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Research Article

Applications of Intuitionistic Fuzzy Sets to Decision-Making Using AG-Groupoids and Klein Four-Group

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Abstract: This paper proposes a novel decision-making framework that integrates algebraic structures within intuitionistic fuzzy logic to address complex real-world decision-making scenarios involving uncertainty. Two models are developed, including the Abel-Grassmann Intuitionistic Fuzzy Decision Matrix (AG-IFDM) for ranking non-associative neural connectivity patterns and the Group-theoretic Intuitionistic Fuzzy Ranking (GIFR) for evaluating agricultural treatment formulations. These models provide a reliable ranking of alternatives in uncertain environments where the presence or absence of associativity influences the outcomes. The integration of empirical intuition with algebraic reasoning enables a more realistic modelling of uncertainty, where theoretical precision is complemented by practical applications. Integrating both theoretical foundation and practical insight, the applications from medical neuroscience and agricultural optimization underscore the versatility of the proposed models in addressing order-sensitive real-world systems, with promising potential for large-scale validation. Furthermore, the practical illustrations of these models confirm their effectiveness, supported by comprehensive robustness, sensitivity and complexity evaluations. This work establishes a flexible and mathematically precise foundation for algebraic decision-making to effectively address order-sensitive problems and enhance the transparency of multi-criteria decisions under uncertain environments.

Keywords: intuitionistic fuzzy sets, Abelian groups, Abel-Grassmann (AG)-groupoids, agriculture sector, decision-making, medical science, score functions

MSC: 08A72, 62C05, 62C86, 68U35, 18B40

1. Introduction

The most important factor in making a decision is the uncertainty or imprecision of data associated with it, which directly impacts the quality of the decision's outcome. Techniques based on usual sets and binary logic generally fail to address these challenges due to their limited ability to manage the ambiguity inherent in real-world scenarios. To overcome this limitation, fuzzy set theory, introduced by Zadeh [1], provides a strong mathematical framework for representing uncertain and imprecise information. By allowing elements to have varying degrees of membership, fuzzy sets better

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reflect the uncertainty inherent in human behaviour. Fuzzy sets allow decision-makers to indicate the degree to which an element belongs to a set by giving membership degrees. This technique allows for more variable characterization of accessible knowledge and explains the lack of obvious boundaries in decision problems. As a result, decision-making processes can better align with the complexities of real-world situations. By incorporating fuzzy sets and fuzzy logic into multi-criteria decision models, decision-makers are empowered to manage multiple factors simultaneously and make well-informed judgements that consider inherent uncertainty. However, relying solely on a single-value membership function has its drawbacks, particularly when it comes to representing both supportive and opposing evidence. To address this limitation, Atanassov introduced intuitionistic fuzzy sets [2], which extend the concept of fuzzy sets by adding a non-membership function. This addition allows for the expression of both uncertainty and hesitation by offering a clearer representation of fuzzy data compared to Zadeh's original model.

This extension of fuzzy sets has played a key role in establishing the foundations of intuitionistic fuzzy numbers and in developing ranking methods that integrate both score and accuracy functions. These methods have led to the formulation of operational laws and a wide range of aggregation operators for decision-making applications, as demonstrated in [3–6]. These include not only averaging, weighted and ordered weighted averaging, and hybrid operators, but also their geometric counterparts, as well as advanced concepts like the intuitionistic fuzzy Bonferroni means and generalized aggregation techniques. Collectively, these contributions have significantly developed the theoretical background and practical methodologies for handling uncertainty and imprecision in complex decision environments. Overall, these foundational studies, together with more recent developments such as those in [7–10], have significantly advanced the understanding of information fusion within intuitionistic fuzzy decision-making and offered practical strategies for real-world implementation. The mathematical framework of intuitionistic fuzzy sets can be naturally extended to abstract algebra, which provides a formal foundation for analysing operations and relationships among elements under uncertainty. This integration opens new avenues for the development of algebraic models capable of capturing both structural properties and degrees of uncertainty for developing new theoretical tools and practical applications.

1.1 *Motivation and primary objectives*

Despite extensive research on the expansion of algebraic structures to incorporate fuzzy set theory, their practical use in decision-making processes remains limited. The motivation behind this study is driven by the need to bridge the gap between abstract mathematical theory and practical uncertainty handling by integrating abstract algebra with intuitionistic fuzzy sets to enhance their application in real-world decision-making problems. Algebraic structures [11] and decision-making through intuitionistic fuzzy sets [12, 13] are two key areas that have each been extensively studied and investigated. Algebraic structures provide a foundational framework for understanding the operations, relationships, and properties within complex mathematical systems. These structures, such as groups and Abel-Grassmann (AG)-groupoids, offer a deeper comprehension of abstract mathematical concepts and their applications. On the other hand, decision-making through intuitionistic fuzzy sets addresses the challenge of handling uncertainty and imprecision in complex decision-making scenarios. A group theory that possesses associativity, along with an AG-groupoid, which possesses non-associativity, makes it more suitable for handling ambiguity in decision-making processes through fuzzy set theory. In practical decision-making problems, there are instances where the associative property holds, meaning the order in which operations are performed does not affect the outcome. On the other hand, there are scenarios where the non-associative property is observed, implying that the order of operations does impact the final result. To address this challenge, we intend to utilize abstract algebraic concepts within the framework of an intuitionistic fuzzy environment that will quantify the accuracy of different alternatives featuring distinct attributes.

The primary contributions of this work are as follows.

1.1.1 *Theoretical novelty of the proposed models*

This paper presents a novel approach by combining an AG-groupoid with intuitionistic fuzzy logic to model systems that involve asymmetry, uncertainty, and non-associative behaviour. The model uses the structure of AG-groupoids through ideal theory and the score (accuracy) functions of intuitionistic fuzzy sets to build a flexible algebraic tool for decision-making. The Klein four-group is extended into an intuitionistic fuzzy framework by applying the group-theoretic congruence concepts and score (accuracy) functions. This extension helps represent symmetric but uncertain compatibility between different alternatives, which allows pairwise interactions to be assessed through fuzzy scores and forms a structured algebraic tool for decision-making. Overall, the proposed framework offers a strong theoretical basis for modelling complex and diverse systems under uncertainty, moving beyond a mere combination of existing concepts, particularly in cases where traditional mathematical models cannot fully capture the nature of real-world interactions.

1.1.2 *Practical novelty of the proposed models*

The proposed models offer a multi-purpose framework with broad applicability to systems characterized by uncertainty and contextual interaction. However, in this paper, their utility is demonstrated through two domain-specific practical problems: (1) the symbolic modelling of coordinated eye-movement neural pathways and (2) the formulation of antibacterial agricultural treatments. In the neuroscience context, the AG-groupoid-based symbolic structure captures neuron-level interactions that reflect vestibulo-ocular dynamics, which serve as a discrete algebraic alternative to traditional models for representing causal and directional signalling. In the agricultural domain, the model supports formulation by ranking ingredient compatibility under uncertainty for reducing formulation failures caused by incompatibilities. Both examples illustrate the practical effectiveness of the models in guiding reasoning and decision-making in systems where relationships are based on context.

1.2 *Structure of the study*

Section I presents the overview, motivation, and primary objectives of this paper. Section II offers a foundational literature review on the introduction of newly developed score and accuracy functions for comparing intuitionistic fuzzy numbers and intuitionistic fuzzy aggregation operators for aggregating intuitionistic fuzzy information, along with some basic results. Sections III and IV develop decision-making models that utilize intuitionistic fuzzy information within the abstract frameworks of AG-groupoids and Abelian groups by addressing decision-making problems in the medical and agricultural sectors, respectively. Section V presents a comparative study of the proposed methodologies with existing approaches from the literature. Section VI presents the sensitivity and robustness analysis of the proposed models, followed by advantages and limitations in Section VII. Concluding remarks and insights are provided in Section VIII.

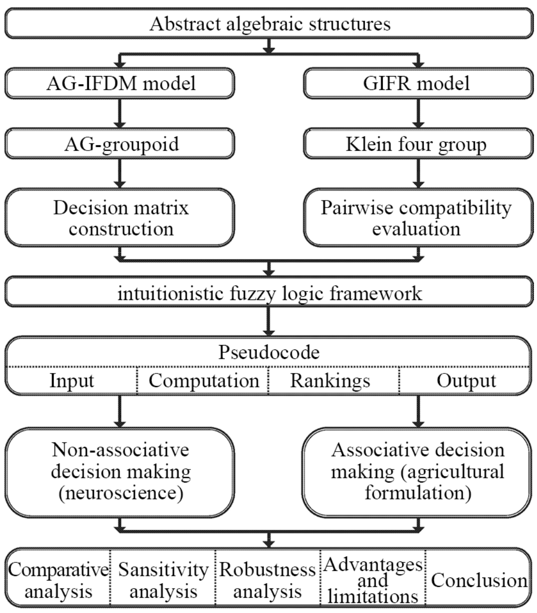

Figure 1. The schematic diagram for the sequence of the study

The detailed schematic diagram (Figure 1) illustrating the theoretical foundations, algorithmic workflow, and applied domains of the two proposed models Abel-Grassmann Intuitionistic Fuzzy Decision Matrix (AG-IFDM) and Group-theoretic Intuitionistic Fuzzy Ranking (GIFR) has been added below. This figure visually distinguishes the non-associative, ideal-based decision matrix construction of AG-IFDM and the associative, congruence-driven pairwise compatibility evaluation of GIFR. It concisely maps the progression from theoretical formulation to computational steps and domain-specific applications (medical neuroscience and agricultural treatment).

2. Conceptual framework

A thorough understanding of the fundamental concepts and terminology is essential for the formulation of decision-making models and their practical implementation. This section focuses on key concepts such as intuitionistic fuzzy sets (numbers), score (accuracy) functions, and intuitionistic fuzzy aggregation operators. To address real-world uncertain and ambiguous problems, the strategies commonly employed in classical mathematics are not always effective. In 1965 [1], Zadeh introduced the concept of fuzzy sets, which allowed elements to have varying degrees of membership rather than being strictly included or excluded. However, in many situations, fuzzy sets alone are not sufficient to capture the hesitation or indecision present in expert evaluations or decision-making processes. To address this limitation, Atanassov extended the fuzzy set theory in 1986 by introducing Intuitionistic Fuzzy Sets (IFSs) [2], which incorporate both membership and non-membership degrees, along with a hesitation margin. For example, in medical treatment, in addition to prescribing or avoiding medication, doctors often choose to monitor symptoms, which acts as a hesitation factor. This approach delays action due to the ambiguity (such as unclear symptoms or overlapping conditions) and prioritizes observation over immediate intervention.

Building upon these foundational developments, recent comprehensive surveys have documented the evolution, theoretical underpinnings, and application domains of fuzzy and intuitionistic fuzzy Multi Criteria Decision Making (MCDM) methods. Zavadskas et al. [14] provide state-of-the-art overviews of MCDM/Multi Attribute Decision Making (MADM) methodologies, while Mardani et al. [15] offer an extensive review of multiple criteria decision-making techniques and their applications. Yager [16] extends the framework through Pythagorean membership grades, enriching the expressiveness of fuzzy MCDM, and Kahraman et al. [17] present a detailed account of five decades of fuzzy set theory and its practical implementations. These works underscore the continuing advancement of fuzzy decision-making and provide the broader methodological context within which the proposed AG-IFDM and GIFR models are situated.

2.1 *Intuitionistic fuzzy sets*

Let S be a universal set. The following are the formal definitions of fuzzy set and intuitionistic fuzzy set.

• A fuzzy set F of S is a class of objects having the form $F = \{(s,\ f(s))/s \in S\}$, where $f : S \to [0,\ 1]$ is called the membership function of the fuzzy set F.

• An Intuitionistic Fuzzy Set (IFS) I of S is an object having the form $I = \{(s,\ f_I(s),\ g_I(s))/s \in S\}$. The functions $f_I : S \longrightarrow [0,\ 1]$ and $g_I : S \longrightarrow [0,\ 1]$ denote the degree of membership and the degree of non-membership of s in I respectively such that for all $s \in S$, we have $0 \leq f_I(s) + g_I(s) \leq 1$.

Moreover, for each IFS I in S, $h_I(s) = 1 - f_I(s) - g_I(s)$ is called the hesitancy degree of s to I. If $h_I(s) = 0$, then the IFS I reduces to a fuzzy set. In [3], Xu and Yager defined each triplet $(f_I(s),\ g_I(s),\ h_I(s))$ as an Intuitionistic Fuzzy Value (IFV) or an Intuitionistic Fuzzy Number (IFN), and denoted it by $I = (f_I,\ g_I,\ h_I)$, where $f_I,\ g_I \geq 0$, $f_I + g_I \leq 1$, $h_I = 1 - f_I - g_I$. Each IFN has a physical meaning; for instance, if $I = (0.5,\ 0.3,\ 0.2)$, it means "0.5 in support of prescribing medication, 0.3 against in favour of avoiding medication, and 0.2 abstaining due to diagnostic ambiguity".

2.2 *Score (accuracy) functions and aggregation operators*

Score and accuracy functions are fundamental components in ranking Intuitionistic Fuzzy Numbers (IFNs). They play a fundamental role in providing a systematic and objective way to evaluate and compare these fuzzy numbers. Chen and Tan [8] were the first to propose the score and accuracy functions for IFNs and utilize them to develop an approach to multi-attribute decision making. Tversky and Kahneman [18] suggested a similar idea for score and accuracy functions. The concept of score and accuracy functions for linear (quadratic) Diophantine fuzzy sets was proposed in [19, 20], and it was then applied to multi-attribute decision-making problems. The relationship between the score function and the accuracy function has been established to be similar to the relationship between the mean and variance in statistics [21]. In statistics, an efficient estimator is characterized by the variance of its sampling distribution; the lower the variance, the more efficient the estimator. By analogy, it is reasonable and appropriate to assert that the higher the degree of accuracy an IFN, the better its quality or performance.

To establish an ordering among the elements of I, we redefine the score and accuracy functions by transforming their respective ranges. The score function is modified from $[-1, 1]$ to $[0, 1]$, and the accuracy function is adjusted from $[0, 1]$ to $[0, 0.5]$. The redefinition of the score and accuracy functions ensures an interpretable and hesitation-sensitive ranking system, guided by decision hierarchy and consistency. The proposed score function maps the net support from the original range $[-1, 1]$ to $[0, 1]$. This eliminates negative values, which ensures that all scores are non-negative and easily interpretable. The accuracy function is compressed to $[0, 0.5]$, which quantifies certainty by reducing scores in response to hesitation.

Definition 1 Let I be the set of all IFNs. A score function on I can be defined by the mapping

$$\Omega : I \to [0,\ 1] \text{ such that } \Omega(I) = \frac{f_I - g_I + 1}{2},$$

where $I = (f_I, g_I)$ for all $s \in S$, Ω is the score function of I, and $\Omega(I)$ is the score of I.

In particular, if $\Omega(I) = 1$, then the IFN I takes the largest value $I^+ = (1, 0)$. If $\Omega(I) = 0$, then the IFN I takes the smallest value $I^- = (0, 1)$.

Example 1 In real-world medical practice, doctors often make decisions under uncertainty, especially when patient symptoms are vague, test results are inconclusive, or multiple treatment paths are available with no clear superior option. To mathematically model such uncertain decisions, let us use IFN $I = (f_I, g_I)$, where f_I represents the degree of confidence that a particular clinical action is appropriate, g_I reflects the degree of doubt or hesitation about the same action, and the remaining component, called the hesitation factor, accounts for uncertainty due to incomplete or inconclusive information. Let us consider four clinical decision scenarios, each represented by the IFNs I_1, I_2, I_3, I_4. To support the prioritization of specific clinical actions, we first needed to compare $I_1 = (0.3, 0.1)$ with $I_2 = (0.2, 0.5)$, and then compare $I_3 = (0.4, 0.3)$ with $I_4 = (0.5, 0.4)$, in order to evaluate which IFN in each pair holds a stronger decision weight.

The corresponding scores are calculated as follows:

$$\Omega(I_1) = 0.6,\ \Omega(I_2) = 0.3 \text{ and } \Omega(I_3) = 0.5 = \Omega(I_4).$$

Since $\Omega(I_1) > \Omega(I_2)$, that is, the score of an IFN I_1 is higher than that of an IFN I_2. This indicates that I_1 should be prioritized for treatment, as it demonstrates stronger clinical support and less opposition as compared to I_2. However, the score function fail to distinguish between I_3 and I_4, indicating its limitation in capturing the underlying clinical uncertainty.

Since IFNs can have similar score values, the score function alone is insufficient for effectively ranking them. To construct the ordering amongst the elements of I when the scores of two IFNs in I are equal, and an accuracy function can be defined as follows:

Definition 2 Let I be the set of all IFNs. An accuracy function on I is a mapping

$$\Pi : I \to [0,\ 0.5] \text{ such that } \Pi(I) = \frac{f_I + g_I}{2},$$

where $I = (f_I, g_I)$ for all $s \in S$, $\Pi(I)$ is an accuracy degree of I.

Now, let us revisit Example 1. The accuracy degrees of I_3 and I_4 are as follows:

$$\Pi(I_3) = 0.3 \text{ and } \Pi(I_4) = 0.4.$$

Since $\Pi(I_4) > \Pi(I_3)$, that is, the accuracy degree of an IFN I_4 is higher than that of an IFN I_3, which indicates that I_4 should be prioritized for treatment.

The relationship between the hesitancy degree and the accuracy degree of I can be shown as follows:

$$\Pi(I_2) = \frac{1 - h_I}{2}.$$

As a result, the smaller the hesitancy degree h_I, the greater the accuracy degree.

Based on the above comprehensive discussion, we can now compare and rank two IFNs using the score and accuracy functions, as demonstrated below.

Definition 3 Let $I_1 = (g_{I_1}, h_{I_1})$ and $I_2 = (g_{I_2}, h_{I_2})$ be two IFNs, $\Omega(I_1)$ and $\Omega(I_2)$ the scores of the IFNs I_1 and I_2 respectively, and $\Pi(I_1)$ and $\Pi(I_2)$ the accuracy degrees of the IFNs I_1 and I_2 respectively. Then

• If $\Omega(I_1) < \Omega(I_2)$, then $I_1 < I_2$.

• If $\Omega(I_1) = \Omega(I_2)$, then

(i) If $\Pi(I_1) < \Pi(I_2)$, then $I_1 < I_2$.

(ii) If $\Pi(I_1) = \Pi(I_2)$, then $I_1 = I_2$.

Keeping in view Definition 3, instead of comparing I_1 with I_2 and I_3 with I_4 in a pairwise manner, we now reframe Example 1 to compare all four IFNs collectively in order to evaluate which holds a stronger decision weight. Based on the preceding calculations and discussion, the resulting ranking is $I_1 > I_4 > I_3 > I_2$.

Theorem 1 below guarantees that ranking remains consistent under any monotonic rescaling of evaluation metrics. This ensures method robustness when measurement units change (e.g., for different normalization schemes).

Theorem 1 The ranking in Definition 3 is invariant under strictly increasing transformations: For any strictly increasing $f, g : \mathbb{R} \to \mathbb{R}$, the ordering of IFNs is preserved when scores Ω and accuracy Π are transformed as $\Omega' = f(\Omega)$ and $\Pi' = g(\Pi)$.

Proof. Let $I_1 > I_2$.

Case 1: If $\Omega(I_1) > \Omega(I_2)$, then $f(\Omega(I_1)) > f(\Omega(I_2))$ by monotonicity.

Case 2: If $\Omega(I_1) = \Omega(I_2)$ and $\Pi(I_1) > \Pi(I_2)$, then $g(\Pi(I_1)) > g(\Pi(I_2))$.

Both preserve $I_1 > I_2$ under transformed values. □

Aggregation operators are of great importance in decision-making processes, particularly in scenarios where multiple criteria, attributes, or sources of information need to be considered. Xu and Yager [22, 23] were the first to describe the algebraic operations and aggregation operators, which are fundamental in integrating information in one form and solving decision-making problems. These comprise the weighted and basic averaging operators, as well as the geometric kinds. Four of them are listed below.

Definition 4 Let $\mathscr{I}_j = (f_{\mathscr{I}_j}, g_{\mathscr{I}_j})$, $(j = 1, 2, \ldots, n)$ be a collection of IFNs, and let w_j be the weight of $\mathscr{I}_j$ satisfying $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$. Then an Intuitionistic Fuzzy Weighted Averaging (IFWA) operator can be defined by

$$\text{IFWA}(\mathscr{I}_1, \mathscr{I}_2, \ldots, \mathscr{I}_n) = w_1\mathscr{I}_1 \oplus w_2\mathscr{I}_2 \oplus \ldots \oplus w_n\mathscr{I}_n,$$

and an Intuitionistic Fuzzy Weighted Geometric (IFWG) operator can be defined by

$$\text{IFWA}(\mathscr{I}_1,\ \mathscr{I}_2,\ ...,\ \mathscr{I}_n) = \mathscr{I}_1^{w_1} \otimes \mathscr{I}_2^{w_2} \otimes ... \otimes \mathscr{I}_n^{w_n}.$$

If $w_i = \dfrac{1}{n}$ $\forall$ i then the IFWA becomes Intuitionistic Fuzzy Averaging (IFA) with

$$\text{IFA}(\mathscr{I}_1,\ \mathscr{I}_2,\ ...,\ \mathscr{I}_n) = \frac{\mathscr{I}_1 \oplus \mathscr{I}_2 \oplus ... \oplus \mathscr{I}_n}{n}.$$

Similarly, the IFWG becomes Intuitionistic Fuzzy Geometric (IFG) with

$$\text{IFG}(\mathscr{I}_1,\ \mathscr{I}_2,\ ...,\ \mathscr{I}_n) = (\mathscr{I}_1 \otimes \mathscr{I}_2 \otimes ... \otimes \mathscr{I}_n)^{\frac{1}{n}}.$$

Definition 5 A score function Ψ is defines as an IF-score left ideal of an AG-groupoid D if it satisfies the condition $\Psi(xy) \geq \Psi(y)$ for all $x,\ y \in D$.

Definition 6 An IF-score congruence is an IF-score equivalence relation $\Psi : G \times G \to [0,\ 1] \times [0,\ 1]$ on G that is also compatible. An IF-score relation is called equivalence if it satisfies, $\Psi(a,\ a) = (1,\ 0)$ (reflexivity) and $\Psi(a,\ b) = \Psi(b,\ a)$ (symmetry) for all $a,\ b \in G$ along with $\Psi \circ \Psi \leq \Psi$ (transitivity). It is said to be compatible if, $\Psi(a,\ b) \wedge \Psi(c,\ d) \leq \Psi(ab,\ cd)$ for all $a,\ b,\ c,\ d \in G$.

To ensure that empirical IF-score evaluations used in later sections obey the required algebraic laws, we state two closure results. They produce the smallest intuitionistic fuzzy structures containing the given data while preserving model interpretability.

Theorem 2 Let $(G,\ \cdot)$ be an AG-groupoid and $\Psi = (f,\ g) : G \to [0,\ 1] \times [0,\ 1]$ an IFS (where $f(x) + g(x) \leq 1$ for all $x \in G$). Define the left-ideal closure $\Psi^\star = (f^\star,\ g^\star)$ as:

$$f^\star(z) = \max\left\{ f(z),\ \sup_{\{y \in G | \exists x \in G : z = xy\}} f(y) \right\},$$

$$g^\star(z) = \min\left\{ g(z),\ \inf_{\{y \in G | \exists x \in G : z = xy\}} g(y) \right\},$$

with the convention that $\sup \varnothing = -\infty$ and $\inf \varnothing = +\infty$ (so $f^\star(z) = f(z)$, $g^\star(z) = g(z)$ if z is not a product).

Then $\Psi^\star$ is the smallest IF-score left ideal containing Ψ, i.e.:

1. $\Psi^\star \geq \Psi$ (component-wise),
2. $\Psi^\star$ satisfies the left-ideal property:

$$f^\star(xy) \geq f^\star(y), \quad g^\star(xy) \leq g^\star(y) \quad \forall x,\ y \in G,$$

3. For any IF-score left ideal $\widetilde{\Psi} \geq \Psi$, $\widetilde{\Psi} \geq \Psi^\star$.

Proof.

1. By definition, $f^\star(z) \geq f(z)$ and $g^\star(z) \leq g(z)$.
2. For $z = ab$,

$$f^\star(ab) \geq \sup_{\{y|\exists x: ab=xy\}} f(y) \geq f(b) \quad (\text{for } x=a,\ y=b),$$

and similarly $g^\star(ab) \leq g(b)$.

3. Let $\widetilde{\Psi} = (\tilde{f},\ \tilde{g})$ be an IF-score left ideal with $\widetilde{\Psi} \geq \Psi$. For any $z \in G$,

$$\tilde{f}(z) \geq f(z), \text{ and } \tilde{f}(z) \geq \tilde{f}(y) \geq f(y)\ \forall y$$

such that $z = xy$, so $\tilde{f}(z) \geq f^\star(z)$. Similarly, $\tilde{g}(z) \leq g^\star(z)$.

For any z, $f^\star(z) + g^\star(z) \leq 1$ since $f(y) + g(y) \leq 1$ for all y in the supremum/infimum. Thus $\Psi^\star$ is the smallest IF-score left ideal containing Ψ. □

Remark 1 When raw Intuitionistic Fuzzy (IF) scores do not satisfy the left-ideal law, we replace Ψ by $\widehat{\Psi}^\star$ to minimally enforce algebraic consistency.

Theorem 3 Let $(G,\ \cdot)$ be a group and $R = (f_R,\ g_R) : G \times G \to [0,\ 1] \times [0,\ 1]$ an IFS relation. The congruence closure $R^\star = (f^\star,\ g^\star)$ is:

$$f^\star = \inf\{S \mid S \text{ is an IF-score congruence},\ S \geq R\},$$

$$g^\star = \sup\{S \mid S \text{ is an IF-score congruence},\ S \geq R\},$$

where the infimum/supremum are taken component-wise over all IF-score congruences $S \geq R$. Then $R^\star$ is the smallest IF-score congruence containing R, i.e.:

1. $R^\star \geq R$,
2. $R^\star$ is reflexive, symmetric, transitive, and compatible:

$$f^\star(ac,\ bd) \geq \min\{f^\star(a,\ b),\ f^\star(c,\ d)\},$$

$$g^\star(ac,\ bd) \leq \max\{g^\star(a,\ b),\ g^\star(c,\ d)\}.$$

3. For any IF-score congruence $\widetilde{R} \geq R$, $\widetilde{R} \geq R^\star$.

Proof. 1. Holds by definition.

2. Reflexivity: $f^\star(a,\ a) = \inf_S f_S(a,\ a) = \inf_S 1 = 1$, $g^\star(a,\ a) = \sup_S g_S(a,\ a) = \sup_S 0 = 0$, Symmetry: follows similarly as $f_S(a,\ b) = f_S(b,\ a)$, $g_S(a,\ b) = g_S(b,\ a)$. Compatibility: For all $a,\ b,\ c,\ d \in G$,

$$\begin{aligned} f^\star(ac,\ bd) &= \inf_S f_S(ac,\ bd) \\ &\geq \inf_S \min\{f_S(a,\ b),\ f_S(c,\ d)\} \\ &\geq \min\{f^\star(a,\ b),\ f^\star(c,\ d)\}, \end{aligned}$$

and similarly for $g^\star$.

Transitivity: for all $a,\ b,\ c \in G$,

$$f^\star(a,\ b) = \inf_S f_S(a,\ b)$$

$$\geq \inf_S \sup_d \min\{f_S(a,\ d),\ f_S(d, b)\}$$

$$\geq \sup_d \min\{f^\star(a,\ d),\ f^\star(d,\ b)\},$$

(analogously for $g^\star$) since each S is transitive.

3. Obviously holds by definition. □

Remark 2 We use $\widehat{R}^\star$ to enforce a valid IF-score congruence before ranking.

Building on this foundational knowledge, the following sections will introduce novel methods that integrate algebraic structures with score and accuracy functions and aggregation techniques. These approaches will be applied to real-world decision-making problems, particularly in the medical and agriculture fields, where information is often incomplete, inconsistent, or obtained from multiple expert sources.

In the medical field, recent work shows practical uses of fuzzy logic. Cui and Tan [24] built a two-level system (Quality Improvement Method using Fuzzy Decision Support (QIM-FDS)) that turns patient feedback into hospital service quality actions. Ambags et al. [25] used fuzzy ideas inside probabilistic decision trees to make clear predictions for thyroid nodules and for the risk of chronic kidney disease progression. Ameen et al. [26] combined expert fuzzy rules on complete blood count tests with a Random Forest to help classify blood disorders. Moreover, in agriculture, fuzzy logic helps with control and planning. Bukhari et al. [27] made an IoT-based irrigation controller that uses a fuzzy rule set with soil moisture and crop factors to open and close valves. Erdoğdu et al. [28] joined fuzzy modelling with a genetic algorithm to balance cost, time, and quality under uncertainty. Sinha and Tiwari [29] reviewed how fuzzy methods are used across farming tasks like monitoring, control, and decision support.

The objective of the proposed work is to bridge the theoretical advancements of the intuitionistic fuzzy framework with practical applications by enhancing the precision, adaptability, and reliability of decisions in the fields of medical and agriculture.

3. From neurons into numbers through an algebraic decision-making model

The human nervous system is a complex network comprising an infinite number of neural patterns. These patterns are the result of the complex interplay of neurons, which are the fundamental building blocks of the nervous system. The complexity of the neural patterns within the nervous system extends well beyond the common notion of a network, which makes their study essential for understanding how the nervous system processes information. To grasp the nature of this system, we must venture into the world of mathematics and neuroscientific inquiry. Investigating deeper into the mathematical aspects of neural patterns allows us to gain the precision with which our nervous system operates. Many researchers have worked on decoding the complex language of neurons by investigating the mathematical patterns that support neural communication [30–33] using abstract algebra. In this section, we utilize an algebraic structure known as an AG-groupoid to optimize communication dynamics among neurons within consolidated neural structures. This approach enhances our understanding of neural network interactions and improves the efficiency of information exchange in complex neural systems.

3.1 *Introduction to AG-groupoids*

P. V. Protic and N. Stevanovic coined the term an Abel-Grassmann's (AG)-groupoid, building on the work of Kazim and Naseeruddin, who first introduced this concept in 1972 [34] under the name of Left Almost (LA)-semigroup. An AG-groupoid $\mathbb{A}_G$ is a type of groupoid that satisfies the left invertive law, which states that $(k_1k_2)k_3 = (k_3k_2)k_1$ for all $k_1,\ k_2,\ k_3 \in \mathbb{A}_G$. An AG-groupoid is a non-associative and non-commutative structure that lies between a groupoid and a commutative semigroup [35]. Kazim and Naseeruddin showed that AG-groupoids are medial, meaning they satisfy $(k_1k_2)(k_3k_4) = (k_1k_3)(k_2k_4)$ for any $k_1,\ k_2,\ k_3,\ k_4 \in \mathbb{A}_G$ [34]. A left identity element in an AG-groupoid may or may not exist, but if it does, it is unique [35]. An AG-groupoid with a left identity element, where each element has an inverse, is referred to as an AG-group [36]. In such structures, the paramedial law holds, expressed as $(k_1k_2)(k_3k_4) = (k_4k_3)(k_2k_1)$ for all $k_1,\ k_2,\ k_3,\ k_4 \in \mathbb{A}_G$. By using the medial law in conjunction with the left identity, the relation $k_1(k_2k_3) = k_2(k_1k_3)$ can be derived for all $k_1,\ k_2,\ k_3 \in \mathbb{A}_G$.

3.2 *The AG-IFDM model*

To rank a collection of alternatives based on a set of attributes and their associated importance, we introduce the Abel-Grassmann Intuitionistic Fuzzy Decision Matrix (AG-IFDM) framework, which integrates the algebraic structure of AG-groupoids with the detailed representation of intuitionistic fuzzy logic to enable effective multi-criteria decision-making. The procedure consists of the following steps:

Preliminary step: Let $\mathscr{D} = \{D_1,\ D_2,\ D_3,\ \ldots,\ D_m\}$ denote a collection of decision alternatives consisting of two or more AG-groupoids defined on the set of attributes $\mathscr{A} = \{A_1,\ A_2,\ A_3,\ \ldots,\ A_n\}$. These attributes serve as the basis on which alternatives are assessed with respect to the weight vector $w = \{w_1,\ w_2,\ w_3,\ \ldots,\ w_n\}$ representing the relative importance of each attribute such that $\sum_{j=1}^{n} w_j = 1$. Each alternative $\mathscr{D}_i$ with respect to the attribute $\mathscr{A}_j$ is evaluated by an IFN $l_{ij} = \left(f_{l_{ij}},\ g_{l_{ij}}\right)$, which is obtained through generating an IF-score left ideal in each alternative $\mathscr{D}_i$. Note that, a score function Ψ is defines as an IF-score left ideal of an AG-groupoid D if it satisfies the condition $\Psi(xy) \geq \Psi(y)$ for all $x,\ y \in D$.

Consequently, an AG-IFDM is constructed to capture and store the characteristics of all alternatives $\mathscr{D}_i$ $(i = 1,\ 2,\ 3,\ \ldots,\ m)$ in relation to the attributes $\mathscr{A}_j$ $(j = 1,\ 2,\ 3,\ \ldots,\ n)$.

Step 1: Take the following information as input of the model.

- Set of alternatives $\mathscr{D} = \{\mathscr{D}_1,\ \mathscr{D}_2,\ \ldots,\ \mathscr{D}_m\}$.
- Set of attributes $\mathscr{A} = \{\mathscr{A}_1,\ \mathscr{A}_2,\ \ldots,\ \mathscr{A}_n\}$.
- Weight vector $w = \{w_1,\ w_2,\ \ldots,\ w_n\}$.
- Expert evaluations or empirical scores (support/rejection) for each pair $(\mathscr{D}_i,\ \mathscr{A}_j)$.

Step 2: For each alternative $\mathscr{D}_i$ and each attribute $\mathscr{A}_j$, compute the IFN $l_{ij} = \left(f_{l_{ij}},\ g_{l_{ij}}\right)$ based on the construction of IF score left ideals within the corresponding $\mathscr{D}_i$. This results in the AG-IFDM, a decision matrix containing all alternative-attribute evaluations.

Step 3: Aggregate the evaluations for each alternative across all attributes using intuitionistic fuzzy aggregation operator(s).

Step 4: Apply the score function to compute a score from each aggregated intuitionistic fuzzy number, which represents the overall strength of each alternative.

Step 5: Rank all alternatives $\mathscr{D}_i$ in descending order based on their respective scores. In cases of identical scores, apply the accuracy function to distinguish between alternatives and refine the ranking.

Step 6: The optimal decision is the alternative $\mathscr{D}_i$ with the highest score (or refined accuracy score), which indicates the most suitable choice under the given criteria and weights.

3.2.1 *Pseudocode for AG-IFDM*

Now we present the pseudocode of the above model. This pseudocode will help the reader to implement the model efficiently by providing a clear, step-by-step representation of its computational logic, which facilitates the reproducibility and further algorithmic development.

Input:

$\mathscr{D} = \{\mathscr{D}_1, \ldots, \mathscr{D}_m\} \to$ alternatives

$\mathscr{A} = \{\mathscr{A}_1, \ldots, \mathscr{A}_m\} \to$ attributes

$w = \{w_1, \ldots, w_n\} \to \Sigma wj = 1$

$E[i, j] \to$ expert support/ rejection for $(\mathscr{D}_i, \mathscr{A}_j)$

$\Psi \to$ IF-score left ideal on $\mathscr{D}_i$ with $\Psi(xy) \geq \Psi(y)$.

Output:

Ranking of $\mathscr{D}_i$ by IF-scores (tie-break by accuracy)

Procedure:

for i in $1, \ldots, m$ and for j in $1, \ldots, n$

$l_{ij} \to$ IFN from Ψ

$L \to [l_{ij}]$ $(m \times n)$ matrix

for i in $1, \ldots, m$

$L_i \to$ IF_aggregate$(\{l_{ij} |\ j = 1..n\}, w)$; $L_i = (f_i, g_i)$

for i in $1, \ldots, m$

$\Omega_i \to (f_i - g_i + 1)/2$

$\Pi_i \to (f_i + g_i)/2$

$\mathscr{D}_i^* \to$argmax$_i$ (Ω_i, then Π_i)

return (ranking, $\mathscr{D}_i^*$).

3.2.2 *Computational complexity analysis*

Let $m =$ number of alternatives and $n =$ number of attributes, then

Table 1. Computational complexity analysis for AG-IFDM

Operation	Complexity	Remarks
IFN construction via left ideals	$O(mn)$	Each pair $(\mathscr{D}_i, \mathscr{A}_j)$ evaluated twice
Aggregation (IFWA/IFWG)	$O(mn)$	Aggregate over n attributes/alternative
Score & accuracy computation	$O(m)$	One per aggregated IFN
Sorting/ranking alternatives	$O(m \log m)$	Standard ranking step
Total complexity	$O(mn + m \log m)$	Linear in n, near-linear in m

From Table 1, it is evident that the AG-IFDM framework is computationally efficient, scaling well for moderate to large decision spaces. The structure is also suitable for algorithmic implementation in Python, MATLAB, or Excel without significant computational overhead.

3.3 *A demonstrative example*

In [37], Tweed et al. demonstrated that the brain's neural computations, especially those underlying spatial orientation and eye movement control through the Vestibulo-Ocular Reflex (VOR) are inherently non-commutative. This ground-breaking result provided compelling evidence that the brain encodes and processes spatial transformations using non-commutative operations, a feature unattainable through traditional commutative integrator models. Their work fundamentally challenged conventional assumptions in sensorimotor modelling and underscored the need for algebraic structures capable of capturing the order-sensitive dynamics of neural processing. Building upon this foundational insight, we propose an algebraic generalization of Tweed's framework by shifting the unit of analysis from head rotations to sequential activations of neuron types involved in eye movement control. Our model formalizes these interactions through non-commutative and non-associative binary operations defined over neuron sets, structured as AG-groupoids, which enable a finer symbolic representation of neuron-to-neuron interactions under sequential signalling.

We investigate three distinct patterns of neural communication, each modelled as an AG-groupoid $(\wp_\tau, \circledast)$, where the neuron set is defined as $\wp_\tau = \{\tau_1, \tau_2, \tau_3, \tau_4, \tau_5\}$, with elements corresponding to functionally significant components of the VOR: τ_1: Vestibular nucleus neuron: sensors from the inner ear that receive head movement input.

τ_2: Abducens motoneuron: controls lateral rectus for horizontal gaze (eye movement).

τ_3: Abducens internuclear neuron: coordinates conjugate horizontal movement.

τ_4: Oculomotor motoneuron: controls medial rectus for conjugate movement.

τ_5: Interstitial nucleus of Cajal neuron: integrates and holds vertical/ torsional eye position. Each AG-groupoid is governed by a distinct binary operation $\circledast \in \{\sharp, \natural, \flat\}$, which determines the interaction dynamics among neurons. These operations are given via Cayley tables, one of which is shown in detail below for the operation $\sharp$:

Table 2. Pattern of neural connection under the rule $\sharp$

$\sharp$	τ_1	τ_2	τ_3	τ_4	τ_5
τ_1	τ_1	τ_1	τ_1	τ_1	τ_1
τ_2	τ_1	τ_5	τ_5	τ_3	τ_5
τ_3	τ_1	τ_5	τ_5	τ_2	τ_5
τ_4	τ_1	τ_2	τ_3	τ_4	τ_5
τ_5	τ_1	τ_5	τ_5	τ_5	τ_5

The binary operation $\sharp$ (see Table 2) defines signal transmission dynamics between specific neuron types involved in VOR. For instance, the transmission of a signal from the neuron τ_4 to τ_5 is the same as the transmission of a signal from τ_5. These can be interpreted functionally as follows:

τ_1 absorbs all incoming signals.

τ_4 routes signals directly without modification.

τ_5 reinforces signals except to τ_1.

τ_2 and τ_3 swap signals via τ_4 and feed into τ_5.

Although not all 25 neuron pairings in the Cayley table are directly supported by known synaptic literature, but each interaction rule $\tau_i \sharp \tau_j = \tau_k$ is grounded in one of three well-accepted neurofunctional interpretations [38]:

(a). Anatomical progression (e.g., $\tau_2 \sharp \tau_3 = \tau_5$)

The vestibular nucleus directly projects to the abducens nucleus, which comprises both motoneuron and internuclear neurons which enables conjugate horizontal eye movements [39].

(b). Functionally abstract (e.g., $\tau_3 \sharp \tau_4 = \tau_2$)

Although abducens internuclear and oculomotor motoneurons lack a direct loop to the abducens motoneuron, they function together within a coordinated horizontal gaze system that supports internuclear activation [40].

(c). Behaviourally emergent (e.g., $\tau_2 \sharp \tau_2 = \tau_5$)

Repetitive or sustained activation of abducens motoneurons leads to gaze-holding behaviour, which is mediated by the interstitial nucleus [41].

This framework demonstrates how algebraic modelling captures both documented and emergent interactions within neural patterns, which enable symbolic interpretation of functional connectivity under uncertain or abstracted conditions.

To evaluate these neural configurations, we assign Intuitionistic Fuzzy Numbers (IFNs) to each neuron based on its excitation and inhibition levels under the binary operation $\sharp$. This is achieved by constructing IF-score left ideals, where the left ideal structure is specifically chosen to reflect directional signal flow, that is, the influence of a leading neuron τ_i on others in the operation $\tau_i \sharp \tau_j$. Each element $\ell_i \in \mathscr{L}_1 = \{\ell_1, \ell_2, \ell_3, \ell_4, \ell_5\}$ corresponds to a neuron $\tau_i \in \wp_\tau$ with its behaviour quantified by a membership function f_{ℓ_i} representing its activation strength and a non-membership function g_{ℓ_i} reflecting its inhibitory response. The assigned values for the IFNs are summarized as follows:

Table 3. Membership and non-membership information on $(\wp_\tau, \sharp)$

τ	IFN ℓ_i	f_{ℓ_i}	g_{ℓ_i}
τ_1 (Vestibular)	ℓ_1	0.4	0.1
τ_2 (Abducens motoneuron)	ℓ_2	0.3	0.2
τ_3 (Abducens internuclear)	ℓ_3	0.3	0.2
τ_4 (Oculomotor motoneuron)	ℓ_4	0.2	0.1
τ_5 (Interstitial nucleus of Cajal)	ℓ_5	0.5	0.3

These values capture both neural responsiveness and resistance under specific operational dynamics. For example, τ_5 exhibits the highest membership score (0.5), consistent with its role as a gaze-stabilizing integrator, while τ_1 has the lowest inhibition (0.1), reflecting its function as an initial signal absorber in vestibular circuits.
In a similar fashion to the aforementioned AG-groupoid, we now examine two additional interaction patterns in form AG-groupoids, each defined by a distinct binary operation over the common set of neurons $\wp_\tau = \{\tau_1,\ \tau_2,\ \tau_3,\ \tau_4,\ \tau_5\}$. For each pattern, we construct corresponding IF-score left ideals and derive Intuitionistic Fuzzy Numbers (IFNs) to quantify neuron activation and inhibition profiles under the respective operation.

The binary operation $\natural$ (see Table 4) encodes an alternative signal transmission logic, characterized by the following Cayley table:

Table 4. Pattern of neural connection under the rule $\natural$

$\natural$	τ_1	τ_2	τ_3	τ_4	τ_5
τ_1	τ_2	τ_1	τ_1	τ_1	τ_1
τ_2	τ_1	τ_2	τ_2	τ_2	τ_2
τ_3	τ_1	τ_2	τ_3	τ_4	τ_5
τ_4	τ_1	τ_2	τ_5	τ_3	τ_4
τ_5	τ_1	τ_2	τ_4	τ_5	τ_3

Neural interpretation of $\natural$:
τ_1 absorbs most signals but activates τ_2 when self-signalling.
τ_2 reinforces itself unless interacting with τ_1.
τ_4 and τ_5 sustain each other and reset to τ_3.
τ_3 facilitates the cycle by passing signals directly.

This structure of neuron interactions likewise satisfies one of the three categories, Anatomical Progression [39], Functional Abstraction [40], or Behavioural Emergence [41], as established in the case of the first pattern of interactions $(\wp_\tau, \sharp)$.

The associated IFNs $\mathscr{L}_2 = \{\ell_1,\ \ell_2,\ \ell_3,\ \ell_4,\ \ell_5\}$ derived from the IF-score left ideal of this structure are:

Table 5. Membership and non-membership information on $(\wp_\tau, \natural)$

τ	ℓ_i	f_{ℓ_i}	g_{ℓ_i}
τ_1	ℓ_1	0.4	0.3
τ_2	ℓ_2	0.4	0.3
τ_3	ℓ_3	0.2	0.6
τ_4	ℓ_4	0.2	0.6
τ_5	ℓ_5	0.2	0.6

Similarly, the operation $\flat$ gives another variation of neuronal interaction, described by Table 6.

Table 6. Pattern of neural connection under the rule $\flat$

$\flat$	τ_1	τ_2	τ_3	τ_4	τ_5
τ_1	τ_1	τ_1	τ_1	τ_1	τ_1
τ_2	τ_1	τ_2	τ_2	τ_2	τ_2
τ_3	τ_1	τ_2	τ_4	τ_5	τ_3
τ_4	τ_1	τ_2	τ_3	τ_4	τ_5
τ_5	τ_1	τ_2	τ_5	τ_3	τ_4

Neural interpretation of $\flat$:

τ_1 absorbs all signals.

τ_2 reinforces itself unless interacting with τ_1.

τ_3, τ_4 and τ_5 supports cyclic structure by sustaining each other, however self signals in τ_3 and τ_5 reset the cycle to τ_4.

τ_4 acts as a neutral by enabling direct signal transmission.

The corresponding IFN set L_3 constructed from the IF-score left ideal is:

Table 7. Membership and non-membership information on $(\wp_\tau, \flat)$

τ	ℓ_i	f_{ℓ_i}	g_{ℓ_i}
τ_1	ℓ_1	0.7	0.2
τ_2	ℓ_2	0.6	0.1
τ_3	ℓ_3	0.7	0.2
τ_4	ℓ_4	0.3	0.7
τ_5	ℓ_5	0.3	0.7

A normalized weight vector is applied to encode the relative importance of each neuron in specific operational contexts. These weights are adaptable and can be recalibrated to reflect varying physiological priorities, such as emphasizing horizontal vs. torsional gaze, depending on task-specific demands or experimental setups.

By aggregating the Intuitionistic Fuzzy Numbers (IFNs) using established fuzzy aggregation operators, we rank the three neural AG-groupoids to determine which configuration exhibits the greatest efficiency, coherence or adaptability in modeling biologically realistic eye movement dynamics. This symbolic algebraic framework thus extends the non-commutative and non-associative neural computation principles originally demonstrated by Tweed et al. [37]. to the level of structured neuron-type interactions. Embedding these relationships within an ideal-theoretic formulation allows for the incorporation of uncertainty, signal directionality and neural dominance by providing a mathematically grounded yet flexible platform.

Now, let us consider a normally distributed weight vector associated with the attributes of $\wp_\tau = \{\tau_1, \tau_2, \tau_3, \tau_4, \tau_5\}$ as $w = \{0.11, 0.24, 0.3, 0.24, 0.11\}$. The combined IFN values listed in Tables 3, 5 and 7 for each pattern across the neuron set $\wp_\tau$ are summarized in the following AG-IFDM:

Table 8. The integrated assessment information by the operators

Patterns	IFA	IFWA	IFG	IFWG
$(\wp_\tau, \sharp)$	(0.34, 0.18)	(0.35, 0.21)	(0.32, 0)	(0.31, 0)
$(\wp_\tau, \natural)$	(0.28, 0.48)	(0.28, 0.48)	(0.25, 0.42)	(0.26, 0.43)
$(\wp_\tau, \flat)$	(0.52, 0.38)	(0.58, 0.34)	(0.52, 0.22)	(0.55, 0.24)

To evaluate the alternatives in Table 8, use the IFWA or IFWG operators on the AG-IFDM in Table 9.

Table 9. AG-intuitionistic fuzzy decision matrix

Patterns	τ_1	τ_2	τ_3	τ_4	τ_5
$(\wp_\tau, \sharp)$	(0.4, 0.1)	(0.3, 0.2)	(0.3, 0.2)	(0.2, 0.1)	(0.5, 0.3)
$(\wp_\tau, \natural)$	(0.4, 0.3)	(0.4, 0.3)	(0.2, 0.6)	(0.2, 0.6)	(0.2, 0.6)
$(\wp_\tau, \flat)$	(0.7, 0.2)	(0.6, 0.1)	(0.7, 0.2)	(0.3, 0.7)	(0.3, 0.7)

Calculate the scores of the overall attribute value of the option given in Table 10 by using the score function.

Table 10. Scores and rankings of alternatives under different aggregation operators

Pattern	IFA		IFWA		IFG		IFWG	
Compositions	Score	Rank	Score	Rank	Score	Rank	Score	Rank
$(\wp_\tau, \sharp)$	0.59	2nd	0.57	2nd	0.57	1st	0.56	1st
$(\wp_\tau, \natural)$	0.41	3rd	0.40	3rd	0.38	3rd	0.37	3rd
$(\wp_\tau, \flat)$	0.63	1st	0.65	1st	0.52	2nd	0.54	2nd

The rankings of the choices for all the operator based on the score function are illustrated in Figure 2 below:

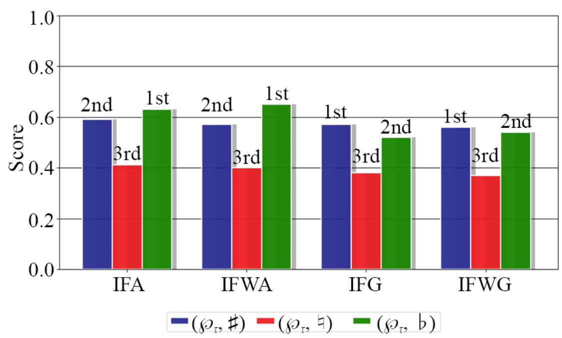

Figure 2. Illustration of scores of various aggregation operators

Note that the normally distributed weight vector (adjustable for varying priorities) implies that operational conditions could shift rankings. Adjustments to the weight vector should align with specific functional goals to refine these rankings further. Based on the current assumed weight vector, the evaluation of three neural pattern models as AG-groupoids; $(\wp_\tau, \sharp)$, $(\wp_\tau, \natural)$, $(\wp_\tau, \flat)\}$, using intuitionistic fuzzy aggregation operators (IFA, IFWA, IFG, IFWG) provides the following insights:

- $(\wp_\tau, \flat)$ exhibits the most efficient and adaptable structure, securing top ranks under IFWA and IFWG. Its cyclic and distributed interaction logic supports flexible signal redirection while maintaining functional integrity.
- $(\wp_\tau, \sharp)$ demonstrates strong activation and consistency, ranking highest in IFA and IFG, though slightly limited in adaptability due to its rigid routing configuration.
- $(\wp_\tau, \natural)$ consistently ranks 3rd with uniformly low membership scores and high non-membership by indicating limited efficiency in dynamic neural interactions.

• $(\wp_\tau, \flat)$ ranks consistently lowest across all operators, reflecting a less effective pattern characterized by weak activation and high inhibition scores.

In summary, the algebraic modelling framework successfully differentiates neural signal patterns in terms of efficiency and adaptability. It extends the non-commutative principle of Tweed et al. [37] to structured neuron-type interactions using symbolic and fuzzy representations, offering a versatile tool for simulating eye movement control under uncertainty.

4. From group-theoretic structure to decision-making in agricultural treatment

Agriculture, the backbone of global food security, is rapidly advancing through the adoption of technologies that increase productivity, sustainability and efficiency. Recent advancements such as precision farming, Internet of Things (IoT) integration, artificial intelligence, and fuzzy logic-based decision systems have shown remarkable promise in transforming traditional agricultural practices into data-driven, optimized operations [42, 43]. One of the most common concerns in modern agriculture is protecting crops from various biological threats, particularly plant diseases, which significantly reduce productivity and quality. The development of smart disease prediction models, sensor-based monitoring systems and intelligent control algorithms offers a proactive approach to plant protection [44–46]. In this developing landscape, the production of antibiotics for agricultural treatment has become a critical component of crop health management. This process is complex and highly regulated and plays an important role in controlling plant pathogens and enhancing plant well-being. A fundamental aspect of this process lies in the collaboration between various active and supportive ingredients during the manufacturing of these treatments. The effectiveness and safety of antibiotic formulations are deeply influenced by the precise interaction of these ingredients.

This section explores the optimization of ingredient interactions in agricultural treatment formulations by focusing on cases where the sequence of mixing components is inconsequential, that is, the process obeys associative and commutative properties. To address this, a group-theoretic framework based on the algebraic structure of the Klein four-group is applied to model ingredient combinations. The Klein four-group, characterized by its commutative and associative nature, serves as an ideal mathematical representation for such mixing processes, where the final product's quality and efficiency depend on functional interplay among ingredients rather than sequential order. This approach is guided through intuitionistic fuzzy congruences, which introduce an innovative way to evaluate uncertainties and partial memberships inherent in real-world ingredient compatibility.

4.1 *The GIFR model*

To facilitate pairwise ranking of alternatives represented as elements of a group, we introduce the Group-theoretic Intuitionistic Fuzzy Ranking (GIFR) model. In this framework, alternatives are evaluated through ordered pairs, which allow for a structured comparison of their mutual compatibility or preference. This approach integrates the abstract concept of group congruences with the analytical depth offered by intuitionistic fuzzy sets that enable effective decision-making in a novel algebraic way. The procedure consists of the following steps:

Preliminary step: Let $G = \{a_1, a_2, a_3, \ldots, a_n\}$ be a finite group, where each element $a_i \in G$ represents an alternative. An intuitionistic fuzzy score (IF-score) congruence $\Psi : G \times G \rightarrow$IFN assigns compatibility degrees to pairs of alternatives based on their group operation outcomes, where an IF-score congruence is an IF-score equivalence relation $\Psi : G \times G \rightarrow [0, 1] \times [0, 1]$ on G that is also compatible.

An IF-score relation on a group G is a mapping $\Psi : G \times G \rightarrow [0, 1] \times [0, 1]$, where Ψ is a score function and is grounded in the original intuitionistic fuzzy structure, because it is a function of membership and non-membership degrees. An IF-score relation called equivalence if it satisfies, $\Psi(a, a) = (1, 0)$ (reflexivity) and $\Psi(a, b) = \Psi(b, a)$ (symmetry) for all $a, b \in G$ along with $\Psi \circ \Psi \leq \Psi$ (transitivity). It is said to be compatible if, $\Psi(a, b) \wedge \Psi(c, d) \leq \Psi(ab, cd)$ for all $a, b, c, d \in G$.

Step 1: Take the following information as input of the model.

• Finite group $G = \{a_1, a_2, \ldots, a_n\}$.

• Cayley table (group operation defined for all pairs).
• For each ordered pair $(a_i,\ a_j)$, obtain:

$$f_{ij} = \text{degree of compatibility} \in [0,\ 1],$$

$$g_{ij} = \text{degree of incompatibility} \in [0,\ 1].$$

Ensure $f_{ij} + g_{ij} \leq 1$. These may be obtained via expert comparison or empirical performance.

Step 2: For each combination of alternatives $(a_i,\ a_j)$, define an IFN $\Psi(a_i,\ a_j) = (f_{ij},\ g_{ij})$ based on the construction of an IF-score congruence within the group G.

Step 3: Eliminate all reflexive pairs $(a_i,\ a_i)$ as they trivially provide an IFN of $(1,\ 0)$ and do not offer meaningful differentiation and correspond in the ranking process.

Step 4: Differentiate each ordered pair of alternatives $(a_i,\ a_j)$ with $i \neq j$ based on their recorded scores and use accuracy if scores are identical.

Step 5: Rank the ordered pairs of alternatives to determine their relative preference in combination.

4.1.1 *Pseudocode for GIFR*

Now we present the pseudocode of the GIFR model. It provides a clear, structured outline of the computational procedure, enabling readers to easily implement, analyse, and extend the model for their own applications.

Input:
$G = \{a_1,\ \ldots,\ a_n\} \rightarrow$ alternatives
Cayley table $\rightarrow$ group operation $(a_i \cdot a_j)$
$(f_{ij},\ g_{ij}) \rightarrow$ expert/ empirical compatibility and incompatibility $(f_{ij} + g_{ij} \leq 1)$
$\Psi \rightarrow$ IF-score congruence on G with $\Psi(a,\ b) = (f_{ij},\ g_{ij})$ and compatibility $\Psi(a,\ b) \wedge \Psi(c,\ d) \leq \Psi(ab,\ cd)$
Output:
Ranking of ordered pairs $(a_i,\ a_j)$ by IF-scores (tie-break by accuracy)
Procedure:
for each $(a_i,\ a_j) \in G \times G$
$\Psi(a_i,\ a_j) \rightarrow (f_{ij},\ g_{ij})$ from expert or empirical evaluation
ensure $\Psi(a_i,\ a_j) = (1,\ 0)$; $\Psi(a_i,\ a_j) = \Psi(a_j,\ a_i)$
remove reflexive pairs $(a_i,\ a_j)$
for each $(a_i,\ a_j)$ with $i \neq j$
$\Omega_{ij} \rightarrow (f_{ij} - g_{ij} + 1)/2$
$\Pi_{ij} \rightarrow (f_{ij} + g_{ij})/2$
sort pairs by (Ω_{ij} desc, Π_{ij} desc)
$(a_i^*,\ a_j^*) \rightarrow$argmax ($\Omega_{ij}$, then Π_{ij})
return (ranking, $(a_i^*,\ a_j^*)$).

4.1.2 *Computational complexity analysis*

Let m = number of alternatives, then

Table 11. Computational complexity analysis for AG-IFDM

Operation	Complexity	Remarks
IFN construction of pairs	$O(m^2)$	All ordered pairs evaluated once
Score & accuracy computation	$O(m^2)$	One per non-reflexive pair
Sorting/ranking pairs	$O(m^2 \log m)$	Full ranking of $m(m-1)$ pairs
Total complexity	$O(m^2 \log m)$	Quadratic in m; logarithmic sorting factor

From Table 11, it is visible that the GIGR model is computationally feasible and efficient for moderately sized decision spaces, making it readily implementable in Python, MATLAB, or other symbolic computation environments.

4.2 *A demonstrative example*

In agricultural formulation development, especially for antibacterial crop treatments, components must be chemically compatible to ensure efficacy, storage stability, and application performance [47]. While standard formulation processes are associative and commutative, meaning mixing order does not affect final composition, pairwise component interactions often dictate whether the mixture remains stable, avoids precipitation, and retains potency under environmental stressors [48]. Farmers and agrochemical producers routinely encounter issues like active degradation, ineffective sprays, or shorter shelf life due to incompatible ingredient combinations. Improving component compatibility is thus essential for effective field deployment and cost-efficient production. In this section, we aim to analyse and improve the formulation process of an antibacterial crop treatment composed of four essential components. Although the mixing of these components generally follows associative and commutative principles, meaning the order of combination does not affect the final outcome, our focus is to prioritize pairwise compatibility among the ingredients. The objective is to identify which combinations yield the most effective interactions, regardless of mixing sequence. The essential components under evaluation in this formulation are as follows:

- Neutral Filler (ε): Substances used to dilute or carry the active ingredient.
- Active Ingredient (k_1): Antimicrobial substances.
- Stabilizer (k_2): Compounds to maintain effectiveness and solubility.
- Solvent (k_3): Water or a water-based solution for dissolving the active ingredient.

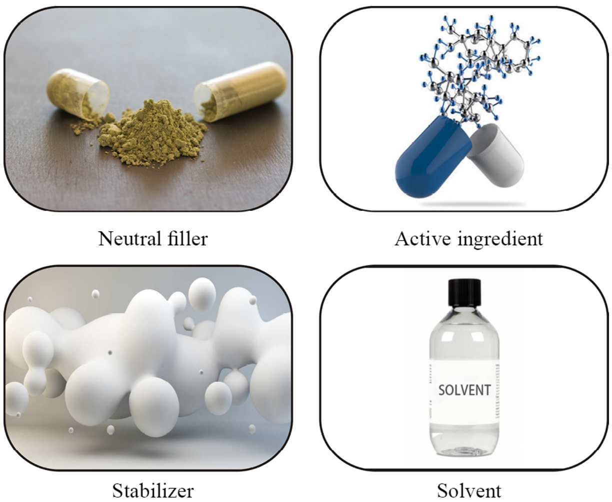

Figure 3. Illustration of ingredients for medicine manufacturing

Figure 3 represents the visual representation of four main ingredients medicine manufacturing. The objective is to compare the compatibility of each pair of ingredients and to provide a mathematically justified recommendation grounded in abstract algebra on the most advantageous sequence for initiating the mixing process. The approach used is purely abstract, which depends on group-theoretic principles, in particular, a symmetric structural framework to model the interactions among the formulation components.

To address this, the components are modelled as elements of a Klein four-group, denoted $K = \{\varepsilon,\ k_1,\ k_2,\ k_3\}$, where $k_3 = k_1 k_2$, which ensures the chemical compatibility of the solvent with both the active ingredient and the stabilizer. The interactions between components are encoded in the Cayley table of the Klein four-group as shown in Table 12:

Table 12. Caylay table of the Klein four group generated for the ingredients

$\odot$	ε	k_1	k_2	k_3
ε	ε	k_1	k_3	k_3
k_1	k_1	ε	k_3	k_2
k_2	k_2	k_3	ε	k_1
k_3	k_3	k_2	k_1	ε

Each of the 16 interactions in the Cayley table is supported by empirical evidence or mechanistic insight [49]:

(a). Neutral filler (e.g., $\varepsilon \odot k_1 = k_1$)

Neutral fillers such as are widely used as inert carriers in pesticide and pharmaceutical formulations to provide physical support and enhancing spray properties, but they do not chemically alter the active ingredient.

(b). Self-mixtures (e.g., $k_1 \odot k_1 = \varepsilon$)

Over-concentration of an active ingredient such as streptomycin can lead to issues like crystallization or reduced solubility, which essentially neutralize its intended effect.

(c). Pairwise cross interactions (e.g., $k_2 \odot k_3 = k_1$)

Combining a stabilizer with a solvent significantly enhances the solubility and bioavailability of active compounds.

These real-world justifications align with the abstract algebraic structure of the Klein four-group, which offers both theoretical clarity and practical applicability to agricultural formulation optimization.

To evaluate the pairwise compatibility of decision alternatives represented as group elements, we define an intuitionistic fuzzy score congruence ϑ, which assigns an IFN to each pair $(x,\ y) \in K \times K$ such that $\vartheta(x,\ y) = (f_{xy},\ g_{xy}) \in [0,\ 1] \times [0,\ 1]$, where f_{xy} is a degree of membership (compatibility), g_{xy} is a degree of non-membership (incompatibility), and $0 \leq f_{xy} + g_{xy} \leq 1$, which ensures that the hesitation degree remains valid. The complete collection of IFNs is given as follows:

$$\vartheta(x,\ y) = \begin{array}{ll} (\varepsilon,\ \varepsilon) \to (1.00,\ 0.00) & (k_2,\ \varepsilon) \to (0.58,\ 0.38) \\ (\varepsilon,\ k_1) \to (0.48,\ 0.28) & (k_2,\ k_1) \to (0.50,\ 0.30) \\ (\varepsilon,\ k_2) \to (0.58,\ 0.38) & (k_2,\ k_2) \to (1.00,\ 0.00) \\ (\varepsilon,\ k_3) \to (0.45,\ 0.25) & (k_2,\ k_3) \to (0.55,\ 0.35) \\ (k_1,\ \varepsilon) \to (0.48,\ 0.28) & (k_3,\ \varepsilon) \to (0.45,\ 0.25) \\ (k_1,\ k_1) \to (1.00,\ 0.00) & (k_3,\ k_1) \to (0.50,\ 0.30) \\ (k_1,\ k_2) \to (0.50,\ 0.30) & (k_3,\ k_2) \to (0.55,\ 0.35) \\ (k_1,\ k_3) \to (0.50,\ 0.30) & (k_3,\ k_3) \to (1.00,\ 0.00) \end{array}$$

The justification for using an intuitionistic fuzzy score congruence lies in the need to model uncertainty, partial truth, and conflicting tendencies in chemical formulations, especially when assessing compatibility between ingredients. Real-world agrochemical interactions are rarely binary (i.e., fully compatible or incompatible); instead, they involve degrees of synergy and hesitancy under various conditions. This evaluation not only reflects expert-informed intuition but also remains algebraically coherent with the underlying Klein four-group structure of the formulation components.

Remove the trivial IFNs of $(1.0,\ 0.0)$, which correspond to self-pairings and offer no value in ranking. Next, compute the score and accuracy for each remaining IFN using the defined functions. Since the decision problem involves determining the most effective or preferable pairs of agricultural ingredients to be mixed, and the mixing operation is inherently commutative (i.e., mixing a_i with a_j gives the same outcome as mixing a_j with a_i), we treat each order pair and its symmetric counterpart as a single entity for the ranking process. However, it is important to note that ranking these pairs does not imply a sequential order of mixing steps, but rather serves as a measure of relative compatibility or priority of combining these ingredients. The ranking reflects which pairs, when mixed, are likely to have more significant or favourable interactions according to the IF-score congruence model. As mixing is associative, the order in which pairs are mixed does not affect the final combined outcome. Thus, the ranking should be interpreted as a guide to prioritize ingredient pairs based on compatibility or impact, not as a directive to mix one pair strictly before another.

All alternative pairs result in an identical score of 0.6, which indicates an equal level of compatibility based on the score function. As a result, further distinction among these pairs is made using the accuracy function, which reflects the degree of uncertainty inherent in each evaluation. The pairs with higher accuracy are thus regarded as more confidently compatible. Table 13 presents the accuracy value for each pair, along with their visualization (Figure 4) for corresponding rankings.

Table 13. Rankings of pairs based on K

Alternative pairs	Accuracy	Rank
$(k_2,\ k_3)$	0.45	2nd
$(\varepsilon,\ k_3)$	0.35	5th
$(k_2,\ k_1),\ (k_1,\ k_3)$	0.40	3rd
$(\varepsilon,\ k_2)$	0.48	1st
$(k_1,\ \varepsilon)$	0.38	4th

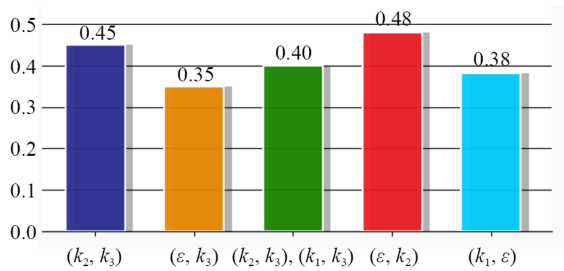

Figure 4. Illustration of the accuracies of ingredients' pairs

The final ranking, derived from the GIFR model, identifies $(\varepsilon,\ k_2)$ (Neutral Filler + Stabilizer) as the most compatible pair for the antibacterial formulation, followed by $(k_2,\ k_3)$ (Stabilizer + Solvent). This result aligns with the problem's algebraic and chemical constraints while resolving ambiguities introduced by commutativity/associativity, and notably it reflects inherent compatibility, not temporal mixing steps. These rankings account for the underlying algebraic and chemical constraints of the problem by resolving ambiguities arising from the commutative and associative nature of the mixing process. They show how well the components work together rather than suggesting the exact order in which they should be mixed.

Figure 5 shows how the ranking of the alternative pairs forms a clear symmetric pattern due to balanced positioning and mirrored ranking levels. This symmetry reflects the algebraic properties of the Klein four-group and shows that the evaluation obeys the natural equivalence between the pairs. As a result, the ranking is not affected by the order in which pairs are considered, which supports the fairness and reliability of the underlying model.

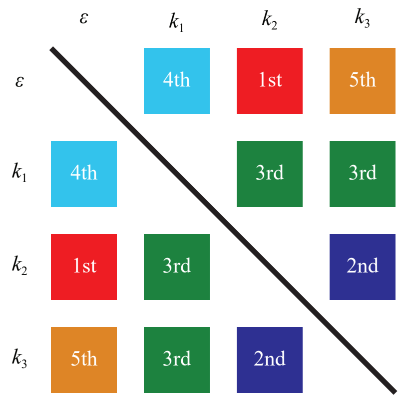

Figure 5. Symmetric ranking of alternative pairs

Figure 6 extends beyond mere compatibility rankings by visualizing the networked confidence among ingredient pairs. Unlike a simple ranking, this diagram highlights a directed compatibility path, where stronger connections form a chain of higher-confidence interactions. This path-oriented structure captures the interdependencies between components, which reveal not just which pairs are individually strong, but how they collectively contribute to a stable and coherent formulation strategy. In practical terms, this implies that even though the operation is associative and commutative, the connectivity of high-confidence pairs provides a rationale for prioritizing certain combination pathways over others when designing strong and stable antibacterial crop formulations.

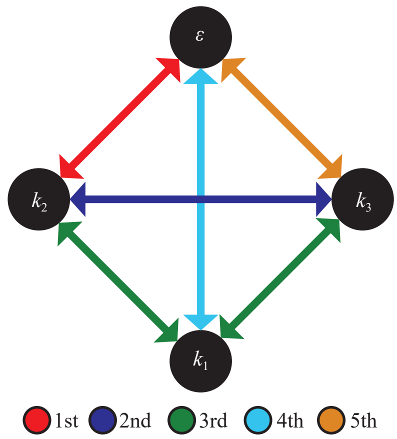

Figure 6. Illustration of ranks of pairwise ingredients

5. Comparative analysis

The integration of algebraic structures with intuitionistic fuzzy logic offers effective frameworks for handling multi-criteria decision-making under uncertainty. This study presents two such models, one based on an AG-groupoid and the other on group theory, specifically the Klein four-group, with both established upon algebraic reasoning but differing significantly in their structural foundations, evaluation mechanisms, and application domains. The AG-IFDM model employs ideal-based intuitionistic fuzzy evaluations to analyse structured alternatives, such as neural activation patterns, by focusing on the non-associative and non-commutative nature of interactions. On the other hand, the Group-theoretic Intuitionistic Fuzzy Ranking (GIFR) model utilizes intuitionistic fuzzy congruences within a finite group to assess pairwise compatibilities, particularly suited for formulations in agricultural antibiotic engineering, where associative and commutative properties exist. These methodologies not only broaden the theoretical background of fuzzy decision-making but also provide domain-specific mechanisms designed to account for whether the sequencing of interactions significantly affects decision outcomes.

In subsequent sections, we conduct a comparative analysis by benchmarking the proposed AG-IFDM against Technique for Order of Preference by Similarity to Ideal Solution (TOPSIS) and the GIFR across multiple options within an intuitionistic fuzzy environment.

5.1 *Evaluating AG-IFDM model with TOPSIS and other MCDM methods*

To demonstrate the effectiveness of the proposed AG-IFDM model, a comparative analysis is presented against an established multi-criteria decision-making method, namely TOPSIS [50]. This section focuses on analysing the consistency, ranking stability, and contextual adaptability of each method in environments characterized by fuzzy and interdependent criteria. The rankings of the patterns $(\wp_\tau, \sharp)$, $(\wp_\tau, \natural)$, and $(\wp_\tau, \flat)$ through AG-IFDM offer a meaningful basis for evaluating the comparative performance of the proposed method against TOPSIS. This comparative analysis employs the same normalized weight vector $w = \{0.11, 0.24, 0.3, 0.24, 0.11\}$ and the AG-IFDM (Table 8) as utilized in subsection 3.3.

The TOPSIS method involves normalization, weighting, identification of ideal solutions, distance calculations, and ranking based on the closeness coefficient. Table 8 is normalized using the Euclidean norm, after which each membership (f_{ij}) and non-membership (g_{ij}) value is multiplied by its corresponding weight w_j. For instance, the weighted membership and non-membership values for the pattern $(\wp_\tau, \sharp)$ under attribute τ_1 are computed as follows:

$$f^w = w_1.f_{11} = 0.11 \times 0.4 = 0.044 \text{ and}$$

$$g^w = w_1.g_{11} = 0.11 \times 0.4 = 0.011.$$

This procedure is repeated for all attributes and patterns to construct the weighted normalized matrix, which is given as in Table 14:

Table 14. The weighted normalized matrix for the neural patterns

Patterns	τ_1	τ_2	τ_3	τ_4	τ_5
$(\wp_\tau, \sharp)$	(0.044, 0.011)	(0.072, 0.048)	(0.09, 0.06)	(0.048, 0.024)	(0.055, 0.033)
$(\wp_\tau, \natural)$	(0.044, 0.033)	(0.096, 0.072)	(0.06, 0.18)	(0.048, 0.144)	(0.022, 0.066)
$(\wp_\tau, \flat)$	(0.077, 0.022)	(0.144, 0.024)	(0.21, 0.06)	(0.072, 0.168)	(0.033, 0.077)

The positive ideal solution (d^+) is defined by the maximum membership f^w_{max} and minimum non-membership g^w_{min} values across all attributes. For instance, attribute τ_1 contributes $(f^w_{\text{max}} = 0.077$ and $g^w_{\text{min}} = 0.011)$, and so on for τ_1 and τ_3. Similarly the negative ideal solution (d^-) is defined by the minimum membership f^w_{min} and maximum non-membership g^w_{max} values for all attributes. Euclidean distances D^+_i and D^-_i are then calculated for all the patterns to compute their closeness coefficient CC_i, where $i = 1, 2, 3$. For $i = 1$, that is, for pattern $(\wp_\tau, \sharp)$, the computations for Euclidean distances are as follows:

$$D^{+} = \sqrt{\begin{pmatrix} (0.044-0.077)^2+(0.072-0.144)^2+ \\ (0.048-0.024)^2+(0.09-0.21)^2+ \\ (0.048-0.072)^2 \end{pmatrix}}$$

$$= 0.147733$$

$$D^{-} = \sqrt{\begin{pmatrix} (0.011-0.033)^2+(0.048-0.072)^2+ \\ (0.09-0.06)^2+(0.06-0.018)^2+ \\ (0.024-0.168)^2+(0.055-0.022)^2+ \\ (0.033-0.077)^2 \end{pmatrix}}$$

$$= 0.200302$$

The CC_1 is calculates as:

$$CC_1 = \frac{D_1^-}{D_1^+ + D_1^-} = \frac{0.200302}{0.0.147733+0.200302}$$

$$= 0.575523 \approx 0.58.$$

Similarly, we can find CC_i for the other two patterns. A higher CC_i value indicates a pattern closer to the ideal solution and, hence, a more favourable option. The final ranking of the neural patterns, based on their respective CC_i values is presented in Table 15:

Table 15. The final TOPSIS ranking

Pattern	CC_i	Rank
$(\wp_N, \sharp)$	0.58	2^{nd}
$(\wp_N, \natural)$	0.13	3^{rd}
$(\wp_N, \flat)$	0.59	1^{st}

The comparative analysis demonstrates a strong alignment between the proposed methodology (using IFA, IFWA, IFG, IFWG operators) and TOPSIS rankings, particularly in identifying the best and least favourable patterns. Both frameworks consistently rank the pattern $(\wp_\tau, \flat)$ as the top performer (ranked 1st under IFA, IFWA and TOPSIS; ranked 2nd under IFG/IFWG, where it remains highly competitive). Similarly, $(\wp_\tau, \natural)$ is uniformly placed last (3rd) across all methods, reflecting its relatively low performance scores. The only difference arises in the middle position where $(\wp_\tau, \sharp)$ ranked 2nd in IFA, IFWA, and TOPSIS, and it is ranked 1st under IFG and IFWG. This variation results from the use of geometric aggregation operators, which help reduce the effect of extreme values and model multiplicative relationships among criteria. Overall, the strong consistency, particularly between TOPSIS and the IFA/IFWA results, supports $(\wp_\tau, \flat)$ as the most appropriate choice across different evaluation approaches.

In addition to the comparison with TOPSIS, it is worth noting that the proposed AG-IFDM and GIFR differ from other established MCDM techniques such as AHP and VIKOR. AHP is effective for hierarchical, consistency-checked pairwise evaluations but assumes transitive preferences, making it less suited to high-hesitation or non-associative contexts where AG-IFDM is applicable. VIKOR seeks compromise solutions under conflicting criteria but operates mainly on crisp or fuzzy data without structural constraints; in contrast, our models embed algebraic relationships (AG-groupoids or groups) to capture both uncertainty and interaction rules. They therefore complement rather than replace established methods, offering advantages in decision problems where algebraic structure and hesitation are integral.

5.2 *Analysis of GIFR model and extended intuitionistic fuzzy models*

The GIFR model is purely abstract and does not incorporate aggregation operators or other specialized methods for ranking alternatives. Therefore, its distinguishing features and advantages arise from its foundational use of intuitionistic fuzzy group-theoretic principles. A comparative summary is presented in Table 16 to highlight the strengths of this approach, particularly in comparison to other extended forms of intuitionistic fuzzy sets, such as Traditional IFS, Pythagorean Fuzzy Sets (PyFS), q-Rung Orthopair Fuzzy Sets (q-ROFS), and Circular IFS (C-IFS), which incorporate dynamic structural constraints and enhanced mechanisms for modelling uncertainty. This summary highlights six important features that focus on the unique capabilities of the GIFR model in addressing decision-making scenarios under uncertainty.

Table 16. Comparison of existing uncertainty-handling methods with proposed method

Feature	Traditional IFS [7]	PyFS [51]	q-ROFS [52]	C-IFS [53]	Proposed method
Handles uncertainty	✓	✓	✓	✓	✓
Algebraic structure (Klein four-group)	✗	✗	✗	✗	✓
Pairwise compatibility	Partial	Partial	✓	✓	✓
Commutativity/associativity guarantees	✗	✗	✗	✗	✓
Dynamic prioritization (scores)	✗	✗	✗	Partial	✓
Parameterized ignorance handling	✗	✓	✗	✓	✓

All methods depend on the foundational constructs of membership and non-membership degrees, which are essential for capturing uncertainty within the intuitionistic fuzzy logic. The IFS model has been progressively extended to enhance its expressive power, first through PyFS, then through q-ROFS, and more recently by C-IFS. Each extension introduces broader ranges or geometric constraints to improve the modelling of ambiguity and hesitation in decision-making contexts. In contrast, the proposed method depends on the classical IFS framework but fundamentally enhances it through a novel algebraic integration of group theory. This integration imposes a well-defined, closed algebraic structure on the set of alternatives, which ensures both commutative and associative interactions that are either absent or only weakly supported in the aforementioned IFS extensions. Such structural consistency is particularly important in applications involving sequential or iterative mixing processes. Although PyFS and q-ROFS provide partial support for pairwise interactions through aggregation operators, they lack a mechanism for enforcing structural rules. In contrast, the group-theoretic foundation of the proposed model offers an algebraically consistent framework for modelling pairwise interactions among alternatives, which ensures structural integrity and interpretability. Moreover, this group-theoretic approach can be extended to incorporate PyFS, q-ROFS, and C-IFS, which enable the development of more effective and algebraically grounded decision-making methods for modelling structured relationships between alternatives.

6. Sensitivity analysis

A sensitivity analysis was conducted to examine the robustness of the proposed decision models to changes in input conditions. For AG-IFDM, the focus is on the effect of varying attribute weights on the ranking of patterns, while for GIFR the effect of perturbing IF-score congruence values for selected unordered pairs is assessed. This combined view allows us to see not only how each model behaves under parameter changes, but also how their stability profiles differ.

6.1 *Sensitivity analysis of AG-IFDM model*

To assess robustness, the three patterns were re-evaluated under four weight schemes: $W_1 = \{0.11, 0.24, 0.30, 0.24, 0.11\}$ (original), $W_2 = \{0.20, 0.20, 0.20, 0.20, 0.20\}$ (uniform), $W_3 = \{0.10, 0.15, 0.50, 0.15, 0.10\}$ (central emphasis), and $W_4 = \{0.30, 0.20, 0.00, 0.20, 0.30\}$ (edge emphasis). Table 17 presents the resulting scores and ranks (higher score is better).

Table 17. Scores and ranks (score [rank]) of AG-IFDM patterns under different weight schemes

W	IFWA			IFWG		
	$\sharp$	$\natural$	$\flat$	$\sharp$	$\natural$	$\flat$
W_1	0.5756 [2]	0.4029 [3]	0.6525 [1]	0.5593 [1]	0.3707 [3]	0.5426 [2]
W_2	0.5919 [2]	0.4161 [3]	0.6333 [1]	0.5705 [1]	0.3821 [3]	0.5185 [2]
W_3	0.5724 [2]	0.3754 [3]	0.6832 [1]	0.5594 [2]	0.3490 [3]	0.5953 [1]
W_4	0.6098 [1]	0.4415 [3]	0.5944 [2]	0.5834 [1]	0.4060 [3]	0.4730 [2]

Under IFWA, the $\flat$ pattern ranks highest in three schemes (W_1, W_2, W_3) and is second only in W_4, while $\natural$ is consistently lowest. Under IFWG, $\sharp$ takes first place in three schemes (W_1, W_2, W_4), with $\flat$ leading only in W_3. These outcomes suggest IFWA yields more stable rankings across weight variations, whereas IFWG is more sensitive to changes in weight distribution. A visual representation of sensitivity analysis for various weights is shown in Figure 7 below.

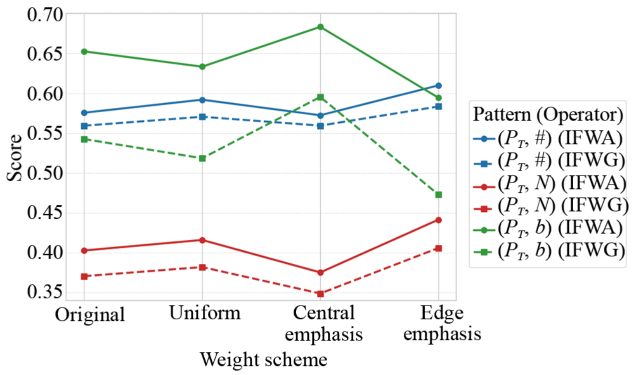

Figure 7. Score sensitivity of neural patterns under IFWA and IFWG

6.2 *Sensitivity analysis of GIFR model*

Following the AG-IFDM evaluation, a similar analysis was applied to GIFR, but here robustness is tested by perturbing the IF-score congruence values of the representative unordered pairs (ε, k_2), (k_2, k_3), (ε, k_3), (k_1, ε), and (k_1, k_2). Three perturbation scenarios were considered: (i) favour solvent-related interactions ($+0.02$ to f, -0.02 to g for pairs involving k_3, within bounds), (ii) increase overall uncertainty ($+0.02$ to both f and g, subject to $f + g \leq 1$), and (iii) penalize filler-related interactions (-0.02 to f, $+0.02$ to g for pairs involving ε, within bounds). Table 18 summarizes the resulting scores, accuracies, and ranks.

Table 18. GIFR sensitivity: Score/Accuracy/Rank for baseline and perturbation scenarios

Pair	Baseline	Scenario (i)	Scenario (ii)	Scenario (iii)
$(\varepsilon,\ k_2)$	0.60/0.48/1	0.60/0.48/3	0.60/0.50/1	0.58/0.48/3
$(k_2,\ k_3)$	0.60/0.45/2	0.62/0.45/1	0.60/0.47/2	0.60/0.45/1
$(k_1,\ k_2)$	0.60/0.40/3	0.60/0.40/4	0.60/0.42/3	0.60/0.40/2
$(k_1,\ \varepsilon)$	0.60/0.38/4	0.60/0.38/5	0.60/0.40/4	0.58/0.38/4
$(\varepsilon,\ k_3)$	0.60/0.35/5	0.62/0.35/2	0.60/0.37/5	0.58/0.35/5

In scenario (i), $(k_2,\ k_3)$ and $(\varepsilon,\ k_3)$ gain higher scores, moving to first and second places respectively, while $(\varepsilon,\ k_2)$ drops to third. Scenario (ii) keeps all scores unchanged, preserving the baseline ranking but with slight accuracy increases. In Scenario (iii), filler-related pairs lose score, allowing $(k_2,\ k_3)$ and $(k_1,\ k_2)$ to take the top two positions. Overall, AG-IFDM shows stability under weight changes with some sensitivity in IFWG, while GIFR remains stable under uniform uncertainty but is more affected by targeted changes to specific interaction types. A visual representation of sensitivity analysis for different scenarios is shown in Figure 8 below.

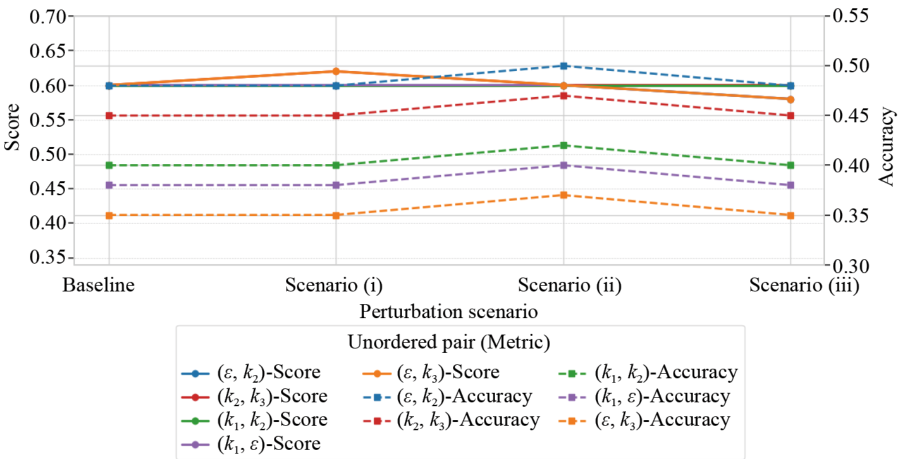

Figure 8. GIFR sensitivity to IF-score congruence perturbations

6.3 *Robustness evaluation*

To assess how stable the rankings remain under the perturbations considered in the sensitivity analysis, we use the Kendall rank correlation coefficient "K". For two total orders on n items, let C and D be the numbers of concordant and discordant pairs among the $\binom{n}{2}$ possible pairs; then

$$K = \frac{C - D}{\binom{n}{2}} \in [-1,\ 1].$$

When ties occur, the tie-adjusted form K_b is used; in our comparisons the orders are total and K suffices.

AG-IFDM Model: Using the baseline weight scheme W_1 and the three variants W_2 - W_4, rankings are compared operator-wise. As an illustration, comparing W_1 and W_4 under IFWA yields one inversion out of $\binom{3}{2} = 3$ pairs, i.e., $C = 2$, $D = 1$, so $K = (2-1)/3 = 0.33$. The complete values are given in Table 19 below:

Table 19. Kendall rank correlation between various weights

Operator	$K(W_1, W_2)$	$K(W_1, W_3)$	$K(W_1, W_4)$
IFWA	1.00	1.00	0.33
IFWG	1.00	0.33	1.00

These values show that the AG-IFDM rankings are *invariant* across most weight changes, with a single swap between the top two patterns in one setting per operator (hence $K \approx 0.33$ there). The bottom alternative remains unchanged throughout.

GIFR Model: Each perturbation of the IF-score congruence is compared to the baseline. For the first perturbation, counting over $\binom{5}{2} = 10$ pairs gives $C = 6$, $D = 4$, hence $K = (6-4)/10 = 0.20$. The summary is given in Table 20.

Table 20. Kendall rank correlation with respect to the baseline

Scenario	(i)	(ii)	(iii)
K	0.20	1.00	0.60

Overall, the GIFR ranking reproduces the baseline under the second perturbation ($K = 1.00$), shows strong agreement in the third ($K = 0.60$), and exhibits the largest yet still positive deviation in the first ($K = 0.20$). In combination with the AG-IFDM results, these outcomes indicate that the proposed methods retain their ordering structure under the tested variations, with only localized swaps at the very top in a few stress configurations. While the above two cases illustrate how the proposed models function in structured decision-making settings, the same principles apply to a broader class of problems where algebraic behaviour, such as order or grouping, significantly affects outcomes.

7. Advantages and limitations

The proposed intuitionistic fuzzy algebraic framework is specifically designed for decision-making environments where the sequence of actions, interdependencies or specific groupings of elements strongly influence outcomes. They are very flexible and transparent as they provide a framework that allows for evaluating alternatives in a single form and as well as in a pairwise form to provide a clear explanation of the decision-making process. Additionally the presented algebraic intuitionistic fuzzy methods by default address conflicting criteria through the simultaneous representation of membership and non-membership values, explicitly capturing trade-offs and hesitations. Utilizing its capacity to model both associative and non-associative, as well as commutative and non-commutative scenarios, the framework extends well beyond the medical and agricultural case studies presented in this work. Its applicability spans a wide range of domains, including logistics and routing, workflow management and approval chains, financial evaluation, public policy formulation, adaptive recommendation systems and other complex processes in which the ordering of operations plays a decisive role.

However, there are some limitations to these approaches, one of which is the computational complexity, which can be high, especially for large datasets or complex decision problems. There is also currently no standard methodology for using intuitionistic fuzzy decision-making through abstract algebra, which can make it difficult to compare results across studies.

8. Conclusion

Far from being purely theoretical, the algebraic structures represent a meaningful shift in how uncertainty can be modelled and managed. This paper fundamentally redefines the role of abstract algebra in addressing real-world decision-making problems under uncertainty, with a particular focus on scenarios where the sequence of operations critically influences outcomes. We introduced a novel framework based on intuitionistic fuzzy AG-groupoids to classify non-commutative and non-associative neural patterns, which offer a powerful tool for handling complex, order-sensitive decisions. For problems grounded in associative behavior, we employed the algebraic structure of the Klein four-group within an intuitionistic fuzzy setting to develop a precise and scalable ranking system for medicine-ingredient combinations. Together, these approaches bridge abstract algebra with computational decision-making, which establishes a strong foundation for adaptive and context-aware reasoning. This advancement represents a pivotal move toward integrating algebraic precision into uncertainty-driven environments, which opens new pathways for the development of intelligent and adaptive decision-making systems.

Building upon this work, the proposed framework can be extended to incorporate more advanced fuzzy paradigms such as Pythagorean fuzzy sets, q-Rung orthopair fuzzy sets, and both linear and quadratic Diophantine fuzzy sets [20]. Incorporating reference parameters within these paradigms would allow the modelling of richer contextual nuances and more refined degrees of uncertainty, thereby enhancing the decision-making process. In addition, the algebraic foundation of the framework can be further enriched by extending its scope from AG-groupoids and the Klein four-group to a wider class of algebraic systems, including semirings, lattices, modules and rings. Such extensions would enable the handling of diverse relational properties (e.g., distributivity, modularity, or closure under scalar multiplication) and more sophisticated algebraic operations. By doing so, the adaptability and robustness of the framework would be significantly enhanced, opening pathways for its application to increasingly complex and multi-layered decision-making environments, such as large-scale optimization, multi-agent negotiations, and context-sensitive recommendation systems.

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Conflict of interest

The authors declare no conflict of interest.

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