

Research Article

Uniqueness Result for Inverse Problem of Determining Coefficient and Source Term in a Two-Term Time Fractional Diffusion Equation

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Abstract: In this paper, a two term time fractional diffusion equation is considered. Based on the technique of eigenfunction expansion and the Laplace transform, a uniqueness result for inverse problem of simultaneously determining coefficient r and source term $f(x)$ is obtained.

Keywords: two-term time fractional diffusion equation, uniqueness, inverse problems, eigenfunction expansion, Mittag-Leffler function

MSC: 35K20, 35Q70

1. Introduction

Numerous natural and engineered diffusion processes including soil contaminant migration, oil movement through porous media, and groundwater pollutant transport over long distances exhibit anomalous diffusion behavior. In these cases, particle dispersion occurs at rates that deviate from predictions made by traditional integer-order diffusion models. Over the past twenty years, researchers have demonstrated that fractional diffusion equations provide effective mathematical frameworks for modeling these non-standard diffusion phenomena (as documented in sources ranging from [1–7]).

The expanding applications of fractional calculus and derivatives across applied sciences have spurred a surge in research on fractional differential equations. There are many numerical methods to solve the fractional differential equation, for example, finite difference method [8–12], weighted average finite difference method [13], fundamental solution method [14, 15], matrix transform method [16, 17] and implicit numerical Euler approximation method [18, 19], etc.

After the first paper on inverse problem for fractional differential equation [20], a lot of articles consider the inverse problem for fractional differential equation from many aspects, for example, Bondarenko et al. [21, 22] consider the numerical treatment of boundary value problem, Luchko [23–25] consider maximum principle, uniqueness and existence result of the solution, Mainardi [26] and Salim et al. [27] consider the fundamental solution, Sakamoto [28] and Xu [29] discuss inverse source problem, Xu et al. [30] obtain stability result using Carleman estimate, etc.

For diffusion equations involving multiple fractional time derivatives, additional relevant works include Jiang et al. [31], Gejji et al [32], and Li et al. [33], along with their references. The study in [34] establishes the unique existence

of solutions, maximum principle, and related properties for cases where the time-derivative coefficients are positive and spatially dependent. The analysis relies on the Fourier method, specifically separation of variables. In [35], the authors prove uniqueness and solution regularity for an initial-boundary value problem involving a symmetric two-term time-fractional diffusion equation, under the assumption of solution existence. Similarly, [33] extends these results to linear non-symmetric diffusion equations with variable fractional time-derivative coefficients which need not be constant or positive. The work in [36] demonstrates uniqueness for two inverse problems involving fractional order identification in multi-term time-fractional diffusion equations using pointwise observations. In [37], the author examines in great detail a reaction-diffusion model with variable coefficients.

The model in this paper often used to describe solute transport in mobile/immobile region [38], where coefficient r stands for the mobile/immobile capacity coefficient and $f(x)$ is the pollutant source. However, r and $f(x)$ are often unknown and hard to measure in general, so determination of the two terms is necessary. In this paper, building upon [36], we prove a uniqueness result for simultaneously determining a coefficient and a space-dependent source term via eigenfunction expansion and Laplace transform. However, unlike prior studies, we focus on a two-term mobile/immobile time-fractional diffusion equation, which models total concentration [38]. Another difference is that [36] gives the uniqueness result of fractional orders and coefficients whereas this paper consider the uniqueness of coefficient and source function.

The rest of this paper is organized as follows. In section 2, we give the formulation of the problem and some preliminary result. In section 3, we give the main result and the proof. Finally, concluding remarks are given.

2. Formulation of the problem and some preliminary results

We consider the following problem:

$$\frac{\partial u}{\partial t} + r \frac{\partial^\alpha u}{\partial t^\alpha} = u_{xx} + f(x), \quad 0 < x < 1, \quad T > t > 0 \quad (1)$$

$$u(x, 0) = \phi(x), \quad 0 \leq x \leq 1 \quad (2)$$

$$u_x(0, t) = u_x(1, t) = 0, \quad T \geq t \geq 0, \quad (3)$$

where $1 > \alpha > 0$ is fractional order, $r > 0$ is a constant. We suppose that $f(x) \in L^2(0, 1)$, $\phi(x) \in L^2(0, 1)$ and $\overline{\text{supp}(f)} \subset (0, 1)$.

Also, to get the uniqueness result, we assume that the intersection of support of $f(x)$ and $\phi(x)$ is an empty set. ∂_t^α denotes the Caputo derivative defined by

$$\frac{\partial^\alpha f}{\partial t^\alpha} = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(s)}{(t-s)^\alpha} ds,$$

and $\Gamma(\cdot)$ denotes the usual Gamma function.

We discuss

Inverse Problem: Let $x_0 = 0$, $x_1 \in [0, 1] \setminus \overline{\text{supp } f}$ lie on the left side of $\overline{\text{supp } f}$. Determine r and $f(x)$ from additional measurements $u(x_0, t)$, $u(x_1, t)$, $0 < t < T$.

To prove the main result, we need to introduce a special function named the multinomial Mittag-Leffler function, which is the natural function that appear in the solutions to multi-term fractional differential equations, playing a role

analogous to the exponential function in integer-order equations. The definition of the multinomial Mittag-Leffler function is as follows:

$$E_{(\alpha_1, \dots, \alpha_n), \beta}(z_1, \dots, z_n) = \sum_{k=0}^{\infty} \sum_{k_1+\dots+k_n=k} (k; k_1, \dots, k_n) \frac{\prod_{j=1}^n z_j^{k_j}}{\Gamma(\beta + \sum_{j=1}^n k_j \alpha_j)},$$

where $0 < \alpha_j < 1$, $0 < \beta < 2$, $z_j \in \mathbb{C}$, $j = 1, \dots, n$ and $(k; k_1, \dots, k_n)$ denotes the multinomial coefficient

$$(k; k_1, \dots, k_n) = \frac{k!}{k_1! \dots k_n!} \text{ with } k = \sum_{j=1}^n k_j.$$

For later use and simplicity, we adopt the abbreviation

$$E_{(\mathbf{r}, \alpha'), 1+\alpha_1}^{(j)}(t) = E_{(\alpha_1, \alpha_1-\alpha_2, \dots, \alpha_1-\alpha_n), 1+\alpha_1}(-\lambda_j t^{\alpha_1}, -r_2 t^{\alpha_1-\alpha_2}, \dots, -r_n t^{\alpha_1-\alpha_n}).$$

To reach the main result, we need to use some properties of the multinomial Mittag-Leffler function, we list these properties in the following Lemma [39]:

Lemma 1 Let $\lambda > 0$, then

$$\frac{d}{dt} \left\{ t^{\alpha_1} E_{(\mathbf{r}, \alpha'), 1+\alpha_1}^{(n)}(t) \right\} = t^{\alpha_1-1} E_{(\mathbf{r}, \alpha'), \alpha_1}^{(n)}.$$

Lemma 2 Let $0 < \beta < 2$, and $1 > \alpha_1 > \dots > \alpha_n > 0$ be given. Assume that $\alpha_1 \pi/2 < \mu < \alpha_1 \pi$, $\mu < |\arg(z)| < \pi$ and there is $K > 0$ such that $-K < z_j < 0$ ($j = 2, \dots, n$). Then there exists a constant $C > 0$ depending on μ , K , α_j ($j = 1, 2, \dots, n$) and β only such that

$$|E_{(\alpha_1, \alpha_1-\alpha_2, \dots, \alpha_1-\alpha_n), \beta}(z_1, \dots, z_n)| \leq \frac{C}{1 + |z_1|}.$$

Lemma 2.2 provides an asymptotic bound that ensures the convergence of the series solutions.

3. Main result and the proof

Before we give the main result, we should give the solution to the forward problem (1)-(3).

The solution is nearly identical to the result in [31], except for the boundary condition shift from Dirichlet to Neumann. This alteration replaces the sine function in [31]'s formula (44) with a cosine. Consequently, we can derive the solution to (1)-(3) analogously to [31] in the following form:

$$u(x, t) = \sum_{n=1}^{\infty} t E_{(1, 1-\alpha), 2}(-\lambda_n t, -rt^{1-\alpha}) (-\lambda_n \phi + f, \cos n\pi x) \cos n\pi x + \sum_{n=1}^{\infty} (\phi, \cos n\pi x) \cos n\pi x.$$

Now we state our main result.

Theorem 1 Let $u(x, t)$ be the weak solution to (1)-(3), and let v be the weak solution of (4)-(6) with the same initial condition and boundary conditions as (1)-(3),

$$\frac{\partial v}{\partial t} + r_1 \frac{\partial^\alpha v}{\partial t^\alpha} = v_{xx} + f_1(x), \quad 0 < x < 1, \quad T > t > 0 \quad (4)$$

$$v(x, 0) = \phi(x), \quad 0 \leq x \leq 1 \quad (5)$$

$$v_x(0, t) = v_x(1, t) = 0, \quad T \geq t \geq 0, \quad (6)$$

where $r_1 > 0$ is a constant and $f_1(x) \in L^2(0, 1)$, $\overline{\text{supp } f_1} \subset (0, 1)$.

Then for $x_0 = 0$, $x_1 \in (0, 1) / (\overline{\text{supp } f} \cup \overline{\text{supp } f_1})$ lying on the left side of $\overline{\text{supp } f} \cup \overline{\text{supp } f_1}$, $u(x_0, t) = v(x_0, t)$, $u(x_1, t) = v(x_1, t)$ imply $r = r_1$ and $f(x) = f_1(x)$, $0 < x < 1$.

Proof. We assume that $x_0 = 0$, and x_1 lies on the left side of $\overline{\text{supp } f} \cup \overline{\text{supp } f_1}$ and $x_0 < x_1$, then $f(x) = f_1(x) = 0$, $x \in (0, x_1)$.

We split the proof into the following two steps:

Step 1 We will prove $r = r_1$.

Denote $u(x_0, t) = \theta_0(t)$, $u(x_1, t) = \theta_1(t)$, then $u(x, t)$ and $v(x, t)$ satisfies the following problem respectively

$$\frac{\partial u}{\partial t} + r \frac{\partial^\alpha u}{\partial t^\alpha} = u_{xx}, \quad 0 < x < x_1, \quad T > t > 0 \quad (7)$$

$$u(x, 0) = \phi(x), \quad 0 \leq x \leq x_1 \quad (8)$$

$$u_x(0, t) = 0, \quad u(x_1, t) = \theta_1(t), \quad T \geq t \geq 0 \quad (9)$$

$$\frac{\partial v}{\partial t} + r_1 \frac{\partial^\alpha v}{\partial t^\alpha} = v_{xx}, \quad 0 < x < x_1, \quad T > t > 0 \quad (10)$$

$$v(x, 0) = \phi(x), \quad 0 \leq x \leq x_1 \quad (11)$$

$$v_x(0, t) = 0, \quad v(x_1, t) = \theta_1(t), \quad T \geq t \geq 0. \quad (12)$$

Similar to the method of [31], we can see that

$$u(x, t) = \sum_{n=1}^{\infty} t E_{(1, 1-\alpha), 2}(-\lambda_n t, -rt^{1-\alpha})(-\lambda_n \phi + f, \cos n\pi x) \cos n\pi x + \sum_{n=1}^{\infty} (\phi, \cos n\pi x) \cos n\pi x$$

$$v(x, t) = \sum_{n=1}^{\infty} t E_{(1, 1-\alpha), 2}(-\lambda_n t, -rt^{1-\alpha})(-\lambda_n \phi + f_1, \cos n\pi x) \cos n\pi x + \sum_{n=1}^{\infty} (\phi, \cos n\pi x) \cos n\pi x.$$

Denote $w = u - v$, then $w(x, t)$ satisfies

$$\frac{\partial w}{\partial t} + r \frac{\partial^\alpha w}{\partial t^\alpha} = w_{xx} + (r_1 - r) \frac{\partial^\alpha v}{\partial t^\alpha}, \quad 0 < x < x_1, \quad T > t > 0 \quad (13)$$

$$w(x, 0) = 0, \quad 0 \leq x \leq x_1 \quad (14)$$

$$w_x(0, t) = 0, \quad w(x_1, t) = 0, \quad T \geq t \geq 0, \quad (15)$$

and $w(x_0, t) = 0$.

Denote $F(x, t) = \frac{\partial^\alpha v}{\partial t^\alpha}$.

$$w(\cdot, t) = \int_0^t \sum_{n=1}^{\infty} E_{(1, 1-\alpha), 1}(-\lambda_n s, -rs^{1-\alpha})(r_1 - r)(F(\cdot, t-s), \phi_n) \phi_n ds,$$

where (\cdot, \cdot) denotes the scalar product in $L^2(0, x_1)$, $\{\lambda_n, \phi_n\}_{n=1}^{\infty}$ be an eigensystem of the following Sturm-Liouville problem

$$X''(x) + \lambda X = 0, \quad 0 < x < x_1$$

$$X'(0) = 0, \quad X(x_1) = 0.$$

If we define

$$U(t)f := \sum_{n=1}^{\infty} E_{(1, 1-\alpha), 1}(-\lambda_n t, -rt^{1-\alpha})(f, \phi_n) \phi_n,$$

then

$$w(\cdot, t) = (r_1 - r) \int_0^t U(s) F(\cdot, t-s) ds.$$

By using Lemma (1) and the formulation of $v(x, t)$, there holds

$$\begin{aligned} F(x, t) &= \frac{\partial^\alpha v}{\partial t^\alpha} = \frac{1}{r_1} \left(v_{xx} - \frac{\partial v}{\partial t} + f_1(x) \right) \\ &= \frac{1}{r_1} \left\{ \sum_{n=1}^{\infty} \left[(1 - \lambda_n t E_{(1, 1-\alpha), 2}(-\lambda_n t, -rt^{1-\alpha})) (\phi, \phi_n) \phi_n'' \right] \right\} \end{aligned}$$

$$\begin{aligned}
& + \sum_{n=1}^{\infty} t E_{(1, 1-\alpha), 2}(-\lambda_n t, -rt^{1-\alpha}) (f_1, \phi_n) \phi_n'' + (f_1, \phi_n) \phi_n \\
& - \sum_{n=1}^{\infty} [-\lambda_n E_{(1, 1-\alpha), 1}(-\lambda_n t, -rt^{1-\alpha}) (\phi, \phi_n) \phi_n + E_{(1, 1-\alpha), 1}(-\lambda_n t, -rt^{1-\alpha}) (f_1, \phi_n) \phi_n] \Big\} \\
& = \frac{1}{r_1} \left\{ \sum_{n=1}^{\infty} [-\lambda_n + \lambda_n E_{(1, 1-\alpha), 1}(-\lambda_n t, -rt^{1-\alpha}) + \lambda_n^2 t E_{(1, 1-\alpha), 2}(-\lambda_n t, -rt^{1-\alpha})] (\phi, \phi_n) \phi_n \right. \\
& \left. - \sum_{n=1}^{\infty} [\lambda_n t E_{(1, 1-\alpha), 2}(-\lambda_n t, -rt^{1-\alpha}) + E_{(1, 1-\alpha), 1}(-\lambda_n t, -rt^{1-\alpha}) - 1] (f_1, \phi_n) \phi_n \right\},
\end{aligned}$$

since the solutions u and v can be analytically extended to $t > 0$ in view of the analyticity of the multinomial Mittag-Leffler function [33], we have $w(x_0, t) = 0$, $t > 0$. So by the Laplace transform we obtain

$$\begin{aligned}
& \frac{r_1 - r}{sr_1} \sum_{n=1}^{\infty} \frac{\phi_n(x_0)}{s + rs^\alpha + \lambda_n} \sum_{n=1}^{\infty} \left[\frac{1}{s} - \frac{1}{s + r_1 s^\alpha + \lambda_n} - \frac{\lambda_n}{s(s + r_1 s^\alpha + \lambda_n)} \right] (-\lambda_n \phi + f_1, \phi_n) (\phi_n, \phi_n) \\
& = \frac{r_1 - r}{sr_1} \sum_{n=1}^{\infty} \frac{\phi_n(x_0)}{s + rs^\alpha + \lambda_n} \sum_{n=1}^{\infty} \frac{r_1 s^\alpha}{s(s + r_1 s^\alpha + \lambda_n)} (-\lambda_n \phi + f_1, \phi_n) (\phi_n, \phi_n) = 0,
\end{aligned}$$

thus there is three possible cases:

- (i) $r_1 - r = 0$;
- (ii) $\sum_{n=1}^{\infty} \frac{\phi_n(x_0)}{s + rs^\alpha + \lambda_n} = 0$;
- (iii) $\sum_{n=1}^{\infty} \frac{r_1 s^\alpha}{s(s + r_1 s^\alpha + \lambda_n)} (-\lambda_n \phi + f_1, \phi_n) (\phi_n, \phi_n) = 0$.

Next we will show that only case (i) holds true.

If case (ii) is true, that is $\sum_{n=1}^{\infty} \frac{\phi_n(x_0)}{s + rs^\alpha + \lambda_n} = 0$.

Denote $\eta = s + rs^\alpha$, then $\sum_{n=1}^{\infty} \frac{\phi_n(x_0)}{\eta + \lambda_n} = 0$ holds for $\eta \in \mathcal{C} \setminus \{\lambda_n\}_{n \geq 1}$. We can take a disk which includes λ_1 and does not include $\{\lambda_n\}_{n \geq 2}$. In the disk, the functions $\frac{\phi_n(x_0)}{\eta + \lambda_n}$, $n \geq 2$ are analytic and their integrals along the disk are zeros. So integration of $\sum_{n=1}^{\infty} \frac{\phi_n(x_0)}{\eta + \lambda_n}$ along the disk equals to $2\pi i \phi_1(x_0)$, which is also equal to 0. Thus, we have $\phi_1(x_0) = 0$.

Take another disk which includes λ_2 but doesn't include $\{\lambda_n\}_{n \geq 3}$, we can get $\phi_2(x_0) = 0$ using similar procedure. Repeating this argument, we can obtain $\phi_n(x_0) = 0$, $n = 3, 4, \dots$, which is impossible in sake of ϕ_n being eigenfunction of Sturm-Liouville problem.

For case (iii):

From $\sum_{n=1}^{\infty} \frac{r_1 s^\alpha}{s(s + r_1 s^\alpha + \lambda_n)} (-\lambda_n \phi + f_1, \phi_n) (\phi_n, \phi_n) = 0$.

Also denote $\eta = s + r_1 s^\alpha$ and take a disk which includes λ_1 and does not include $\{\lambda_n\}_{n \geq 2}$. By Cauchy integral theorem, integrating along the disk, we have

$$2\pi i(-\lambda_1\phi + f_1, \phi_1)(\phi_1, \phi_1)[r_1 s^\alpha]|_{\eta=-\lambda_1} = 0,$$

considering $r_1 s^\alpha = \eta - s$ and $(\phi_1, \phi_1) > 0$, we have

$$(-\lambda_1\phi + f_1, \phi_1) = 0.$$

Repeating this argument, we can obtain

$$(-\lambda_n\phi + f_1, \phi_n) = 0, n = 2, 3, \dots$$

However, we know that $f_1(x) = 0$ for $x \in (0, x_1)$ and (\cdot, \cdot) denotes inner product on the interval $(0, x_1)$, so $(f_1, \phi_n) = 0, n = 1, 2, \dots$, then we can get $(\phi, \phi_n) = 0, n = 1, 2, \dots$ indicates that $\phi(x) = 0, x \in (0, x_1)$ for arbitrary initial function $\phi(x)$. That's impossible.

so we can conclude that only case (i) holds true, i.e., $r = r_1$.

Step 2 we will prove $f(x) = f_1(x)$.

We turn back to (1)-(3) and (4)-(6).

Denote $\varphi_n = 2 \int_0^1 \phi(x) \cos(n\pi x) dx$, $f_n = 2 \int_0^1 f(x) \cos(n\pi x) dx$ and $f_{1n} = 2 \int_0^1 f_1(x) \cos(n\pi x) dx$.

By analytic continuation we can get $u(x_0, t) = v(x_0, t)$, $t > 0$ from $u(x_0, t) = v(x_0, t)$, $T > t > 0$. Then

$$\begin{aligned} & \int_0^t \sum_{n=1}^{\infty} E_{(1, 1-\alpha), 2}(-\mu_n s, -rs^{1-\alpha}) f_n ds \cos(n\pi x_0) \\ &= \int_0^t \sum_{n=1}^{\infty} E_{(1, 1-\alpha), 2}(-\mu_n s, -rs^{1-\alpha}) f_{1n} ds \cos(n\pi x_0), \end{aligned}$$

where $\mu_n = n^2\pi^2$. Apply Laplace transform on the above equation, considering the property of $\mathcal{L}(\int_0^t f(\tau) d\tau) = \frac{1}{s} \mathcal{L}(f(t))$ of Laplace transform, then we can get

$$\frac{1}{s} \sum_{n=1}^{\infty} \frac{f_n \cos(n\pi x_0)}{\eta + r\eta^\alpha + \mu_n} = \frac{1}{s} \sum_{n=1}^{\infty} \frac{f_{1n} \cos(n\pi x_0)}{\eta + r\eta^\alpha + \mu_n},$$

which implies

$$\sum_{n=1}^{\infty} \frac{a_n \cos(n\pi x_0)}{z + \mu_n} = 0,$$

where z represents $\eta + r\eta^\alpha$ and $a_n = f_n - f_{1n}$.

We can take a disk which includes μ_1 and does not include $\{n^2\pi^2\}_{n \geq 2}$. By Cauchy integral theorem, integrating along the disk, we have $a_1 \cos(\pi x_0) = 0$, which gives $f_1 = f_{11}$. Repeating this argument, we can obtain

$$f_n = f_{1n}, n = 2, 3, \dots$$

Thus we can conclude that $f(x) = f_1(x)$.

That's end the proof. □

4. Conclusion

In this paper, we consider the two term fractional diffusion equation, which can be used to simulate mobile/immobile diffusion process in porous media. However, the coefficient r and the source term $f(x)$ may be unknown, so determination of these two terms is necessary. We prove the uniqueness result of simultaneously determining coefficient and source term from two points observation, and the proof relies on eigenfunction expansion and the Laplace transform. Actually, it seems easy to determine r or $f(x)$, however, it is difficult to determine the two terms simultaneously since r and $f(x)$ may belong to different function space.

We also need to point out that the result can be extended to multi-term time fractional diffusion equations through the similar technique. In the future work, we may investigate the stability of the inverse problem and the numerical algorithm based on the presented theory.

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Conflict of interest

The authors declare no competing financial interest.

References

- [1] Gelhar LW, Welty C, Rehfeldt KR. A critical review of data on field-scale dispersion in aquifers. *Water Resources Research*. 1992; 28(7): 1955-1974. Available from: <https://doi.org/10.1029/92WR00607>.
- [2] Hatano Y, Hatano N. Dispersive transport of ions in column experiments: an explanation of long-tailed profiles. *Water Resources Research*. 1998; 34: 1027-1033. Available from: <https://doi.org/10.1029/98WR00214>.
- [3] Benson DA, Wheatcraft SW, Meerschaert MM. The fractional advection-dispersion equation: development and application. *Water Resources Research*. 2000; 36(6): 1403-1412. Available from: <https://doi.org/10.1029/2000WR900031>.
- [4] Pachepsky Y, Benson D, Rawls W. Simulating scale-dependent solute transport in soils with the fractional advective-dispersive equation. *Soil Science Society of America Journal*. 2000; 64(4): 1234-1243. Available from: <https://doi.org/10.2136/sssaj2000.6441234x>.
- [5] Berkowitz B, Scher H, Silliman SE. Anomalous transport in laboratory-scale heterogeneous porous media. *Water Resources Research*. 2000; 36: 149-158. Available from: <https://doi.org/10.1029/1999wr900295>.
- [6] Zhou L, Selim HM. Application of the fractional advection-dispersion equation in porous media. *Soil Science Society of America Journal*. 2003; 67: 1079-1084. Available from: <https://doi.org/10.2136/sssaj2003.1079>.

[7] Xiong Y, Huang G, Huang Q. Modeling solute transport in one-dimensional homogeneous and heterogeneous soil columns with continuous time random walk. *Journal of Contaminant Hydrology*. 2006; 86: 163-175. Available from: <https://doi.org/10.1016/j.jconhyd.2006.03.001>.

[8] Gao GH, Sun ZZ. A compact difference scheme for the fractional sub-diffusion equations. *Journal of Computational Physics*. 2011; 230: 586-595. Available from: <https://doi.org/10.1016/j.jcp.2010.10.007>.

[9] Liu F, Zhuang P, Burrage K. Numerical methods and analysis for a class of fractional advection-dispersion models. *Computers & Mathematics with Applications*. 2012; 64: 2990-3007. Available from: <https://doi.org/10.1016/j.camwa.2012.01.020>.

[10] Schere R, Kalla SL, Boyadjiev L. Numerical treatment of fractional heat equations. *Applied Numerical Mathematics*. 2008; 58: 1212-1223. Available from: <https://doi.org/10.1016/j.apnum.2007.06.003>.

[11] Shen S, Liu F, Anh V. Numerical approximations and solution techniques for the space-time Riesz-Caputo fractional advection-diffusion equation. *Numerical Algorithms*. 2011; 56: 383-404. Available from: <https://doi.org/10.1007/s11075-010-9393-x>.

[12] Zhang Y. A finite difference method for fractional partial differential equation. *Applied Mathematics and Computation*. 2009; 215: 524-529. Available from: <https://doi.org/10.1016/j.amc.2009.05.018>.

[13] Yuste SB. Weighted average finite difference methods for fractional diffusion equations. *Journal of Computational Physics*. 2006; 216: 264-274. Available from: <https://doi.org/10.1016/j.jcp.2005.12.006>.

[14] Shen S, Liu F, Anh V. Fundamental solution and discrete random walk model for time-space fractional diffusion equation. *Journal of Applied Mathematics & Computing*. 2008; 28: 147-164. Available from: <https://doi.org/10.1007/s12190-008-0084-x>.

[15] Shen S, Liu F, Anh V. The fundamental solution and numerical solution of the Riesz fractional advection dispersion equation. *IMA Journal of Applied Mathematics*. 2008; 73: 850-872. Available from: <https://doi.org/10.1093/imamat/hxn033>.

[16] Yang Q, Liu F, Turner I. Numerical methods for fractional partial differential equations with Riesz space fractional derivatives. *Applied Mathematical Modelling*. 2010; 34: 200-218. Available from: <https://doi.org/10.1016/j.apm.2009.04.006>.

[17] Yang Q, Turner I, Liu F, Ilić M. Novel numerical methods for solving the time-space fractional diffusion equation in two dimensions. *SIAM Journal on Scientific Computing*. 2011; 33: 1159-1180. Available from: <https://doi.org/10.1137/100800634>.

[18] Zhuang P, Liu F, Anh V, Turner I. New solution and analytical techniques of the implicit numerical method for the anomalous subdiffusion equation. *SIAM Journal on Numerical Analysis*. 2008; 46: 1079-1095. Available from: <https://doi.org/10.1137/060673114>.

[19] Zhuang P, Liu L, Anh V, Turner I. Numerical methods for the variable-order fractional advection-diffusion with a nonlinear source term. *SIAM Journal on Numerical Analysis*. 2009; 47: 1760-1781. Available from: <https://doi.org/10.1137/080730597>.

[20] Cheng J, Nakagawa J, Yamamoto M. Uniqueness in an inverse problem for a one-dimensional fractional diffusion equation. *Inverse Problems*. 2009; 25(11): 115002. Available from: <https://doi.org/10.1088/0266-5611/25/11/115002>.

[21] Bondarenko AN, Ivaschenko DS. Generalized sommerfeld problem for time fractional diffusion equation: analytical and numerical approach. *Journal of Inverse and Ill-posed Problems*. 2009; 17(4): 321-335. Available from: <https://doi.org/10.1515/JIIP.2009.022>.

[22] Bondarenko AN, Ivaschenko DS. Numerical methods for solving inverse problems for time fractional diffusion equation with variable coefficient. *Journal of Inverse and Ill-posed Problems*. 2009; 17(7): 419-440. Available from: <https://doi.org/10.1515/JIIP.2009.028>.

[23] Luchko Y. Boundary value problems for the generalized time-fractional diffusion equation of distributed order. *Fractional Calculus and Applied Analysis*. 2009; 12: 409-422. Available from: <https://doi.org/10.1016/j.camwa.2009.08.015>.

[24] Luchko Y. Maximum principle for the generalized time-fractional diffusion equation. *Journal of Mathematical Analysis and Applications*. 2009; 351(1): 218-223. Available from: <https://doi.org/10.1016/j.jmaa.2008.10.018>.

[25] Luchko Y. Initial-boundary-value problems for the generalized multi-term time-fractional diffusion equation. *Journal of Mathematical Analysis and Applications*. 2011; 374(2): 538-548. Available from: <https://doi.org/10.1016/j.jmaa.2010.08.048>.

[26] Mainardi F. The fundamental solutions for the fractional diffusion-wave equation. *Applied Mathematics Letters*. 1996; 9(6): 23-28. Available from: [https://doi.org/10.1016/0893-9659\(96\)00089-4](https://doi.org/10.1016/0893-9659(96)00089-4).

[27] Salim TO, El Kahlout A. Analytical solution of time-fractional advection dispersion equation. *Applications and Applied Mathematics*. 2009; 4(1): 176-188. Available from: <https://doi.org/10.1016/j.aej.2020.10.043>.

[28] Sakamoto K, Yamamoto M. Inverse source problem with a final overdetermination for a fractional diffusion equation. *Mathematical Control and Related Fields*. 2011; 1(4): 509-518. Available from: <https://doi.org/10.3934/mcrf.2011.1.509>.

[29] Zhang Y, Xu X. Inverse source problem for a fractional diffusion equation. *Inverse Problems*. 2011; 27(3): 035010. Available from: <https://doi.org/10.1088/0266-5611/27/3/035010>.

[30] Xu X, Cheng J, Yamamoto M. Carleman estimate for a fractional diffusion equation with half order and application. *Applicable Analysis*. 2011; 90(9): 1355-1371. Available from: <https://doi.org/10.1080/00036811.2010.507199>.

[31] Jiang H, Liu F, Turner I, Burrage K. Analytical solutions for the multi-term time-space caputo-riesz fractional advection-diffusion equations on a finite domain. *Journal of Mathematical Analysis and Applications*. 2012; 389: 1117-1127. Available from: <https://doi.org/10.1016/j.jmaa.2011.12.055>.

[32] Daftardar-Gejji V, Bhalekar S. Boundary value problems for multi-term fractional differential equations. *Journal of Mathematical Analysis and Applications*. 2008; 345: 754-765. Available from: <https://doi.org/10.1016/j.jmaa.2008.04.065>.

[33] Li ZY, Yamamoto M. Initial-boundary value problems for linear diffusion equation with multiple time-fractional derivatives. *arXiv:1306.2778v2*. 2013. Available from: <https://doi.org/10.48550/arXiv.1306.2778>.

[34] Luchko Y. Some uniqueness and existence results for the initial-boundary-value problems for the generalized time-fractional diffusion equation. *Computers and Mathematics with Applications*. 2010; 59: 1766-1772. Available from: <https://doi.org/10.1016/j.camwa.2009.08.015>.

[35] Beckers S, Yamamoto M. Regularity and unique existence of solution to linear diffusion equation with multiple time-fractional derivatives. In: Bredies K, Clason C, Kunisch K, Von Winckel G. (eds.) *Control and Optimization with PDE Constraints*. Basel: Birkhäuser; 2013.

[36] Li ZY, Yamamoto M. Uniqueness for inverse problems of determining orders of multi-term time-fractional derivatives of diffusion equation. *Applicable Analysis*. 2015; 94(3): 570-579. Available from: <https://doi.org/10.1080/00036811.2014.926335>.

[37] Baranovskii ES, Brizitskii RV, Saritskaia ZY. Optimal control problems for the reaction-diffusion convection equation with variable coefficients. *Nonlinear Analysis: Real World Applications*. 2024; 75: 103979. Available from: <https://doi.org/10.1016/j.nonrwa.2023.103979>.

[38] Schumer R, Benson DA, Meerschaert MM, Baeumer B. Fractal mobile/immobile solute transport. *Water Resources Research*. 2003; 39: 1296-1307. Available from: <https://doi.org/10.1029/2003WR002141>.

[39] Li ZY, Liu YK, Yamamoto M. Initial-boundary value problems for multi-term time-fractional diffusion equations with positive constant coefficients. *Applied Mathematics and Computation*. 2015; 257: 381-397. Available from: <https://doi.org/10.1016/j.amc.2014.11.073>.