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Research Article

Almost Periodic Uniform Stability of Delayed Fractional-Order Quaternion-Valued Fuzzy Neural Networks with Stochastic Terms Via a Direct Method

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Abstract: Fractional-order neural networks are vital for modeling neuronal interactions in information processing, and exploring their existence and stability remains a key research challenge. This paper aims to achieve almost periodic uniform stability for fractional-order stochastic fuzzy neural networks in the quaternion field, which is crucial for ensuring reliable and sustained periodic performance in dynamic environments, essential for applications like simulating biological neural systems. We employ a direct method without decomposing quaternion-valued networks into real-valued counterparts, and by leveraging fixed point theorem, fundamental fractional calculus properties, and various inequality techniques, we establish the existence of a unique almost periodic solution under specific conditions. Additionally, by constructing an error system and applying uniform stability definitions with inequality methods, we derive conditions for almost periodic uniform stability. Finally, MATLAB simulations validate the feasibility and accuracy of our results.

***Keywords*:** fractional-order fuzzy neural networks, quaternions, stochastic neural networks, almost periodic, uniform stability

MSC: 34K50, 34K20, 34K14

1. Introduction

As part of the study of the human brain, neural networks have a history spanning hundreds of years. Initially, Cohen and Grossberg, along with Hopfield and Tank, proposed recurrent neural networks, which are significant for hardware implementation due to their capability to address optimization and computational challenges [1–3]. In recent decades, the rapid advancement of science and technology has led to the widespread application of neural networks across various fields, including image processing, associative memory, pattern recognition, and the resolution of nonlinear algebraic equations. Consequently, the dynamic properties of traditional neural networks have emerged as a prominent and active area of research, attracting significant attention from numerous scholars. Notably, substantial theoretical advancements have been made, particularly regarding the existence and stability of the solution, attitude control, linear programming and so on [4–8].

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Fractional calculus, originating approximately 300 years ago as a generalized form of integral calculus, has made remarkable theoretical advances. Leveraging its infinite memory property, it offers more precise tools for modeling real-world phenomena such as biological and viscoelastic systems. The significance of fractional-order neural networks lies in their capacity to capture complex dynamics and long-term memory effects in natural and engineered systems. They excel in describing neuronal activities and brain signal transmission in biological systems, while providing enhanced flexibility and accuracy in modeling and controlling complex systems with fractional-order dynamics in engineering applications. Consequently, they hold immense value for developing efficient and robust solutions across diverse fields, including biomedical engineering, robotics, and beyond [9–11].

The enhanced precision in data processing capabilities exhibited by Fractional Order Neural Networks (FONNs) in artificial network computations has garnered considerable attention from researchers, leading to a surge in qualitative analysis and control studies of these networks [12–15]. However, existing research predominantly focuses on real/complex-valued FONNs, creating a critical knowledge gap in three aspects: (1) the unique algebraic properties of Quaternion-valued FONNs (QFONNs) for multidimensional feature encoding, (2) the interplay between fractional dynamics and quaternion algebra in stability analysis, and (3) synchronization protocols for delayed QFONNs considering rotational invariance. In 1843, the Irish mathematician William Rowan Hamilton introduced the concept of quaternions [16]. Unlike traditional real/complex-valued networks that require separate processing channels for multidimensional data (e.g., Red, Green, Blue (RGB) channels in color images), QFONNs leverage quaternion's 4D algebraic structure to achieve simultaneous encoding of interdependent features through a single neuron. This dimensionality reduction capability, mathematically represented as $\mathbf{q} = r + \mathbf{i}x + \mathbf{j}y + \mathbf{k}z$ (where $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are imaginary units), preserves spatial relationships while reducing computational complexity by 75% compared to real-valued counterparts. Consequently, they have been widely adopted in various fields, including computer vision, celestial mechanics, signal processing, and color image analysis.

The stability of neural networks underpins their practical applications; thus, the dynamic behavior of quaternion numerical neural networks has garnered significant interest from researchers [17–19]. While previous studies established stability criteria for discontinuous random neural networks [17] and finite-time synchronization protocols for delayed QFONNs [18], they overlooked two fundamental challenges: (a) the memory effects of fractional-order dynamics ($\mathrm{D}^{\alpha}x(t)$ with $0 < \alpha < 1$) that require modified Lyapunov functions, and (b) the rotational invariance of quaternion algebra ($q \cdot p \neq p \cdot q$) demanding non-commutative stability analysis. For instance, Xiang et al. proposed a novel method employing linear operators to tackle the formal problem of solutions when analyzing the dynamics of discontinuous random neural networks, establishing criteria for the existence and stability of solutions in memristive random neural networks based on quaternions [17]. Recently, researchers such as Song have utilized techniques including the homeomorphism principle, matrix inequality methods, and Lyapunov stability theory to address the stability issues of quaternion-valued neural networks with mixed delays [19].

Fuzzy neural networks are a special type of neural network that integrates fuzzy logic. They employ fuzzy input signals and fuzzy weights, enabling them to accomplish complex evaluation and reasoning tasks through the neural network architecture. This allows them to leverage the parallel processing advantages of neural networks while also utilizing the reasoning strengths of fuzzy systems. In 1996, Yang innovatively proposed a fuzzy neural network model that introduced fuzzy logic and fuzzy rules into the framework of neural networks. Research has shown that fuzzy neural networks play a significant role in fields such as image processing and pattern recognition [20, 21]. Given that the stability and synchronization of fuzzy neural networks are crucial in these dynamic application scenarios, it is essential to conduct in-depth research on their dynamic behaviors.

In recent years, significant achievements have been made in the research on the stability of neural networks. Fractional-order fuzzy neural networks, owing to their unique advantages in modeling complex dynamic systems, have witnessed remarkable progress in the analysis of the existence and stability of solutions: In 2022, Liu et al. targeted fuzzy inertial neural networks with specific time delays and activation functions, and for the first time, proved the existence conditions of fixed-time stability and achieved finite-time synchronization using specific methods Liu et al. [22]. In 2023, Du and Lu extended the research to fractional-order time-delay fuzzy neural networks with uncertain parameters, established finite-time synchronization criteria, and implicitly verified the uniqueness of solutions Du and Lu [23]. In the same year, Gong et al. proposed a time-delay-independent nonlinear fuzzy control method for Takagi-Sugeno (T-S) fuzzy

memory neural networks Gong et al. [24]. In 2024, Narayanan et al. focused on discrete fractional-order systems and provided theoretical support for biomolecular models Narayanan et al. [25]. Meanwhile, notable outcomes have also emerged in the stability research of Quaternion-Valued Neural Networks (QVNNs) and stochastic fuzzy neural networks: In 2009, Reference [26] proposed global stability criteria for time-delay uncertain fuzzy Hopfield neural networks. In 2021, Reference [27] proposed stability criteria for time-varying delays in fractional-order fuzzy cellular neural networks. These studies, by integrating multiple theories, offer crucial theoretical support for image processing, encrypted communication, and fractional-order system control.

Synaptic delays during information transfer within neural networks can result in oscillations, subpar functioning, and noise disturbances, potentially leading to instability, chaotic behavior, diminished performance, and unpredictable variations in the network. Recently, the dynamic characteristics of stochastic neural networks, including stability, robust estimation, solution existence, and synchronization, have garnered significant interest from researchers [28–30].

Inspired by the preceding discussions, this research delves into the almost periodic uniform stability of delayed fractional-order quaternion-valued stochastic fuzzy neural networks. The innovative aspects and contributions of this research are outlined below:

(1) Building on prior research [31], our work represents the inaugural investigation into the almost periodic uniform stability of delayed fractional-order quaternion-valued fuzzy neural networks.

(2) Contrary to previous approaches, our methodology avoids breaking down the quaternion-valued fuzzy system into four real-valued fuzzy systems. By considering the quaternion-valued fuzzy system in its entirety, we employ a non-decomposition strategy that substantially diminishes computational complexity and the difficulties linked to theoretical analysis.

(3) Moreover, our approach holds potential for application in exploring the almost periodic uniformly stable behavior of other delayed fractional-order quaternion-valued neural networks under the influence of stochastic disturbances.

Notations: Let $\mathbb{N} = \{1, 2, \cdots, n\}$, $\mathbb{R}$ and $\mathbb{R}^+$ represent the set of natural numbers, real numbers, and positive real numbers, respectively. The symbol $\mathscr{H}$ denotes the set of quaternions, and $\mathscr{H}^n$ is the n-dimensional quaternion vector space. In quaternions, the skew field is expressed as $\mathscr{H} := \{x | x = x^R + ix^I + jx^J + kx^K\}$, where $x^R, x^I, x^J, x^K \in \mathbb{R}$ and i, j, k adhere to the Hamilton multiplication rules defined by $i^2 = j^2 = k^2 = ijk = -1$, $ij = -ji = k$, $jk = -kj = i$, $ki = -ik = j$. The norms are defined as $\|x_p\|_{\mathscr{H}} = \{\sqrt{x_p \bar{x}_p}\}$ and $\|x_p\|^2_{\mathscr{H}} = \bar{x}_p x_p = x_p \bar{x}_p = (x_p^R)^2 + (x_p^I)^2 + (x_p^J)^2 + (x_p^K)^2$, where $\bar{x}_p = x_p^R - ix_p^I - jx_p^J - kx_p^K$. For any $x = (x_1, x_2, \ldots, x_n)^T \in \mathscr{H}^n$, we define $\|x\|_0 = \max\limits_{p\in\mathbb{N}}\{\|x_p\|_{\mathscr{H}}\}$. Consider $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, P)$ as a complete probability space with a natural filtration $\{\mathscr{F}_t\}_{t\geq 0}$ that satisfies the standard conditions. Let E denote the expectation operator concerning the probability measure. Define $\mathscr{L}^2(P, \mathscr{H}^n)$ as the space of all $\mathscr{H}^n$-valued random variables such that $E(\|X\|_0^2) = \int_\Omega \|X\|_0^2 dP < \infty$. For $X \in \mathscr{L}^2(P, \mathscr{H}^n)$, we denote $\|X\|^2_{\mathscr{H}} = \left(\int_\Omega \|X\|_0^2 dP\right)^{\frac{1}{2}}$, $E\|X\|_0^2 = \int_\Omega \|X\|_0^2 dP$, where $\|X\|_0 = \max\limits_{p\in\mathbb{N}}\{\|X_p\|_{\mathscr{H}}\}$. The family of all bounded, $\mathscr{F}_0$-measurable, $C([-\theta, 0], \mathscr{H}^n)$-valued random variables is denoted by $C^B_{\mathscr{F}_0}([-\theta, 0], \mathscr{H}^n)$.

This paper focuses on the issue of almost periodic uniform stability in distribution for the following quaternion-valued fractional-order stochastic fuzzy neural network with delays:

$$
\begin{aligned}
{}^{C}_{t_0}D^{\lambda}_t \zeta_p(t) = & -\mathscr{C}_p(t)\zeta_p(t) + \sum_{q=1}^{n} \mathscr{A}_{pq}(t) f_q(\zeta_q(t-\theta_q)) + \sum_{q=1}^{n} \mathscr{B}_{pq}(t)\mu_q(t) + u_p(t) \\
& + \bigwedge_{q=1}^{n} \alpha_{pq}(t) g_q(\zeta_q(t-\theta_q)) + \bigvee_{q=1}^{n} \beta_{pq}(t) h_q(\zeta_q(t-\theta_q)) + \bigwedge_{q=1}^{n} \mathscr{T}_{pq}(t)\mu_q(t) \\
& + \bigvee_{q=1}^{n} \mathscr{S}_{pq}(t)\mu_q(t) + \sum_{q=1}^{n} \mathscr{D}_{pq}(t)\delta_{pq}(\zeta_q(t-\theta_q))\frac{d\omega_q(t)}{dt},
\end{aligned} \tag{1}
$$

In the above equation, ${}_{t_0}^{C}D_t^{\lambda}$ represents the Caputo fractional derivative of order $\lambda(0 < \lambda < 1)$; $p \in \mathbb{N}$, where n denotes the number of neurons in the layer; $\zeta_p \in \mathscr{H}$, $u_q \in \mathscr{H}$, $\mu_p \in \mathscr{H}$ are the state, input, and bias of the pth neuron at time t; $\mathscr{C}_p(t) \in \mathbb{R}^+$ represents the self-feedback connection weight; $\mathscr{A}_{pq}(\cdot) \in \mathscr{H}$ is the delayed connection weight from neuron q to neuron p; $\alpha_{pq}(\cdot) \in \mathscr{H}$, $\beta_{pq}(\cdot) \in \mathscr{H}$, $\mathscr{T}_{pq}(\cdot) \in \mathscr{H}$, $\mathscr{S}_{pq}(\cdot) \in \mathscr{H}$ are the elements of the fuzzy feedback minimum template, fuzzy feedback maximum template, fuzzy feedforward minimum template, and fuzzy feedforward maximum template, respectively; $\mathscr{B}_{pq}(t) \in \mathscr{H}$ is feed-forward template; $\bigwedge$ and $\bigvee$ denote the fuzzy AND and fuzzy OR operation, respectively (usually, for $a,\ b \in \mathbb{R}$, $a \bigwedge b = \min\{a,\ b\}$, $a \bigvee b = \max\{a,\ b\}$, $a,\ b \in \mathbb{R}$); θ_q is the transmission delay satisfying $0 \le \theta_q \le \theta(\theta \in \mathbb{R}^+)$; the functions $f_q,\ g_q,\ h_q : \mathscr{H} \to \mathscr{H}$ serve as the activation functions for signal transmission; the vector $\omega(t) = \big(\omega_1(t),\ \omega_2(t), \ldots, \omega_n(t)\big)^T$ represents an n-dimensional Brownian motion defined over a complete probability space. Additionally, δ_{pq} is a Borel-measurable function mapping from $\mathscr{H}$ to $\mathscr{H}$.

The initial condition for system (1) is specified as follows:

$$\zeta_p(s) = \varphi_p(s) \in C^B_{\mathscr{F}_0}([-\theta,\ 0],\ \mathscr{H}^n),\ p \in \mathbb{N},\ s \in [-\theta,\ 0]. \tag{2}$$

Remark 1 This paper proceeds under the assumption that the following equality is valid:

$$\bigwedge_{p=1}^{n}(\zeta_p^R + i\zeta_p^I + j\zeta_p^J + k\zeta_p^K) = \bigwedge_{p=1}^{n}\zeta_p^R + i\bigwedge_{p=1}^{n}\zeta_p^I + j\bigwedge_{p=1}^{n}\zeta_p^J + k\bigwedge_{p=1}^{n}\zeta_p^K.$$

The organization of this paper is outlined below. In section 2, we present the relevant definitions of fractional calculus, several preliminary lemmas, and square-mean almost periodic functions in the quaternion domain, followed by an examination of some properties of square-mean almost periodic functions. Section 3 is dedicated to exploring the existence of the square-mean almost periodic solution for system (1). In section 4, we delve into the almost periodic uniform stability of the square-mean almost periodic solution derived in the previous section. Section 5 provides a numerical example to demonstrate the feasibility of our obtained results. Finally, a concise conclusion is presented in section 6.

2. Preliminaries

Within this section, our initial step involves revisiting the definition and associated properties.

Definition 1 [28] The fractional integral of a function $f(t)$ is defined as:

$$I^{\lambda}f(t) = \frac{1}{\Gamma(\lambda)}\int_{t_0}^{t}(t-s)^{\lambda-1}f(s)ds.$$

The Caputo derivative of order λ for a function $f : [0,\ \infty) \to \mathbb{R}$, which is at least one-times differentiable can be defined as

$${}_{t_0}^{C}D_t^{\lambda}f(t) = \frac{1}{\Gamma(1-\lambda)}\int_{t_0}^{t}\frac{g^{'}(s)}{(t-s)^{\lambda}}ds,$$

for $0 \le \lambda < 1$, where $\Gamma(\cdot)$ is Gamma function defined by $\Gamma(\lambda) = \int_0^{+\infty} t^{\lambda-1}e^{-t}dt$.

Lemma 1 [29, 30] For $q \in \mathbb{N}$, let $x,\ y$ be two states, and $g_q,\ h_q : \mathbb{R} \to \mathbb{R}$ be continuous function, $\alpha_{pq},\ \beta_{pq} \in C(\mathbb{R},\ \mathbb{R})$, we obtain

$$\left| \bigwedge_{q=1}^{n} \alpha_{pq}(t) g_q(x) - \bigwedge_{q=1}^{n} \alpha_{pq}(t) g_q(y) \right| \leq \sum_{q=1}^{n} |\alpha_{pq}(t)||g_q(x) - g_q(y)|,$$

$$\left| \bigvee_{q=1}^{n} \beta_{pq}(t) h_q(v) - \bigvee_{q=1}^{n} \beta_{pq}(t) h_q(x) \right| \leq \sum_{q=1}^{n} |\beta_{pq}(t)||h_q(x) - h_q(y)|.$$

Similarly, we give the following lemma.

Lemma 2 For $q \in \mathbb{N}$, let $\zeta,\ \bar{\zeta}$ be two states of the system (1) and $g_q,\ h_q : \mathscr{H} \to \mathscr{H}$ be continuous function, $\alpha_{pq},\ \beta_{pq} \in C(\mathbb{R},\ \mathscr{H})$, we obtain

$$\left\| \bigwedge_{q=1}^{n} \alpha_{pq}(t) g_q(\zeta) - \bigwedge_{q=1}^{n} \alpha_{pq}(t) g_q(\bar{\zeta}) \right\|_{\mathscr{H}} \leq \sum_{q=1}^{n} \|\alpha_{pq}(t)\|_{\mathscr{H}} \|g_q(\zeta) - g_q(\bar{\zeta})\|_{\mathscr{H}},$$

$$\left\| \bigvee_{q=1}^{n} \beta_{pq}(t) g_q(\zeta) - \bigvee_{q=1}^{n} \beta_{pq}(t) h_q(\bar{\zeta}) \right\|_{\mathscr{H}} \leq \sum_{q=1}^{n} \|\beta_{pq}(t)\|_{\mathscr{H}} \|h_q(\zeta) - h_q(\bar{\zeta})\|_{\mathscr{H}}.$$

Proof. Let $q \in \mathbb{N}$, ζ and $\bar{\zeta}$ be two states of the system (1.1), $g_q,\ h_q : \mathscr{H} \to \mathscr{H}$ be continuous functions, and $\alpha_{pq},\ \beta_{pq} \in C(\mathbb{R},\ \mathscr{H})$. For $\left\| \bigwedge_{q=1}^{n} \alpha_{pq}(t) g_q(\zeta) - \bigwedge_{q=1}^{n} \alpha_{pq}(t) g_q(\bar{\zeta}) \right\|_{\mathscr{H}}$, according to the properties of fuzzy-AND operations and relevant inequalities, we have:

$$\left\| \bigwedge_{q=1}^{n} \alpha_{pq}(t) g_q(\zeta) - \bigwedge_{q=1}^{n} \alpha_{pq}(t) g_q(\bar{\zeta}) \right\|_{\mathscr{H}} \leq \sum_{q=1}^{n} \left\| \alpha_{pq}(t) \right\|_{\mathscr{H}} \left\| g_q(\zeta) - g_q(\bar{\zeta}) \right\|_{\mathscr{H}}$$

Similarly, for $\left\| \bigvee_{q=1}^{n} \beta_{pq}(t) g_q(\zeta) - \bigvee_{q=1}^{n} \beta_{pq}(t) h_q(\bar{\zeta}) \right\|_{\mathscr{H}}$, we can get:

$$\left\| \bigvee_{q=1}^{n} \beta_{pq}(t) g_q(\zeta) - \bigvee_{q=1}^{n} \beta_{pq}(t) h_q(\bar{\zeta}) \right\|_{\mathscr{H}} \leq \sum_{q=1}^{n} \left\| \beta_{pq}(t) \right\|_{\mathscr{H}} \left\| h_q(\zeta) - h_q(\bar{\zeta}) \right\|_{\mathscr{H}}.$$ □

Definition 2 [17] A stochastic process $X : \mathbb{R} \to \mathscr{L}^2_{\mathscr{F}_0}(P,\ \mathscr{H}^n)$ is said to be $\mathscr{L}^2$-continuous if for any $s \in \mathbb{R}$

$$\lim_{t \to s} E\|X(t) - X(s)\|^2_{\mathscr{H}^n} = 0.$$

It is $\mathscr{L}^2$-bounded if $\sup_{t \in \mathbb{R}} E\|X(t)\|_0^2 < \infty$.

Referring to the relevant definitions and lemmas in Reference [17], this paper will present the following definitions:

Definition 3 In the context of stochastic processes, an $\mathscr{L}^2$-continuous stochastic process $X : \mathbb{R} \to \mathscr{L}^2(P,\ \mathscr{H}^n)$ is termed square-mean almost periodic in the Bohr sense if, for every $\varepsilon > 0$, there exists a positive number $l(\varepsilon)$ such that for any real number $a \in \mathbb{R}$, there exists a shift $\tau = \tau(\varepsilon) \in [a,\ a + l]$ fulfilling the condition $E\|f(t+\tau) - f(t)\|_0^2 < \varepsilon$, for all $t \in \mathbb{R}$. The shift τ is referred to as the ε-translation number of f.

The set of all such functions will be denoted by $AP(\mathbb{R},\ \mathscr{L}^2(P,\ \mathscr{H}^n))$.

Definition 4 For $t \geq t_0$, a $\mathscr{F}_t$-measurable stochastic process $\zeta(t) = (\zeta_1(t),\ \zeta_2(t),\ \ldots,\ \zeta_n(t))^T_{t\in\mathbb{R}}$ is called a mild solution of system (1), if $x(t)$ satisfies the following integral equation:

$$\begin{aligned}\zeta_p(t) = F_p(t-t_0)\zeta_p(t_0) + \int_{t_0}^{t}(t-s)^{\lambda-1}\eta_p(t-s)\Bigg[\sum_{q=1}^{n}\mathscr{A}_{pq}(s)f_q(\zeta_q(s-\theta_q))\\ + \sum_{q=1}^{n}\mathscr{B}_{pq}(s)\mu_q(s) + \bigwedge_{q=1}^{n}\alpha_{pq}(s)g_q(\zeta_q(s-\theta_q)) + \bigvee_{q=1}^{n}\beta_{pq}(s)h_q(\zeta_q(s-\theta_q))\\ + \bigwedge_{q=1}^{n}\mathscr{T}_{pq}(s)\mu_q(s) + \bigvee_{q=1}^{n}\mathscr{S}_{pq}(s)\mu_q(s) + u_p(s)\Bigg]ds\\ + \int_{t_0}^{t}(t-s)^{\lambda-1}\eta_p(t-s)\sum_{q=1}^{n}\mathscr{D}_{pq}(s)\delta_{pq}(\zeta_q(s-\theta_q))d\omega_q(s),\end{aligned} \tag{3}$$

where $\zeta_p(t_0) = \varphi(t_0)$, $F_p(t) = \int_0^{+\infty}\xi_\lambda(v)e^{-\mathscr{C}_p t^\lambda v}dv$, $\eta_p(t) = \lambda\int_0^{+\infty}v\xi_\lambda(v)e^{-\mathscr{C}_p t^\lambda v}dv$, $\xi_\lambda(v) = \frac{1}{\lambda}v^{-1-\frac{1}{\lambda}}\rho_\lambda(v^{-\frac{1}{\lambda}}) \geq 0$, $\rho_\lambda = \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}v^{-n\lambda-1}\frac{\Gamma(n\lambda+1)}{n!}\sin(n\pi\lambda)$, $v \in (0,\ +\infty)$, ξ_λ is the probability density function defined on $(0,\ +\infty)$, namely, $\xi_\lambda(v) \geq 0$, $v \in (0,\ +\infty)$, $\int_0^{+\infty}\xi_\lambda(v)dv = 1$ and $\int_0^{+\infty}v\xi_\lambda(v)dv = \frac{1}{\Gamma(\lambda+1)}$.

Lemma 3 [17] Suppose $f_p,\ g_p \in AP(\mathbb{R},\ \mathscr{L}^2(P,\ \mathscr{H}))$, then $f_p \pm g_p,\ f_p \cdot g_p \in AP(\mathbb{R},\ \mathscr{L}^2(P,\ \mathscr{H}))$.

Lemma 4 [17] A function $f \in AP(\mathbb{R},\ \mathscr{H})$ if and only if, $f^b \in AP(\mathbb{R},\ \mathscr{L}^2_{\mathscr{F}_0}(P,\ \mathscr{H}))$, where f^b is the Bochner's transform of f, and is defined as $f^b(t) = \{f(t+s) : s \in [0,\ 1]\}$, $t \in \mathbb{R}$.

To facilitate our discussion, we hereby introduce the following notations:

$$\mathscr{C}_p^+ = \sup_{t\in\mathbb{R}}\|\mathscr{C}_p(t)\|_{\mathscr{H}},\ \mathscr{C}_p = \inf_{t\in\mathbb{R}}\|\mathscr{C}_p(t)\|_{\mathscr{H}},\ \bar{\mathscr{C}} = \max_{p\in\mathbb{N}}\mathscr{C}_p,\ \theta = \max_{q\in\mathbb{N}}\{\theta_q\},\ u_p^+ = \sup_{t\in\mathbb{R}}\|u_p(t)\|_{\mathscr{H}},$$

$$\mathscr{A}_{pq}^+ = \sup_{t\in\mathbb{R}}\|\mathscr{A}_{pq}(t)\|_{\mathscr{H}},\ \mathscr{B}_{pq}^+ = \sup_{t\in\mathbb{R}}\|\mathscr{B}_{pq}(t)\|_{\mathscr{H}},\ \mathscr{D}_{pq}^+ = \sup_{t\in\mathbb{R}}\|\mathscr{D}_{pq}(t)\|_{\mathscr{H}},\ \mathscr{S}_{pq}^+ = \sup_{t\in\mathbb{R}}\|\mathscr{S}_{pq}(t)\|_{\mathscr{H}},$$

$$\alpha_{pq}^+ = \sup_{t\in\mathbb{R}}\|\alpha_{pq}(t)\|_{\mathscr{H}},\ \beta_{pq}^+ = \sup_{t\in\mathbb{R}}\|\beta_{pq}(t)\|_{\mathscr{H}},\ \mu_p^+ = \sup_{t\in\mathbb{R}}\|\mu_p(t)\|_{\mathscr{H}},\ \mathscr{T}_{pq}^+ = \sup_{t\in\mathbb{R}}\|\mathscr{T}_{pq}(t)\|_{\mathscr{H}}.$$

In the entirety of this paper, we operate under the premise that the activation functions adhere to the subsequent assumptions.

(H_1) For all $p,\ q \in \mathbb{N}$, $\mathscr{A}_{pq}, \mathscr{B}_{pq}, \alpha_{pq}, \beta_{pq}, \mathscr{T}_{pq}, \mathscr{S}_{pq}, \mu_q, u_p$ are almost periodic functions in distribution sense.

(H_2) For $q \in \mathbb{N}$, there exist constants $L_q^f, L_q^g, L_q^h, L_{pq}^\delta > 0$, and positive constants $M_q^f, M_q^g, M_q^h, M_q^\mu, M_q^u, M_{pq}^\delta, M_{pq}^{\mathscr{T}}, M_{pq}^{\mathscr{S}}$ such that

$$\|f_q(\zeta) - f_q(\bar{\zeta})\|_{\mathscr{H}} \leq L_q^f \|\zeta - \bar{\zeta}\|_{\mathscr{H}},$$

$$\|g_q(\zeta) - g_q(\bar{\zeta})\|_{\mathscr{H}} \leq L_q^g \|\zeta - \bar{\zeta}\|_{\mathscr{H}},$$

$$\|h_q(\zeta) - h_q(\bar{\zeta})\|_{\mathscr{H}} \leq L_q^h \|\zeta - \bar{\zeta}\|_{\mathscr{H}},$$

$$\|\delta_{pq}(\zeta) - \delta_{pq}(\bar{\zeta})\|_{\mathscr{H}} \leq L_{pq}^{\delta} \|\zeta - \bar{\zeta}\|_{\mathscr{H}},$$

$$\|f_q(\zeta)\|_{\mathscr{H}} \leq M_q^f,\ \|g_q(\zeta)\|_{\mathscr{H}} \leq M_q^g,\ \|h_q(\zeta)\|_{\mathscr{H}} \leq M_q^h,\ \|\delta_{pq}(\zeta)\|_{\mathscr{H}} \leq M_{pq}^{\delta},$$

$$\|\mu_q(\zeta)\|_{\mathscr{H}} \leq M_q^{\mu},\ \|u_q(\zeta)\|_{\mathscr{H}} \leq M_q^u,\ \|\mathscr{T}_{pq}(\zeta)\|_{\mathscr{H}} \leq M_{pq}^{\mathscr{T}},\ \|\mathscr{S}_{pq}(\zeta)\|_{\mathscr{H}} \leq M_{pq}^{\mathscr{S}},$$

for all $\zeta,\ \bar{\zeta} \in \mathscr{H}$.

(H_3) For $p, q \in \mathbb{N}$, the following inequalities

$$\Delta := \max_{p\in\mathbb{N}} \left\{ \frac{16}{\mathscr{C}_p^2} \left[\sum_{q=1}^{n} (\mathscr{A}_{pq}^{+})^2 \sum_{q=1}^{n} (L_q^f)^2 + \sum_{q=1}^{n} (\alpha_{pq}^{+})^2 \sum_{q=1}^{n} (L_q^g)^2 + \sum_{q=1}^{n} (\beta_{pq}^{+})^2 \sum_{q=1}^{n} (L_q^h)^2 \right.\right.$$

$$\left.\left. + \frac{\mathscr{C}_p}{2} \sum_{q=1}^{n} (\mathscr{D}_{pq}^{+})^2 \sum_{q=1}^{n} (L_{pq}^{\delta})^2 \right] \right\} \leq 1$$

hold.

3. The existence of almost periodic solutions

In this section, our objective is to address the existence problem pertaining to the solutions of system (1) and derive sufficient conditions that guarantee the existence of the almost periodic solution.

To establish the existence of the almost periodic solution, we will invoke the fixed point theorem, alongside certain definitions, lemmas, and inequality techniques. Initially, we denote by $UCB(\mathbb{R},\ \mathscr{L}^2(P,\ \mathscr{H}^n))$ the space encompassing all bounded and uniformly continuous functions originating from $\mathbb{R}$ to $\mathscr{L}^2(P,\ \mathscr{H}^n)$.

Subsequently, we define the subsequent Banach space. Let

$$\mathscr{X} = \{\varphi = (\varphi_1,\ \varphi_2,\ \ldots,\ \varphi_n)^T \in UCB(\mathbb{R},\ \mathscr{L}^2(P,\ \mathscr{H}^n))\},$$

endowed with the norm $\|\varphi\|_{\mathscr{X}} = \sup_{t\in\mathbb{R}} \{E\|\varphi(t)\|_0^2\}^{\frac{1}{2}}$. Under this norm, $\mathscr{X}$ constitutes a Banach space.

Set $\varphi^0 = (\varphi_1^0,\ \varphi_2^0,\ \ldots,\ \varphi_n^0)^T$, where

$$\varphi_p^0(t) = \int_{-\infty}^{t} (t-s)^{\lambda-1}\eta_p(t-s)\Bigg[\sum_{q=1}^{n}\mathscr{B}_{pq}(s)\mu_q(s) + \bigwedge_{q=1}^{n}\mathscr{T}_{pq}(s)\mu_q(s)$$

$$+ \bigvee_{q=1}^{n}\mathscr{S}_{pq}(s)\mu_q(s) + u_p(s)\Bigg]ds,\ p \in \mathbb{N},$$

and there exists a constant $\mathscr{K}$, such that $\|\varphi^0\|_{\mathscr{X}} \leq \mathscr{K}$.

Definition 5 A stochastic process $\zeta(t) = (\zeta_1(t),\ \zeta_2(t),\ \ldots,\ \zeta_n(t))_{t\in\mathbb{R}}^T$ that is measurable with respect to the filtration $\mathscr{F}_t$ is referred to as a mild solution of system (1) if $x(t)$ adheres to the subsequent stochastic integral equation:

$$\zeta_p(t) = \int_{-\infty}^{t} (t-s)^{\lambda-1}\eta_p(t-s)\Bigg[\sum_{q=1}^{n}\mathscr{A}_{pq}(s)f_q(\zeta_q(s-\theta_q))$$

$$+ \sum_{q=1}^{n}\mathscr{B}_{pq}(s)\mu_q(s) + + \bigwedge_{q=1}^{n}\alpha_{pq}(s)g_q(\zeta_q(s-\theta_q)) + \bigvee_{q=1}^{n}\beta_{pq}(s)h_q(\zeta_q(s-\theta_q))$$

$$+ \bigwedge_{q=1}^{n}\mathscr{T}_{pq}(s)\mu_q(s) + \bigvee_{q=1}^{n}\mathscr{S}_{pq}(s)\mu_q(s) + u_p(s)\Bigg]ds$$

$$+ \int_{-\infty}^{t} (t-s)^{\lambda-1}\eta_p(t-s)\sum_{q=1}^{n}\mathscr{D}_{pq}(s)\delta_{pq}(\zeta_q(s-\theta_q))d\omega_q(s). \tag{4}$$

Proof. From the Definition 4, letting $t_0 \to -\infty$, we get

$$\zeta_p(t) = \int_{-\infty}^{t} (t-s)^{\lambda-1}\eta_p(t-s)\Bigg[\sum_{q=1}^{n}\mathscr{A}_{pq}(s)f_q(\zeta_q(s-\theta_q))$$

$$+ \sum_{q=1}^{n}\mathscr{B}_{pq}(s)\mu_q(s) + \bigwedge_{q=1}^{n}\alpha_{pq}(s)g_q(\zeta_q(s-\theta_q)) + \bigvee_{q=1}^{n}\beta_{pq}(s)h_q(\zeta_q(s-\theta_q))$$

$$+ \bigwedge_{q=1}^{n}\mathscr{T}_{pq}(s)\mu_q(s) + \bigvee_{q=1}^{n}\mathscr{S}_{pq}(s)\mu_q(s) + u_p(s)\Bigg]ds$$

$$+ \int_{-\infty}^{t} (t-s)^{\lambda-1}\eta_p(t-s)\sum_{q=1}^{n}\mathscr{D}_{pq}(s)\delta_{pq}(\zeta_q(s-\theta_q))d\omega_q(s),\ p,\ q \in \mathbb{N},$$

acting as a mild solution for the system (1). □

Theorem 1 Provided that conditions (H_1)-(H_3) are satisfied, the system (1) possesses at least one square-mean almost periodic solution within the closed ball $\mathscr{X}^* = \{\varphi \in \mathscr{X} | \|\varphi - \varphi^0\|_{\mathscr{X}} \leq \mathscr{K}\}$.

Proof. We introduce a nonlinear operator $\Upsilon : \mathscr{X}^* \to \mathscr{X}^*$, which is defined by mapping

$$(\varphi_1, \varphi_2, \ldots, \varphi_n) \to (\zeta_1^{\varphi}, \zeta_2^{\varphi}, \ldots, \zeta_n^{\varphi}),$$

where $(\varphi_1, \varphi_2, \ldots, \varphi_n) \in \mathscr{X}$, $p, q \in \mathbb{N}$,

$$\begin{aligned}
\zeta_p^{\varphi}(t) = & \int_{-\infty}^{t} (t-s)^{\lambda-1} \eta_p(t-s) \Bigg[\sum_{q=1}^{n} \mathscr{A}_{pq}(s) f_q(\varphi_q(s-\theta_q)) \\
& + \sum_{q=1}^{n} \mathscr{B}_{pq}(s) \mu_q(s) + \bigwedge_{q=1}^{n} \alpha_{pq}(s) g_q(\varphi_q(s-\theta_q)) + \bigvee_{q=1}^{n} \beta_{pq}(s) h_q(\varphi_q(s-\theta_q)) \\
& + \bigwedge_{q=1}^{n} \mathscr{T}_{pq}(s) \mu_q(s) + \bigvee_{q=1}^{n} \mathscr{S}_{pq}(s) \mu_q(s) + u_p(s) \Bigg] ds \\
& + \int_{-\infty}^{t} (t-s)^{\lambda-1} \eta_p(t-s) \sum_{q=1}^{n} \mathscr{D}_{pq}(s) \delta_{pq}(\varphi_q(s-\theta_q)) d\omega_q(s) \\
:= & \int_{-\infty}^{t} (t-s)^{\lambda-1} \eta_p(t-s) \mathscr{G}_p(s) ds + \int_{-\infty}^{t} (t-s)^{\lambda-1} \eta_p(t-s) \Pi_p(s) d\omega_q(s),
\end{aligned} \tag{5}$$

where

$$\begin{aligned}
\mathscr{G}_p = & \sum_{q=1}^{n} \mathscr{A}_{pq}(s) f_q(\varphi_q(s-\theta_q)) + \sum_{q=1}^{n} \mathscr{B}_{pq}(s) \mu_q(s) \\
& + \bigwedge_{q=1}^{n} \alpha_{pq}(s) g_q(\varphi_q(s-\theta_q)) + \bigvee_{q=1}^{n} \beta_{pq}(s) h_q(\varphi_q(s-\theta_q)) \\
& + \bigwedge_{q=1}^{n} \mathscr{T}_{pq}(s) \mu_q(s) + \bigvee_{q=1}^{n} \mathscr{S}_{pq}(s) \mu_q(s) + u_p(s),
\end{aligned}$$

$$\Pi_p = \sum_{q=1}^{n} \mathscr{D}_{pq}(s) \delta_{pq}(\varphi_q(s-\theta_q)).$$

Now, we will prove that $\Upsilon\varphi$ is well defined. Actually, by (H_1)-(H_3) and (4), for $p, q \in \mathbb{N}$, one deduces that

$$
\begin{aligned}
E\|\Upsilon\varphi(t)\|_0^2 = \max_{p\in\mathbb{N}}\Bigg\{ & E\Bigg\| \int_{-\infty}^{t} (t-s)^{\lambda-1}\eta_p(t-s)\Bigg[\sum_{q=1}^{n}\mathscr{A}_{pq}(s)f_q(\varphi_q(s-\theta_q)) \\
& + \sum_{q=1}^{n}\mathscr{B}_{pq}(s)\mu_q(s) + \bigwedge_{q=1}^{n}\alpha_{pq}(s)g_q(\varphi_q(s-\theta_q)) + \bigvee_{q=1}^{n}\beta_{pq}(s)h_q(\varphi_q(s-\theta_q)) \\
& + \bigwedge_{q=1}^{n}\mathscr{T}_{pq}(s)\mu_q(s) + \bigvee_{q=1}^{n}\mathscr{S}_{pq}(s)\mu_q(s) + u_p(s)\Bigg]ds \\
& + \int_{-\infty}^{t}(t-s)^{\lambda-1}\eta_p(t-s)\sum_{q=1}^{n}\mathscr{D}_{pq}(s)\delta_{pq}(\varphi_q(s-\theta_q))d\omega_q(s)\|_{\mathscr{H}}^2\Bigg\} \\
\leq \max_{p\in\mathbb{N}}\Bigg\{ & \frac{4}{\mathscr{C}_p^2}\Bigg[\sum_{q=1}^{n}(\mathscr{A}_{pq}^{+})^2\sum_{q=1}^{n}(L_q^f)^2 + \sum_{q=1}^{n}(\mathscr{B}_{pq}^{+})^2\sum_{q=1}^{n}(M_q^{\mu})^2 + \sum_{q=1}^{n}(\alpha_{pq}^{+})^2\sum_{q=1}^{n}(L_q^g)^2 \\
& + \sum_{q=1}^{n}(\beta_{pq}^{+})^2\sum_{q=1}^{n}(L_q^h)^2 + \sum_{q=1}^{n}(\mathscr{T}_{pq}^{+})^2\sum_{q=1}^{n}(M_q^{\mu})^2 + \sum_{q=1}^{n}(\mathscr{S}_{pq}^{+})^2\sum_{q=1}^{n}(M_q^{\mu})^2 \\
& + \frac{\mathscr{C}_p}{2}\sum_{q=1}^{n}(\mathscr{D}_{pq}^{+})^2\sum_{q=1}^{n}(L_{pq}^{\delta})^2\Bigg]\Bigg\} < +\infty.
\end{aligned}
$$

That is, $\Upsilon\varphi$ is well defined.

Secondly, our aim is to demonstrate that Υ constitutes a self-mapping from $\mathscr{X}^*$ to $\mathscr{X}^*$. For any arbitrary $\varphi \in \mathscr{X}^*$, it holds that

$$
\|\varphi\|_{\mathscr{X}} \leq \|\varphi^0\|_{\mathscr{X}} + \|\varphi - \varphi^0\|_{\mathscr{X}} \leq 2\mathscr{K}.
$$

Consequently,

$$
\begin{aligned}
\|\Upsilon\varphi - \varphi^0\|_{\mathscr{X}}^2 = & \sup_{t\in\mathbb{R}} E\|(\Upsilon\varphi)(t) - (\Upsilon\varphi^0)(t)\|_0^2 \\
= & \sup_{t\in\mathbb{R}}\max_{p\in\mathbb{N}}\Bigg\{ E\Bigg\| \int_{-\infty}^{t}(t-s)^{\lambda-1}\eta_p(t-s)\Bigg[\sum_{q=1}^{n}\mathscr{A}_{pq}(s)f_q(\varphi_q(s-\theta_q)) \\
& + \bigwedge_{q=1}^{n}\alpha_{pq}(s)g_q(\varphi_q(s-\theta_q)) + \bigvee_{q=1}^{n}\beta_{pq}(s)h_q(\varphi_q(s-\theta_q))\Bigg]ds
\end{aligned}
$$

$$+\int_{-\infty}^{t}(t-s)^{\lambda-1}\eta_p(t-s)\sum_{q=1}^{n}\mathscr{D}_{pq}(s)\delta_{pq}(\varphi_q(s-\theta_q))d\omega_q(s)\Big\|_{\mathscr{H}}^2\Big\}$$

$$\leq 4\sup_{t\in\mathbb{R}}\max_{p\in\mathbb{N}}\Big\{E\Big\|\int_{-\infty}^{t}(t-s)^{\lambda-1}\eta_p(t-s)\sum_{q=1}^{n}\mathscr{A}_{pq}(s)f_q(\varphi_q(s-\theta_q))ds\Big\|_{\mathscr{H}}^2\Big\}$$

$$+4\sup_{t\in\mathbb{R}}\max_{p\in\mathbb{N}}\Big\{E\Big\|\int_{-\infty}^{t}(t-s)^{\lambda-1}\eta_p(t-s)\bigwedge_{q=1}^{n}\alpha_{pq}(s)g_q(\varphi_q(s-\theta_q))ds\Big\|_{\mathscr{H}}^2\Big\}$$

$$+4\sup_{t\in\mathbb{R}}\max_{p\in\mathbb{N}}\Big\{E\Big\|\int_{-\infty}^{t}(t-s)^{\lambda-1}\eta_p(t-s)\bigvee_{q=1}^{n}\beta_{pq}(s)h_q(\varphi_q(s-\theta_q))ds\Big\|_{\mathscr{H}}^2\Big\}$$

$$+4\sup_{t\in\mathbb{R}}\max_{p\in\mathbb{N}}\Big\{E\Big\|\int_{-\infty}^{t}(t-s)^{\lambda-1}\eta_p(t-s)\sum_{q=1}^{n}\mathscr{D}_{pq}(s)\delta_{pq}(\varphi_q(s-\theta_q))d\omega_q(s)\Big\|_{\mathscr{H}}^2\Big\}$$

$$:=\overline{\mathscr{M}}_1+\overline{\mathscr{M}}_2+\overline{\mathscr{M}}_3+\overline{\mathscr{M}}_4. \tag{6}$$

Applying the Cauchy-Schwartz inequality, we obtain

$$\overline{\mathscr{M}}_1=4\sup_{t\in\mathbb{R}}\max_{p\in\mathbb{N}}\Big\{E\Big\|\int_{-\infty}^{t}(t-s)^{\lambda-1}\eta_p(t-s)\sum_{q=1}^{n}\mathscr{A}_{pq}(s)f_q(\varphi_q(s-\theta_q))ds\Big\|_{\mathscr{H}}^2\Big\}$$

$$\leq 4\sup_{t\in\mathbb{R}}\max_{p\in\mathbb{N}}\Big\{E\Big\|\Big[\int_{-\infty}^{t}(t-s)^{\lambda-1}\eta_p(t-s)ds\Big]$$

$$\times\int_{-\infty}^{t}(t-s)^{\lambda-1}\eta_p(t-s)\Big[\sum_{q=1}^{n}\mathscr{A}_{pq}(s)f_q(\varphi_q(s-\theta_q))\Big]^2 ds\Big\|_{\mathscr{H}}\Big\}$$

$$\leq \max_{p\in\mathbb{N}}\frac{4}{\mathscr{C}_p^2}\Big[\sum_{q=1}^{n}(\mathscr{A}_{pq}^{+})^2\sum_{q=1}^{n}(L_q^f)^2\Big]\|\varphi\|_{\mathscr{X}}^2, \tag{7}$$

similarly, we get

$$\overline{\mathscr{M}}_2\leq\max_{p\in\mathbb{N}}\frac{4}{c_p^2}\Big[\sum_{q=1}^{n}(\alpha_{pq}^{+})^2\sum_{q=1}^{n}(L_q^g)^2\Big]\|\varphi\|_{\mathscr{X}}^2, \tag{8}$$

$$\overline{\mathscr{M}}_3\leq\max_{p\in\mathbb{N}}\frac{4}{c_p^2}\Big[\sum_{q=1}^{n}(\beta_{pq}^{+})^2\sum_{q=1}^{n}(L_q^h)^2\Big]\|\varphi\|_{\mathscr{X}}^2. \tag{9}$$

Moreover, by the Itô isometric property, we obtain

$$\overline{\mathscr{M}}_4 = 4\sup_{t\in\mathbb{R}}\max_{p\in\mathbb{N}}\left\{E\left\|\int_{-\infty}^{t}(t-s)^{\lambda-1}\eta_p(t-s)\sum_{q=1}^{n}\mathscr{D}_{pq}(s)\delta_{pq}(\varphi_q(s-\theta_q))d\omega_q(s)\right\|_{\mathscr{H}}^2\right\}$$

$$= 4\sup_{t\in\mathbb{R}}\max_{p\in\mathbb{N}}\left\{\left[\int_{-\infty}^{t}(t-s)^{\lambda-1}\eta_p(t-s)\right]^2\times E\left\|\sum_{q=1}^{n}\mathscr{D}_{pq}(s)\delta_{pq}(\varphi_q(s-\theta_q))\right\|_{\mathscr{H}}^2 ds\right\}$$

$$\leq \max_{p\in\mathbb{N}}\frac{2}{\mathscr{C}_p}\sum_{q=1}^{n}(\mathscr{D}_{pq}^{+})^2\sum_{q=1}^{n}(L_{pq}^{\delta})^2\|\varphi\|_{\mathscr{X}}^2. \tag{10}$$

By assuming (H_3) and substituting equations (7)–(10) into equation (6), by the assumption (H_3), we obtain

$$\|\Upsilon\varphi-\varphi^0\|_{\mathscr{X}}^2 \leq \max_{p\in\mathbb{N}}\left\{\frac{4}{\mathscr{C}_p^2}\left[\sum_{q=1}^{n}(\mathscr{A}_{pq}^{+})^2\sum_{q=1}^{n}(L_q^f)^2+\sum_{q=1}^{n}(\alpha_{pq}^{+})^2\sum_{q=1}^{n}(L_q^g)^2+\sum_{q=1}^{n}(\beta_{pq}^{+})^2\sum_{q=1}^{n}(L_q^h)^2\right.\right.$$

$$\left.\left.+\frac{\mathscr{C}_p}{2}\sum_{q=1}^{n}(\mathscr{D}_{pq}^{+})^2\sum_{q=1}^{n}(L_{pq}^{\delta})^2\right]\right\}\|\varphi\|_{\mathscr{X}}^2\leq\mathscr{K}^2,$$

which implies that $\|\Upsilon\varphi-\varphi^0\|_{\mathscr{X}}\leq\mathscr{K}$, $\Upsilon\varphi\in\mathscr{X}^*$.

Since $\mu_p, u_p, \mathscr{T}_{pq}, \mathscr{S}_{pq}$ are almost periodic, we can deduce that $\mu_p, u_p, \mathscr{T}_{pq}, \mathscr{S}_{pq}$ are bounded. Specifically, there exist positive constants $\overline{M}_p^u, \overline{M}_p^{\mu}, \overline{M}_{pq}^{\mathscr{T}}, \overline{M}_{pq}^{\mathscr{S}}$, such that

$$\|\mu_q(t)\|_{\mathscr{H}}\leq\overline{M}_q^{\mu},\ \|u_p(t)\|_{\mathscr{H}}\leq\overline{M}_p^{u},\ \|\mathscr{T}_{pq}(t)\|_{\mathscr{H}}\leq\overline{M}_{pq}^{\mathscr{T}},\ \|\mathscr{S}_{pq}(t)\|_{\mathscr{H}}\leq\overline{M}_{pq}^{\mathscr{S}}.$$

For any $\varphi\in\mathscr{X}^*$, for each $t_1, t_2\in\mathbb{R}$ and $t_1>t_2$, we can get

$$E\|\Upsilon\varphi(t_1)-\Upsilon\varphi(t_2)\|_0^2$$

$$= \max_{p\in\mathbb{N}}\left\{E\left\|\int_{-\infty}^{t_1}(t_1-s)^{\lambda-1}\eta_p(t_1-s)\mathscr{G}_p(s)ds+\int_{-\infty}^{t_1}(t_1-s)^{\lambda-1}\eta_p(t_1-s)\Pi_p(s)d\omega_q(s)\right.\right.$$

$$\left.\left.-\int_{-\infty}^{t_2}(t_2-s)^{\lambda-1}\eta_p(t_2-s)\mathscr{G}_p(s)ds\int_{-\infty}^{t_2}(t_2-s)^{\lambda-1}\eta_p(t_2-s)\Pi_p(s)d\omega_q(s)\right\|_{\mathscr{H}}^2\right\}$$

$$= \max_{p\in\mathbb{N}}\left\{E\left\|\int_{-\infty}^{t_2}\left[(t_1-s)^{\lambda-1}\eta_p(t_1-s)\mathscr{G}_p(s)-(t_2-s)^{\lambda-1}\eta_p(t_2-s)\mathscr{G}_p(s)\right]ds\right.\right.$$

$$\left.\left.+\int_{-\infty}^{t_2}\left[(t_1-s)^{\lambda-1}\eta_p(t_1-s)\Pi_p(s)-(t_2-s)^{\lambda-1}\eta_p(t_2-s)\Pi_p(s)\right]d\omega_q(s)\right.\right.$$

$$+\int_{t_2}^{t_1}(t_1-s)^{\lambda-1}\eta_p(t_1-s)\mathscr{G}_p(s)ds+\int_{t_2}^{t_1}(t_1-s)^{\lambda-1}\eta_p(t_1-s)\Pi_p(s)d\omega_q(s)\Big\|_{\mathscr{H}}^2$$

$$\leq 4\max_{p\in\mathbb{N}}E\Big\|\int_{-\infty}^{t_2}\left[(t_1-s)^{\lambda-1}\eta_p(t_1-s)-(t_2-s)^{\lambda-1}\eta_p(t_2-s)\right]\mathscr{G}_p(s)ds\Big\|_{\mathscr{H}}^2$$

$$+4\max_{p\in\mathbb{N}}E\Big\|\int_{-\infty}^{t_2}\left[(t_1-s)^{\lambda-1}\eta_p(t_1-s)-(t_2-s)^{\lambda-1}\eta_p(t_2-s)\right]\Pi_p(s)d\omega_q(s)\Big\|_{\mathscr{H}}^2$$

$$+4\max_{p\in\mathbb{N}}E\Big\|\int_{t_2}^{t_1}(t_1-s)^{\lambda-1}\eta_p(t_1-s)\mathscr{G}_p(s)ds\Big\|_{\mathscr{H}}^2$$

$$+4\max_{p\in\mathbb{N}}E\Big\|\int_{t_2}^{t_1}(t_1-s)^{\lambda-1}\eta_p(t_1-s)\Pi_p(s)d\omega_q(s)\Big\|_{\mathscr{H}}^2$$

$$:=4(\widetilde{\mathscr{M}_1}+\widetilde{\mathscr{M}_2}+\widetilde{\mathscr{M}_3}+\widetilde{\mathscr{M}_4}). \tag{11}$$

By the Cauchy-Schwartz inequality, we have

$$\widetilde{\mathscr{M}_1}=\max_{p\in\mathbb{N}}E\Big\|\int_{-\infty}^{t_2}\left[(t_1-s)^{\lambda-1}\eta_p(t_1-s)-(t_2-s)^{\lambda-1}\eta_p(t_2-s)\right]\mathscr{G}_p(s)ds\Big\|_{\mathscr{H}}^2$$

$$=\max_{p\in\mathbb{N}}E\Big\|\int_{-\infty}^{t_2}\left[(t_1-s)^{\lambda-1}\eta_p(t_1-s)-(t_2-s)^{\lambda-1}\eta_p(t_2-s)\right]$$

$$\times\left[\sum_{q=1}^{n}\mathscr{A}_{pq}(s)f_q(\varphi_q(s-\theta_q))+\sum_{q=1}^{n}\mathscr{B}_{pq}(s)\mu_q(s)+\bigwedge_{q=1}^{n}\alpha_{pq}(s)g_q(\varphi_q(s-\theta_q))\right.$$

$$\left.+\bigvee_{q=1}^{n}\beta_{pq}(s)h_q(\varphi_q(s-\theta_q))+\bigwedge_{q=1}^{n}\mathscr{T}_{pq}(s)\mu_q(s)+\bigvee_{q=1}^{n}\mathscr{S}_{pq}(s)\mu_q(s)+u_p(s)\right]ds\Big\|_{\mathscr{H}}^2$$

$$\leq 7\max_{p\in\mathbb{N}}\left\{\left[\int_{-\infty}^{t_2}|(t_1-s)^{\lambda-1}\eta_p(t_1-s)-(t_2-s)^{\lambda-1}\eta_p(t_2-s)|ds\right]\right.$$

$$\times\left[\int_{-\infty}^{t_2}|(t_1-s)^{\lambda-1}\eta_p(t_1-s)-(t_2-s)^{\lambda-1}\eta_p(t_2-s)|\right.$$

$$\left.\left.\times E\Big\|\sum_{q=1}^{n}\mathscr{A}_{pq}(s)f_q(\varphi_q(s-\theta_q))\Big\|_{\mathscr{H}}^2ds\right]\right\}$$

$$+7\max_{p\in\mathbb{N}}\left\{\left[\int_{-\infty}^{t_2}|(t_1-s)^{\lambda-1}\eta_p(t_1-s)-(t_2-s)^{\lambda-1}\eta_p(t_2-s)|ds\right]\right.$$

$$\left.\times\left[\int_{-\infty}^{t_2}|(t_1-s)^{\lambda-1}\eta_p(t_1-s)-(t_2-s)^{\lambda-1}\eta_p(t_2-s)|\times E\left\|\sum_{q=1}^{n}\mathscr{B}_{pq}(s)\mu_q(s)\right\|_{\mathscr{H}}^2 ds\right]\right\}$$

$$+7\max_{p\in\mathbb{N}}\left\{\left[\int_{-\infty}^{t_2}|(t_1-s)^{\lambda-1}\eta_p(t_1-s)-(t_2-s)^{\lambda-1}\eta_p(t_2-s)|ds\right]\right.$$

$$\times\left[\int_{-\infty}^{t_2}|(t_1-s)^{\lambda-1}\eta_p(t_1-s)-(t_2-s)^{\lambda-1}\eta_p(t_2-s)|\right.$$

$$\left.\left.\times E\left\|\bigwedge_{q=1}^{n}\alpha_{pq}(s)g_q(\varphi_q(s-\theta_q))\right\|_{\mathscr{H}}^2 ds\right]\right\}$$

$$+7\max_{p\in\mathbb{N}}\left\{\left[\int_{-\infty}^{t_2}|(t_1-s)^{\lambda-1}\eta_p(t_1-s)-(t_2-s)^{\lambda-1}\eta_p(t_2-s)|ds\right]\right.$$

$$\times\left[\int_{-\infty}^{t_2}|(t_1-s)^{\lambda-1}\eta_p(t_1-s)-(t_2-s)^{\lambda-1}\eta_p(t_2-s)|\right.$$

$$\left.\left.\times E\left\|\bigvee_{q=1}^{n}\beta_{pq}(s)h_q(\varphi_q(s-\theta_q))\right\|_{\mathscr{H}}^2 ds\right]\right\}$$

$$+7\max_{p\in\mathbb{N}}\left\{\left[\int_{-\infty}^{t_2}|(t_1-s)^{\lambda-1}\eta_p(t_1-s)-(t_2-s)^{\lambda-1}\eta_p(t_2-s)|ds\right]\right.$$

$$\left.\times\left[\int_{-\infty}^{t_2}|(t_1-s)^{\lambda-1}\eta_p(t_1-s)-(t_2-s)^{\lambda-1}\eta_p(t_2-s)|\times E\left\|\bigwedge_{q=1}^{n}\mathscr{T}_{pq}(s)\mu_q(s)\right\|_{\mathscr{H}}^2 ds\right]\right\}$$

$$+7\max_{p\in\mathbb{N}}\left\{\left[\int_{-\infty}^{t_2}|(t_1-s)^{\lambda-1}\eta_p(t_1-s)-(t_2-s)^{\lambda-1}\eta_p(t_2-s)|ds\right]\right.$$

$$\left.\times\left[\int_{-\infty}^{t_2}|(t_1-s)^{\lambda-1}\eta_p(t_1-s)-(t_2-s)^{\lambda-1}\eta_p(t_2-s)|\times E\left\|\bigwedge_{q=1}^{n}\mathscr{S}_{pq}(s)\mu_q(s)\right\|_{\mathscr{H}}^2 ds\right]\right\}$$

$$+7\max_{p\in\mathbb{N}}\left\{\left[\int_{-\infty}^{t_2}|(t_1-s)^{\lambda-1}\eta_p(t_1-s)-(t_2-s)^{\lambda-1}\eta_p(t_2-s)|ds\right]\right.$$

$$\left.\times\left[\int_{-\infty}^{t_2}|(t_1-s)^{\lambda-1}\eta_p(t_1-s)-(t_2-s)^{\lambda-1}\eta_p(t_2-s)|\times E\|u_p(s)\|_{\mathscr{H}}^2 ds\right]\right\}$$

$$\leq 7\max_{p\in\mathbb{N}}\sum_{q=1}^{n}\sum_{q=1}^{n}\Big[(\mathscr{A}_{pq}^{+})^2(L_q^f)^2(2\mathscr{K})^2+(\alpha_{pq}^{+})^2(L_q^g)^2(2\mathscr{K})^2+(\mathscr{B}_{pq}^{+})^2(\overline{M}_q^{\mu})^2$$

$$+(\beta_{pq}^{+})^2(L_q^h)^2(2\mathscr{K})^2+(\mathscr{T}_{pq}^{+})^2(\overline{M}_q^{\mu})^2+(\mathscr{S}_{pq}^{+})^2(\overline{M}_q^{\mu})^2\Big]+(\overline{M}_q^{u})^2$$

$$\times\left(\int_{-\infty}^{t_2}|(t_1-s)^{\lambda-1}\eta_p(t_1-s)-(t_2-s)^{\lambda-1}\eta_p(t_2-s)|ds\right)^2$$

$$\leq 7\max_{p\in\mathbb{N}}\Bigg[\sum_{q=1}^{n}(\mathscr{A}_{pq}^{+})^2\sum_{q=1}^{n}(L_q^f)^2(2\mathscr{K})^2+\sum_{q=1}^{n}(\alpha_{pq}^{+})^2\sum_{q=1}^{n}(L_q^g)^2(2\mathscr{K})^2+\sum_{q=1}^{n}(\mathscr{B}_{pq}^{+})^2\sum_{q=1}^{n}(\overline{M}_q^{\mu})^2$$

$$+\sum_{q=1}^{n}(\beta_{pq}^{+})^2\sum_{q=1}^{n}(L_q^h)^2(2\mathscr{K})^2+\sum_{q=1}^{n}(\mathscr{T}_{pq}^{+})^2\sum_{q=1}^{n}(\overline{M}_q^{\mu})^2+\sum_{q=1}^{n}(\mathscr{S}_{pq}^{+})^2\sum_{q=1}^{n}(\overline{M}_q^{\mu})^2+(\overline{M}_q^{u})^2\Bigg]$$

$$\times\frac{2}{\mathscr{C}_p^2}|t_1-t_2|^2. \tag{12}$$

By employing the Cauchy-Schwartz inequality and the Itô isometric property, we derive

$$\widetilde{\mathscr{M}_2}=\max_{p\in\mathbb{N}}E\Bigg\|\int_{-\infty}^{t_2}\left[(t_1-s)^{\lambda-1}\eta_p(t_1-s)-(t_2-s)^{\lambda-1}\eta_p(t_2-s)\right]$$

$$\times\sum_{q=1}^{n}\mathscr{D}_{pq}(s)\delta_{pq}(\varphi_q(s-\theta_q))d\omega_q(s)\Bigg\|_{\mathscr{H}}^2$$

$$\leq\max_{p\in\mathbb{N}}\frac{1}{\mathscr{C}_p}\sum_{q=1}^{n}(\mathscr{D}_{pq}^{+})^2\sum_{q=1}^{n}(L_{pq}^{\delta})^2(2\mathscr{K})^2|t_1-t_2|^2. \tag{13}$$

Similarly, we get

$$\widetilde{\mathscr{M}_3}\leq 7\max_{p\in\mathbb{N}}\Big[\sum_{q=1}^{n}(\mathscr{A}_{pq}^{+})^2\sum_{q=1}^{n}(L_q^f)^2(2\mathscr{K})^2+\sum_{q=1}^{n}(\alpha_{pq}^{+})^2\sum_{q=1}^{n}(L_q^g)^2(2\mathscr{K})^2+\sum_{q=1}^{n}(\mathscr{B}_{pq}^{+})^2\sum_{q=1}^{n}(\overline{M}_q^{\mu})^2$$

$$+\sum_{q=1}^{n}(\beta_{pq}^{+})^2\sum_{q=1}^{n}(L_q^h)^2(2\mathscr{K})^2+\sum_{q=1}^{n}(\mathscr{T}_{pq}^{+})^2\sum_{q=1}^{n}(\overline{M}_q^{\mu})^2+\sum_{q=1}^{n}(\mathscr{S}_{pq}^{+})^2\sum_{q=1}^{n}(\overline{M}_q^{\mu})^2+(\overline{M}_q^{u})^2\Big]$$

$$\times\frac{1}{\mathscr{C}_p^2}|t_1-t_2|^2, \tag{14}$$

$$\widetilde{\mathscr{M}_4} \leq \max_{p\in\mathbb{N}} \frac{1}{2\mathscr{C}_p} \sum_{q=1}^{n} (\mathscr{D}_{pq}^{+})^2 \sum_{q=1}^{n} (L_{pq}^{\delta})^2 (2\mathscr{K})^2 |t_1 - t_2|^2. \tag{15}$$

Substituting (12)–(15) into (11), we obtain

$$\begin{aligned}
&E\|\Upsilon\varphi(t_1) - \Upsilon\varphi(t_2)\|_0^2 \\
&\leq \max_{p\in\mathbb{N}} \Bigg\{ 28 \Bigg[\sum_{q=1}^{n} (\mathscr{A}_{pq}^{+})^2 \sum_{q=1}^{n} (L_q^f)^2 (2\mathscr{K})^2 + \sum_{q=1}^{n} (\alpha_{pq}^{+})^2 \sum_{q=1}^{n} (L_q^g)^2 (2\mathscr{K})^2 + \sum_{q=1}^{n} (\mathscr{B}_{pq}^{+})^2 \sum_{q=1}^{n} (\overline{M}_q^{\mu})^2 \\
&\quad + \sum_{q=1}^{n} (\beta_{pq}^{+})^2 \sum_{q=1}^{n} (L_q^h)^2 (2\mathscr{K})^2 + \sum_{q=1}^{n} (\mathscr{T}_{pq}^{+})^2 \sum_{q=1}^{n} (\overline{M}_q^{\mu})^2 + \sum_{q=1}^{n} (\mathscr{S}_{pq}^{+})^2 \sum_{q=1}^{n} (\overline{M}_q^{\mu})^2 + (\overline{M}_q^{\mu})^2 \Bigg] \\
&\quad \times \left[\frac{2}{\mathscr{C}_p^2} |t_1 - t_2|^2 + \frac{1}{\mathscr{C}_p^2} |t_1 - t_2|^2 \right] + \frac{4}{\mathscr{C}_p} \sum_{q=1}^{n} (\mathscr{D}_{pq}^{+})^2 \sum_{q=1}^{n} (L_{pq}^{\delta})^2 (2\mathscr{K})^2 |t_1 - t_2|^2 \\
&\quad + \frac{2}{\mathscr{C}_p} \sum_{q=1}^{n} (\mathscr{D}_{pq}^{+})^2 \sum_{q=1}^{n} (L_{pq}^{\delta})^2 (2\mathscr{K})^2 |t_1 - t_2|^2 \Bigg\}.
\end{aligned} \tag{16}$$

Consequently, based on inequality (16), we infer that as $t_1 \to t_2$, $E\|\Upsilon\varphi(t_1) - \Upsilon\varphi(t_2)\|_0^2 \to 0$, indicating that $\Upsilon\varphi$ is uniformly continuous. Hence, it follows that $\Upsilon(\mathscr{X}^*) \subset \mathscr{X}^*$.

Next, we aim to demonstrate that Υ acts as a contraction mapping within $\mathscr{X}^*$. For any $\varphi, \psi \in \mathscr{X}^*$ and $p, q \in \mathbb{N}$, leveraging the aforementioned inequalities, we proceed as follows:

$$\begin{aligned}
&\|\Upsilon\varphi - \Upsilon\psi\|_{\mathscr{X}}^2 \\
&= \sup_{t\in\mathbb{R}} E\|(\Upsilon\varphi)(t) - (\Upsilon\psi)(t)\|_0^2 \\
&= \sup_{t\in\mathbb{R}} \max_{p\in\mathbb{N}} \Bigg\{ E \Bigg\| \int_{-\infty}^{t} (t-s)^{\lambda-1} \eta_p(t-s) \Bigg[\sum_{q=1}^{n} \mathscr{A}_{pq}(s) [f_q(\varphi_q(s-\theta_q)) - f_q(\psi_q(s-\theta_q))] \\
&\quad + \bigwedge_{q=1}^{n} \alpha_{pq}(s) [g_q(\varphi_q(s-\theta_q)) - g_q(\psi_q(s-\theta_q))] \\
&\quad + \bigvee_{q=1}^{n} \beta_{pq}(s) [h_q(\varphi_q(s-\theta_q)) - h_q(\psi_q(s-\theta_q))] \Bigg] ds
\end{aligned}$$

$$+\int_{-\infty}^{t}(t-s)^{\lambda-1}\eta_p(t-s)\sum_{q=1}^{n}\mathscr{D}_{pq}(s)[\delta_{pq}(\varphi_q(s-\theta_q))-\delta_{pq}(\psi_q(s-\theta_q))]d\omega_q(s)\Big\|_{\mathscr{H}}^2\Big\}$$

$$\leq 4\sup_{t\in\mathbb{R}}\max_{p\in\mathbb{N}}\Big\{E\Big\|\int_{-\infty}^{t}(t-s)^{\lambda-1}\eta_p(t-s)\sum_{q=1}^{n}\mathscr{A}_{pq}(s)[f_q(\varphi_q(s-\theta_q))-f_q(\psi_q(s-\theta_q))]ds\Big\|_{\mathscr{H}}^2$$

$$+E\Big\|\int_{-\infty}^{t}(t-s)^{\lambda-1}\eta_p(t-s)\bigwedge_{q=1}^{n}\alpha_{pq}(s)\times[g_q(\varphi_q(s-\theta_q))-g_q(\psi_q(s-\theta_q))]ds\Big\|_{\mathscr{H}}^2$$

$$+E\Big\|\int_{-\infty}^{t}(t-s)^{\lambda-1}\eta_p(t-s)\bigvee_{q=1}^{n}\beta_{pq}(s)\times[h_q(\varphi_q(s-\theta_q))-h_q(\psi_q(s-\theta_q))]ds\Big\|_{\mathscr{H}}^2$$

$$+E\Big\|\int_{-\infty}^{t}(t-s)^{\lambda-1}\eta_p(t-s)\sum_{q=1}^{n}\mathscr{D}_{pq}(s)\times[\delta_{pq}(\varphi_q(s-\theta_q))-\delta_{pq}(\psi_q(s-\theta_q))]d\omega_q(s)\Big\|_{\mathscr{H}}^2\Big\}$$

$$\leq \max_{p\in\mathbb{N}}\Big\{\frac{4}{\mathscr{C}_p^2}\Big[\sum_{q=1}^{n}(\mathscr{A}_{pq}^{+})^2\sum_{q=1}^{n}(L_q^f)^2+\sum_{q=1}^{n}(\alpha_{pq}^{+})^2\sum_{q=1}^{n}(L_q^g)^2+\sum_{q=1}^{n}(\beta_{pq}^{+})^2\sum_{q=1}^{n}(L_q^h)^2$$

$$+\frac{\mathscr{C}_p}{2}\sum_{q=1}^{n}(\mathscr{D}_{pq}^{+})^2\sum_{q=1}^{n}(L_{pq}^{\delta})^2\Big]\Big\}\|\varphi-\psi\|_{\mathscr{X}}^2$$

$$:=\Pi\|\varphi-\psi\|_{\mathscr{X}}^2,$$

where

$$\Pi=\max_{p\in\mathbb{N}}\Big\{\frac{4}{\mathscr{C}_p^2}\Big[\sum_{q=1}^{n}(\mathscr{A}_{pq}^{+})^2\sum_{q=1}^{n}(L_q^f)^2+\sum_{q=1}^{n}(\alpha_{pq}^{+})^2\sum_{q=1}^{n}(L_q^g)^2+\sum_{q=1}^{n}(\beta_{pq}^{+})^2\sum_{q=1}^{n}(L_q^h)^2$$

$$+\frac{\mathscr{C}_p}{2}\sum_{q=1}^{n}(\mathscr{D}_{pq}^{+})^2\sum_{q=1}^{n}(L_{pq}^{\delta})^2\Big]\Big\}.$$

Hence, according to (H_3), we obtain $\|\Upsilon\varphi-\Upsilon\psi\|_{\mathscr{X}}^2\leq\Pi\|\varphi-\psi\|_{\mathscr{X}}^2\leq\|\varphi-\psi\|_{\mathscr{X}}^2$. Therefore, Υ is a contraction mapping.

In fact, for every $\varphi\in\mathscr{X}$,

$$E\|\mathscr{G}_p(s+\tau)-\mathscr{G}_p(s)\|_{\mathscr{H}}^2$$

$$=E\|\sum_{q=1}^{n}\mathscr{A}_{pq}(s+\tau)f_q(\varphi_q(s+\tau-\theta_q))+\sum_{q=1}^{n}\mathscr{B}_{pq}(s+\tau)\mu_q(s+\tau)$$

$$
\begin{aligned}
&+\bigwedge_{q=1}^{n}\alpha_{pq}(s+\tau)g_q(\varphi_q(s+\tau-\theta_q))+\bigvee_{q=1}^{n}\beta_{pq}(s)h_q(\varphi_q(s+\tau-\theta_q))\\
&+\bigwedge_{q=1}^{n}\mathscr{T}_{pq}(s+\tau)\mu_q(s+\tau)+\bigvee_{q=1}^{n}\mathscr{S}_{pq}(s+\tau)\mu_q(s)+u_p(s+\tau)\\
&-\sum_{q=1}^{n}\mathscr{A}_{pq}(s)f_q(\varphi_q(s-\theta_q))-\sum_{q=1}^{n}\mathscr{B}_{pq}(s)\mu_q(s)\\
&-\bigwedge_{q=1}^{n}\alpha_{pq}(s)g_q(\varphi_q(s-\theta_q))-\bigvee_{q=1}^{n}\beta_{pq}(s)h_q(\varphi_q(s-\theta_q))\\
&-\bigwedge_{q=1}^{n}\mathscr{T}_{pq}(s)\mu_q(s)-\bigvee_{q=1}^{n}\mathscr{S}_{pq}(s)\mu_q(s)-u_p(s)\|_{\mathscr{H}}^2\\
\le\ & 7E\left\|\sum_{q=1}^{n}\mathscr{A}_{pq}(s+\tau)f_q(\varphi_q(s+\tau-\theta_q))-\sum_{q=1}^{n}\mathscr{A}_{pq}(s)f_q(\varphi_q(s-\theta_q))\right\|_{\mathscr{H}}^2\\
&+7E\left\|\sum_{q=1}^{n}\mathscr{B}_{pq}(s+\tau)\mu_q(s+\tau)-\sum_{q=1}^{n}\mathscr{B}_{pq}(s)\mu_q(s)\right\|_{\mathscr{H}}^2\\
&+7E\left\|\bigwedge_{q=1}^{n}\alpha_{pq}(s+\tau)g_q(\varphi_q(s+\tau-\theta_q))-\sum_{q=1}^{n}\mathscr{A}_{pq}(s)f_q(\varphi_q(s-\theta_q))\right\|_{\mathscr{H}}^2\\
&+7E\left\|\bigvee_{q=1}^{n}\beta_{pq}(s)h_q(\varphi_q(s+\tau-\theta_q))-\bigvee_{q=1}^{n}\beta_{pq}(s)h_q(\varphi_q(s-\theta_q))\right\|_{\mathscr{H}}^2\\
&+7E\left\|\bigwedge_{q=1}^{n}\mathscr{T}_{pq}(s+\tau)\mu_q(s+\tau)-\bigwedge_{q=1}^{n}\mathscr{T}_{pq}(s)\mu_q(s)\right\|_{\mathscr{H}}^2\\
&+7E\left\|\bigvee_{q=1}^{n}\mathscr{S}_{pq}(s+\tau)\mu_q(s)-\bigvee_{q=1}^{n}\mathscr{S}_{pq}(s)\mu_q(s)\right\|_{\mathscr{H}}^2\\
&+7E\|u_p(s+\tau)-u_p(s))\|_{\mathscr{H}}^2:=7\sum_{i=1}^{7}\mathscr{M}_i, \qquad (17)
\end{aligned}
$$

by the assumption (H_1)–(H_2), we have

$$\mathscr{M}_1 = E\Big\| \sum_{q=1}^{n} \mathscr{A}_{pq}(s+\tau) f_q(\varphi_q(s+\tau-\theta_q)) - \sum_{q=1}^{n} \mathscr{A}_{pq}(s) f_q(\varphi_q(s-\theta_q)) \Big\|_{\mathscr{H}}^2$$

$$\leq 2E\Big\| \sum_{q=1}^{n} \mathscr{A}_{pq}(s+\tau) f_q(\varphi_q(s+\tau-\theta_q)) - \sum_{q=1}^{n} \mathscr{A}_{pq}(s+\tau) f_q(\varphi_q(s-\theta_q)) \Big\|_{\mathscr{H}}^2$$

$$+ 2E\Big\| \sum_{q=1}^{n} \mathscr{A}_{pq}(s+\tau) f_q(\varphi_q(s-\theta_q)) - \sum_{q=1}^{n} \mathscr{A}_{pq}(s) f_q(\varphi_q(s-\theta_q)) \Big\|_{\mathscr{H}}^2$$

$$\leq 2E\Big\| \sum_{q=1}^{n} \mathscr{A}_{pq}(s+\tau) \big[f_q(\varphi_q(s+\tau-\theta_q)) - f_q(\varphi_q(s-\theta_q)) \big] \Big\|_{\mathscr{H}}^2$$

$$+ 2E\Big\| \big[\sum_{q=1}^{n} \mathscr{A}_{pq}(s+\tau) - \sum_{q=1}^{n} \mathscr{A}_{pq}(s) \big] f_q(\varphi_q(s-\theta_q)) \Big\|_{\mathscr{H}}^2$$

$$\leq 2 \sum_{q=1}^{n} (\mathscr{A}_{pq}^{+})^2 \sum_{q=1}^{n} (L_q^f)^2 E\|\varphi_q(s+\tau-\theta_q) - \varphi_q(s-\theta_q)\|_{\mathscr{H}}^2$$

$$+ 2n \sum_{q=1}^{n} (M_q^f)^2 E\|\mathscr{A}_{pq}(s+\tau) - \mathscr{A}_{pq}(s)\|_{\mathscr{H}}^2. \tag{18}$$

Similarly, we get

$$\mathscr{M}_2 \leq 2n \sum_{q=1}^{n} (\mathscr{B}_{pq}^{+})^2 E\|\mu_q(s+\tau) - \mu_q(s)\|_{\mathscr{H}}^2$$

$$+ 2n \sum_{q=1}^{n} (M_q^{\mu})^2 E\|\mathscr{B}_{pq}(s+\tau) - \mathscr{B}_{pq}(s)\|_{\mathscr{H}}^2 \tag{19}$$

$$\mathscr{M}_3 \leq 2 \sum_{q=1}^{n} (\alpha_{pq}^{+})^2 \sum_{q=1}^{n} (L_q^g)^2 E\|\varphi_q(s+\tau-\theta_q) - \varphi_q(s-\theta_q)\|_{\mathscr{H}}^2$$

$$+ 2n \sum_{q=1}^{n} (M_q^g)^2 E\|\alpha_{pq}(s+\tau) - \alpha_{pq}(s)\|_{\mathscr{H}}^2 \tag{20}$$

$$\mathscr{M}_4 \leq 2 \sum_{q=1}^{n} (\beta_{pq}^{+})^2 \sum_{q=1}^{n} (L_q^h)^2 E\|\varphi_q(s+\tau-\theta_q) - \varphi_q(s-\theta_q)\|_{\mathscr{H}}^2$$

$$+2n\sum_{q=1}^{n}(M_q^h)^2E\|\beta_{pq}(s+\tau)-\beta_{pq}(s)\|_{\mathscr{H}}^2 \tag{21}$$

$$\mathscr{M}_5\le 2n\sum_{q=1}^{n}(M_{pq}^{\mathscr{T}})^2E\|\mu_q(s+\tau)-\mu_q(s)\|_{\mathscr{H}}^2$$

$$+2n\sum_{q=1}^{n}(M_q^{\mu})^2E\|\mathscr{T}_{pq}(s+\tau)-\mathscr{T}_{pq}(s)\|_{\mathscr{H}}^2 \tag{22}$$

$$\mathscr{M}_6\le 2n\sum_{q=1}^{n}(M_{pq}^{\mathscr{S}})^2E\|\mu_q(s+\tau)-\mu_q(s)\|_{\mathscr{H}}^2$$

$$+2n\sum_{q=1}^{n}(M_q^{\mu})^2E\|\mathscr{S}_{pq}(s+\tau)-\mathscr{S}_{pq}(s)\|_{\mathscr{H}}^2. \tag{23}$$

Substituting inequalities (18)–(23) into the inequality (17), we obtain

$$E\|\mathscr{G}_p(s+\tau)-\mathscr{G}_p(s)\|_{\mathscr{H}}^2$$

$$\le 14\sum_{q=1}^{n}(\mathscr{A}_{pq}^{+})^2\sum_{q=1}^{n}(L_q^f)^2E\|\varphi_q(s+\tau-\theta_q)-\varphi_q(s-\theta_q)\|_{\mathscr{H}}^2$$

$$+14n\sum_{q=1}^{n}(M_q^f)^2E\|\mathscr{A}_{pq}(s+\tau)-\mathscr{A}_{pq}(s)\|_{\mathscr{H}}^2$$

$$+14n\sum_{q=1}^{n}(\mathscr{B}_{pq}^{+})^2E\|\mu_q(s+\tau)-\mu_q(s)\|_{\mathscr{H}}^2+14n\sum_{q=1}^{n}(M_q^{\mu})^2E\|\mathscr{B}_{pq}(s+\tau)-\mathscr{B}_{pq}(s)\|_{\mathscr{H}}^2$$

$$+14\sum_{q=1}^{n}(\alpha_{pq}^{+})^2\sum_{q=1}^{n}(L_q^g)^2E\|\varphi_q(s+\tau-\theta_q)-\varphi_q(s-\theta_q)\|_{\mathscr{H}}^2$$

$$+14n\sum_{q=1}^{n}(M_q^g)^2E\|\alpha_{pq}(s+\tau)-\alpha_{pq}(s)\|_{\mathscr{H}}^2+14n\sum_{q=1}^{n}(M_q^h)^2E\|\beta_{pq}(s+\tau)-\beta_{pq}(s)\|_{\mathscr{H}}^2$$

$$+14\sum_{q=1}^{n}(\beta_{pq}^{+})^2\sum_{q=1}^{n}(L_q^h)^2E\|\varphi_q(s+\tau-\theta_q)-\varphi_q(s-\theta_q)\|_{\mathscr{H}}^2$$

$$+14n\sum_{q=1}^{n}(M_{pq}^{\mathscr{T}})^2E\|\mu_q(s+\tau)-\mu_q(s)\|_{\mathscr{H}}^2+14n\sum_{q=1}^{n}(M_q^{\mu})^2E\|\mathscr{T}_{pq}(s+\tau)-\mathscr{T}_{pq}(s)\|_{\mathscr{H}}^2$$

$$+14n\sum_{q=1}^{n}(M_{pq}^{\mathscr{S}})^2E\|\mu_q(s+\tau)-\mu_q(s)\|_{\mathscr{H}}^2+14n\sum_{q=1}^{n}(M_q^{\mu})^2E\|\mathscr{S}_{pq}(s+\tau)-\mathscr{S}_{pq}(s)\|_{\mathscr{H}}^2.$$

Similarly, for every $\varphi\in\mathscr{X}$, by the assumption (H_1)–(H_2), we get

$$E\|\Pi_p(s+\tau)-\Pi_p(s)\|_{\mathscr{H}}^2$$

$$=E\|\sum_{q=1}^{n}[\mathscr{D}_{pq}(s+\tau)\delta_{pq}(\varphi_q(s+\tau-\theta_q))-d_{pq}(s)\delta_{pq}(\varphi_q(s-\theta_q))]\|_{\mathscr{H}}^2$$

$$\leq 2\sum_{q=1}^{n}(\mathscr{D}_{pq}^{+})^2\sum_{q=1}^{n}(L_q^{\delta})^2E\|\varphi_q(s+\tau-\theta_q)-\varphi_q(s-\theta_q)\|_{\mathscr{H}}^2$$

$$+2n\sum_{q=1}^{n}(M_{pq}^{\delta})^2E\|\mathscr{D}_{pq}(s+\tau)-\mathscr{D}_{pq}(s)\|_{\mathscr{H}}^2.$$

According to Lemma 4, the functions $\mathscr{G}_p$, $\Pi_p$exhibit almost periodicity in distribution. Consequently, for every $\varepsilon>0$, $p\in\mathbb{N}$, there exists a positive constant $L=L(\varepsilon)$ such that every interval of length L contains a constant τ satisfying

$$E\|\mathscr{G}_p(s+\tau)-\mathscr{G}_p(s)\|_{\mathscr{H}}^2\leq\varepsilon,$$

$$E\|\Pi_p(s+\tau)-\Pi_p(s)\|_{\mathscr{H}}^2\leq\varepsilon.$$

Therefore, we obtain

$$E\|(\Upsilon\varphi)_p(t+\tau)-(\Upsilon\varphi)_p(t)\|_{\mathscr{H}}^2$$

$$=E\Bigg\|\int_{-\infty}^{t+\tau}(t+\tau-s)^{\lambda-1}\eta_p(t+\tau-s)\mathscr{G}_p(s)ds$$

$$+\int_{-\infty}^{t+\tau}(t+\tau-s)^{\lambda-1}\eta_p(t+\tau-s)\Pi_p(s)d\omega_q(s)$$

$$-\int_{-\infty}^{t}(t-s)^{\lambda-1}\eta_p(t-s)\mathscr{G}_p(s)ds-\int_{-\infty}^{t+\tau}(t-s)^{\lambda-1}\eta_p(t-s)\Pi_p(s)d\omega_q(s)\Bigg\|_{\mathscr{H}}^2$$

$$\leq 2E\Bigg\|\int_{-\infty}^{t+\tau}(t+\tau-s)^{\lambda-1}\eta_p(t+\tau-s)\mathscr{G}_p(s)ds-\int_{-\infty}^{t}(t-s)^{\lambda-1}\eta_p(t-s)\mathscr{G}_p(s)ds\Bigg\|_{\mathscr{H}}^2$$

$$+2E\left\|\int_{-\infty}^{t+\tau}(t+\tau-s)^{\lambda-1}\eta_p(t+\tau-s)\Pi_p(s)d\omega_q(s)\right.$$

$$\left.-\int_{-\infty}^{t}(t-s)^{\lambda-1}\eta_p(t-s)\Pi_p(s)d\omega_q(s)\right\|_{\mathscr{H}}^2$$

$$:=\mathscr{I}_1+\mathscr{I}_2. \tag{24}$$

By the Itô isometric properties, the assumption (H_1)–(H_2), and the above inequalities, we get

$$\mathscr{I}_1=2E\left\|\int_{-\infty}^{t+\tau}(t+\tau-s)^{\lambda-1}\eta_p(t+\tau-s)\mathscr{G}_p(s)ds-\int_{-\infty}^{t}(t-s)^{\lambda-1}\eta_p(t-s)\mathscr{G}_p(s)ds\right\|_{\mathscr{H}}^2$$

$$=2E\left\|\int_{-\infty}^{t}(t-s)^{\lambda-1}\eta_p(t-s)\mathscr{G}_p(s+\tau)ds-\int_{-\infty}^{t}(t-s)^{\lambda-1}\eta_p(t-s)\mathscr{G}_p(s)ds\right\|_{\mathscr{H}}^2$$

$$=2E\left\|\int_{-\infty}^{t}(t-s)^{\lambda-1}\eta_p(t-s)\big[\mathscr{G}_p(s+\tau)-\mathscr{G}_p(s)\big]ds\right\|_{\mathscr{H}}^2$$

$$\leq 2\int_{-\infty}^{t}(t-s)^{\lambda-1}\eta_p(t-s)ds\times\int_{-\infty}^{t}(t-s)^{\lambda-1}\eta_p(t-s)E\|\mathscr{G}_p(s+\tau)-\mathscr{G}_p(s)\|_{\mathscr{H}}^2ds$$

$$\leq 2\varepsilon\int_{-\infty}^{t}(t-s)^{\lambda-1}\lambda\int_0^{+\infty}v\xi_\lambda(v)e^{-\mathscr{C}_p(t-s)^\lambda v}dvds$$

$$\times\int_{-\infty}^{t}(t-s)^{\lambda-1}\lambda\int_0^{+\infty}v\xi_\lambda(v)e^{-\mathscr{C}_p(t-s)^\lambda v}dvds$$

$$\leq 2\varepsilon\left(\int_0^{+\infty}\int_0^{+\infty}\lambda v\xi_\lambda(v)\bar{w}^{\lambda-1}e^{-\mathscr{C}_p(t-s)\bar{w}^\lambda v}d\bar{w}dv\right)^2\leq\frac{2\varepsilon}{\mathscr{C}_p^2}, \tag{25}$$

$$\mathscr{I}_2=2E\left\|\int_{-\infty}^{t+\tau}(t+\tau-s)^{\lambda-1}\eta_p(t+\tau-s)\Pi_p(s)d\omega_q(s)\right.$$

$$\left.-\int_{-\infty}^{t}(t-s)^{\lambda-1}\eta_p(t-s)\Pi_p(s)d\omega_q(s)\right\|_{\mathscr{H}}^2$$

$$=2E\left\|\int_{-\infty}^{t}(t-s)^{\lambda-1}\eta_p(t-s)\Pi_p(s+\tau)d[\omega_q(s+\tau)-\omega_q(s)]\right.$$

$$\left.-\int_{-\infty}^{t}(t-s)^{\lambda-1}\eta_p(t-s)\Pi_p(s)d\omega_q(s)\right\|_{\mathscr{H}}^2,\ p\in\mathbb{N},$$

where $\omega_q(s+\tau)-\omega_q(s)$ represents a Brownian motion sharing the same distribution as $\omega_q(s)$, with $\bar{w}=t-s$. Consequently, the aforementioned inequality can be reformulated as:

$$\mathscr{I}_2 \leq 2\int_{-\infty}^{t}(t-s)^{\lambda-1}\eta_p(t-s)ds$$

$$\times\int_{-\infty}^{t}(t-s)^{\lambda-1}\eta_p(t-s)E\|\Pi_p(s+\tau)-\Pi_p(s)\|^2_{\mathscr{H}}ds \leq \frac{2\varepsilon}{\mathscr{C}_p^2}. \tag{26}$$

By substituting equations (25)–(26) into equation (24), we obtain

$$E\|(\Upsilon\varphi)_p(t+\tau)-(\Upsilon\varphi)_p(t)\|^2_{\mathscr{H}} \leq \frac{4\varepsilon}{\mathscr{C}_p^2},$$

which indicates that $\Upsilon\varphi$ is an almost periodic function in terms of distribution.

In summary, Υ possesses a unique fixed point within $\mathscr{X}^*$. Consequently, we can infer that system (3) admits at least one almost periodic solution in distribution. The proof is now complete. □

4. Uniform stability

Definition 6 Let $\zeta(t)=(\zeta_1(t),\ \zeta_2(t),\ \ldots,\ \zeta_n(t))^T$ represent an almost periodic solution in distribution for system (3), with the initial value $\varphi(t)=(\varphi_1(t),\ \varphi_2(t),\ \ldots,\ \varphi_n(t))^T \in C^B_{\mathscr{F}_0}([-\theta,\ 0],\ \mathscr{H}^n)$. Similarly, let $\bar{\zeta}(t)=(\bar{\zeta}_1(t),\ \bar{\zeta}_2(t),\ \ldots,\ \bar{\zeta}_n(t))^T$ denote an arbitrary solution of system (3) with the initial value $\psi(t)=(\psi_1(t),\ \psi_2(t),\ \ldots,\ \psi_n(t))^T \in C^B_{\mathscr{F}_0}([-\theta,\ 0],\ \mathscr{H}^n)$. The solution of system (3) is considered stable if, for any $\varepsilon>0$, there exists a $\delta(t_0,\ \varepsilon)>0$ such that $t\geq t_0\geq 0$, $E\|\psi(t)-\varphi(t)\|^2_\theta<\delta$ implies $E\|\bar{\zeta}(t,\ t_0,\ \psi)-\zeta(t,\ t_0,\ \varphi)\|^2_{\mathscr{X}}<\varepsilon$, where

$$E\|\bar{\zeta}(t)-\zeta(t)\|^2_{\mathscr{X}}=\sup_{t\in\mathbb{R}}\max_{p\in\mathbb{N}}\{E\|\bar{\zeta}_p(t)-\zeta_p(t)\|^2_{\mathscr{H}}\},$$

$$E\|\psi(t)-\varphi(t)\|^2_\theta=\sup_{t\in[-\theta,\ 0]}\max_{p\in\mathbb{N}}\{E\|\psi_p(t)-\varphi_p(t)\|^2_{\mathscr{H}}\}.$$

The system exhibits uniform stability in distribution if the above δ is independent of t_0.

Theorem 2 Under assumptions (H_1)–(H_3), the system (3) exhibits almost periodic uniform stability in distribution.

Proof. In accordance with Definition 4, the solution to system (3) can be formulated as follows:

$$\zeta_p(t)=F_p(t-t_0)\zeta_p(t_0)+\int_{t_0}^{t}(t-s)^{\lambda-1}\eta_p(t-s)\Bigg[\sum_{q=1}^{n}\mathscr{A}_{pq}(s)f_q(\zeta_q(s-\theta_q))$$

$$+\sum_{q=1}^{n}\mathscr{B}_{pq}(s)\mu_q(s)+\bigwedge_{q=1}^{n}\alpha_{pq}(s)g_q(\zeta_q(s-\theta_q))+\bigvee_{q=1}^{n}\beta_{pq}(s)h_q(x_q(s-\theta_q))$$

$$+\bigwedge_{q=1}^{n}\mathscr{T}_{pq}(s)\mu_q(s)+\bigvee_{q=1}^{n}\mathscr{S}_{pq}(s)\mu_q(s)+u_p(s)\Big]ds$$

$$+\int_{t_0}^{t}(t-s)^{\lambda-1}\eta_p(t-s)\sum_{q=1}^{n}\mathscr{D}_{pq}(s)\delta_{pq}(\zeta_q(s-\theta_q))d\omega_q(s). \tag{27}$$

From above inequality, we can get

$$\bar{\zeta}_p(t)-\zeta_p(t)=F_p(t)(\bar{\zeta}_p(0)-\zeta_p(0))+\int_0^t(t-s)^{\lambda-1}\eta_p(t-s)$$

$$\times\Big\{\sum_{q=1}^{n}\mathscr{A}_{pq}(s)[f_q(\bar{\zeta}_q(s-\theta_q))-f_q(\zeta_q(s-\theta_q))]$$

$$+\bigwedge_{q=1}^{n}\alpha_{pq}(s)[g_q(\bar{\zeta}_q(s-\theta_q))-g_q(\zeta_q(s-\theta_q))]$$

$$+\bigvee_{q=1}^{n}\beta_{pq}(s)[h_q(\bar{\zeta}_q(s-\theta_q))-h_q(\zeta_q(s-\theta_q))]\Big\}ds$$

$$+\int_0^t(t-s)^{\lambda-1}\eta_p(t-s)\sum_{q=1}^{n}\mathscr{D}_{pq}(s)[\delta_{pq}(\bar{\zeta}_q(s-\theta_q))-\delta_{pq}(\zeta_q(s-\theta_q))]d\omega_q(s).$$

Since $(a+b+c)^2\le 3a^2+3b^2+3c^2$, and using the inequality technique and Itô isometric property, we have

$$E\|\bar{\zeta}_p(t)-\zeta_p(t)\|^2_{\mathscr{H}}$$

$$=E\Big\|F_p(t)(\bar{\zeta}_p(0)-\zeta_p(0))+\int_0^t(t-s)^{\lambda-1}\eta_p(t-s)$$

$$\times\Big\{\sum_{q=1}^{n}\mathscr{A}_{pq}(s)[f_q(\bar{\zeta}_q(s-\theta_q))-f_q(\zeta_q(s-\theta_q))]$$

$$+\bigwedge_{q=1}^{n}\alpha_{pq}(s)[g_q(\bar{\zeta}_q(s-\theta_q))-g_q(\zeta_q(s-\theta_q))]$$

$$+\bigvee_{q=1}^{n}\beta_{pq}(s)[h_q(\bar{\zeta}_q(s-\theta_q))-h_q(\zeta_q(s-\theta_q))]\Big\}ds$$

$$+\int_0^t (t-s)^{\lambda-1}\eta_p(t-s)\sum_{q=1}^{n}\mathscr{D}_{pq}(s)\times[\delta_{pq}(\bar{\zeta}_q(s-\theta_q))-\delta_{pq}(\zeta_q(s-\theta_q))]d\omega_q(s)\big\|_{\mathscr{H}}^2$$

$$\leq 5E\|F_p(t)(\bar{\zeta}_p(0)-\zeta_p(0))\|_{\mathscr{H}}^2$$

$$+5E\Big\|\int_0^t (t-s)^{\lambda-1}\eta_p(t-s)\times\sum_{q=1}^{n}\mathscr{A}_{pq}(s)[f_q(\bar{\zeta}_q(s-\theta_q))-f_q(\zeta_q(s-\theta_q))]ds\Big\|_{\mathscr{H}}^2$$

$$+5E\Big\|\int_0^t (t-s)^{\lambda-1}\eta_p(t-s)\times\bigwedge_{q=1}^{n}\alpha_{pq}(s)[g_q(\bar{\zeta}_q(s-\theta_q))-g_q(\zeta_q(s-\theta_q))]ds\Big\|_{\mathscr{H}}^2$$

$$+5E\Big\|\int_0^t (t-s)^{\lambda-1}\eta_p(t-s)\times\bigvee_{q=1}^{n}\beta_{pq}(s)[h_q(\bar{\zeta}_q(s-\theta_q))-h_q(\zeta_q(s-\theta_q))]ds\Big\|_{\mathscr{H}}^2$$

$$+5E\Big\|\int_0^t (t-s)^{\lambda-1}\eta_p(t-s)\times\sum_{q=1}^{n}\mathscr{D}_{pq}(s)[\delta_{pq}(\bar{\zeta}_q(s-\theta_q))-\delta_{pq}(\zeta_q(s-\theta_q))]d\omega_q(s)\|_{\mathscr{H}}^2$$

$$\leq\frac{5}{\mathscr{C}_p^2}E\|\varphi_p(0)-\psi_p(0)\|_{\mathscr{H}}^2+\frac{5}{\mathscr{C}_p^2}\Bigg[\sum_{q=1}^{n}(\mathscr{A}_{pq}^{+})^2\sum_{q=1}^{n}(L_q^f)^2+\sum_{q=1}^{n}(\alpha_{pq}^{+})^2\sum_{q=1}^{n}(L_q^g)^2$$

$$+\sum_{q=1}^{n}(\beta_{pq}^{+})^2\sum_{q=1}^{n}(L_q^h)^2+\frac{\mathscr{C}_p}{2}\sum_{q=1}^{n}(\mathscr{D}_{pq}^{+})^2\sum_{q=1}^{n}(L_{pq}^{\delta})^2\Bigg]E\|\bar{\zeta}_q(s-\theta_q)-\zeta_q(s-\theta_q)\|_{\mathscr{H}}^2. \tag{28}$$

From the preceding inequality, it can be deduced that

$$E\|\bar{\zeta}(t)-\zeta(t)\|_{\mathscr{X}}^2$$

$$\leq\max_{p\in\mathbb{N}}\left\{\frac{5}{\mathscr{C}_p^2}E\|\psi_p(t)-\varphi_p(t)\|_{\theta}^2\right\}+\max_{p\in\mathbb{N}}\Bigg\{\frac{5}{\mathscr{C}_p^2}\Bigg[\sum_{q=1}^{n}(\mathscr{A}_{pq}^{+})^2\sum_{q=1}^{n}(L_q^f)^2+\sum_{q=1}^{n}(\alpha_{pq}^{+})^2\sum_{q=1}^{n}(L_q^g)^2$$

$$+\sum_{q=1}^{n}(\beta_{pq}^{+})^2\sum_{q=1}^{n}(L_q^h)^2+\frac{\mathscr{C}_p}{2}\sum_{q=1}^{n}(\mathscr{D}_{pq}^{+})^2\sum_{q=1}^{n}(L_{pq}^{\delta})^2\Bigg]\Bigg\}\sup_{t\in\mathbb{R}}E\|\bar{\zeta}(s-\theta_q)-\zeta(s-\theta_q)\|_0^2$$

$$\leq\max_{p\in\mathbb{N}}\left\{\frac{5}{\mathscr{C}_p^2}E\|\psi_p(t)-\varphi_p(t)\|_{\theta}^2\right\}+\max_{p\in\mathbb{N}}\Bigg\{\frac{5}{\mathscr{C}_p^2}\Bigg[\sum_{q=1}^{n}(\mathscr{A}_{pq}^{+})^2\sum_{q=1}^{n}(L_q^f)^2+\sum_{q=1}^{n}(\alpha_{pq}^{+})^2\sum_{q=1}^{n}(L_q^g)^2$$

$$+\sum_{q=1}^{n}(\beta_{pq}^{+})^2\sum_{q=1}^{n}(L_q^h)^2+\frac{\mathscr{C}_p}{2}\sum_{q=1}^{n}(\mathscr{D}_{pq}^{+})^2\sum_{q=1}^{n}(L_{pq}^{\delta})^2\Bigg]\Bigg\}E\|\bar{\zeta}(t)-\zeta(t)\|_{\mathscr{X}}^2. \tag{29}$$

Thus, we can obtain

$$(1-\Xi)E\|\bar{\zeta}(t)-\zeta(t)\|^2_{\mathscr{Z}} \leq \max_{p\in\mathbb{N}}\left\{\frac{5}{\mathscr{C}_p^2}\right\}E\|\psi_p(t)-\varphi_p(t)\|^2_\theta,$$

$$E\|\bar{\zeta}(t)-\zeta(t)\|^2_{\mathscr{Z}} \leq \max_{p\in\mathbb{N}}\left\{\frac{5}{\mathscr{C}_p^2(1-\Xi)}\right\}E\|\psi_p(t)-\varphi_p(t)\|^2_\theta,$$

where

$$\Xi = \max_{p\in\mathbb{N}}\left\{\frac{5}{\mathscr{C}_p^2}\left[\sum_{q=1}^{n}(\mathscr{A}_{pq}^{+})^2\sum_{q=1}^{n}(L_q^f)^2+\sum_{q=1}^{n}(\alpha_{pq}^{+})^2\sum_{q=1}^{n}(L_q^g)^2\right.\right.$$

$$\left.\left.+\sum_{q=1}^{n}(\beta_{pq}^{+})^2\sum_{q=1}^{n}(L_q^h)^2+\frac{\mathscr{C}_p}{2}\sum_{q=1}^{n}(\mathscr{D}_{pq}^{+})^2\sum_{q=1}^{n}(L_{pq}^{\delta})^2\right]\right\},$$

by the assumption (H_3), we have $\Xi<1$. If we take $E\|\psi-\varphi\|^2_\theta \leq \max_{p\in\mathbb{N}}\left\{\frac{\varepsilon\mathscr{C}_p^2(1-\Xi)}{5}\right\}<\delta$, then we can derive

$$E\|\bar{\zeta}(t)-\zeta(t)\|^2_{\mathscr{Z}} \leq \max_{p\in\mathbb{N}}\left\{\frac{5}{\mathscr{C}_p^2(1-\Xi)}\right\}E\|\psi(t)-\varphi(t)\|^2_\theta<\varepsilon.$$

From the inequalities presented above, it can be concluded that for any $\varepsilon>0$, there exists a $\delta(\varepsilon)>0$ such that $E\|\bar{\zeta}(t,\ t_0,\ \psi)-\zeta(t,\ t_0,\ \varphi)\|^2_{\mathscr{Z}}<\varepsilon$ whenever $E\|\psi(t)-\varphi(t)\|^2_\theta<\delta$. Consequently, the solution $\zeta(t)$ exhibits almost periodic uniform stability. This completes the proof. □

5. Example

Example 1 Let $p,\ q=1,\ 2$, we examine the subsequent fractional-order quaternion-valued stochastic fuzzy neural network:

$$\begin{aligned}{}_{t_0}^{C}D_t^{\lambda}x_p(t) =& -\mathscr{C}_p(t)x_p(t)+\sum_{q=1}^{n}\mathscr{A}_{pq}(t)f_q(x_q(t-\theta_q))+\sum_{q=1}^{n}\mathscr{B}_{pq}(t)\mu_q(t)+u_p(t)\\&+\bigwedge_{q=1}^{n}\alpha_{pq}(t)g_q(x_q(t-\theta_q))+\bigvee_{q=1}^{n}\beta_{pq}(t)h_q(x_q(t-\theta_q))+\bigwedge_{q=1}^{n}\mathscr{T}_{pq}(t)\mu_q(t)\\&+\bigvee_{q=1}^{n}\mathscr{S}_{pq}(t)\mu_q(t)+\sum_{q=1}^{n}\mathscr{D}_{pq}(t)\delta_{pq}(x_q(t-\theta_q))\frac{d\omega_q(t)}{dt},\end{aligned}$$

where $\lambda = 0.6$, $\theta_1 = 0.05$, $\theta_2 = 0.03$, $x_p = x_p^R + ix_p^I + jx_p^J + kx_p^K$, $\mathscr{C}_1(t) = 4.8 + \cos t$, $\mathscr{C}_2(t) = 4.2 + \sin t$, and the coefficients are taken as follows:

$$\mathscr{A}_{pq} = 0.03\sin\sqrt{2}t + i0.1\sin\sqrt{3}t + j0.2\cos\sqrt{2}t + k0.28\cos 2t,$$

$$\mathscr{B}_{pq} = 0.3\cos\sqrt{2}t + i0.2\sin\sqrt{3}t + j0.2\cos 3t + k0.18\sin 2t,$$

$$\alpha_{pq} = 0.12\cos\sqrt{3}t + i0.16\sin\sqrt{3}t + j0.12\cos 3t + k0.18\cos 2t,$$

$$\mathscr{D}_{pq} = 0.13\cos\sqrt{2}t + i0.12\sin\sqrt{3}t + j0.1\cos 3t + k0.15\sin 2t,$$

$$\alpha_{pq} = 0.12\cos\sqrt{3}t + i0.16\sin\sqrt{3}t + j0.12\cos 3t + k0.18\cos 2t,$$

$$\beta_{pq} = 0.1\sin\sqrt{3}t + i0.23\cos\sqrt{2}t + j0.2\cos 3t + k0.3\sin 2t,$$

$$\mathscr{T}_{pq} = 0.1\sin\sqrt{5}t + i0.2\sin\sqrt{3}t + j0.12\sin 2t + k0.2\sin 3t,$$

$$\mathscr{S}_{pq} = 0.15\sin\sqrt{3}t + i0.2\cos\sqrt{2}t + j0.1\cos 3t + k0.3\sin 2t,$$

$$u_1(t) = 0.5\sin\sqrt{5}t + i0.6\sin\sqrt{2}t + j0.25\cos\sqrt{2}t + k0.2\cos 2t,$$

$$u_2(t) = 0.35\cos\sqrt{5}t + i0.4\sin\sqrt{2}t + j0.2\cos\sqrt{2}t + k0.2\cos 2t,$$

$$\mu_q(t) = 0.12\cos\sqrt{2}t + i0.1\sin\sqrt{2}t + j0.1\cos 3t + k0.12\cos 2t,$$

$$f_q(x_q) = 0.1\sin x_q^R + i\frac{1}{18}\cos x_q^I + j\frac{1}{15}\sin x_q^J + k\frac{1}{16}\cos x_q^K,$$

$$g_q(x_q) = \frac{1}{15}\sin x_q^R + i\frac{1}{16}\sin x_q^I + j\frac{1}{15}\cos x_q^J + k\frac{1}{18}\cos x_q^K,$$

$$h_q(x_q) = 0.06\cos x_q^R + i\frac{1}{14}\sin x_q^I + j\frac{1}{17}\sin x_q^J + k\frac{1}{19}\cos x_q^K,$$

$$7\delta_{pq}(x_q) = 0.3\cos x_q^R + i\frac{1}{6}\sin x_q^I + j\frac{1}{7}\cos x_q^J + k\frac{1}{8}\sin x_q^K.$$

In fact, by simple calculation, we can get

$$\mathscr{C}_1 = 3.8, \mathscr{C}_2 = 3.2, L_q^f \le 0.1464, L_q^g \le 0.1260, L_q^h \le 0.1222, L_{pq}^\delta \le 0.3922,$$

$$(\mathscr{A}_{pq}^{+})^2 \leq 0.1293,\ (\mathscr{B}_{pq}^{+})^2 \leq 0.2024,\ (\mathscr{D}_{pq}^{+})^2 \leq 0.0638,\ (\alpha_{pq}^{+})^2 \leq 0.0868,\ (\beta_{pq}^{+})^2 \leq 0.1929.$$

It is easily found that Assumption (H_1)–(H_3) hold

$$\Delta = \max_{p=1,2}\left\{\frac{16}{\mathscr{C}_p^2}\left[\sum_{q=1}^{n}(\mathscr{A}_{pq}^{+})^2\sum_{q=1}^{n}(L_q^f)^2+\sum_{q=1}^{n}(\alpha_{pq}^{+})^2\sum_{q=1}^{n}(L_q^g)^2+\sum_{q=1}^{n}(\beta_{pq}^{+})^2\sum_{q=1}^{n}(L_q^h)^2\right.\right.$$

$$\left.\left.+\frac{\mathscr{C}_p}{2}\sum_{q=1}^{n}(\mathscr{D}_{pq}^{+})^2\sum_{q=1}^{n}(L_{pq}^{\delta})^2\right]\right\} \cong 0.061 \leq 1.$$

Therefore, all conditions outlined in Theorem 1 and Theorem 2 are fulfilled, indicating that system (1) achieves almost periodic uniform stability in distribution.

Remark 2 Figure 1 shows that states $(x_1^R,\ x_1^I,\ x_1^J,\ x_1^K)^T$ and $(x_2^R,\ x_2^I,\ x_2^J,\ x_2^K)^T$ of (1). x_1^R with the initial value $x_1^R(t) = 0.3$, x_1^I with the initial value $x_1^I(s) = 0.5$, x_1^J with the initial value $x_1^J(t) = 0.6$, x_1^K with the initial value $x_1^K(t) = -0.4$; x_2^R with the initial value $x_2^R(t) = -0.3$, x_2^I with the initial value $x_2^I(t) = 0.4$, x_2^J with the initial value $x_2^J(t) = -0.2$, x_2^K with the initial value $x_2^K(t) = 0.7$. Figures 2 and 3 show that the system (1) with initial values $(x_1^R,\ x_1^I,\ x_1^J,\ x_1^K)^T = (0.3,\ 0.5,\ 0.6,\ -0.4)^T$, $(x_2^R,\ x_2^I,\ x_2^J,\ x_2^K)^T = (-0.3,\ 0.4,\ -0.2,\ 0.7)^T$ has at least one almost periodic solution in distribution that is the uniform stability.

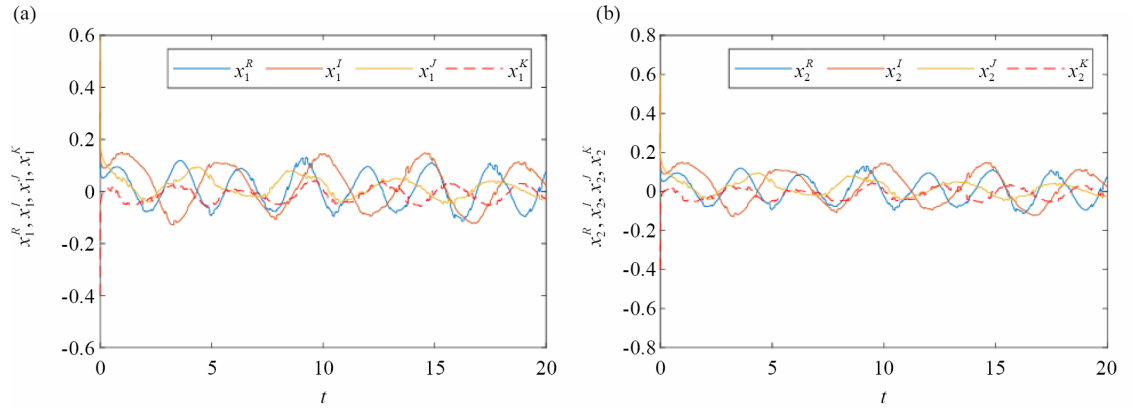
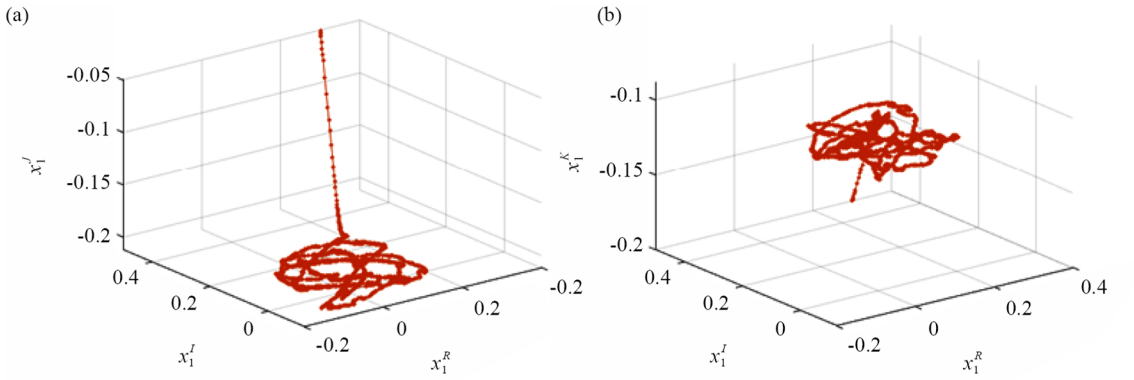


Figure 1. The state trajectories of the system (1) in Example 5.1, $x_1(t) = (x_1^R(t),\ x_1^I(t),\ x_1^J(t),\ x_1^K(t))(t)^T$, $x_2(t) = (x_2^R(t),\ x_2^I(t),\ x_2^J(t),\ x_2^K(t))^T$



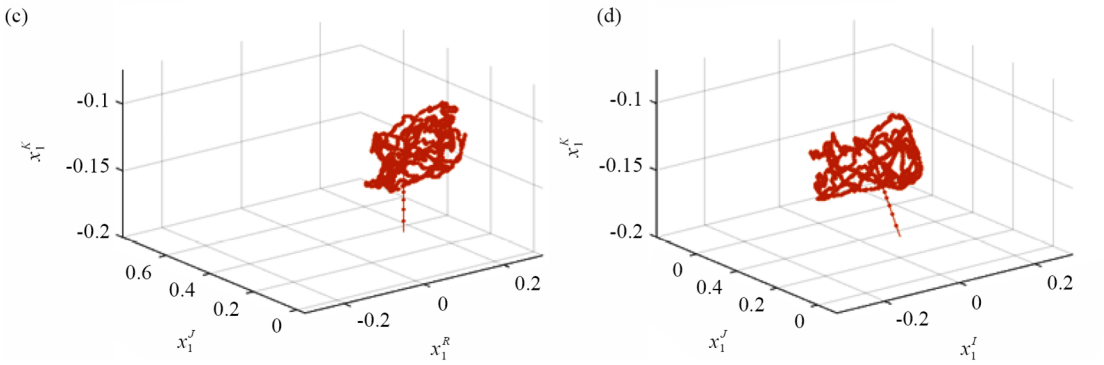


Figure 2. Stability of the system (1) in Example 5.1, $x_1(t) = (x_1^R(t), x_1^I(t), x_1^J(t), x_1^K(t))^T$

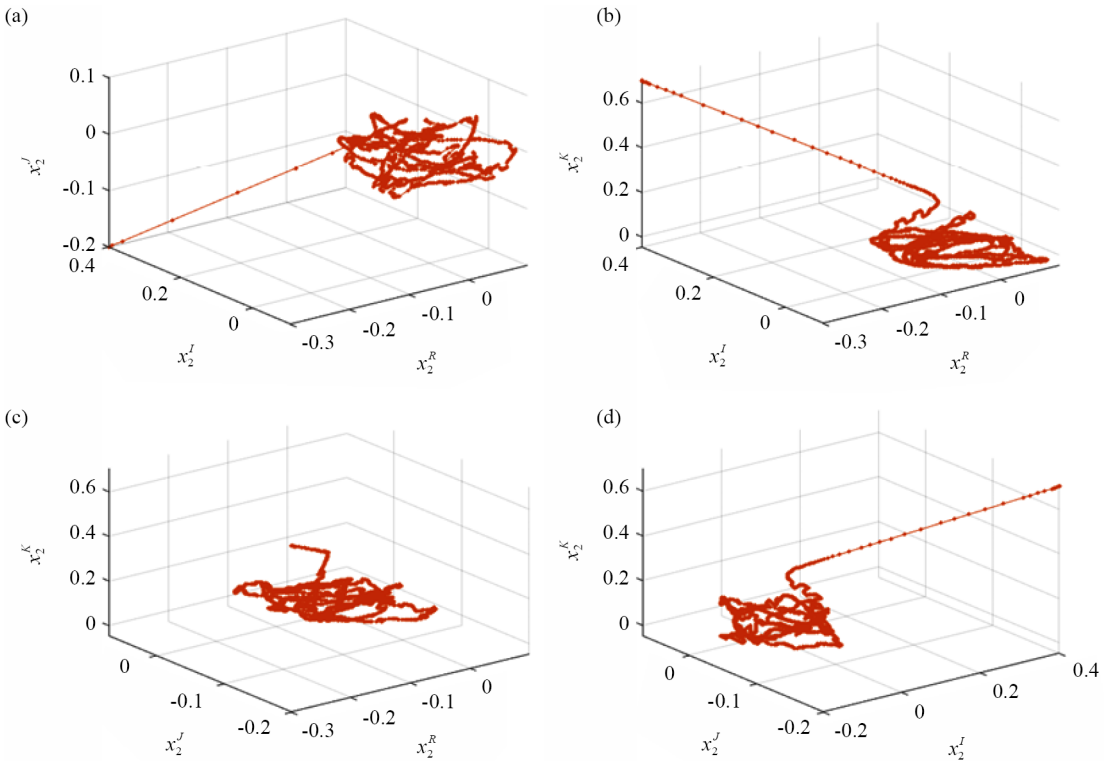


Figure 3. Stability of the system (1) in Example 5.1, $x_2(t) = (x_2^R(t), x_2^I(t), x_2^J(t), x_2^K(t))^T$

6. Conclusion

This study has established the existence and uniform stability, in the distribution sense, of almost periodic solutions for quaternion-valued fractional-order stochastic fuzzy neural networks incorporating distributed delays. By applying the Banach fixed point theorem, we have derived several criteria to ensure the uniform stability of the system and the existence along with uniqueness of the almost periodic solution via a direct approach. It is noteworthy that quaternion-valued fractional-order stochastic fuzzy neural networks represent an extension of both real-valued and complex-valued neural networks. The methodologies and findings presented herein can be effectively employed to investigate the challenges associated with real-valued or complex-valued fractional-order stochastic neural networks that incorporate delays.

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Conflict of interest

The authors declare no competing financial interest.

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