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Research Article

Uniqueness Result for Inverse Problem of Determining Coefficient and Source Term in a Two-Term Time Fractional Diffusion Equation

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Abstract: In this paper, a two term time fractional diffusion equation is considered. Based on the technique of eigenfunction expansion and the Laplace transform, a uniqueness result for inverse problem of simultaneously determining coefficient r and source term $f(x)$ is obtained.

Keywords: two-term time fractional diffusion equation, uniqueness, inverse problems, eigenfunction expansion, Mittag-Leffler function

MSC: 35K20, 35Q70

1. Introduction

Numerous natural and engineered diffusion processes including soil contaminant migration, oil movement through porous media, and groundwater pollutant transport over long distances exhibit anomalous diffusion behavior. In these cases, particle dispersion occurs at rates that deviate from predictions made by traditional integer-order diffusion models. Over the past twenty years, researchers have demonstrated that fractional diffusion equations provide effective mathematical frameworks for modeling these non-standard diffusion phenomena (as documented in sources ranging from [1–7]).

The expanding applications of fractional calculus and derivatives across applied sciences have spurred a surge in research on fractional differential equations. There are many numerical methods to solve the fractional differential equation, for example, finite difference method [8–12], weighted average finite difference method [13], fundamental solution method [14, 15], matrix transform method [16, 17] and implicit numerical Euler approximation method [18, 19], etc.

After the first paper on inverse problem for fractional differential equation [20], a lot of articles consider the inverse problem for fractional differential equation from many aspects, for example, Bondarenko et al. [21, 22] consider the numerical treatment of boundary value problem, Luchko [23–25] consider maximum principle, uniqueness and existence result of the solution, Mainardi [26] and Salim et al. [27] consider the fundamental solution, Sakamoto [28] and Xu [29] discuss inverse source problem, Xu et al. [30] obtain stability result using Carleman estimate, etc.

For diffusion equations involving multiple fractional time derivatives, additional relevant works include Jiang et al. [31], Gejji et al [32], and Li et al. [33], along with their references. The study in [34] establishes the unique existence

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of solutions, maximum principle, and related properties for cases where the time-derivative coefficients are positive and spatially dependent. The analysis relies on the Fourier method, specifically separation of variables. In [35], the authors prove uniqueness and solution regularity for an initial-boundary value problem involving a symmetric two-term time-fractional diffusion equation, under the assumption of solution existence. Similarly, [33] extends these results to linear non-symmetric diffusion equations with variable fractional time-derivative coefficients which need not be constant or positive. The work in [36] demonstrates uniqueness for two inverse problems involving fractional order identification in multi-term time-fractional diffusion equations using pointwise observations. In [37], the author examines in great detail a reaction-diffusion model with variable coefficients.

The model in this paper often used to describe solute transport in mobile/immobile region [38], where coefficient r stands for o the mobile/immobile capacity coefficient and $f(x)$ is the pollutant source. However, r and $f(x)$ are often unknown and hardly to measure in general, so determination of the two terms is necessary. In this paper, building upon [36], we prove a uniqueness result for simultaneously determining a coefficient and a space-dependent source term via eigenfunction expansion and Laplace transform. However, unlike prior studies, we focus on a two-term mobile/immobile time-fractional diffusion equation, which models total concentration [38]. Another difference is that [36] gives the uniqueness result of fractional orders and coefficients whereas this paper consider the uniqueness of coefficient and source function.

The rest of this paper is organized as follows. In section 2, we give the formulation of the problem and some preliminary result. In section 3, we give the main result and the proof. Finally, concluding remarks are given.

2. Formulation of the problem and some preliminary results

We consider the following problem:

$$\frac{\partial u}{\partial t} + r\frac{\partial^\alpha u}{\partial t^\alpha} = u_{xx} + f(x),\ 0 < x < 1,\ T > t > 0 \tag{1}$$

$$u(x,\ 0) = \phi(x),\ 0 \le x \le 1 \tag{2}$$

$$u_x(0,\ t) = u_x(1,\ t) = 0,\ T \ge t \ge 0, \tag{3}$$

where $1 > \alpha > 0$ is fractional order, $r > 0$ is a constant. We suppose that $f(x) \in L^2(0,\ 1),\ \phi(x) \in L^2(0,\ 1)$ and $\overline{supp(f)} \subset (0,\ 1)$.

Also, to get the uniqueness result, we assume that the intersection of support of $f(x)$ and $\phi(x)$ is an empty set.

∂_t^α denotes the Caputo derivative defined by

$$\frac{\partial^\alpha f}{\partial t^\alpha} = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{f'(s)}{(t-s)^\alpha}ds,$$

and $\Gamma(\cdot)$ denotes the usual Gamma function.

We discuss

Inverse Problem: Let $x_0 = 0,\ x_1 \in [0,\ 1]/\overline{suppf}$ lie on the left side of $\overline{suppf}$. Determine r and $f(x)$ from additional measurements $u(x_0,\ t),\ u(x_1,\ t),\ 0 < t < T$.

To prove the main result, we need to introduce a special function named the multinomial Mittag-Leffler function, which is the natural function that appear in the solutions to multi-term fractional differential equations, playing a role analogous to the exponential function in integer-order equations. The definition of the mlutinomial Mittag-Leffler function is as follows:

$$E_{(\alpha_1,\, \cdots,\, \alpha_n),\, \beta}(z_1,\, \cdots,\, z_n) = \sum_{k=0}^{\infty} \sum_{k_1+\cdots+k_n=k} (k;\, k_1,\, \cdots k_n) \frac{\prod_{j=1}^{n} z_j^{k_j}}{\Gamma(\beta+\sum_{j=1}^{n} k_j \alpha_j)},$$

where $0<\alpha_j<1$, $0<\beta<2$, $z_j \in \mathbb{C}$, $j=1,\cdots,n$ and $(k;\, k_1,\, \cdots k_n)$ denotes the multinomial coefficient

$$(k;\, k_1,\, \cdots k_n) = \frac{k!}{k_1!\cdots k_n!} \text{ with } k=\sum_{j=1}^{n} k_j.$$

For later use and simplicity, we adopt the abbreviation

$$E^{(j)}_{(\mathbf{r},\, \alpha'),\, 1+\alpha_1}(t) = E_{(\alpha_1,\, \alpha_1-\alpha_2,\, \cdots,\, \alpha_1-\alpha_n),\, 1+\alpha_1}\left(-\lambda_j t^{\alpha_1},\, -r_2 t^{\alpha_1-\alpha_2} \cdots,\, -r_n t^{\alpha_1-\alpha_n}\right).$$

To reach the main result, we need to use some properties of the multinomial Mittag-Leffler function, we list these properties in the following Lemma [39]:

Lemma 1 Let $\lambda>0$, then

$$\frac{d}{dt}\left\{t^{\alpha_1} E^{(n)}_{(\mathbf{r},\, \alpha'),\, 1+\alpha_1}(t)\right\} = t^{\alpha_1-1} E^{(n)}_{(\mathbf{r},\, \alpha'),\, \alpha_1}.$$

Lemma 2 Let $0<\beta<2$, and $1>\alpha_1>\cdots>\alpha_n>0$ be given. Assume that $\alpha_1\pi/2<\mu<\alpha_1\pi$, $\mu<|arg(z)|<\pi$ and there is $K>0$ such that $-K<z_j<0(j=2,\, \cdots n)$. Then there exists a constant $C>0$ depending on $\mu,\, K,\, \alpha_j\ (j=1,\, 2,\, \cdots,\, n)$ and β only such that

$$|E_{(\alpha_1,\, \alpha_1-\alpha_2,\, \cdots,\, \alpha_1-\alpha_n),\, \beta}(z_1,\, \cdots,\, z_n)| \le \frac{C}{1+|z_1|}.$$

Lemma 2.2 provides an asymptotic bound that ensures the convergence of the series solutions.

3. Main result and the proof

Before we give the main result, we should give the solution to the forward problem (1)-(3).

The solution is nearly identical to the result in [31], except for the boundary condition shift from Dirichlet to Neumann. This alteration replaces the sine function in [31]'s formula (44) with a cosine. Consequently, we can derive the solution to (1)-(3) analogously to [31] in the following form:

$$u(x,\, t) = \sum_{n=1}^{\infty} tE_{(1,\, 1-\alpha),\, 2}(-\lambda_n t,\, -rt^{1-\alpha})(-\lambda_n\phi+f,\, \cos n\pi x)\cos n\pi x + \sum_{n=1}^{\infty}(\phi,\, \cos n\pi x)\cos n\pi x.$$

Now we state our main result.

Theorem 1 Let $u(x,\ t)$ be the weak solution to (1)-(3), and let v be the weak solution of (4)-(6) with the same initial condition and boundary conditions as (1)-(3),

$$\frac{\partial v}{\partial t}+r_1\frac{\partial^\alpha v}{\partial t^\alpha}=v_{xx}+f_1(x),\ 0<x<1,\ T>t>0 \tag{4}$$

$$v(x,\ 0)=\phi(x),\ 0\le x\le 1 \tag{5}$$

$$v_x(0,\ t)=v_x(1,\ t)=0,\ T\ge t\ge 0, \tag{6}$$

where $r_1>0$ is a constant and $f_1(x)\in L^2(0,\ 1)$, $\overline{suppf_1}\subset(0,\ 1)$.

Then for $x_0=0,\ x_1\in(0,\ 1)/(\overline{suppf}\ \bigcup\ \overline{suppf_1})$ lying on the left side of $\overline{suppf}\ \bigcup\ \overline{suppf_1}$, $u(x_0,\ t)=v(x_0,\ t)$, $u(x_1,\ t)=v(x_1,\ t)$ imply $r=r_1$ and $f(x)=f_1(x)$, $0<x<1$.

Proof. We assume that $x_0=0$, and x_1 lies on the left side of $\overline{suppf}\ \bigcup\ \overline{suppf_1}$ and $x_0<x_1$, then $f(x)=f_1(x)=0,\ x\in(0,\ x_1)$.

We split the proof into the following two steps:

Step 1 We will prove $r=r_1$.

Denote $u(x_0,\ t)=\theta_0(t)$, $u(x_1,\ t)=\theta_1(t)$, then $u(x,\ t)$ and $v(x,\ t)$ satisfies the following problem respectively

$$\frac{\partial u}{\partial t}+r\frac{\partial^\alpha u}{\partial t^\alpha}=u_{xx},\ 0<x<x_1,\ T>t>0 \tag{7}$$

$$u(x,\ 0)=\phi(x),\ 0\le x\le x_1 \tag{8}$$

$$u_x(0,\ t)=0,\ u(x_1,\ t)=\theta_1(t),\ T\ge t\ge 0 \tag{9}$$

$$\frac{\partial v}{\partial t}+r_1\frac{\partial^\alpha v}{\partial t^\alpha}=v_{xx},\ 0<x<x_1,\ T>t>0 \tag{10}$$

$$v(x,\ 0)=\phi(x),\ 0\le x\le x_1 \tag{11}$$

$$v_x(0,\ t)=0,\ v(x_1,\ t)=\theta_1(t),\ T\ge t\ge 0. \tag{12}$$

Similar to the method of [31], we can see that

$$u(x,\ t)=\sum_{n=1}^{\infty}tE_{(1,\ 1-\alpha),\ 2}(-\lambda_n t,\ -rt^{1-\alpha})(-\lambda_n\phi+f,\ \cos n\pi x)\cos n\pi x+\sum_{n=1}^{\infty}(\phi,\ \cos n\pi x)\cos n\pi x$$

$$v(x,\ t)=\sum_{n=1}^{\infty}tE_{(1,\ 1-\alpha),\ 2}(-\lambda_n t,\ -rt^{1-\alpha})(-\lambda_n\phi+f_1,\ \cos n\pi x)\cos n\pi x+\sum_{n=1}^{\infty}(\phi,\ \cos n\pi x)\cos n\pi x.$$

Denote $w = u - v$, then $w(x,\ t)$ satisfies

$$\frac{\partial w}{\partial t} + r\frac{\partial^\alpha w}{\partial t^\alpha} = w_{xx} + (r_1 - r)\frac{\partial^\alpha v}{\partial t^\alpha},\ 0 < x < x_1,\ T > t > 0 \tag{13}$$

$$w(x,\ 0) = 0,\ 0 \le x \le x_1 \tag{14}$$

$$w_x(0,\ t) = 0,\ w(x_1,\ t) = 0,\ T \ge t \ge 0, \tag{15}$$

and $w(x_0,\ t) = 0$.

Denote $F(x,\ t) = \dfrac{\partial^\alpha v}{\partial t^\alpha}$.

$$w(\cdot,\ t) = \int_0^t \sum_{n=1}^{\infty} E_{(1,\ 1-\alpha),\ 1}(-\lambda_n s,\ -rs^{1-\alpha})(r_1 - r)(F(\cdot,\ t-s),\ \phi_n)\phi_n ds,$$

where $(\cdot,\ \cdot)$ denotes the scalar product in $L^2(0,\ x_1)$, $\{\lambda_n,\ \phi_n\}_{n=1}^{\infty}$ be an eigensystem of the following Sturm-Liouville problem

$$X''(x) + \lambda X = 0,\ 0 < x < x_1$$

$$X'(0) = 0,\ X(x_1) = 0.$$

If we define

$$U(t)f := \sum_{n=1}^{\infty} E_{(1,\ 1-\alpha),\ 1}(-\lambda_n t,\ -rt^{1-\alpha})(f,\ \phi_n)\phi_n,$$

then

$$w(\cdot,\ t) = (r_1 - r)\int_0^t U(s)F(\cdot,\ t-s)ds.$$

By using Lemma (1) and the formulation of $v(x,\ t)$, there holds

$$F(x,\ t) = \frac{\partial^\alpha v}{\partial t^\alpha} = \frac{1}{r_1}\left(v_{xx} - \frac{\partial v}{\partial t} + f_1(x)\right)$$

$$= \frac{1}{r_1}\left\{\sum_{n=1}^{\infty}\left[\left(1 - \lambda_n t E_{(1,\ 1-\alpha),\ 2}(-\lambda_n t,\ -rt^{1-\alpha})\right)(\phi,\ \phi_n)\phi_n''\right.\right.$$

$$+\sum_{n=1}^{\infty} tE_{(1,\ 1-\alpha),\ 2}(-\lambda_n t,\ -rt^{1-\alpha})(f_1,\ \phi_n)\phi_n'' + (f_1,\ \phi_n)\phi_n\Bigg]$$

$$-\sum_{n=1}^{\infty}\left[-\lambda_n E_{(1,\ 1-\alpha),\ 1}(-\lambda_n t,\ -rt^{1-\alpha})(\phi,\ \phi_n)\phi_n + E_{(1,\ 1-\alpha),\ 1}(-\lambda_n t,\ -rt^{1-\alpha})(f_1,\ \phi_n)\phi_n\right]\Bigg\}$$

$$=\frac{1}{r_1}\left\{\sum_{n=1}^{\infty}\left[-\lambda_n + \lambda_n E_{(1,\ 1-\alpha),\ 1}(-\lambda_n t,\ -rt^{1-\alpha}) + \lambda_n^2 tE_{(1,\ 1-\alpha),\ 2}(-\lambda_n t,\ -rt^{1-\alpha})\right](\phi,\ \phi_n)\phi_n\right.$$

$$\left.-\sum_{n=1}^{\infty}\left[\lambda_n tE_{(1,\ 1-\alpha),\ 2}(-\lambda_n t,\ -rt^{1-\alpha}) + E_{(1,\ 1-\alpha),\ 1}(-\lambda_n t,\ -rt^{1-\alpha}) - 1\right](f_1,\ \phi_n)\phi_n\right\},$$

since the solutions u and v can be analytically extended to $t > 0$ in view of the analyticity of the multinomial Mittag-Leffler function [33], we have $w(x_0,\ t) = 0,\ t > 0$. So by the Laplace transform we obtain

$$\frac{r_1 - r}{sr_1}\sum_{n=1}^{\infty}\frac{\phi_n(x_0)}{s + rs^{\alpha} + \lambda_n}\sum_{n=1}^{\infty}\left[\frac{1}{s} - \frac{1}{s + r_1 s^{\alpha} + \lambda_n} - \frac{\lambda_n}{s(s + r_1 s^{\alpha} + \lambda_n)}\right](-\lambda_n\phi + f_1,\ \phi_n)(\phi_n,\ \phi_n)$$

$$=\frac{r_1 - r}{sr_1}\sum_{n=1}^{\infty}\frac{\phi_n(x_0)}{s + rs^{\alpha} + \lambda_n}\sum_{n=1}^{\infty}\frac{r_1 s^{\alpha}}{s(s + r_1 s^{\alpha} + \lambda_n)}(-\lambda_n\phi + f_1,\ \phi_n)(\phi_n,\ \phi_n) = 0,$$

thus there is three possible cases:

(i) $r_1 - r = 0$;

(ii) $\sum_{n=1}^{\infty}\frac{\phi_n(x_0)}{s + rs^{\alpha} + \lambda_n} = 0$;

(iii) $\sum_{n=1}^{\infty}\frac{r_1 s^{\alpha}}{s(s + r_1 s^{\alpha} + \lambda_n)}(-\lambda_n\phi + f_1,\ \phi_n)(\phi_n,\ \phi_n) = 0$.

Next we will show that only case (i) holds true.

If case (ii) is true, that is $\sum_{n=1}^{\infty}\frac{\phi_n(x_0)}{s + rs^{\alpha} + \lambda_n} = 0$.

Denote $\eta = s + rs^{\alpha}$, then $\sum_{n=1}^{\infty}\frac{\phi_n(x_0)}{\eta + \lambda_n} = 0$ holds for $\eta \in \mathscr{C}\ \{\lambda_n\}_{n\geq 1}$. We can take a disk which includes λ_1 and does not include $\{\lambda_n\}_{n\geq 2}$. In the disk, the functions $\frac{\phi_n(x_0)}{\eta + \lambda_n}$, $n \geq 2$ are analytic and their integrals along the disk are zeros. So integration of $\sum_{n=1}^{\infty}\frac{\phi_n(x_0)}{\eta + \lambda_n}$ along the disk equals to $2\pi i\phi_1(x_0)$, which is also equal to 0. Thus, we have $\phi_1(x_0) = 0$.

Take another disk which includes λ_2 but doesn't include $\{\lambda_n\}_{n\geq 3}$, we can get $\phi_2(x_0) = 0$ using similar procedure. Repeating this argument, we can obtain $\phi_n(x_0) = 0,\ n = 3,\ 4,\ \cdot$, which is impossible in sake of ϕ_n being eigenfunction of Sturm-Liouville problem.

For case (iii):

From $\sum_{n=1}^{\infty}\frac{r_1 s^{\alpha}}{s(s + r_1 s^{\alpha} + \lambda_n)}(-\lambda_n\phi + f_1,\ \phi_n)(\phi_n,\ \phi_n) = 0$.

Also denote $\eta = s + r_1 s^{\alpha}$ and take a disk which includes λ_1 and does not include $\{\lambda_n\}_{n\geq 2}$. By Cauchy integral theorem, integrating along the disk, we have

$$2\pi i(-\lambda_1\phi + f_1,\ \phi_1)(\phi_1,\ \phi_1)[r_1 s^{\alpha}]|_{\eta=-\lambda_1} = 0,$$

considering $r_1 s^{\alpha} = \eta - s$ and $(\phi_1,\ \phi_1) > 0$, we have

$$(-\lambda_1\phi + f_1,\ \phi_1) = 0.$$

Repeating this argument, we can obtain

$$(-\lambda_n\phi + f_1,\ \phi_n) = 0,\ n = 2,\ 3,\ \cdots.$$

However, we know that $f_1(x) = 0$ for $x \in (0,\ x_1)$ and $(\cdot,\ \cdot)$ denotes inner product on the interval $(0,\ x_1)$, so $(f_1,\ \phi_n) = 0,\ n = 1,\ 2,\ \cdots$, then we can get $(\phi,\ \phi_n) = 0,\ n = 1,\ 2,\ \cdots$ indicates that $\phi(x) = 0,\ x \in (0,\ x_1)$ for arbitrary initial function $\phi(x)$. That's impossible.

so we can conclude that only case (i) holds true, i.e., $r = r_1$.

Step 2 we will prove $f(x) = f_1(x)$.

We turn back to (1)-(3) and (4)-(6).

Denote $\varphi_n = 2\int_0^1 phi(x)\cos(n\pi x)dx$, $f_n = 2\int_0^1 f(x)\cos(n\pi x)dx$ and $f_{1n} = 2\int_0^1 f_1(x)\cos(n\pi x)dx$.

By analytic continuation we can get $u(x_0,\ t) = v(x_0,\ t),\ t > 0$ from $u(x_0,\ t) = v(x_0,\ t),\ T > t > 0$. Then

$$\int_0^t \sum_{n=1}^{\infty} E_{(1,\ 1-\alpha),\ 2}\left(-\mu_n s,\ -rs^{1-\alpha}\right) f_n ds \cos(n\pi x_0)$$

$$= \int_0^t \sum_{n=1}^{\infty} E_{(1,\ 1-\alpha),\ 2}\left(-\mu_n s,\ -rs^{1-\alpha}\right) f_{1n} ds \cos(n\pi x_0),$$

where $\mu_n = n^2\pi^2$. Apply Laplace transform on the above equation, considering the property of $\mathscr{L}(\int_0^t f(\tau)d\tau) = \frac{1}{s}\mathscr{L}(f(t))$ of Laplace transform, then we can get

$$\frac{1}{s}\sum_{n=1}^{\infty} \frac{f_n \cos(n\pi x_0)}{\eta + r\eta^{\alpha} + \mu_n} = \frac{1}{s}\sum_{n=1}^{\infty} \frac{f_{1n}\cos(n\pi x_0)}{\eta + r\eta^{\alpha} + \mu_n},$$

which implies

$$\sum_{n=1}^{\infty} \frac{a_n \cos(n\pi x_0)}{z + \mu_n} = 0,$$

where z represents $\eta + r\eta^{\alpha}$ and $a_n = f_n - f_{1n}$.

We can take a disk which includes μ_1 and does not include $\{n^2\pi^2\}_{n\geq 2}$. By Cauchy integral theorem, integrating along the disk, we have $a_1\cos(\pi x_0) = 0$, which gives $f_1 = f_{11}$. Repeating this argument, we can obtain

$$f_n = f_{1n},\ n = 2,\ 3,\ \cdots.$$

Thus we can conclude that $f(x) = f_1(x)$.
That's end the proof. □

4. Conclusion

In this paper, we consider the two term fractional diffusion equation, which can be used to simulate mobile/immobile diffusion process in porus media. However, the coefficient r and the source term $f(x)$ may be unknown, so determination of these two terms is necessary. We prove the uniqueness result of simultaneously determining coefficient and source term from two points observation, and the proof relies on eigenfunction expansion and the Laplace transform. Actually, it seems easy to determine r or $f(x)$, however, it is difficult to determine the two terms simultaneously since r and $f(x)$ may belong to different function space.

We also need to point out that the result can be extended to multi-term time fractional diffusion equations through the similar technique. In the future work, we may investigate the stability of the inverse problem and the numerical algorithm based on the presented theory.

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Conflict of interest

The authors declare no competing financial interest.

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