mathfrak{q}}}{[\ell]_{\mathfrak{q}}+\lambda([\nu+\ell+1]_{\mathfrak{q}}-[\ell]_{\mathfrak{q}})}\right)^{r}\right]^{2}\frac{(\mathfrak{q}[\nu]_{\mathfrak{q}}+\rho)}{(1-\rho)}$$

$$\leq\frac{(\mathfrak{q}[\nu]_{\mathfrak{q}}+\Theta)^{2}}{2(1-\Theta)^{2}}\left(\frac{[\ell]_{\mathfrak{q}}}{[\ell]_{\mathfrak{q}}+\lambda([\nu+\ell+1]_{\mathfrak{q}}-[\ell]_{\mathfrak{q}})}\right)^{r}\frac{(1-\rho)}{(\mathfrak{q}[\nu]_{\mathfrak{q}}+\rho)}$$

$$\geq\frac{2(1-\Theta)^{2}}{(\mathfrak{q}[\nu]_{\mathfrak{q}}+\Theta)^{2}\left(\frac{[\ell]_{\mathfrak{q}}}{[\ell]_{\mathfrak{q}}+\lambda([\nu+\ell+1]_{\mathfrak{q}}-[\ell]_{\mathfrak{q}})}\right)^{r}}=\psi(\nu),$$

where $\psi(\nu)$ is decreasing of ν and has a maximum value $\psi(\nu)=\frac{2(1-\Theta)^{2}}{(\mathfrak{q}+\Theta)^{2}\left(\frac{[\ell]_{\mathfrak{q}}}{[\ell]_{\mathfrak{q}}+\lambda([\ell+2]_{\mathfrak{q}}-[\ell]_{\mathfrak{q}})}\right)^{r}}$ attains at $\nu=1$.

$$\frac{(1-\rho)}{(\mathfrak{q}+\rho)}\geq\frac{2(1-\Theta)^{2}}{(\mathfrak{q}+\Theta)^{2}\left(\frac{[\ell]_{\mathfrak{q}}}{[\ell]_{\mathfrak{q}}+\lambda([\ell+2]_{\mathfrak{q}}-[\ell]_{\mathfrak{q}})}\right)^{r}},$$

$$(1-\rho)(\mathfrak{q}+\Theta)^{2}\left(\frac{[\ell]_{\mathfrak{q}}}{[\ell]_{\mathfrak{q}}+\lambda([\ell+2]_{\mathfrak{q}}-[\ell]_{\mathfrak{q}})}\right)^{r}\geq 2(1-\Theta)^{2}(\mathfrak{q}+\rho),$$

on simplification, we get

$$\rho\leq 1-\frac{2(1-\Theta)^{2}(\mathfrak{q}+1)}{2(1-\Theta)^{2}+(\mathfrak{q}+\Theta)^{2}\left(\frac{[\ell]_{\mathfrak{q}}}{[\ell]_{\mathfrak{q}}+\lambda([\ell+2]_{\mathfrak{q}}-[\ell]_{\mathfrak{q}})}\right)^{r}}.$$

This completes the proof. □

3. The q-Bernardi integral operator

Bernardi q-meromorphic refers to a class of complex functions that are both meromorphic (analytic except for poles) and exhibit q-analogue properties, often studied in the context of q-difference equations or q-integral operators like the generalized q-Bernardi integral operator. Researchers use these operators to define and study new subclasses of these functions, such as meromorphic q-starlike and q-convex functions, examining their coefficient estimates, inclusion properties, and other analytical characteristics.

The q-Bernardi integral operator generalizes the classical Bernardi operator into the framework of q-calculus, acting as a linear coefficient multiplier operator on analytic functions, with wide applications in q-versions of GFT.

For a function $\mathfrak{f}\in\Sigma$, we denote by $\mathfrak{I}_{\rho,\,\mathfrak{q}}$ the q-Bernardi integral operator $\mathfrak{I}_{\rho,\,\mathfrak{q}}$ defined by (see [17, 26, 27]).

$$\mathfrak{F}_{\mathrm{q}}(\zeta) = \mathfrak{I}_{\rho,\,\mathrm{q}}[\mathfrak{f}(\zeta)] = \frac{[\rho]_{\mathrm{q}}}{\zeta^{\rho+1}} \int_0^{\zeta} \tau^{\rho}\, \mathfrak{f}(\tau)\, d_{\mathrm{q}}\tau \qquad (\rho \in \mathbb{N}). \tag{18}$$

The q-Bernardi integral operator $\mathfrak{I}_{\rho,\,q}\colon \Sigma \to \Sigma$, defined in (18), satisfies the following relationship:

$$\mathrm{q}^{\rho+1}\zeta\partial_{\mathrm{q}}\mathfrak{D}_{\mathrm{q}}^{r}(\lambda,\,\ell)\mathfrak{F}_{\mathrm{q}}(\zeta) = [\rho]_{\mathrm{q}}\mathfrak{D}_{\mathrm{q}}^{r}(\lambda,\,\ell)\mathfrak{f}(\zeta) - [\rho+1]_{\mathrm{q}}\mathfrak{D}_{\mathrm{q}}^{r}(\lambda,\,\ell)\mathfrak{F}_{\mathrm{q}}(\zeta). \tag{19}$$

We now state and prove the following result.

Theorem 6 If $\mathfrak{f} \in \Sigma$ defined by (1) is in the function class $\Sigma_{\mathrm{q}}^{*,\,r}(\Theta,\,\lambda,\,\ell)$, and $Re\{-\frac{1}{\mathrm{q}}\varphi + \frac{[\rho+1]_{\mathrm{q}}}{\mathrm{q}^{\rho+1}}\} > 0$ then $\mathfrak{F}_{\mathrm{q}}(\zeta)$ defined by (18) also belongs to the class $\Sigma_{\mathrm{q}}^{*,\,r}(\Theta,\,\lambda,\,\ell)$.

Proof. Let $\mathfrak{f} \in \Sigma_{\mathrm{q}}^{*,\,r}(\Theta,\,\lambda,\,\ell)$, we put

$$\omega(\zeta) = \frac{-\mathrm{q}\zeta\partial_{\mathrm{q}}(\mathfrak{D}_{\mathrm{q}}^{r}(\lambda,\,\ell)\mathfrak{F}_{\mathrm{q}}(\zeta))}{\mathfrak{D}_{\mathrm{q}}^{r}(\lambda,\,\ell)\mathfrak{F}_{\mathrm{q}}(\zeta)}, \tag{20}$$

where $\omega(\zeta)$ is analytic in $\mathbb{U}$ with $\omega(0) = 1$.

From (19), we show that

$$\omega(\zeta) = -\frac{[\rho]_{\mathrm{q}}}{\mathrm{q}^{\rho}} \frac{\mathfrak{D}_{\mathrm{q}}^{r}(\lambda,\,\ell)\mathfrak{f}(\zeta)}{\mathfrak{D}_{\mathrm{q}}^{r}(\lambda,\,\ell)\mathfrak{F}_{\mathrm{q}}(\zeta)} + \frac{[\rho+1]_{\mathrm{q}}}{\mathrm{q}^{\rho}}.$$

On q-logarithmic differentiation we get

$$\omega(\zeta) + \frac{\zeta\partial_{\mathrm{q}}\omega(\zeta)}{-\frac{1}{\mathrm{q}}\omega(\zeta) + \frac{[\rho+1]_{\mathrm{q}}}{\mathrm{q}^{\rho+1}}} = \frac{-\mathrm{q}\zeta\partial_{\mathrm{q}}(\mathfrak{D}_{\mathrm{q}}^{r}(\lambda,\,\ell)\mathfrak{f}(\zeta))}{\mathfrak{D}_{\mathrm{q}}^{r}(\lambda,\,\ell)\mathfrak{f}(\zeta)} \prec \varphi(\zeta). \tag{21}$$

Since $\mathfrak{f} \in \Sigma_{\mathrm{q}}^{*,\,r}(\Theta,\,\lambda,\,\ell)$, we can revise (21) as

$$\omega(\zeta) + \frac{\zeta\partial_{\mathrm{q}}\omega(\zeta)}{-\frac{1}{\mathrm{q}}\omega(\zeta) + \frac{[\rho+1]_{\mathrm{q}}}{\mathrm{q}^{\rho}}} \prec \varphi(\zeta).$$

Now, by using Lemma 1, we conclude $\omega(\zeta) \prec \varphi(\zeta)$. Consequently $\frac{-\mathrm{q}\zeta\partial_{\mathrm{q}}(\mathfrak{D}_{\mathrm{q}}^{r}(\lambda,\,\ell)\mathfrak{F}_{\mathrm{q}}(\zeta))}{\mathfrak{D}_{\mathrm{q}}^{r}(\lambda,\,\ell)\mathfrak{F}_{\mathrm{q}}(\zeta)} \prec \varphi(\zeta)$. Hence $\mathfrak{F}_{\mathrm{q}}(\zeta) \in \Sigma_{\mathrm{q}}^{*,\,r}(\Theta,\,\lambda,\,\ell)$. □

4. Neighborhoods on $\Sigma_{\mathrm{q}}^{\varepsilon*,\,r}(\Theta,\,\lambda,\,\ell)$

In the field of complex analysis, understanding neighborhoods is essential as they play a key role in studying functions of complex numbers. A neighborhood in this context refers to a set of points surrounding a particular point in the complex plane. Just like in our physical neighborhood, where we interact with nearby places, in complex analysis, a neighborhood allows mathematicians to examine the behavior of complex functions near specific points. This concept helps to explore important ideas like limits, continuity, and differentiability, which are fundamental to the study of complex systems.

Following the earlier works by Goodman [28], Ruscheweyh [29] and Liu and Srivastava [30], we define the α-neighborhood of a function $\mathfrak{f} \in \Sigma$ by

$$N_\alpha(\mathfrak{f}) = \left\{ \mathfrak{h} \in \Sigma \colon \mathfrak{h}(\zeta) = \frac{1}{\zeta} + \sum_{\nu=1}^{\infty} \gamma_\nu \zeta^\nu \quad \text{and} \quad \sum_{\nu=1}^{\infty} \nu \, |a_\nu - \gamma_\nu| \leq \alpha, \quad 0 \leq \alpha < 1 \right\}. \tag{22}$$

For the identity function $I(\zeta) = \zeta$, we have

$$N_\alpha(I) = \left\{ \mathfrak{h} \in \Sigma \colon \mathfrak{h}(\zeta) = \frac{1}{\zeta} + \sum_{\nu=1}^{\infty} \gamma_\nu \zeta^\nu \quad \text{and} \quad \sum_{\nu=1}^{\infty} \nu \, |\gamma_\nu| \leq \alpha \right\}. \tag{23}$$

Definition 3 If $\mathfrak{f} \in \Sigma$ of the form (1) be in the class $\Sigma_{\mathfrak{q}}^{\varepsilon *, r}(\Theta, \lambda, \ell)$ if there exist $\mathfrak{h} \in \Sigma_{\mathfrak{q}}^{*, r}(\Theta, \lambda, \ell)$ such that

$$\left| \frac{\mathfrak{f}(\zeta)}{\mathfrak{h}(\zeta)} - 1 \right| < 1 - \varepsilon, \quad (\zeta \in \mathbb{U}^*, \ 0 \leq \varepsilon < 1).$$

Theorem 7 If $\mathfrak{h} \in \Sigma_{\mathfrak{q}}^{*, r}(\Theta, \lambda, \ell)$ and

$$\varepsilon = 1 - \frac{\alpha(\mathfrak{q} + \Theta) \left([\ell]_{\mathfrak{q}} \right)^r}{(\mathfrak{q} + \Theta) \left([\ell]_{\mathfrak{q}} \right)^r - \left([\ell]_{\mathfrak{q}} + \lambda([\ell + 2]_{\mathfrak{q}} - [\ell]_{\mathfrak{q}} \right)^r (1 - \Theta)}, \tag{24}$$

then $N_\alpha(\mathfrak{h}) \subset \Sigma_{\mathfrak{q}}^{\varepsilon *, r}(\Theta, \lambda, \ell)$.

Proof. Consider $\mathfrak{f} \in N_\alpha(\mathfrak{h})$, then from (22), we have

$$\sum_{\nu=1}^{\infty} [\nu]_{\mathfrak{q}} \, |a_\nu - \gamma_\nu| \leq \alpha \Rightarrow \sum_{\nu=1}^{\infty} |a_\nu - \gamma_\nu| \leq \alpha, \ (\nu \in \mathbb{N}).$$

Since $\mathfrak{h} \in \Sigma_{\mathfrak{q}}^{*, r}(\Theta, \lambda, \ell)$, then from Theorem 1 we have

$$(\mathfrak{q} + \Theta) \left(\frac{[\ell]_{\mathfrak{q}}}{[\ell]_{\mathfrak{q}} + \lambda([\ell + 2]_{\mathfrak{q}} - [\ell]_{\mathfrak{q}})} \right)^r \sum_{\nu=1}^{\infty} |\gamma_\nu|$$

$$\leq \sum_{\nu=1}^{\infty} (\mathfrak{q}[\nu]_{\mathfrak{q}} + \Theta) \left(\frac{[\ell]_{\mathfrak{q}}}{[\ell]_{\mathfrak{q}} + \lambda([\nu + \ell + 1]_{\mathfrak{q}} - [\ell]_{\mathfrak{q}})} \right)^r |\gamma_\nu|$$

$$\leq 1 - \Theta$$

$$\sum_{\nu=1}^{\infty}|\gamma_\nu| \leq \frac{1-\Theta}{(\mathfrak{q}+\Theta)\left(\frac{[\ell]_\mathfrak{q}}{[\ell]_\mathfrak{q}+\lambda([\ell+2]_\mathfrak{q}-[\ell]_\mathfrak{q})}\right)^r}$$

$$\sum_{\nu=1}^{\infty}|\gamma_\nu| \leq \frac{\left([\ell]_\mathfrak{q}+\lambda([\ell+2]_\mathfrak{q}-[\ell]_\mathfrak{q})\right)^r(1-\Theta)}{(\mathfrak{q}+\Theta)\left([\ell]_\mathfrak{q}\right)^r},$$

and hence

$$\left|\frac{\mathfrak{f}(\zeta)}{\mathfrak{h}(\zeta)}-1\right| < \frac{\sum_{\nu=1}^{\infty}|a_\nu-\gamma_\nu|}{1-\sum_{\nu=1}^{\infty}|\gamma_\nu|}$$

$$\leq \frac{\alpha(\mathfrak{q}+\Theta)\left([\ell]_\mathfrak{q}\right)^r}{(\mathfrak{q}+\Theta)\left([\ell]_\mathfrak{q}\right)^r-\left([\ell]_\mathfrak{q}+\lambda([\ell+2]_\mathfrak{q}-[\ell]_\mathfrak{q})\right)^r(1-\Theta)}$$

$$=1-\varepsilon.$$

Thus, for given ε in (24) and by definition (22) we have $\mathfrak{f} \in \Sigma_\mathfrak{q}^{\varepsilon *, r}(\Theta, \lambda, \ell)$, and the proof is complete. □

5. Conclusion

This study introduces a new subclass of meromorphic functions in the punctured unit disk $\mathbb{U}^*$ by employing the $\mathfrak{q}$-analogue of a multiplier operator associated with the family of linear operators $\Sigma_\mathfrak{q}^{*, r}(\Theta, \lambda, \ell)$. The structure of this operator enables a systematic analysis of the geometric properties of the proposed class. For functions belonging to this class, we have established coefficient conditions and investigated neighborhood properties, convolution relations, and a closure theorem involving convex combinations. In several cases, the sharpness of the obtained results has been demonstrated, underscoring both the theoretical robustness and the applicability of the developed approach.

The flexibility of the operator framework provides a foundation for generating additional subclasses of meromorphic functions, opening avenues for further research in geometric function theory. The findings presented here are new and are expected to stimulate continued exploration of related operators and their analytic implications.

Data availability statement

The data used to support the findings of this study are included within the article.

Conflict of interest

The authors declare no competing financial interest.

References

[1] Miller SS, Mocanu PT. Differential subordinations and univalent functions. *Michigan Mathematical Journal*. 1981; 28: 157-171. Available from: https://doi.org/10.1307/mmj/1029002507.

[2] Miller SS, Mocanu PT. *Differential Subordinations: Theory and Applications*. CRC Press; 2000. Available from: https://doi.org/10.1201/9781482289817.

[3] Bulboaca T. *Differential Subordinations and Superordinations, Recent Results*. House of Scientific Book Publication: Cluj-Napoca, Romania; 2005.

[4] Exton H. *q-Hypergeometric Functions and Applications*. Hastead Press: New York, NY, USA; 1983.

[5] Gasper G, Rahman M. Some systems of multivariable orthogonal q-Racah polynomials. *Ramanujan Journal*. 2007; 13: 389-405.

[6] Frenkel E, Mukhin E. Combinatorics of q-characters of finite-dimensional representations of quantum affine algebras. *Communications in Mathematical Physics*. 2001; 216: 23-57.

[7] Srivastava HM. A survey of some recent developments on higher transcendental functions of analytic number theory and applied mathematics. *Symmetry*. 2021; 13: 2294. Available from: https://doi.org/10.3390/sym13122294.

[8] Aral A, Gupta V, Agarwal RP. *Applications of q-Calculus in Operator Theory*. Springer Science + Business Media: New York, NY, USA; 2013.

[9] Srivastava HM. Univalent functions, fractional calculus and associated generalized hypergeometric functions. In: Srivastava HM, Owa S. (eds.) *Univalent Functions, Fractional Calculus, and Their Applications*. Chichester, UK: Ellis Horwood Limited; 1989. p.329-354.

[10] Matthew OO, Moses OB, Davids EO. On geometric properties of certain class of analytic functions connected with a multiplier transformation. *International Journal of Development Mathematics*. 2025; 2: 190-201. Available from: https://doi.10.62054/ijdm/0202.10.

[11] Ismail MEH, Merkes E, Styer D. A generalization of starlike functions. *Complex Variables, Theory and Application*. 1990; 14: 77-84.

[12] Jackson FH. On q-functions and a certain difference operator. *Transactions of the Royal Society of Edinburgh*. 1908; 46: 253-281.

[13] Jackson FH. On q-definite integrals. *Quarterly Journal of Pure and Applied Mathematics*. 1910; 41: 193-203.

[14] Aral A, Gupta V. Generalized q-Baskakov operators. *Mathematics Slovaca*. 2016; 61: 619-634.

[15] Elhaddad S, Darus M. On meromorphic functions defined by a new operator containing the Mittag-Leffler function. *Symmetry*. 2019; 11: 210.

[16] Ali EE, Oros GI, El-Ashwah RM, Kota WY, Albalahi AM. Geometric properties connected with a certain multiplier integral q-analogue operator. *Symmetry*. 2024; 16: 863.

[17] Ali EE, Srivastava HM, Lashin AY, Albalahi AM. Applications of some subclasses of meromorphic functions associated with the q-derivatives of the q-Binomials. *Mathematics*. 2023; 11: 2496.

[18] Ali EE, El-Ashwah RM, Albalahi AM, Sidaoui R, Ennaceur M, Cortez MV. Convolution results with subclasses of p-valent meromorphic function connected with q-difference operator. *Mathematics*. 2024; 12: 3548.

[19] Kota WY, El-Ashwah RM. Some application of subordination theorems associated with fractional q-calculus operator. *Mathematica Bohemica*. 2023; 148(2): 131-148.

[20] Wang B, Srivastava R, Liu JL. A certain subclass of multivalent analytic functions defined by the q-difference operator related to the Janowski functions. *Mathematics*. 2021; 9: 1706.

[21] Ali EE, Srivastava HM, Albalahi AM. Subclasses of p-valent κ-uniformly convex and starlike functions defined by the q-derivative operator. *Mathematics*. 2023; 11: 2578.

[22] Bulboaca T, Aouf MK, El-Ashwah RM. Convolution properties for subclasses of meromorphic univalent functions of complex order. *Filomat*. 2012; 26: 153-163.

[23] El-Ashwah RM. Argument properties for p-valent meromorphic functions defined by differintegral operator. *Southeast Asian Bulletin of Mathematics*. 2018; 42: 359-365.

[24] Pommerenke C. On meromorphic starlike functions. *Pacific Journal of Mathematics*. 1963: 221-235.

[25] Shamsan H, Latha S. On generalized bounded Mocanu variation related to q-derivative and conic regions. *Annals of Pure and Applied Mathematics*. 2018; 17: 67-83.

[26] Long PH, Tang H, Wang WS. Functional inequalities for several classes of q-starlike and q-convex type analytic and multivalent functions using a generalized Bernardi integral operator. *AIMS Mathematics*. 2021; 6: 1191-1208.

[27] Srivastava HM, Hadi SH, Darus M. Some subclasses of p-valent γ-uniformly type q-starlike and q-convex functions defined by using a certain generalized q-Bernardi integral operator. *Journal of the Royal Academy of Exact, Physical, and Natural Sciences Series A Mathematics*. 2023; 117(1): 50.

[28] Goodman AW. Univalent functions and nonanalytic curves. *Proceedings of the American Mathematical Society*. 1957; (8)3: 598-601.

[29] Ruscheweyh S. Neighborhoods of univalent functions. *Proceedings of the American Mathematical Society*. 1981; 81(4): 521-527.

[30] Liu JL, Srivastava HM. A linear operator and associated families of meromorphically multivalent functions. *Journal of Mathematical Analysis and Applications*. 2001; 259: 566-581.