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Research Article

Existence and Stability of Nonlinear Langevin FDEs with Mixed ψ-Hilfer and ψ-Caputo Derivatives Under Nonlocal Boundary Conditions

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Abstract: In this paper, we investigate the existence, uniqueness, and Hyers-Ulam stability of a nonlinear fractional Langevin equation involving a generalized fractional derivative that unifies the ψ-Hilfer and ψ-Caputo types. The problem incorporates mixed and nonlocal boundary conditions. Existence results are derived using Krasnoselskii's fixed-point theorem, while uniqueness is established through Banach's contraction principle. Moreover, Ulam-type stability is analyzed under both Ulam-Hyers and Ulam-Hyers-Rassias criteria. Finally, a numerical example is presented, and the computed results are summarized in a table, showing the influence of different ψ-functions on the stability constants and confirming the consistency between theoretical and numerical findings.

Keywords: ψ-Hilfer derivative, ψ-Caputo derivative, langevin equation, nonlocal boundary condition, fixed point theorems, Ulam stability

MSC: 34A08, 34A12, 47H10

Abbreviation

NLFDEs	Nonlinear Langevin Fractional Differential Equations
UHS	Ulam-Hyers Stability
UHRS	Ulam-Hyers-Rassias Stability

1. Introduction

Fractional calculus, which is based on fractional derivatives, is a relatively young but rapidly developing field of study. It has a wider applicability both in time and in space compared to traditional integer-order calculus, allowing a better description of real-world problems and playing an increasingly important role in modeling complex processes in science and engineering [1]. Its applications extend to diverse disciplines such as physics [2], epidemiology [3], virology

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[4], ecology [5], and economics [6]. From a theoretical perspective, new classes of fractional derivatives that involve an auxiliary function have been developed as generalizations of the classical Riemann-Liouville derivative [7–9]. These fractional derivatives differ from traditional ones because their integration kernels are expressed in terms of another function, denoted by ψ; hence, they are called ψ-fractional derivatives. Recently, Almeida [10, 11] introduced a Caputo-type regularization for these operators and established several important analytical properties, providing a solid foundation for ψ-fractional calculus.

The Langevin equation was first introduced by Paul Langevin, a prominent French physicist, in 1908 [12], as a nonlinear differential equation providing a precise mathematical description of Brownian motion. Since then, it has been widely applied to model diverse physical phenomena, including oscillators, financial market dynamics [13], evacuation processes [14], particle suspensions in fluids [15], self-organization in complex systems [16], photon-induced electron counting [17], and protein dynamics [18]. Other researchers have extended its applications to polymer rheology and stochastic simulations. For instance, Lim et al. [19] later generalized the Langevin equation to fractional orders, introducing two distinct fractional derivatives to describe viscoelastic anomalous diffusion in complex fluids, an approach that inspired further studies on fractional Langevin models in polymer rheology, viscoelasticity, and stochastic systems (see [13, 20–23]).

In our previous work [24], we investigated Nonlinear Langevin Fractional Differential Equations (NLFDEs) with mixed nonlocal boundary conditions, comprising multi-point, fractional integral, and fractional derivative types, using the Caputo exponential operator. Related studies have also extended the Langevin framework: Almaghamsi et al. [25] established the existence of solutions for a Langevin-type equation involving the ψ-Hilfer fractional derivative via a variational approach based on the mountain pass theorem; Boutiara et al. [26] analyzed Langevin equations with Caputo function-dependent-kernel fractional derivatives under antiperiodic boundary conditions; and Ali and Abdo [27] provided a qualitative study of multiterm Langevin systems governed by generalized Caputo fractional operators of different orders.

The stability analysis of differential equations plays a crucial role in understanding how solutions respond to small perturbations in initial conditions. The concept was first proposed by Ulam in 1940 [28] and later refined by Hyers and Rassias, leading to what are now known as Ulam-Hyers Stability (UHS) and Ulam-Hyers-Rassias Stability (UHRS); see [29–32] for related developments.

In the context, Chen et al. [30], studied the existence and Ulam's type stability of higher-order NLFDEs:

$$\begin{cases} {}^{C}D^{r}_{0,\,t}(D^2+\kappa_i)v_i(t)=\phi_i(t,\,v_i(t),\,{}^{C}D^{\theta}_{0,\,t}v_i(t)),\, t\in[0,\,\tau_i],\ \ i=1,\,2,\,\dots,\,k,\\[2ex] v_i'(0)=v_i(\tau_i)=0,\ \ \sum_{i=1}^{k}v_i''(\tau_i)=0,\ \ i=1,\,2,\,\dots,\,k,\\[2ex] v_i''(\tau_i)=v_j''(\tau_j),\, i,\, j=1,\,2,\,\dots,\,k,\ \ i\neq j,\end{cases}$$

where $r\in(0,\,1)$ and $\theta\in(0,\,r)$, $\lambda_i\in\mathbb{R}^+$, D^2 is the ordinary second-order derivative, $\phi_i\in C([0,\,\tau_i]\times\mathbb{R},\,\mathbb{R})$, ${}^{C}D^{r}_{0}$ and ${}^{C}D^{\theta}_{0}$ denote the Caputo fractional derivatives.

Recently, in 2025, Ullah Khan et al. [33], proposed a new variant of the Langevin equation using generalized Liouville-Caputo derivatives and studied its existence, uniqueness, and stability of the solution to the problem:

$$\begin{cases} {}^{C}D^{r^\star,\,\beta}_{b^+}({}^{C}D^{\gamma,\,\beta}_{b^+}+\kappa)v(t)=\phi(t,\,v(t),\,v(\kappa t),\,{}^{C}D^{\gamma,\,\beta}_{b^+}v(t)),\ \ t\in[b,\,d],\ \ \kappa\in(0,\,1),\\[2ex] v(b)=0,\ \ v(\theta)=0,\ \ v(t)=\rho I^{\delta,\,\beta}_{b^+}v(\zeta),\ \ 0<\theta<\zeta<d,\end{cases}$$

where the Liouville-Caputo-type generalized fractional derivatives of orders $r^\star\in(1,\,2]$, $\gamma\in(0,\,1)$, and $\beta>0$ are denoted by ${}^{C}D^{r^\star,\,\beta}_{b^+}$ and ${}^{C}D^{\gamma,\,\beta}_{b^+}$, respectively. The operator $I^{\delta,\,\beta}_{b^+}$ represents the corresponding generalized fractional integral of order δ with $\beta>0$, and ϕ: $[b,\,d]\times\mathbb{R}^3\to\mathbb{R}$ is a given continuous function.

In 2025, Dhaniya et al. [34], investigated the existence, uniqueness, and stability of solutions for the Langevin equation involving the ϕ-Caputo fractional operator with nonlocal boundary conditions:

$$\begin{cases} {}^{c}D_{a^+}^{\beta_1,\,\psi}({}^{c}D_{a^+}^{\beta_2,\,\psi}+\kappa)v(t)=\phi(t,\,v(t)),\ t\in S:=[a,\,T],\\ v(a)=0,\ v'(a)=0,\ {}^{c}D_{a^+}^{\beta_2,\,\psi}v(T)+\lambda v(T)=0, \end{cases}$$

where ${}^{c}D_{a^+}^{\alpha,\,\psi}$ denotes the ψ-Caputo fractional derivative of order $\alpha\in\{\beta_1,\,\beta_2\}$ such that $0<\beta_1\le 1$, $1<\beta_2\le 2$, ϕ: $S\times\mathbb{R}\longrightarrow\mathbb{R}$ is a continuous function, and κ, $\lambda\in\mathbb{R}$.

In this paper, we study a NLFDE that incorporates both ψ-Hilfer and ψ-Caputo fractional derivatives, subject to nonlocal mixed boundary conditions:

$$\begin{cases} {}^{H}D_{b}^{r,\,\beta;\,\psi}({}^{C}D_{b}^{\gamma;\,\psi}+\kappa)v(t)=\phi\big(t,\,v(t),\,I_b^{\delta;\,\psi}v(t),\,{}^{C}D_b^{\theta;\,\psi}v(t)\big),\quad t\in[b,\,T],\ T>0, & (1)\\ v(b)=0,\ \rho I_b^{q_1;\,\psi}v(\eta)+\xi\,{}^{C}D_b^{q_2;\,\psi}v(T)=0, & (2) \end{cases}$$

where $1<r<2$, $0<q_2<\theta<\gamma<\beta<1$, δ, $q_1>0$, ${}^{H}D_b^{r,\,\beta;\,\psi}$, ${}^{C}D_b^{i;\,\psi}$ denote the ψ-Hilfer and ψ-Caputo fractional derivative operators of orders r, $i\in\{\gamma,\,\theta,\,q_2\}$, respectively, $I_b^{j;\,\psi}$ is the Riemann-Liouville fractional integral operator of order $j\in\{\delta,\,q_1\}$, ψ: $[b,\,T]\to\mathbb{R}$ is an increasing function satisfy $\psi'(t)>0$, $\forall t\in[b,\,T]$, $\phi\in C([b,\,T]\times\mathbb{R}^3,\,\mathbb{R})$, ρ, ξ, $\kappa\in\mathbb{R}^+$, and $\eta\in(b,\,T)$.

This work formulates the problem (1)-(2) involving a mixed ψ-Hilfer and ψ-Caputo derivative under nonlocal mixed boundary conditions. Existence of solutions is established using Krasnoselskii's fixed-point theorem, and uniqueness is proven via Banach's contraction principle. Stability analysis, including both UHS and UHRS stability, is investigated for the proposed system. Finally, a numerical example illustrates the theoretical results, with comparisons of different ψ-functions demonstrating the influence of fractional operators and boundary parameters on solution behavior, confirming the accuracy of the analytical findings.

The structure of the paper is as follows: Section 2 presents essential preliminaries, including fundamental definitions, lemmas, and theorems. In Section 3, the existence and uniqueness of solutions are established using Krasnoselskii's and Banach's fixed point theorems. Moreover, we discuss the stability of solutions in the frameworks of UHS and UHRS. Section 4 provides an illustrative example to justify the theoretical findings, and the final section summarizes the main contributions and conclusions.

2. Preliminaries

Here, we provide several definitions, lemmas, and key theorems that will be utilized in the subsequent analysis.

Definition 1 ([8, 25]) For $r>0$, the ψ-Riemann-Liouville fractional integral of order α of an integrable function v: $[b,\,T]\longrightarrow\mathbb{R}$ with respect to a function ψ: $[b,\,T]\longrightarrow\mathbb{R}$, which is continuously differentiable and strictly increasing with $\psi'(t)\neq 0$ for all $t\in[b,\,T]$, is defined by

$$I_{b^+}^{r;\,\psi}v(t)=\frac{1}{\Gamma(r)}\int_b^t\psi'(s)(\psi(t)-\psi(s))^{r-1}v(s)ds,$$

where Γ is the gamma function.

Definition 2 ([8, 25]) Let $n \in \mathbb{N}$, and let $v,\ \psi \in C^n([b,\, T],\, \mathbb{R})$ be two functions with v increasing and $\psi'(t) \neq 0$ for all $t \in [b,\, T]$. The left-sided ψ-Riemann-Liouville fractional derivative of v of order r is defined as follows

$$D_{b^+}^{r;\,\psi} v(t) = \left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n I_{b^+}^{n-r;\,\psi} v(t)$$

$$= \frac{1}{\Gamma(n-r)}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n \int_b^t \psi'(s)(\psi(t)-\psi(s))^{n-r-1} v(s)ds,$$

where $n = [r] + 1$.

Definition 3 ([35]) Let $\gamma = r + \beta(n-r)$, where $r \in (n-1,\, n)$ and $n \in \mathbb{N}$. Consider $v \in C^n([b,\, T],\, \mathbb{R})$ and an increasing function $\psi \in C^1([b,\, T],\, \mathbb{R})$ with $\psi'(t) \neq 0$ for all $t \in [b,\, T]$. The ψ-Hilfer fractional derivative of order r and type $\beta \in [0,\, 1]$ of v, with respect to ψ, is then defined by:

$${}^{H}D_{b^+}^{r,\,\beta;\,\psi} v(t) = I_{b^+}^{\beta(n-r);\,\psi}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n I_{b^+}^{(1-\beta)(n-r);\,\psi} v(t) = I_{b^+}^{\gamma-r;\,\psi} D_{b^+}^{\gamma;\,\psi} v(t),$$

where $D_{b^+}^{\gamma;\,\psi} v(t) = D_{b^+}^{n;\,\psi} I_{b^+}^{(1-\beta)(n-r);\,\psi} v(t)$.

Definition 4 ([11]) Let $n \in \mathbb{N}$, and let $\psi \in C^n([b,\, T],\, \mathbb{R})$ be an increasing function with $\psi'(t) \neq 0$ for all $t \in [b,\, T]$. Suppose $v \in C^{n-1}([b,\, T],\, \mathbb{R})$. Then, the ψ-Caputo fractional derivative of v of order r is defined as follows:

$${}^{C}D_{b^+}^{r;\,\psi} v(t) = D_{b^+}^{\alpha;\,\psi}\left[v(t) - \sum_{k=0}^{n-1} \frac{v_{\psi}^{[k]}(b)}{k!}(\psi(t)-\psi(b))^k\right],$$

where

$$n = [r] + 1 \text{ for } r \notin \mathbb{N},\ \ n = r \text{ for } r \in \mathbb{N},$$

and

$$v_{\psi}^{[n]}(t) = \left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n v(t).$$

If $v \in C^n[b,\, T]$, then

$${}^{C}D_{b^+}^{r;\,\psi} v(t) = I_{b^+}^{n-r;\,\psi}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n v(t).$$

Thus,

$$
{}^{C}D_{b^+}^{r;\,\psi}v(t)=\begin{cases}\displaystyle\int_b^t\frac{\psi'(s)(\psi(t)-\psi(s))^{n-r-1}}{\Gamma(n-r)}v_\psi^{[n]}(s)ds, & \text{if } r\notin\mathbb{N},\\ v_\psi^{[n]}, & \text{if } r\in\mathbb{N}.\end{cases}
$$

Lemma 1 ([11]) Let $v,\ \psi$: $[b,\ T]\longrightarrow\mathbb{R}$ and $r>0$, the following properties hold:

i) If $v\in C[b,\ T]$, then ${}^{C}D_{b^+}^{r;\,\psi}I_{b^+}^{r;\,\psi}v(t)=v(t)$.

ii) If $\psi,\ v\in C^{n-1}[b,\ T]$, then $I_{b^+}^{r;\,\psi C}D_{b^+}^{r;\,\psi}v(t)=v(t)-\sum_{k=0}^{n-1}\frac{v_\psi^{[k]}(b)}{k!}(\psi(t)-\psi(b))^k$.

Lemma 2 ([8]) Let $r>0$ and $v,\ \psi\in C([b,\ T])$. Then,

$$
\|I_{b^+,\,t}^{r;\,\psi}[v]\|_C\leq K_{r,\,1}^{\psi}(T)\|v\|_C,\ \ K_{r,\,1}^{\psi}(T)=\frac{1}{\Gamma(r+1)}(\psi(T)-\psi(b))^r.
$$

For all $n-1<r<n$,

$$
\|{}^{C}D_{b^+,\,t}^{r;\,\psi}[v]\|_C\leq K_{n-r,\,1}^{\psi}(T)\|v\|_{C^{[n]}},\ \ K_{n-r,\,1}^{\psi}(T)=\frac{1}{\Gamma(n-r+1)}(\psi(T)-\psi(b))^{n-r},
$$

where $\|.\|_C$, $\|.\|_{C^{[n]}}$ are the norms on $C([b,\ T])$, $C^{[n]}([b,\ T])$ respectively.

Lemma 3 ([8]) Let $r,\ \beta>0$, we have

$$
I_{b^+,\,t}^{r;\,\psi}(\psi(s)-\psi(b))^{\beta-1}=\frac{1}{\Gamma(r+\beta)}(\psi(t)-\psi(b))^{r+\beta-1},
$$

and

$$
D_{b^+,\,t}^{r;\,\psi}(\psi(s)-\psi(b))^{\beta-1}=\frac{1}{\Gamma(\beta-r)}(\psi(t)-\psi(b))^{\beta-r-1},
$$

Moreover, $D_{b^+,\,t}^{r;\,\psi}(\psi(s)-\psi(b))^k=0,\ $ for $k=0,\ \ldots,\ n-1$.

Theorem 1 ([36]) Let $\mathfrak{D}$ be a non-empty closed subset of a Banach space $\mathscr{V}$. Then any contraction mapping Π from $\mathfrak{D}$ into itself has a unique fixed point in $\mathscr{V}$.

Theorem 2 ([37]) Let $\mathscr{S}$ be a closed, bounded, convex, and nonempty subset of a Banach space $\mathscr{V}$. Let Π_1, Π_2 be the operators such that

1. $\Pi_1u+\Pi_2v\in\mathscr{S}$ for all $u,\ v\in\mathscr{S}$
2. Π_1 is compact and continuous.
3. Π_2 is a contraction mapping. Then, there exists $w\in\mathscr{S}$ such that $w=\Pi_1u+\Pi_2v$.

3. Main results

This section presents the results of the existence, uniqueness, and stability results to the problem (1)-(2).

Definition 5 A function $v \in C^2([b, T], \mathbb{R})$ is said to be a solution of the problem (1)-(2) if v satisfies the equation ${}^{H}D_b^{r,\,\beta;\,\psi}({}^{C}D_b^{\beta,\,\psi}+\kappa)v(t) = \phi(t, v(t), I_b^{\beta;\,\psi}v(t), {}^{C}D_b^{\beta;\,\psi}v(t))$ and the conditions $v(b) = 0,\ \kappa I_b^{q_1;\,\psi}v(\eta)+\xi\,{}^{C}D_b^{q_2;\,\psi}v(T) = 0$ on $[b, T]$.

Lemma 4 Let $1 < r < 2$, $0 < q_2 < \gamma < \beta < 1$, $q_1 > 0$, $\vartheta = r+\beta(1-r)$ and let $h \in C([b, T], \mathbb{R})$. The unique solution of the linear problem

$$\begin{cases} {}^{H}D_b^{r,\,\beta;\,\psi}({}^{C}D_b^{\gamma,\,\psi}+\kappa)v(t) = h(t),\ t \in [b, T], & (3) \\ v(b) = 0,\ \rho I_b^{q_1;\,\psi}v(\eta)+\xi\,{}^{C}D_b^{q_2;\,\psi}v(T) = 0, & (4) \end{cases}$$

is given by

$$\begin{aligned} v(t) = I_b^{r+\gamma;\,\psi}h(t)-\kappa I_b^{\gamma;\,\psi}v(t)+\frac{K_{\gamma,\,\vartheta}^{\psi}(t)}{\Xi}\Big[\rho\kappa I_b^{\gamma+q_1;\,\psi}v(\eta)+\xi\lambda I_b^{\gamma-q_2;\,\psi}v(T) \\ -\rho I_b^{r+\gamma+q_1;\,\psi}h(\eta)-\xi I_b^{r+\gamma-q_2;\,\psi}h(T)\Big], \end{aligned} \tag{5}$$

where $\Xi = \rho K_{\gamma+q_1,\,\vartheta}^{\psi}(\eta)+\xi K_{\gamma-q_2,\,\vartheta}^{\psi}(T) \neq 0$ and $K_{\gamma,\,\vartheta}^{\psi}(t) = \dfrac{\Gamma(\vartheta)}{\Gamma(\gamma+\vartheta)}(\psi(t)-\psi(b))^{\vartheta+\gamma-1}$.

Proof. Since $1 < r < 2$ and $0 < \gamma < 1$, the derivative ${}^{H}D_b^{r,\,\beta;\,\psi}({}^{C}D_b^{\gamma,\,\psi}+\kappa)v(t)$ is well-defined and applies to a twice-differentiable function, the order sequence $0 < q_2 < \theta < \gamma < \beta < 1$ ensures that the derivatives ${}^{C}D_b^{\gamma;\,\psi}$, ${}^{C}D_b^{\theta;\,\psi}$, ${}^{C}D_b^{q_2;\,\psi}$ are well-defined and Gamma functions remain finite, and we use the conditions $\delta > 0$, $q_1 > 0$ to ensure that the operators $I_b^{\delta;\,\psi}$, $I_b^{q_1;\,\psi}$ are well-defined on the space $C([b, T])$.

By applying $I_b^{r;\,\psi}$ to both sides of (3) and then using Lemma 1, we have

$$({}^{C}D_b^{\gamma;\,\psi}+\kappa)v(t) = I_b^{r}h(t)+c_1(\psi(t)-\psi(b))^{\vartheta-1}+c_2(\psi(t)-\psi(b))^{\vartheta-2}, \tag{6}$$

or

$${}^{C}D_b^{\gamma;\,\psi}v(t) = I_b^{r}h(t)-\kappa v(t)+c_1(\psi(t)-\psi(b))^{\vartheta-1}+c_2(\psi(t)-\psi(b))^{\vartheta-2}. \tag{7}$$

Once more, applying $I_b^{\gamma;\,\psi}$ to both sides of (7), then using Lemma 1 and Lemma 3, we get

$$v(t) = I_b^{r+\gamma;\,\psi}h(t)-\kappa I_b^{\gamma;\,\psi}v(t)+\frac{c_1\Gamma(\vartheta)}{\Gamma(\vartheta+\gamma)}(\psi(t)-\psi(b))^{\vartheta+\gamma-1}+\frac{c_2\Gamma(\vartheta-1)}{\Gamma(\vartheta+\gamma-1)}(\psi(t)-\psi(b))^{\vartheta+\gamma-2}+c_3, \tag{8}$$

where $c_1, c_2, c_3 \in \mathbb{R}$. Using the first condition of (4), we find that $c_2 = c_3 = 0$ and from the second condition, we find that

$$c_1 = \frac{1}{\Xi}\Big[-\rho I_b^{r+\gamma+q_1;\,\psi} h(\eta) - \xi I_b^{r+\gamma-q_2;\,\psi} h(T) + \rho\kappa I_b^{\gamma+q_1;\,\psi} v(\eta) + \xi\kappa I_b^{\gamma-q_2;\,\psi} v(T)\Big],$$

where $\Xi = \rho K^{\psi}_{\gamma+q_1,\,\vartheta}(\eta) + \xi K^{\psi}_{\gamma-q_2,\,\vartheta}(T) \neq 0$.

Finally, replacing these values of c_1, c_2 and c_3 in (8), we obtain the solution (5).

Conversely, assume that v satisfies the equation (5), by applying the two operators ${}^{H}D_b^{r,\,\beta;\,\psi}$ and ${}^{C}D_b^{\gamma;\,\psi}$, to the equation (5), we obtain (3), we simultaneously verify that the condition (4) is satisfied. □

Let $C([b,\,T],\,\mathbb{R})$ be a Banach space of all continuous functions defined on $[b,\,T]$ endowed with the usual supremum norm. Define the Banach space

$$\mathscr{V} = \{v\colon\ v \in C^2([b,\,T],\,\mathbb{R}),\ {}^{C}D_b^{\theta} v \in C([b,\,T],\,\mathbb{R}),\ 0 < \theta < 1\},$$

equipped with the norm

$$\|v\|_{\mathscr{V}} = \max\{|v(t)|\colon t \in [b,\,T]\} + \max\{|{}^{C}D_b^{\theta} v(t)|\colon t \in [b,\,T]\}.$$

To study the fixed point problem associated with problem (1)-(2), we define the integral operator $\Pi\colon \mathscr{V} \longrightarrow \mathscr{V}$ as follows:

$$\begin{aligned}(\Pi v)(t) =& I_b^{r+\gamma;\,\psi}\phi(t,\,v(t),\,I_b^{\delta;\,\psi}v(t),\,{}^{C}D_b^{\theta;\,\psi}v(t)) - \kappa I_b^{\gamma;\,\psi}v(t) + \frac{K^{\psi}_{\gamma,\,\vartheta}(t)}{\Xi}\Big[\rho\kappa I_a^{\gamma+q_1;\,\psi}v(\eta)\\ &+ \xi\kappa I_b^{\gamma-q_2;\,\psi}v(T) - \rho I_b^{r+\gamma+q_1;\,\psi}\phi(\eta,\,v(\eta),\,I_b^{\delta;\,\psi}v(\eta),\,{}^{C}D_b^{\theta;\,\psi}v(\eta))\\ &- \xi I_b^{r+\gamma-q_2;\,\psi}\phi(T,\,v(T),\,I_b^{\delta;\,\psi}v(T),\,{}^{C}D_b^{\theta;\,\psi}v(T))\Big],\end{aligned} \tag{9}$$

where $\Xi = \rho K^{\psi}_{\gamma+q_1,\,\vartheta}(\eta) + \xi K^{\psi}_{\gamma-q_2,\,\vartheta}(T) \neq 0$ and $K^{\psi}_{\gamma,\,\vartheta}(t) = \dfrac{\Gamma(\vartheta)}{\Gamma(\gamma+\vartheta)}(\psi(t)-\psi(b))^{\vartheta+\gamma-1}$.

3.1 *Uniqueness result via Banach fixed point theorem*

Accordingly, we assume the following conditions:

(C1) $\phi\colon [b,\,T] \times \mathbb{R}^3 \longrightarrow \mathbb{R}$ is a continuous function.

(C2) There exists a constant $L \in \mathbb{R}_+$ such that

$$|\phi(t,\,u,\,v,\,w) - \phi(t,\,\overline{u},\,\overline{v},\,\overline{w})| \leq L(|u-\overline{u}| + |v-\overline{v}| + |w-\overline{w}|),$$

for any u, v, w, $\overline{u}$, $\overline{v}$ and $\overline{w} \in \mathbb{R}$, for a.e., $t \in [b,\,T]$.

For simplicity and brevity, we use the following notations:

$$\rho = 1 + \frac{(\rho + \xi)}{|\Xi|} K^{\psi}_{\gamma,\,\vartheta}(T), \tag{10}$$

$$\tilde{\rho} = 1 + \frac{(\rho + \xi)}{|\Xi|} K^{\psi}_{\gamma-\theta,\,\vartheta}(T), \tag{11}$$

and

$$\Delta = \max\left\{L\Lambda(\Omega_{r+\gamma} + \tilde{\Omega}_{r+\gamma}) + \kappa(\Omega_{\gamma} + \tilde{\Omega}_{\gamma}),\ L(\Omega_{r+\gamma} + \tilde{\Omega}_{r+\gamma})\right\}$$

$$= L\Lambda(\Omega_{r+\gamma} + \tilde{\Omega}_{r+\gamma}) + \kappa(\Omega_{\gamma} + \tilde{\Omega}_{\gamma}), \tag{12}$$

where

$$\Omega_{\omega} = K^{\psi}_{\omega,\,1}(T) + \frac{K^{\psi}_{\gamma,\,\vartheta}(T)}{|\Xi|}\Big(\rho K^{\psi}_{\omega+q_1,\,1}(T) + \xi K^{\psi}_{\omega-q_2,\,1}(T)\Big), \tag{13}$$

$$\tilde{\Omega}_{\omega} = K^{\psi}_{\omega-\theta,\,1}(T) + \frac{K^{\psi}_{\gamma-\theta,\,\vartheta}(T)}{|\Xi|}\Big(\rho K^{\psi}_{\omega+q_1,\,1}(T) + \xi K^{\psi}_{\omega-q_2,\,1}(T)\Big),\ \omega \in \{\gamma,\, r+\gamma\}, \tag{14}$$

and

$$\Lambda = 1 + K^{\psi}_{\delta,\,1}(T). \tag{15}$$

Theorem 3 Assume that the hypotheses (C1), (C2) are satisfied and the condition

$$\Delta < 1, \tag{16}$$

then problem (1)-(2) has a unique solution in the space $\mathscr{V}$.

Proof. Let Π: $\mathscr{V} \longrightarrow \mathscr{V}$ be an operator defined by (9). The fixed points of this operator are necessarily the solutions of problem (1)-(2). Setting $M_1 = \sup_{t\in[b,\,T]} |\phi(t,\, 0,\, 0)|$ and choosing:

$$R \geq \frac{(\Omega_{r+\gamma} + \tilde{\Omega}_{r+\gamma})M_1}{1-\Delta}.$$

Note that $\mathfrak{B}_R = \{v \in \mathscr{V}\colon \|v\| \leq R\}$ is bounded, closed, and convex subset of $\mathscr{V}$. The proof will be presented in two steps:

Step 1: $\Pi\mathfrak{B}_R \subset \mathfrak{B}_R$. For $v \in \mathfrak{B}_R$, we have from (C1), (C2) and Lemma 2 that

$$\begin{aligned}|\phi(t,v(t),I_b^{\delta;\psi}v(t),{}^CD_b^{\theta;\psi}v(t))| \leq & |\phi(t,v(t),I_b^{\delta;\psi}v(t),{}^CD_b^{\theta;\psi}v(t))-\phi(t,0,0,0)|+|\phi(t,0,0,0)| \\ \leq & L(|v(t)|+|I_b^{\delta;\psi}v(t)|+|{}^CD_b^{\theta;\psi}v(t)|)+M_1 \\ \leq & L(\|v\|+K_{\delta,1}^{\psi}(T)\|v\|+\|{}^CD_b^{\theta;\psi}v\|)+M_1 \\ \leq & L\Lambda\|v\|+L\|{}^CD_b^{\theta;\psi}v\|+M_1, \end{aligned} \tag{17}$$

where Λ is given by (15).

Substituting (17) into (9), we find

$$\begin{aligned}|(\Pi v)(t)| \leq & (I_b^{r+\gamma;\psi})|\phi(t,v(t),I_b^{\delta;\psi}v(t),{}^CD_b^{\theta;\psi}v(t))|+\kappa(I_b^{\gamma;\psi})|v(t)|+\frac{K_{\gamma,\vartheta}^{\psi}(t)}{|\Xi|}\Big[\rho(I_b^{r+\gamma+q_1;\psi}) \\ & \times|\phi(\eta,v(\eta),I_b^{\delta;\psi}v(\eta),{}^CD_b^{\theta;\psi}v(\eta))|+\xi(I_b^{r+\gamma-q_2})|\phi(T,v(T),I_b^{\delta;\psi}v(T),{}^CD_b^{\theta;\psi}v(T))| \\ & +\rho\kappa(I_b^{\gamma+q_1;\psi})|v(\eta)|+\xi\kappa(I_b^{\gamma-q_2;\psi})|v(T)|\Big] \\ \leq & (K_{r+\gamma,1}^{\psi}(T))(L\|v\|+LK_{\delta,1}^{\psi}(T)\|v\|+L\|{}^CD_b^{\theta;\psi}v\|+M_1)+\kappa(K_{\gamma,1}^{\psi}(T))\|v\| \\ & +\frac{K_{\gamma,\vartheta}^{\psi}(t)}{|\Xi|}\Big[\rho(K_{r+\gamma+q_1,1}^{\psi}(T))(L\|v\|+LK_{\delta,1}^{\psi}(T)\|v\|+L\|{}^CD_b^{\theta;\psi}v\|+M_1) \\ & +\xi(K_{r+\gamma-q_2,1}^{\psi}(T))(L\|v\|+LK_{\delta,1}^{\psi}(T)\|v\|+L\|{}^CD_b^{\theta;\psi}v\|+M_1) \\ & +\rho\kappa(K_{\gamma+q_1,1}^{\psi}(T))\|v\|+\xi\kappa(K_{\gamma-q_2,1}^{\psi}(T))\|v\|\Big] \\ \leq & \Bigg[L\Lambda\Bigg(K_{r+\gamma,1}^{\psi}(T)+\frac{K_{\gamma,\vartheta}^{\psi}(T)}{|\Xi|}\Big(\rho K_{r+\gamma+q_1,1}^{\psi}(T)+\xi K_{r+\gamma-q_2,1}^{\psi}(T)\Big)\Bigg) \\ & +\kappa\Bigg(K_{\gamma,1}^{\psi}(T)+\frac{K_{\gamma,\vartheta}^{\psi}(T)}{|\Xi|}\Big(\rho K_{\gamma+q_1,1}^{\psi}(T)+\xi K_{\gamma-q_2,1}^{\psi}(T)\Big)\Bigg)\Bigg]\|v\| \\ & +\Bigg[L\Bigg(K_{r+\gamma,1}^{\psi}(T)+\frac{K_{\gamma,\vartheta}^{\psi}(T)}{|\Xi|}\Big(\rho K_{r+\gamma+q_1,1}^{\psi}(T)+\xi K_{r+\gamma-q_2,1}^{\psi}(T)\Big)\Bigg)\Bigg]\|{}^CD_b^{\theta;\psi}v\| \end{aligned}$$

$$+\left[K^{\psi}_{r+\gamma,\,1}(T)+\frac{K^{\psi}_{\gamma,\,\vartheta}(T)}{|\Xi|}\Big(\rho K^{\psi}_{r+\gamma+q_1,\,1}(T)+\xi K^{\psi}_{r+\gamma-q_2,\,1}(T)\Big)\right]M_1$$

$$\leq\left(L\Lambda\Omega_{r+\gamma}+\kappa\Omega_{\gamma}\right)\|v\|+L\Omega_{r+\gamma}\|^{C}D_b^{\theta;\,\psi}v\|+\Omega_{r+\gamma}M_1. \tag{18}$$

On the other hand, we have

$$|^{C}D_b^{\theta;\,\psi}(\Pi v)(t)|\leq(I_b^{r+\gamma-\theta;\,\psi})|\phi(t,\,v(t),\,I_b^{\delta;\,\psi}v(t),\,^{C}D_b^{\theta;\,\psi}v(t))|+\kappa(I_b^{\gamma-\theta;\,\psi})|v(t)|+\frac{K^{\psi}_{\gamma-\theta,\,\vartheta}(t)}{|\Xi|}\Big[\rho(I_b^{r+\gamma+q_1;\,\psi})$$

$$\times|\phi(\eta,\,v(\eta),\,I_b^{\delta;\,\psi}v(\eta),\,^{C}D_b^{\theta;\,\psi}v(\eta))|+\xi(I_b^{r+\gamma-q_2})|\phi(T,\,v(T),\,I_b^{\delta;\,\psi}v(T),\,^{C}D_b^{\theta;\,\psi}v(T))|$$

$$+\rho\kappa(I_b^{\gamma+q_1;\,\psi})|v(\eta)|+\xi\kappa(I_b^{\gamma-q_2;\,\psi})|v(T)|\Big]$$

$$\leq(K^{\psi}_{r+\gamma-\theta,\,1}(T))(L\|v\|+LK^{\psi}_{\delta,\,1}(T)\|v\|+L\|^{C}D_b^{\theta;\,\psi}v\|+M_1)+\kappa(K^{\psi}_{\gamma-\theta,\,1}(T))\|v\|$$

$$+\frac{K^{\psi}_{\gamma-\theta,\,\vartheta}(t)}{|\Xi|}\Big[\rho(K^{\psi}_{r+\gamma+q_1,\,1}(T))(L\|v\|+LK^{\psi}_{\delta,\,1}(T)\|v\|+L\|^{C}D_b^{\theta;\,\psi}v\|+M_1)$$

$$+\xi(K^{\psi}_{r+\gamma-q_2,\,1}(T))(L\|v\|+LK^{\psi}_{\delta,\,1}(T)\|v\|+L\|^{C}D_b^{\theta;\,\psi}v\|+M_1)$$

$$+\rho\kappa(K^{\psi}_{\gamma+q_1,\,1}(T))\|v\|+\xi\kappa(K^{\psi}_{\gamma-q_2,\,1}(T))\|v\|\Big]$$

$$\leq\left(L\Lambda\tilde{\Omega}_{r+\gamma}+\kappa\tilde{\Omega}_{\gamma}\right)\|v\|+L\tilde{\Omega}_{r+\gamma}\|^{C}D_b^{\theta;\,\psi}v\|+\tilde{\Omega}_{r+\gamma}M_1. \tag{19}$$

Consequently, by (18) and (19), we have

$$\|(\Pi v)\|_{\mathscr{V}}\leq\left(L\Lambda(\Omega_{r+\gamma}+\tilde{\Omega}_{r+\gamma-\theta})+\kappa(\Omega_{\gamma}+\tilde{\Omega}_{\gamma-\theta})\right)\|v\|_{\infty}+L(\Omega_{r+\gamma}+\tilde{\Omega}_{r+\gamma-\theta})\|^{C}D_b^{\theta;\,\psi}v\|_{\infty}+(\Omega_{r+\gamma}+\tilde{\Omega}_{r+\gamma-\theta})M_1$$

$$\leq\left(L\Lambda(\Omega_{r+\gamma}+\tilde{\Omega}_{r+\gamma})+\kappa(\Omega_{\gamma}+\tilde{\Omega}_{\gamma})\right)(\|v\|_{\infty}+\|^{C}D_b^{\theta;\,\psi}v\|_{\infty})+(\Omega_{r+\gamma}+\tilde{\Omega}_{r+\gamma})M_1$$

$$\leq\Delta\|v\|_{\mathscr{V}}+(\Omega_{r+\gamma}+\tilde{\Omega}_{r+\gamma})M_1$$

$$\leq \Delta R + (\Omega_{r+\gamma} + \tilde{\Omega}_{r+\gamma})M_1 \leq R.$$

Thus, Π maps bounded sets into bounded sets in $\mathscr{V}$.
Step 2: Π: $\mathscr{V} \longrightarrow \mathscr{V}$ is a contraction. Let $u, v \in \mathscr{V}$ and $t \in [b, T]$, we have

$$\begin{aligned}(\Pi u)(t) - (\Pi v)(t) =& (I_b^{r+\gamma;\psi})(\phi(t, u(t), I_b^{\delta;\psi}u(t), {}^C D_b^{\theta;\psi}u(t)) - \phi(t, v(t), I_b^{\delta;\psi}v(t), {}^C D_b^{\theta;\psi}v(t)))\\ & - \kappa(I_b^{\gamma;\psi})(u(t) - v(t)) + \frac{K_{\gamma,\vartheta}^{\psi}(t)}{\Xi}\Big[-\rho(I_b^{r+\gamma+q_1;\psi})(\phi(\eta, u(\eta), I_b^{\delta;\psi}u(\eta), {}^C D_b^{\theta;\psi}u(\eta))\\ & - \phi(\eta, v(\eta), I_b^{\delta;\psi}v(\eta), {}^C D_b^{\theta;\psi}v(\eta))) - \xi(I_b^{r+\gamma-q_2;\psi})(\phi(T, u(T), I_b^{\delta;\psi}u(T), {}^C D_b^{\theta;\psi}u(T))\\ & - \phi(T, v(T), I_b^{\delta;\psi}v(T), {}^C D_b^{\theta;\psi}v(T))) + \rho\kappa(I_b^{\gamma+q_1;\psi})(u(\eta) - v(\eta))\\ & + \xi\kappa(I_b^{\gamma-q_2;\psi})(u(T) - v(T))\Big].\end{aligned}$$

By (C2), we have

$$\begin{aligned}|(\Pi u)(t) - (\Pi v)(t)| \leq& (I_b^{r+\gamma;\psi})(L(\|u-v\| + K_{\delta;\psi}(T)\|u-v\| + \|{}^C D_b^{\theta;\psi}u - {}^C D_b^{\theta;\psi}v\|)) + \kappa(I_b^{\gamma;\psi})\|u-v\|\\ & + \frac{K_{\gamma,\vartheta}^{\psi}(t)}{|\Xi|}\Big[\rho(I_b^{r+\gamma+q_1;\psi})((L\|u-v\| + K_{\delta,1}^{\psi}(T)\|u-v\| + \|{}^C D_b^{\theta;\psi}u - {}^C D_b^{\theta;\psi}v\|))\\ & + \xi(I_b^{r+\gamma-q_2;\psi})(L(\|u-v\| + K_{\delta,1}^{\psi}(T)\|u-v\| + \|{}^C D_b^{\theta;\psi}u - {}^C D_b^{\theta;\psi}v\|))\\ & + \rho\kappa(I_b^{\gamma+q_1;\psi})\|u-v\| + \xi\kappa(I_b^{\gamma-q_2;\psi})\|u-v\|\Big]\\ \leq& \Bigg\{L\Lambda(I_b^{r+\gamma;\psi})[1] + \kappa(I_b^{\gamma;\psi})[1] + \frac{K_{\gamma,\vartheta}^{\psi}(T)}{|\Xi|}\Big[L\Lambda\rho(I_b^{r+\gamma+q_1;\psi})[1] + L\Lambda\xi(I_b^{r+\gamma-q_2,\psi})[1]\\ & + \rho\kappa(I_b^{\gamma+q_1;\psi})[1] + \xi\kappa(I_b^{\gamma-q_2;\psi})[1]\Big]\Bigg\}\|u-v\|\\ & + \Bigg\{L(I_b^{r+\gamma;\psi})[1] + \frac{K_{\gamma,\vartheta}^{\psi}(T)}{|\Xi|}\Big[\rho L(I_b^{r+\gamma+q_1;\psi})[1] + \xi L(I_b^{r+\gamma-q_2;\psi})[1]\Big]\Bigg\}\|{}^C D_b^{\theta;\psi}u - {}^C D_b^{\theta;\psi}v\|\end{aligned}$$

$$\leq \left(L\Lambda\Omega_{r+\gamma} + \kappa\Omega_{\gamma} \right) \|u - v\| + L\Omega_{r+\gamma} \|{}^C D_b^{\theta;\,\psi} u - {}^C D_b^{\theta;\,\psi} v\|. \tag{20}$$

Similarly, we have

$$\begin{aligned}
&{}^C D_b^{\theta;\,\psi}(\Pi u)(t) - {}^C D_b^{\theta;\,\psi}(\Pi v)(t) \\
=&(I_b^{r+\gamma-\theta;\,\psi})(\phi(t,\, u(t),\, I_b^{\delta;\,\psi} u(t),\, {}^C D_b^{\theta;\,\psi} u(t)) - \phi(t,\, v(t),\, I_b^{\delta;\,\psi} v(t),\, {}^C D_b^{\theta;\,\psi} v(t))) \\
&- \kappa(I_b^{\gamma-\theta;\,\psi})(u(t) - v(t)) + \frac{K^{\psi}_{\gamma-\theta,\,\vartheta}(t)}{\Xi} \Big[- \rho(I_b^{r+\gamma+q_1;\,\psi})(\phi(\eta,\, u(\eta),\, I_b^{\delta;\,\psi} u(\eta),\, {}^C D_b^{\theta;\,\psi} u(\eta)) \\
&- \phi(\eta,\, v(\eta),\, I_b^{\delta;\,\psi} v(\eta),\, {}^C D_b^{\theta;\,\psi} v(\eta))) - \xi(I_b^{r+\gamma-q_2;\,\psi})(\phi(T,\, u(T),\, I_b^{\delta;\,\psi} u(T),\, {}^C D_b^{\theta;\,\psi} u(T)) \\
&- \phi(T,\, v(T),\, I_b^{\delta;\,\psi} v(T),\, {}^C D_b^{\theta;\,\psi} v(T))) + \rho\kappa(I_b^{\gamma+q_1;\,\psi})(u(\eta) - v(\eta)) \\
&+ \xi\kappa(I_b^{\gamma-q_2;\,\psi})(u(T) - v(T)) \Big].
\end{aligned}$$

By (C2), we have

$$|{}^C D_b^{\theta;\,\psi}(\Pi u)(t) - {}^C D_b^{\theta;\,\psi}(\Pi v)(t)| \leq \left(L\Lambda\tilde{\Omega}_{r+\gamma} + \kappa\tilde{\Omega}_{\gamma} \right) \|u - v\| + L\tilde{\Omega}_{r+\gamma} \|{}^C D_b^{\theta;\,\psi} u - {}^C D_b^{\theta;\,\psi} v\|. \tag{21}$$

Combining (20) and (21), we obtain

$$\begin{aligned}
\|(\Pi u) - (\Pi v)\|_{\mathscr{V}} \leq& \left(L\Lambda(\Omega_{r+\gamma} + \tilde{\Omega}_{r+\gamma}) + \kappa(\Omega_{\gamma} + \tilde{\Omega}_{\gamma}) \right) \|u - v\| + L(\Omega_{r+\gamma} + \tilde{\Omega}_{r+\gamma}) \|{}^C D_b^{\theta;\,\psi} u - {}^C D_b^{\theta;\,\psi} v\| \\
\leq& \left(L\Lambda(\Omega_{r+\gamma} + \tilde{\Omega}_{r+\gamma}) + \kappa(\Omega_{\gamma} + \tilde{\Omega}_{\gamma}) \right) (\|u - v\|_{\infty} + \|{}^C D_b^{\theta;\,\psi} u - {}^C D_b^{\theta;\,\psi} v\|_{\infty}) \\
\leq& \Delta \|u - v\|_{\mathscr{V}}.
\end{aligned} \tag{22}$$

From (16), Π is a contraction. It follows that Π has a unique fixed point in $\mathscr{V}$, and hence, the problem (1)-(2) has a unique solution on $[b,\, T]$. $\square$

3.2 *Existence result via Krasnoselskii's fixed point theorem*

Theorem 4 Assume that conditions (C1) and (C2) are satisfied. Furthermore, suppose that the function ϕ fulfills the following condition:

(C3) There exists a function $\sigma(t) \in C([b, T], \mathbb{R}_+)$, such that

$$|\phi(t, u, v, w)| \leq \sigma(t) \text{ for all } (t, u, v, w) \in [b, T] \times \mathbb{R}^3.$$

If $\kappa\Omega_\gamma < 1$, then the problem (1)-(2) has at least one solution $v(t)$ on $[b, T]$.

Proof. Setting $\sup_{t\in[b,T]} \sigma(t) = \|\sigma\|$, and choosing

$$\tilde{R} \geq \frac{(\Omega_{r+\gamma} + \tilde{\Omega}_{r+\gamma-\theta})\|\sigma\|}{1 - \kappa(\Omega_\gamma + \tilde{\Omega}_{\gamma-\theta})}.$$

Let $\mathfrak{B}_{\tilde{R}} = \{v \in \mathscr{V} : \|v\| \leq \tilde{R}\}$ and define the operators Π_1 and Π_2 on $\mathfrak{B}_{\tilde{R}}$ by

$$(\Pi_1 v)(t) = I_b^{r+\gamma;\psi}\phi(t, v(t), I_b^{\delta;\psi}v(t), {}^C D_b^{\theta;\psi}v(t)) - \frac{K_{\gamma,\vartheta}^{\psi}(t)}{\Xi}\Big[\rho I_b^{r+\gamma+q_1;\psi}\phi(\eta, v(\eta), I_b^{\delta;\psi}v(\eta), {}^C D_b^{\theta;\psi}v(\eta))$$

$$+\xi I_b^{r+\gamma-q_2;\psi}\phi(T, v(T), I_b^{\delta;\psi}v(T), {}^C D_b^{\theta;\psi}v(T))\Big],$$

$$(\Pi_2 v)(t) = -\kappa I_b^{\gamma;\psi}v(t) + \frac{K_{\gamma,\vartheta}^{\psi}(t)}{\Xi}\Big[\rho\kappa I_b^{\gamma+q_1;\psi}v(\eta) + \xi\kappa I_v^{\gamma-q_2;\psi}v(T)\Big],$$

for all $t \in [b, T]$ and $v \in \mathfrak{B}_{\tilde{R}}$.

We claim that the proof is achieved in the following steps:

Step 1: $\Pi_1 u + \Pi_2 v \in \mathfrak{B}_{\tilde{R}}$. For any $u, v \in \mathfrak{B}_{\tilde{R}}$ and using (C3), we have

$$|(\Pi_1 u)(t) - (\Pi_2 v)(t)| \leq (I_b^{r+\gamma;\psi})|\phi(t, u(t), I_b^{\delta;\psi}u(t), {}^C D_b^{\theta;\psi}u(t))| +$$

$$+\frac{K_{\gamma,\vartheta}^{\psi}(t)}{|\Xi|}\Big[\rho(I_a^{r+\gamma+q_1;\psi})|\phi(\eta, u(\eta), I_b^{\delta;\psi}u(\eta), {}^C D_b^{\theta;\psi}u(\eta))|$$

$$+\xi(I_b^{r+\gamma-q_2;\psi})|\phi(T, u(T), I_b^{\delta;\psi}u(T), {}^C D_b^{\theta;\psi}u(T))|\Big]$$

$$+\kappa(I_b^{\gamma;\psi})|v(t)| + \frac{K_{\gamma,\vartheta}^{\psi}(t)}{|\Xi|}\Big[\rho\kappa(I_b^{\gamma+q_1;\psi})|v(\eta)| + \xi\kappa(I_a^{\gamma-q_2;\psi})|v(T)|\Big]$$

$$\leq (I_b^{r+\gamma;\psi})[1]\|\sigma\| + \frac{K_{\gamma,\vartheta}^{\psi}(t)}{|\Xi|}\Big[\rho(I_b^{r+\gamma+q_1;\psi})[1]\|\sigma\| + \xi(I_a^{r+\gamma-q_2;\psi})[1]\|\sigma\|\Big]$$

$$+\kappa(I_b^{\gamma;\psi})[1]\|v\|+\frac{K^{\psi}_{\gamma,\vartheta}(t)}{|\Xi|}\Big[\rho\kappa(I_b^{\gamma+q_1;\psi})[1]\|v\|+\xi\kappa(I_b^{\gamma-q_2;\psi})[1]\|v\|\Big]$$

$$\leq K^{\psi}_{r+\gamma,1}(T)\|\sigma\|+\frac{K^{\psi}_{\gamma,\vartheta}(T)}{|\Xi|}\Big[\rho K^{\psi}_{r+\gamma+q_1,1}(T)\|\sigma\|+\xi K^{\psi}_{r+\gamma-q_2,1}(T)\|\sigma\|\Big]$$

$$+\kappa K^{\psi}_{\gamma,1}(T)\tilde{R}+\frac{K^{\psi}_{\gamma,\vartheta}(T)}{|\Xi|}\Big[\rho\kappa K^{\psi}_{\gamma+q_1,1}(T)\tilde{R}+\xi\kappa K^{\psi}_{\gamma-q_2,1}(T)\tilde{R}\Big]$$

$$\leq\Big[K^{\psi}_{r+\gamma,1}(T)+\frac{\rho}{|\Xi|}K^{\psi}_{\gamma,\vartheta}(T)K^{\psi}_{r+\gamma+q_1,1}(T)+\frac{\xi}{|\Xi|}K^{\psi}_{\gamma,\vartheta}(T)K^{\psi}_{r+\gamma-q_2,1}(T)\Big]\|\sigma\|$$

$$+\kappa\Big[K^{\psi}_{\gamma,1}(T)+\frac{\rho}{|\Xi|}K^{\psi}_{\gamma,\vartheta}(T)K^{\psi}_{\gamma+q_1,1}(T)+\frac{\xi}{|\Xi|}K^{\psi}_{\gamma,\vartheta}(T)K^{\psi}_{\gamma-q_2,1}(T)\Big]\tilde{R}$$

$$\leq\Omega_{r+\gamma}\|\sigma\|+\kappa\Omega_\gamma\tilde{R}. \tag{23}$$

On the other hand, we have

$$|{}^{C}D_b^{\theta;\psi}(\Pi_1u)(t)-{}^{C}D_b^{\theta;\psi}(\Pi_2v)(t)|\leq(I_b^{r+\gamma-\theta;\psi})|\phi(t,u(t),I_b^{\delta;\psi}u(t),{}^{C}D_b^{\theta;\psi}u(t))|+$$

$$+\frac{K^{\psi}_{\gamma-\theta,\vartheta}(t)}{|\Xi|}\Big[\rho(I_a^{r+\gamma+q_1;\psi})|\phi(\eta,u(\eta),I_b^{\delta;\psi}u(\eta),{}^{C}D_b^{\theta;\psi}u(\eta))|$$

$$+\xi(I_b^{r+\gamma-q_2;\psi})|\phi(T,u(T),I_b^{\delta;\psi}u(T),{}^{C}D_b^{\theta;\psi}u(T))|\Big]$$

$$+\kappa(I_b^{\gamma-\theta;\psi})|v(t)|+\frac{K^{\psi}_{\gamma-\theta,\vartheta}(t)}{|\Xi|}\Big[\rho\kappa(I_b^{\gamma+q_1;\psi})|v(\eta)|+\xi\kappa(I_a^{\gamma-q_2;\psi})|v(T)|\Big]$$

$$\leq(I_b^{r+\gamma-\theta;\psi})[1]\|\sigma\|+\frac{K^{\psi}_{\gamma-\theta,\vartheta}(t)}{|\Xi|}\Big[\rho(I_b^{r+\gamma+q_1;\psi})[1]\|\sigma\|+\xi(I_a^{r+\gamma-q_2;\psi})[1]\|\sigma\|\Big]$$

$$+\kappa(I_b^{\gamma-\theta;\psi})[1]\|v\|+\frac{K^{\psi}_{\gamma-\theta,\vartheta}(t)}{|\Xi|}\Big[\rho\kappa(I_b^{\gamma+q_1;\psi})[1]\|v\|+\xi\kappa(I_b^{\gamma-q_2;\psi})[1]\|v\|\Big]$$

$$\leq\tilde{\Omega}_{r+\gamma}\|\sigma\|+\kappa\tilde{\Omega}_\gamma\tilde{R}. \tag{24}$$

Combining (23) and (24), we obtain

$$\|(\Pi_1 u)-(\Pi_2 v)\|_{\mathscr{V}} \leq (\Omega_{r+\gamma}+\tilde{\Omega}_{r+\gamma})\|\sigma\|+\kappa(\Omega_\gamma+\tilde{\Omega}_\gamma)\tilde{R}, \tag{25}$$

which implies that $\|(\Pi_1 u)-(\Pi_2 v)\| \leq \tilde{R}$. Thus, $\Pi_1 u+\Pi_2 v \in \mathfrak{B}_{\tilde{R}}$.

Step 2: Π is a contration on $\mathfrak{B}_{\tilde{R}}$. In view of Theorem 3, Π_2 is also a contraction mapping.

Step 3: Π_1 is completely continuous on $\mathfrak{B}_{\tilde{R}}$. From the continuity of $\phi(t,.,.,.)$, it follows that Π_1 is continuous, and using (C3), we have

$$\begin{aligned}
|\Pi_1 v(t)| \leq & (I_b^{r+\gamma;\psi})|\phi(t, v(t), I_b^{\delta;\psi}v(t), {}^C D_b^{\theta;\psi}v(t))| + \frac{K^{\psi}_{\gamma,\vartheta}(t)}{|\Xi|}\Big[\rho(I_b^{r+\gamma+q_1;\psi})|\phi(\eta, v(\eta), I_b^{\delta;\psi}v(\eta), {}^C D_b^{\theta;\psi}v(\eta))| \\
& + \xi(I_b^{r+\gamma-q_2;\psi})|\phi(T, x(T), I_b^{\delta;\psi}v(T), {}^C D_b^{\theta;\psi}v(T))|\Big] \\
\leq & (I_b^{r+\gamma;\psi})[1]\|\sigma\| + \frac{K^{\psi}_{\gamma,\vartheta}(T)}{|\Xi|}\Big[\rho(I_b^{r+\gamma+q_1;\psi})[1]\|\sigma\| + \xi(I_b^{r+\gamma-q_2;\psi})[1]\|\sigma\|\Big] \\
\leq & \left[(I_b^{r+\gamma;\psi})[1] + \frac{K^{\psi}_{\gamma,\vartheta}(T)}{|\Xi|}\Big[\rho(I_b^{r+\gamma+q_1;\psi})[1] + \xi(I_b^{r+\gamma-q_2;\psi})[1]\Big]\right]\|\sigma\| \\
\leq & \Omega_{r+\gamma}\|\sigma\|.
\end{aligned} \tag{26}$$

Beside, we have

$$\begin{aligned}
& |{}^C D_b^{\theta;\psi}(\Pi_1 v)(t)| \\
\leq & (I_b^{r+\gamma-\theta;\psi})|\phi(t, v(t), I_b^{\delta;\psi}v(t), {}^C D_b^{\theta;\psi}v(t))| + \frac{K^{\psi}_{\gamma-\theta,\vartheta}(t)}{|\Xi|}\Big[\rho(I_b^{r+\gamma+q_1;\psi})|\phi(\eta, v(\eta), I_b^{\delta;\psi}v(\eta), {}^C D_b^{\theta;\psi}v(\eta))| \\
& + \xi(I_b^{r+\gamma-q_2;\psi})|\phi(T, x(T), I_b^{\delta;\psi}v(T), {}^C D_b^{\theta;\psi}v(T))|\Big] \\
\leq & (I_b^{r+\gamma-\theta;\psi})[1]\|\sigma\| + \frac{K^{\psi}_{\gamma-\theta,\vartheta}(T)}{|\Xi|}\Big[\rho(I_b^{r+\gamma+q_1;\psi})[1]\|\sigma\| + \xi(I_b^{r+\gamma-q_2;\psi})[1]\|\sigma\|\Big] \\
\leq & \left[(I_b^{r+\gamma-\theta;\psi})[1] + \frac{K^{\psi}_{\gamma-\theta,\vartheta}(T)}{|\Xi|}\Big[\rho(I_b^{r+\gamma+q_1;\psi})[1] + \xi(I_b^{r+\gamma-q_2;\psi})[1]\Big]\right]\|\sigma\| \\
\leq & \tilde{\Omega}_{r+\gamma}\|\sigma\|,
\end{aligned} \tag{27}$$

which underscore that Π_1 uniformly bounded on $\mathfrak{B}_{\tilde{R}}$.

Step 4: Π_1 is compact. Let us assume that $b \leq t_1 \leq t_2 \leq T$. For $v \in \mathfrak{B}_{\tilde{R}}$, we have from (C3), Lemma 2 and Lemma 3 that

$$\begin{aligned}
|(\Pi_1 v)(t_2) - (\Pi_1 v)(t_1)| \leq & \frac{1}{\Gamma(r+\gamma)}\left[\int_b^{t_1} \psi'(s)\Big[(\psi(t_2)-\psi(s))^{r+\gamma-1} - (\psi(t_1)-\psi(s))^{r+\gamma-1}\Big]\right. \\
& \times |\phi(s, v(s), I_b^{\delta;\psi} v(s), {}^C D_b^{\theta;\psi} v(s))| ds + \int_{t_1}^{t_2} \psi'(s)(\psi(t_2)-\psi(s))^{r+\gamma-1} \\
& \left.\times |\phi(s, v(s), I_b^{\delta;\psi} v(s), {}^C D_b^{\theta;\psi} v(s))| ds\right] + \frac{K_{\gamma,\vartheta}^{\psi}(t_2) - K_{\gamma,\vartheta}^{\psi}(t_1)}{|\Xi|} \\
& \times \Big[I^{r+\gamma+q_1;\psi}[1] |\phi(\eta, v(\eta), I_b^{\delta,\psi} v(\eta), {}^C D_b^{\theta;\psi} v(\eta))| + I^{r+\gamma-q_2,\psi}[1] \\
& \times |\phi(T, v(T), I_b^{\delta;\psi} v(T), {}^C D_b^{\theta;\psi} v(T))|\Big], \\
\leq & \frac{2\|\sigma\|}{\Gamma(r+\gamma+1)}(\psi(t_2)-\psi(t_1))^{r+\gamma} + \frac{\|\sigma\|}{|\Xi|}\Big[K_{r+\gamma+q_1,1}^{\psi}(T) + K_{r+\gamma-q_2,1}^{\psi}(T)\Big] \\
& \times (K_{\gamma,\vartheta}^{\psi}(t_2) - K_{\gamma,\vartheta}^{\psi}(t_1)).
\end{aligned}$$

Hence

$$\|(\Pi_1 v) - (\Pi_1 v)\|_\infty \longrightarrow 0 \text{ as } t_2 \longrightarrow t_1. \tag{28}$$

On the other hand, we have

$$\begin{aligned}
|{}^C D_b^{\theta;\psi}(\Pi_1 v)(t_2) - {}^C D_b^{\theta;\psi}(\Pi_1 v)(t_1)| \leq & \frac{1}{\Gamma(r+\gamma-\theta)}\left[\int_b^{t_1} \psi'(s)\Big[(\psi(t_2)-\psi(s))^{r+\gamma-\theta-1} - (\psi(t_1)-\psi(s))^{r+\gamma-\theta-1}\Big]\right. \\
& \times |\phi(s, v(s), I_b^{\delta;\psi} v(s), {}^C D_b^{\theta;\psi} v(s))| ds + \int_{t_1}^{t_2} \psi'(s)(\psi(t_2)-\psi(s))^{r+\gamma-\theta-1} \\
& \left.\times |\phi(s, v(s), I_b^{\delta;\psi} v(s), {}^C D_b^{\theta;\psi} v(s))| ds\right] + \frac{K_{\gamma-\theta,\vartheta}^{\psi}(t_2) - K_{\gamma-\theta,\vartheta}^{\psi}(t_1)}{|\Xi|} \\
& \times \Big[I^{r+\gamma+q_1;\psi}[1] |\phi(\eta, v(\eta), I_b^{\delta,\psi} v(\eta), {}^C D_b^{\theta;\psi} v(\eta))| + I^{r+\gamma-q_2,\psi}[1]
\end{aligned}$$

$$\times |\phi(T, v(T), I_b^{\delta;\psi}v(T), {}^C D_b^{\theta;\psi}v(T))|\Big],$$

$$\leq \frac{2\|\sigma\|}{\Gamma(r+\gamma-\theta+1)}(\psi(t_2)-\psi(t_1))^{r+\gamma-\theta} + \frac{\|\sigma\|}{|\Xi|}\Big[K^{\psi}_{r+\gamma+q_1,\,1}(T) + K^{\psi}_{r+\gamma-q_2,\,1}(T)\Big]$$

$$\times (K^{\psi}_{\gamma-\theta,\,\vartheta}(t_2) - K^{\psi}_{\gamma-\theta,\,\vartheta}(t_1)).$$

Consequently

$$\|{}^C D_b^{\theta;\psi}(\Pi_1 v) - {}^C D_b^{\theta;\psi}(\Pi_1 v)\|_\infty \longrightarrow 0 \text{ as } t_2 \longrightarrow t_1. \tag{29}$$

Using (28) and (29), we get

$$\|(\Pi_1 v) - (\Pi_1 v)\|_{\mathscr{V}} \longrightarrow 0 \text{ as } t_2 \longrightarrow t_1.$$

It is uniformly bounded with respect to v and converges as $t_2 \longrightarrow t_1$ to 0. Hence, Π_1 is relatively compact in $\mathfrak{B}_{\tilde{R}}$. By the Arzelà-Ascoli theorem, Π_1 is compact. Consequently, all the assumptions of Theorem 4 are satisfied, and the problem (1)-(2) admits at least one solution in $\mathfrak{B}_{\tilde{R}}$. □

3.3 *Type stability*

To conclude this subsection, we present the definitions of (UHS) and (UHRS), followed by a discussion of the corresponding results.

Definition 6 The problem (1)-(2) is called UHS if there exists a positive number C_ϕ such that for each $\varepsilon > 0$ and for each solution $v^* \in \mathscr{V}$ of the following inequality

$$|{}^H D_b^{r,\beta;\psi}({}^C D_b^{\gamma;\psi} + \lambda)v^*(t) - \phi(t, v^*(t), I_b^{\delta;\psi}v^*(t), {}^C D_b^{\theta;\psi}v^*(t))| \leq \varepsilon, \quad t \in [b, T], \tag{30}$$

there exists a solution $v \in \mathscr{V}$ of the problem (1)-(2) such that

$$\|v^* - v\|_{\mathscr{V}} \leq C_\phi \varepsilon. \tag{31}$$

Remark 1 A function v^* is a solution of inequality (30) if there exists a function $\varphi_1 \in \mathscr{V}$ (which depends on v^*) such that

i) $|\varphi_1(t)| \leq \varepsilon, \quad b \leq t \leq T.$

ii) ${}^H D_b^{r,\beta;\psi}({}^C D_b^{\gamma;\psi} + \kappa)v^*(t) = \phi(t, v^*(t), I_b^{\delta;\psi}v^*(t), {}^C D_b^{\theta;\psi}v^*(t)) + \varphi_1(t), b \leq t \leq T.$

Definition 7 The problem (1)-(2) is called UHRS with respect to $\chi \in C([b, T], \mathbb{R}^+)$ if there exists a positive number $\tilde{C}_\phi$ such that for each $\varepsilon > 0$ and for each solution $v^* \in \mathscr{V}$ of the following inequality

$$|{}^{H}D_{b}^{r,\,\beta;\,\psi}({}^{C}D_{b}^{\gamma;\,\psi}+\kappa)v^{*}(t)-\phi(t,\,v^{*}(t),\,I_{b}^{\delta;\,\psi}v^{*}(t),\,{}^{C}D_{b}^{\theta;\,\psi}v^{*}(t))|\leq\chi(t)\varepsilon,\quad t\in[b,\,T],\tag{32}$$

there exists a solution $v\in\mathscr{V}$ of the problem (1)-(2) such that

$$\|v^{*}-v\|_{\mathscr{V}}\leq\tilde{C}_{\phi}\chi(t)\varepsilon.\tag{33}$$

Remark 2 A function v^{*} is a solution of inequality (32) if there exists a function $\varphi_{2}\in\mathscr{V}$ (which depends on v^{*}) such that

i) $|\varphi_{2}(t)|\leq\varepsilon\chi(t),\quad b\leq t\leq T$.

ii) ${}^{H}D_{b}^{r,\,\beta;\,\psi}({}^{C}D_{b}^{\gamma;\,\psi}+\kappa)v^{*}(t)=\phi(t,\,v^{*}(t),\,I_{b}^{\delta;\,\psi}v^{*}(t),\,{}^{C}D_{b}^{\theta;\,\psi}v^{*}(t))+\varphi_{2}(t),\,b\leq t\leq T$.

To establish the following lemmas, we consider the problem

$${}^{H}D_{b}^{r,\,\beta;\,\psi}({}^{C}D_{b}^{\gamma;\,\psi}+\kappa)v^{*}(t)=\phi(t,\,v^{*}(t),\,I_{b}^{\delta;\,\psi}v^{*}(t),\,{}^{C}D_{b}^{\theta;\,\psi}v^{*}(t))+\varphi_{i}(t),\ t\in[b,\,T],\ T>0,\tag{34}$$

$$v^{*}(b)=0,\ \rho I_{b}^{q_{1};\,\psi}v^{*}(\eta)+\xi\,{}^{C}D_{b}^{q_{2};\,\psi}v^{*}(T)=0,\tag{35}$$

where $i\in\{1,\,2\}$.

Lemma 5 Let $1<r<2$, $0<q_{2}<\gamma<\beta<1$, $q_{1}>0$, $\vartheta=r+\beta(1-r)$ and let $v^{*}\in\mathscr{V}$ be a solution of the inequality (30). Then v^{*} is a solution of the following integral inequality:

$$|v^{*}(t)-\omega(t)|\leq\Omega_{r+\gamma}\varepsilon,$$

and

$$|{}^{C}D_{b}^{\theta;\,\psi}(v^{*}(t))-{}^{C}D_{b}^{\theta;\,\psi}(\omega(t))|\leq\tilde{\Omega}_{r+\gamma}\varepsilon,$$

where

$$\begin{aligned}\omega(t)=&I_{b}^{r+\gamma;\,\psi}\phi(t,\,v^{*}(t),\,I_{b}^{\delta;\,\psi}v^{*}(t),\,{}^{C}D_{b}^{\theta;\,\psi}v^{*}(t))-\kappa I_{b}^{\gamma;\,\psi}v^{*}(t)+\frac{K_{\gamma,\,\vartheta}^{\psi}(t)}{\Xi}\Big[\rho\kappa I_{a}^{\gamma+q_{1};\,\psi}v^{*}(\eta)\\&+\xi\kappa I_{b}^{\gamma-q_{2};\,\psi}v^{*}(T)-\rho I_{b}^{r+\gamma+q_{1};\,\psi}\phi(\eta,\,v^{*}(\eta),\,I_{b}^{\delta;\,\psi}v^{*}(\eta),\,{}^{C}D_{b}^{\theta;\,\psi}v^{*}(\eta))\\&-\xi I_{b}^{r+\gamma-q_{2};\,\psi}\phi(T,\,v^{*}(T),\,I_{b}^{\delta;\,\psi}v^{*}(T),\,{}^{C}D_{b}^{\theta;\,\psi}v^{*}(T))\Big],\end{aligned}\tag{36}$$

and $\Omega_{r+\gamma}$, $\tilde{\Omega}_{r+\gamma}$ are given by (13) and (14).

Proof. Given v^* is the solution of the inequality (30), then according to Remark 1 (ii) and by Lemma 4, the solution of (34)-(35) such that $i = 1$, is given by

$$v^*(t) = I_b^{r+\gamma;\,\psi}[\phi(t,\, v^*(t),\, I_b^{\delta;\,\psi}v^*(t),\, {}^CD_b^{\theta;\,\psi}v^*(t)) + \varphi_1(t)] - \kappa I_b^{\gamma;\,\psi}v^*(t) + \frac{K_{\gamma,\,\vartheta}^{\psi}(t)}{\Xi}\Big[\rho\kappa I_b^{\gamma+q_1;\,\psi}v^*(\eta)$$

$$+\xi\kappa I_b^{\gamma-q_2;\,\psi}v^*(T) - \rho I_b^{r+\gamma+q_1;\,\psi}[\phi(\eta,\, v^*(\eta),\, I_b^{\delta;\,\psi}v^*(\eta),\, {}^CD_b^{\theta;\,\psi}v^*(\eta)) + \varphi_1(t)]$$

$$-\xi I_b^{r+\gamma-q_2;\,\psi}[\phi(T,\, v^*(T),\, I_b^{\delta;\,\psi}v^*(T),\, {}^CD_b^{\theta;\,\psi}v^*(T)) + \varphi_1(t)]\Big]. \tag{37}$$

By applying Remark 1 (i), we get

$$|v^*(t) - \omega(t)|$$

$$= \Big|v^*(t) - I_b^{r+\gamma;\,\psi}\varphi(t,\, v^*(t),\, I_b^{\delta;\,\psi}v^*(t),\, {}^CD_b^{\theta;\,\psi}v^*(t)) + \kappa I_b^{\gamma;\,\psi}v^*(t) - \frac{K_{\gamma,\,\vartheta}^{\psi}(t)}{\Xi}\Big[\rho\kappa I_b^{\gamma+q_1;\,\psi}v^*(\eta) + \xi\kappa I_b^{\gamma-q_2;\,\psi}v^*(T)$$

$$-\rho I_b^{r+\gamma+q_1;\,\psi}\phi(\eta,\, v^*(\eta),\, I_b^{\delta;\,\psi}v^*(\eta),\, {}^CD_b^{\theta;\,\psi}v^*(\eta)) - \xi I_b^{r+\gamma-q_2;\,\psi}\phi(T,\, v^*(T),\, I_b^{\delta;\,\psi}v^*(T),\, {}^CD_b^{\theta;\,\psi}v^*(T))\Big]\Big|$$

$$\leq (I_b^{r+\gamma;\,\psi})[1]|\varphi_1(t)| + \frac{K_{\gamma,\,\vartheta}^{\psi}(T)}{|\Xi|}\Big[\rho(I_b^{r+\gamma+q_1;\,\psi})[1]|\varphi_1(t)| + \xi(I_b^{r+\gamma-q_2;\,\psi})[1]|\varphi_1(t)|\Big]$$

$$\leq \Big[K_{r+\gamma,\,1}^{\psi}(T) + \frac{\rho}{|\Xi|}K_{\gamma,\,\vartheta}^{\psi}(T)K_{r+\gamma+q_1,\,1}^{\psi}(T) + \frac{\xi}{|\Xi|}K_{\gamma,\,\vartheta}^{\psi}(T)K_{r+\gamma-q_2,\,1}^{\psi}(T)\Big]|\varphi_1(t)|$$

$$\leq \Omega_{r+\gamma}\varepsilon,$$

where $\Omega_{r+\gamma}$ is given in (11).

Similarly, applying the operator ${}^CD_b^{\theta;\,\psi}$ on (36) and (37), we have

$$|{}^CD_b^{\theta;\,\psi}v^*(t) - {}^CD_b^{\theta;\,\psi}\omega(t)|$$

$$= \Big|{}^CD_b^{\theta;\,\psi}v^*(t) - I_b^{r+\gamma-\theta;\,\psi}\varphi(t,\, v^*(t),\, I_b^{\delta;\,\psi}v^*(t),\, {}^CD_b^{\theta;\,\psi}v^*(t)) + \kappa I_b^{\gamma-\theta;\,\psi}v^*(t) - \frac{K_{\gamma-\theta,\,\vartheta}^{\psi}(t)}{\Xi}\Big[\rho\kappa I_b^{\gamma+q_1;\,\psi}v^*(\eta)$$

$$+\xi\kappa I_b^{\gamma-q_2;\,\psi}v^*(T) - \rho I_b^{r+\gamma+q_1;\,\psi}\phi(\eta,\, v^*(\eta),\, I_b^{\delta;\,\psi}v^*(\eta),\, {}^CD_b^{\theta;\,\psi}v^*(\eta))$$

$$-\xi I_b^{r+\gamma-q_2;\,\psi}\phi(T,\, v^*(T),\, I_b^{\delta;\,\psi}v^*(T),\, {}^CD_b^{\theta;\,\psi}v^*(T))\Big|\Big]$$

$$\leq (I_b^{r+\gamma-\theta;\,\psi})[1]|\varphi_1(t)| + \frac{K^{\psi}_{\gamma-\theta,\,\vartheta}(T)}{|\Xi|}\Big[\rho(I_b^{r+\gamma+q_1;\,\psi})[1]|\varphi_1(t)| + \xi(I_b^{r+\gamma-q_2;\,\psi})[1]|\varphi_1(t)|\Big]$$

$$\leq \Big[K^{\psi}_{r+\gamma-\theta,\,1}(T) + \frac{\rho}{|\Xi|}K^{\psi}_{\gamma-\theta,\,\vartheta}(T)K^{\psi}_{r+\gamma+q_1,\,1}(T) + \frac{\xi}{|\Xi|}K^{\psi}_{\gamma,\,\vartheta}(T)K^{\psi}_{r+\gamma-q_2,\,1}(T)\Big]|\varphi_1(t)|$$

$$\leq \tilde{\Omega}_{r+\gamma}\varepsilon,$$

where $\tilde{\Omega}_{r+\gamma}$ is given in (14). □

Theorem 5 If (C1) and (C2) are satisfied, then the problem (1)-(2) is UHS, namely, assume that $v^* \in \mathscr{V}$ is a solution of inequality (30) and $v \in \mathscr{V}$ is a unique solution of the problem (1)-(2) such that

$$\|v^* - v\|_{\mathscr{V}} \leq C_{\phi}\varepsilon. \tag{38}$$

Proof. From assumptions (C1)-(C2), we can conclude by Theorem 3 that the problem (1)-(2) has a unique solution $v \in \mathscr{V}$, that satisfies the following integral equation:

$$\begin{aligned} v(t) =& I_b^{r+\gamma;\,\psi}\phi(t,\, v(t),\, I_b^{\delta;\,\psi}v(t),\, {}^{C}D_b^{\theta;\,\psi}v(t)) - \kappa I_b^{\gamma;\,\psi}v(t) + \frac{K^{\psi}_{\gamma,\,\vartheta}(t)}{\Xi}\Big[\rho\kappa I_b^{\gamma+q_1;\,\psi}v(\eta) + \xi\kappa I_b^{\gamma-q_2;\,\psi}v(T) \\ & - \rho I_b^{r+\gamma+q_1;\,\psi}\phi(\eta,\, v(\eta),\, I_b^{\delta;\,\psi}v(\eta),\, {}^{C}D_b^{\theta;\,\psi}v(\eta)) - \xi I_b^{r+\gamma-q_2;\,\psi}\phi(T,\, v(T),\, I_b^{\delta;\,\psi}v(T),\, {}^{C}D_b^{\theta;\,\psi}v(T))\Big]. \end{aligned} \tag{39}$$

Assume that v^* is a solution of the inequality (30). Then, for every $\varepsilon > 0$, it follows from Remark 1 (ii) that v^* satisfies the following integral equation

$$\begin{aligned} v^*(t) =& I_b^{r+\gamma;\,\psi}[\phi(t,\, v^*(t),\, I_b^{\delta;\,\psi}v^*(t),\, {}^{C}D_b^{\theta;\,\psi}v^*(t)) + \varphi_1(t)] - \kappa I_b^{\gamma;\,\psi}v^*(t) \\ & + \frac{K^{\psi}_{\gamma,\,\vartheta}(t)}{\Xi}\Big[\rho\kappa I_b^{\gamma+q_1;\,\psi}v^*(\eta) + \xi\kappa I_b^{\gamma-q_2;\,\psi}v^*(T) \\ & - \rho I_b^{r+\gamma+q_1;\,\psi}[\phi(\eta,\, v^*(\eta),\, I_b^{\delta;\,\psi}v^*(\eta),\, {}^{C}D_b^{\theta;\,\psi}v^*(\eta)) + \varphi_1(t)] \\ & - \xi I_b^{r+\gamma-q_2;\,\psi}[\phi(T,\, v^*(T),\, I_b^{\delta;\,\psi}v^*(T),\, {}^{C}D_b^{\theta;\,\psi}v^*(T)) + \varphi_1(t)]\Big]. \end{aligned} \tag{40}$$

Therefore, we have from Lemma 5 and (C2) that

$$
\begin{aligned}
|v^*(t)-v(t)| \leq & |v^*(t)-\omega(t)|+|\omega(t)-v(t)| \\
\leq & \Omega_{r+\gamma}\varepsilon + I_b^{r+\gamma;\,\psi}|\phi(t,\, v^*(t),\, I_b^{\delta;\,\psi}v^*(t),\, {}^C D_b^{\theta;\,\psi}v^*(t)) - \phi(t,\, v(t),\, I_b^{\delta;\,\psi}v(t),\, {}^C D_b^{\theta;\,\psi}v(t))| \\
& + \kappa I_b^{\gamma;\,\psi}|v^*(t)-v(t)| + \frac{K_{\gamma,\,\vartheta}^{\psi}(t)}{|\Xi|}\Big[\rho\kappa I_b^{\gamma+q_1;\,\psi}|v^*(\eta)-v(\eta)| + \xi\kappa I_b^{\gamma-q_2;\,\psi}|v^*(T)-v(T)| \\
& + \rho I_b^{r+\gamma+q_1;\,\psi}|\phi(\eta,\, v^*(\eta),\, I_b^{\delta;\,\psi}\eta^*(T),\, {}^C D_b^{\theta;\,\psi}\eta^*(T)) - \phi(\eta,\, v(\eta),\, I_b^{\delta;\,\psi}v(\eta),\, {}^C D_b^{\theta;\,\psi}v(\eta))| \\
& + \xi I_b^{r+\gamma-q_2;\,\psi}|\phi(T,\, v^*(T),\, I_b^{\delta;\,\psi}v^*(T),\, {}^C D_b^{\theta;\,\psi}v^*(T)) - \varphi(T,\, v(T),\, I_b^{\delta;\,\psi}v(T),\, {}^C D_b^{\theta;\,\psi}v(T))| \\
\leq & \Omega_{r+\gamma}\varepsilon + (I_b^{r+\gamma;\,\psi})(L(\|v^*-v\| + K_{\delta;\,1}^{\psi}(T)\|v^*-v\| + \|{}^C D_b^{\theta;\,\psi}v^* - {}^C D_b^{\theta;\,\psi}v\|)) + \kappa(I_b^{\gamma;\,\psi})\|v^*-v\| \\
& + \frac{K_{\gamma,\,\vartheta}^{\psi}(t)}{|\Xi|}\Big[\rho(I_b^{r+\gamma+q_1;\,\psi})((L\|v^*-v\| + K_{\delta,\,1}^{\psi}(T)\|v^*-v\| + \|{}^C D_b^{\theta;\,\psi}v^* - {}^C D_b^{\theta;\,\psi}v\|)) \\
& + \xi(I_b^{r+\gamma-q_2;\,\psi})(L(\|v^*-v\| + K_{\delta,\,1}^{\psi}(T)\|v^*-v\| + \|{}^C D_b^{\theta;\,\psi}v^* - {}^C D_b^{\theta;\,\psi}v\|)) \\
& + \rho\kappa(I_b^{\gamma+q_1;\,\psi})\|v^*-v\| + \xi\kappa(I_b^{\gamma-q_2;\,\psi})\|v^*-v\|\Big] \\
\leq & \Omega_{r+\gamma}\varepsilon + \Bigg\{L\Lambda(I_b^{r+\gamma;\,\psi})[1] + \kappa(I_b^{\gamma;\,\psi})[1] + \frac{K_{\gamma,\,\vartheta}^{\psi}(T)}{|\Xi|}\Big[L\Lambda\rho(I_b^{r+\gamma+q_1;\,\psi})[1] + L\Lambda\xi(I_b^{r+\gamma-q_2,\,\psi})[1] \\
& + \rho\kappa(I_b^{\gamma+q_1;\,\psi})[1] + \xi\kappa(I_b^{\gamma-q_2;\,\psi})[1]\Big]\Bigg\}\|u-v\| \\
& + \Bigg\{L(I_b^{r+\gamma;\,\psi})[1] + \frac{K_{\gamma,\,\vartheta}^{\psi}(T)}{|\Xi|}\Big[\rho L(I_b^{r+\gamma+q_1;\,\psi})[1] + \xi L(I_b^{r+\gamma-q_2;\,\psi})[1]\Big]\Bigg\}\|{}^C D_b^{\theta;\,\psi}u - {}^C D_b^{\theta;\,\psi}v\| \\
\leq & \Omega_{r+\gamma}\varepsilon + \Bigg(L\Lambda\Omega_{r+\gamma} + \kappa\Omega_{\gamma}\Bigg)\|v^*-v\| + L\Omega_{r+\gamma}\|{}^C D_b^{\theta;\,\psi}v^* - {}^C D_b^{\theta;\,\psi}v\|. \qquad (41)
\end{aligned}
$$

Also,

$$
\begin{aligned}
&|{}^{C}D_{b}^{\theta;\,\psi}v^{*}(t)-{}^{C}D_{b}^{\theta;\,\psi}v(t)|\\
\leq&|{}^{C}D_{b}^{\theta;\,\psi}v^{*}(t)-{}^{C}D_{b}^{\theta;\,\psi}\omega(t)|+|{}^{C}D_{b}^{\theta;\,\psi}\omega(t)-{}^{C}D_{b}^{\theta;\,\psi}v(t)|\\
\leq&\tilde{\Omega}_{r+\gamma}\varepsilon+I_{b}^{r+\gamma-\theta;\,\psi}|\phi(t,\,v^{*}(t),\,I_{b}^{\delta;\,\psi}v^{*}(t),\,{}^{C}D_{b}^{\theta;\,\psi}v^{*}(t))-\phi(t,\,v(t),\,I_{b}^{\delta;\,\psi}v(t),\,{}^{C}D_{b}^{\theta;\,\psi}v(t))|\\
&+\kappa I_{b}^{\gamma-\theta;\,\psi}|v^{*}(t)-v(t)|+\frac{K_{\gamma-\theta,\,\vartheta}^{\psi}(t)}{|\Xi|}\Big[\rho\kappa I_{b}^{\gamma+q_{1};\,\psi}|v^{*}(\eta)-v(\eta)|+\xi\kappa I_{b}^{\gamma-q_{2};\,\psi}|v^{*}(T)-v(T)|\\
&+\rho I_{b}^{r+\gamma+q_{1};\,\psi}|\phi(\eta,\,v^{*}(\eta),\,I_{b}^{\delta;\,\psi}\eta^{*}(T),\,{}^{C}D_{b}^{\theta;\,\psi}\eta^{*}(T))-\phi(\eta,\,v(\eta),\,I_{b}^{\delta;\,\psi}v(\eta),\,{}^{C}D_{b}^{\theta;\,\psi}v(\eta))|\\
&+\xi I_{b}^{r+\gamma-q_{2};\,\psi}|\phi(T,\,v^{*}(T),\,I_{b}^{\delta;\,\psi}v^{*}(T),\,{}^{C}D_{b}^{\theta;\,\psi}v^{*}(T))-\varphi(T,\,v(T),\,I_{b}^{\delta;\,\psi}v(T),\,{}^{C}D_{b}^{\theta;\,\psi}v(T))|\\
\leq&\tilde{\Omega}_{r+\gamma}\varepsilon+(I_{b}^{r+\gamma-\theta;\,\psi})(L(\|v^{*}-v\|+K_{\delta;\,1}^{\psi}(T)\|v^{*}-v\|+\|{}^{C}D_{b}^{\theta;\,\psi}v^{*}-{}^{C}D_{b}^{\theta;\,\psi}v\|))+\kappa(I_{b}^{\gamma-\theta;\,\psi})\|v^{*}-v\|\\
&+\frac{K_{\gamma-\theta,\,\vartheta}^{\psi}(t)}{|\Xi|}\Big[\rho(I_{b}^{r+\gamma+q_{1};\,\psi})((L\|v^{*}-v\|+K_{\delta,\,1}^{\psi}(T)\|v^{*}-v\|+\|{}^{C}D_{b}^{\theta;\,\psi}v^{*}-{}^{C}D_{b}^{\theta;\,\psi}v\|))\\
&+\xi(I_{b}^{r+\gamma-q_{2};\,\psi})(L(\|v^{*}-v\|+K_{\delta,\,1}^{\psi}(T)\|v^{*}-v\|+\|{}^{C}D_{b}^{\theta;\,\psi}v^{*}-{}^{C}D_{b}^{\theta;\,\psi}v\|))\\
&+\rho\kappa(I_{b}^{\gamma+q_{1};\,\psi})\|v^{*}-v\|+\xi\kappa(I_{b}^{\gamma-q_{2};\,\psi})\|v^{*}-v\|\Big]\\
\leq&\tilde{\Omega}_{r+\gamma}\varepsilon+\Bigg\{L\Lambda(I_{b}^{r+\gamma;\,\psi})[1]+\kappa(I_{b}^{\gamma;\,\psi})[1]+\frac{K_{\gamma-\theta,\,\vartheta}^{\psi}(T)}{|\Xi|}\Big[L\Lambda\rho(I_{b}^{r+\gamma+q_{1};\,\psi})[1]+L\Lambda\xi(I_{b}^{r+\gamma-q_{2},\,\psi})[1]\\
&+\rho\kappa(I_{b}^{\gamma+q_{1};\,\psi})[1]+\xi\kappa(I_{b}^{\gamma-q_{2};\,\psi})[1]\Big]\Bigg\}\|u-v\|\\
&+\Bigg\{L(I_{b}^{r+\gamma;\,\psi})[1]+\frac{K_{\gamma,\,\vartheta}^{\psi}(T)}{|\Xi|}\Big[\rho L(I_{b}^{r+\gamma+q_{1};\,\psi})[1]+\xi L(I_{b}^{r+\gamma-q_{2};\,\psi})[1]\Big]\Bigg\}\|{}^{C}D_{b}^{\theta;\,\psi}v^{*}-{}^{C}D_{b}^{\theta;\,\psi}v\|\\
\leq&\tilde{\Omega}_{r+\gamma}\varepsilon+\Bigg(L\Lambda\tilde{\Omega}_{r+\gamma}+\kappa\tilde{\Omega}_{\gamma}\Bigg)\|v^{*}-v\|+L\tilde{\Omega}_{r+\gamma}\|{}^{C}D_{b}^{\theta;\,\psi}v^{*}-{}^{C}D_{b}^{\theta;\,\psi}v\|. \qquad (42)
\end{aligned}
$$

From (41) and (42), we get

$$\|v^*-v\|_{\mathscr{V}} \leq (\Omega_{r+\gamma}+\tilde{\Omega}_{r+\gamma})\varepsilon + \left(L\Lambda(\Omega_{r+\gamma}+\tilde{\Omega}_{r+\gamma})+\kappa(\Omega_\gamma+\tilde{\Omega}_\gamma)\right)\|v^*-v\| + L(\Omega_{r+\gamma}+\tilde{\Omega}_{r+\gamma})\|{}^C D_b^{\theta;\,\psi}v^* - {}^C D_b^{\theta;\,\psi}v\|$$

$$\leq (\Omega_{r+\gamma}+\tilde{\Omega}_{r+\gamma})\varepsilon + \left(L\Lambda(\Omega_{r+\gamma}+\tilde{\Omega}_{r+\gamma})+\kappa(\Omega_\gamma+\tilde{\Omega}_\gamma)\right)(\|v^*-v\|_\infty + \|{}^C D_b^{\theta;\,\psi}v^* - {}^C D_b^{\theta;\,\psi}v\|_\infty)$$

$$\leq (\Omega_{r+\gamma}+\tilde{\Omega}_{r+\gamma})\varepsilon + \Delta\|v^*-v\|_{\mathscr{V}}. \tag{43}$$

According to (43), we find

$$\|v^*-v\|_{\mathscr{V}} \leq C_\phi \varepsilon,$$

where $C_\phi = \dfrac{\Omega_{r+\gamma}+\tilde{\Omega}_{r+\gamma}}{1-\Delta}$. Hence, we conclude that the problem (1)-(2) is UHS. □

Now, we make the following assumption for the proof of our next Lemma.

(C4) There exists a nondecreasing function $\varphi_2 \in C([b,\, T],\, \mathbb{R}^+)$ and there exists $L_\psi > 0$, such that

$$I_b^{\alpha;\,\psi}\varphi_2(t) \leq L_\psi \varphi_2(t), \;\; \alpha \in S = \{r+\gamma-\theta,\, r+\gamma-q_2,\, r+\gamma,\, r+\gamma+q_1\}, \;\;\; t\in[b,\, T].$$

Lemma 6 Let $1<r<2$, $0<q_2<\gamma<\beta<1$, $q_1>0$, $\vartheta = r+\beta(1-r)$. Assume that hypothesis (C4) holds, and let $v^* \in \mathscr{V}$ be a solution of inequality (32). Then v^* is a solution of the following integral inequality:

$$|v^*(t)-\omega(t)| \leq \rho L_\psi \chi(t)\varepsilon, \tag{44}$$

and

$$|{}^C D_b^{\theta;\,\psi}v^*(t) - {}^C D_b^{\theta;\,\psi}\omega(t)| \leq \tilde{\rho} L_\psi \chi(t)\varepsilon, \tag{45}$$

where, ρ and $\tilde{\rho}$ are given by (10) and (11) respectively.

Proof. Let v^* denote a solution of the inequality (32). Then, using Remark 2(ii) together with Lemma 4, the solution of problem (34)-(35) for $i=2$ can be expressed as follows:

$$v^*(t) = I_b^{r+\gamma;\,\psi}[\phi(t,\, v^*(t),\, I_b^{\delta;\,\psi}v^*(t),\, {}^C D_b^{\theta;\,\psi}v^*(t)) + \varphi_2(t)] - \kappa I_b^{\gamma;\,\psi}v^*(t)$$

$$+\frac{K_{\gamma,\,\vartheta}^{\psi}(t)}{\Xi}\Big[\rho\kappa I_b^{\gamma+q_1;\,\psi}v^*(\eta) + \xi\kappa I_b^{\gamma-q_2;\,\psi}v^*(T)$$

$$-\rho I_b^{r+\gamma+q_1;\,\psi}[\phi(\eta,\,v^*(\eta),\,I_b^{\delta;\,\psi}v^*(\eta),\,{}^CD_b^{\theta;\,\psi}v^*(\eta))+\varphi_2(t)]$$

$$-\xi I_b^{r+\gamma-q_2;\,\psi}[\phi(T,\,v^*(T),\,I_b^{\delta;\,\psi}v^*(T),\,{}^CD_b^{\theta;\,\psi}v^*(T))+\varphi_2(t)]\Big]. \tag{46}$$

From Remark 2 (i) and (C4), we obtain

$$\begin{aligned}
&|v^*(t)-\omega(t)|\\
=&\Big|v^*(t)-I_b^{r+\gamma;\,\psi}\phi(t,\,v^*(t),\,I_b^{\delta;\,\psi}v^*(t),\,{}^CD_b^{\theta;\,\psi}v^*(t))+\kappa I_b^{\gamma;\,\psi}v^*(t)-\frac{K_{\gamma,\,\vartheta}^{\psi}(t)}{\Xi}\Big[\rho\kappa I_b^{\gamma+q_1;\,\psi}v^*(\eta)+\xi\kappa I_b^{\gamma-q_2;\,\psi}v^*(T)\\
&-\rho I_b^{r+\gamma+q_1;\,\psi}\phi(\eta,\,v^*(\eta),\,I_b^{\delta;\,\psi}v^*(\eta),\,{}^CD_b^{\theta;\,\psi}v^*(\eta))-\xi I_b^{r+\gamma-q_2;\,\psi}\phi(T,\,v^*(T),\,I_b^{\delta;\,\psi}v^*(T),\,{}^CD_b^{\theta;\,\psi}v^*(T))\Big]\Big|\\
\le&|(I_b^{r+\gamma;\,\psi})\varphi_2(t)|+\frac{K_{\gamma,\,\vartheta}^{\psi}(T)}{|\Xi|}\Big[|\rho(I_b^{r+\gamma+q_1;\,\psi})\varphi_2(t)|+|\xi(I_b^{r+\gamma-q_2;\,\psi})\varphi_2(t)|\Big]\\
\le&L_\psi|\varphi_2(t)|+\frac{K_{\gamma,\,\vartheta}^{\psi}(T)}{|\Xi|}\Big[\rho L_\psi|\varphi_2(t)|+\xi L_\psi|\varphi_2(t)|\Big]\\
\le&L_\psi\chi(t)\varepsilon+\frac{K_{\gamma,\,\vartheta}^{\psi}(T)}{|\Xi|}\Big[\rho L_\psi\chi(t)\varepsilon+\xi L_\psi\chi(t)\varepsilon\Big]\\
\le&\Big(1+\frac{(\rho+\xi)}{|\Xi|}K_{\gamma,\,\vartheta}^{\psi}(T)\Big)L_\psi\chi(t)\varepsilon\\
\le&\rho L_\psi\chi(t)\varepsilon,
\end{aligned} \tag{47}$$

where ρ is given in (10), and

$$\begin{aligned}
&|{}^CD_b^{\theta;\,\psi}v^*(t)-{}^CD_b^{\theta;\,\psi}\omega(t)|\\
\le&\Big|{}^CD_b^{\theta;\,\psi}v^*(t)-I_b^{r+\gamma-\theta;\,\psi}\phi(t,\,v^*(t),\,I_b^{\delta;\,\psi}v^*(t),\,{}^CD_b^{\theta;\,\psi}v^*(t))+\kappa I_b^{\gamma-\theta;\,\psi}v^*(t)-\frac{K_{\gamma-\theta,\,\vartheta}^{\psi}(t)}{\Xi}\Big[\rho\kappa I_b^{\gamma+q_1;\,\psi}v^*(\eta)\\
&+\xi\kappa I_b^{\gamma-q_2;\,\psi}v^*(T)-\rho I_b^{r+\gamma+q_1;\,\psi}\phi(\eta,\,v^*(\eta),\,I_b^{\delta;\,\psi}v^*(\eta),\,{}^CD_b^{\theta;\,\psi}v^*(\eta))\\
&-\xi I_b^{r+\gamma-q_2;\,\psi}\phi(T,\,v^*(T),\,I_b^{\delta;\,\psi}v^*(T),\,{}^CD_b^{\theta;\,\psi}v^*(T))\Big]\Big|
\end{aligned}$$

$$\leq |(I_b^{r+\gamma-\theta;\,\psi})\varphi_2(t)| + \frac{K^{\psi}_{\gamma-\theta,\,\vartheta}(T)}{|\Xi|}\Big[|\rho(I_b^{r+\gamma+q_1;\,\psi})\varphi_2(t)| + |\xi(I_b^{r+\gamma-q_2;\,\psi})\varphi_2(t)|\Big]$$

$$\leq L_\psi|\varphi_2(t)| + \frac{K^{\psi}_{\gamma-\theta,\,\vartheta}(T)}{|\Xi|}\Big[\rho L_\psi|\varphi_2(t)| + \xi L_\psi|\varphi_2(t)|\Big]$$

$$\leq L_\psi\chi(t)\varepsilon + \frac{K^{\psi}_{\gamma-\theta,\,\vartheta}(T)}{|\Xi|}\Big[\rho L_\psi\chi(t)\varepsilon + \xi L_\phi\chi(t)\varepsilon\Big]$$

$$\leq\Big[1 + \frac{(\rho+\xi)}{|\Xi|}K^{\psi}_{\gamma-\theta,\,\vartheta}(T)\Big]L_\psi\chi(t)\varepsilon$$

$$\leq\tilde{\rho}L_\psi\chi(t)\varepsilon, \tag{48}$$

where $\tilde{\rho}$ is given in (11). □

Theorem 6 If (C1)-(C2) are satisfied, then the problem (1)-(2) is UHRS, namely, assume that $v^* \in \mathscr{V}$ is a solution of inequality (32) and $v \in \mathscr{V}$ is a unique solution of the problem (1)-(2), such that

$$\|v^* - v\|_{\mathscr{V}} \leq \tilde{C}_\phi\chi(t)\varepsilon. \tag{49}$$

Proof. Under the assumptions (C1) and (C2), Theorem 3 ensures that the problem (1)-(2) admits a unique solution $v \in \mathscr{V}$ satisfying the integral equation (39). Let v^* be a solution of the inequality (32) as before. Then, for any $\varepsilon > 0$, it follows from Remark 2(ii) that v^* satisfies the integral equation given by

$$\begin{aligned} v^*(t) =& I_b^{r+\gamma;\,\psi}[\phi(t,\, v^*(t),\, I_b^{\delta;\,\psi}v^*(t),\, {}^{C}D_b^{\theta;\,\psi}v^*(t)) + \varphi_2(t)] - \kappa I_b^{\gamma;\,\psi}v^*(t) + \frac{K^{\psi}_{\gamma,\,\vartheta}(t)}{\Xi}\Big[\rho\kappa I_b^{\gamma+q_1;\,\psi}v^*(\eta) \\ &+ \xi\kappa I_b^{\gamma-q_2;\,\psi}v^*(T) - \rho I_b^{r+\gamma+q_1;\,\psi}[\phi(\eta,\, v^*(\eta),\, I_b^{\delta;\,\psi}v^*(\eta),\, {}^{C}D_b^{\theta;\,\psi}v^*(\eta)) + \varphi_2(t)] \\ &- \xi I_b^{r+\gamma-q_2;\,\psi}[\phi(T,\, v^*(T),\, I_b^{\delta;\,\psi}v^*(T),\, {}^{C}D_b^{\theta;\,\psi}v^*(T)) + \varphi_2(t)]\Big]. \end{aligned} \tag{50}$$

By Lemma 6, we can derive the following inequality:

$$\begin{aligned} |v^*(t) - v(t)| \leq& |v^*(t) - \omega(t)| + |\omega(t) - v(t)| \\ \leq& \rho L_\psi\chi(t)\varepsilon + I_b^{r+\gamma;\,\psi}|\phi(t,\, v^*(t),\, I_b^{\delta;\,\psi}v^*(t),\, {}^{C}D_b^{\theta;\,\psi}v^*(t)) - \phi(t,\, v(t),\, I_b^{\delta;\,\psi}v(t),\, {}^{C}D_b^{\theta;\,\psi}v(t))| \end{aligned}$$

$$
\begin{aligned}
&+\kappa I_b^{\gamma;\,\psi}|v^*(t)-v(t)|+\frac{K^{\psi}_{\gamma,\,\vartheta}(t)}{|\Xi|}\Big[\rho\kappa I_b^{\gamma+q_1;\,\psi}|v^*(\eta)-v(\eta)|+\xi\kappa I_b^{\gamma-q_2;\,\psi}|v^*(T)-v(T)|\\
&+\rho I_b^{r+\gamma+q_1;\,\psi}|\phi(\eta,\,v^*(\eta),\,I_b^{\delta;\,\psi}\eta^*(T),\,{}^{C}D_b^{\theta;\,\psi}\eta^*(T))-\phi(\eta,\,v(\eta),\,I_b^{\delta;\,\psi}v(\eta),\,{}^{C}D_b^{\theta;\,\psi}v(\eta))|\\
&+\xi I_b^{r+\gamma-q_2;\,\psi}|\phi(T,\,v^*(T),\,I_b^{\delta;\,\psi}v^*(T),\,{}^{C}D_b^{\theta;\,\psi}v^*(T))-\phi(T,\,v(T),\,I_b^{\delta;\,\psi}v(T),\,{}^{C}D_b^{\theta;\,\psi}v(T))|\\
\le&\rho L_\psi\chi(t)\varepsilon+(I_b^{r+\gamma;\,\psi})(L(\|v^*-v\|+K^{\psi}_{\delta;\,1}(T)\|v^*-v\|+\|{}^{C}D_b^{\theta;\,\psi}v^*-{}^{C}D_b^{\theta;\,\psi}v\|))+\kappa(I_b^{\gamma;\,\psi})\|v^*-v\|\\
&+\frac{K^{\psi}_{\gamma,\,\vartheta}(t)}{|\Xi|}\Big[\rho(I_b^{r+\gamma+q_1;\,\psi})((L\|v^*-v\|+K^{\psi}_{\delta,\,1}(T)\|v^*-v\|+\|{}^{C}D_b^{\theta;\,\psi}v^*-{}^{C}D_b^{\theta;\,\psi}v\|))\\
&+\xi(I_b^{r+\gamma-q_2;\,\psi})(L(\|v^*-v\|+K^{\psi}_{\delta,\,1}(T)\|v^*-v\|+\|{}^{C}D_b^{\theta;\,\psi}v^*-{}^{C}D_b^{\theta;\,\psi}v\|))\\
&+\rho\kappa(I_b^{\gamma+q_1;\,\psi})\|v^*-v\|+\xi\kappa(I_b^{\gamma-q_2;\,\psi})\|v^*-v\|\Big]\\
\le&\rho L_\psi\chi(t)\varepsilon+\Bigg\{L\Lambda(I_b^{r+\gamma;\,\psi})[1]+\kappa(I_b^{\gamma;\,\psi})[1]+\frac{K^{\psi}_{\gamma,\,\vartheta}(T)}{|\Xi|}\Big[L\Lambda\rho(I_b^{r+\gamma+q_1;\,\psi})[1]+L\Lambda\xi(I_b^{r+\gamma-q_2,\,\psi})[1]\\
&+\rho\kappa(I_b^{\gamma+q_1;\,\psi})[1]+\xi\kappa(I_b^{\gamma-q_2;\,\psi})[1]\Big]\Bigg\}\|u-v\|\\
&+\Bigg\{L(I_b^{r+\gamma;\,\psi})[1]+\frac{K^{\psi}_{\gamma,\,\vartheta}(T)}{|\Xi|}\Big[\rho L(I_b^{r+\gamma+q_1;\,\psi})[1]+\xi L(I_b^{r+\gamma-q_2;\,\psi})[1]\Big]\Bigg\}\|{}^{C}D_b^{\theta;\,\psi}u-{}^{C}D_b^{\theta;\,\psi}v\|\\
\le&\rho L_\psi\chi(t)\varepsilon+\Bigg(L\Lambda\Omega_{r+\gamma}+\kappa\Omega_\gamma\Bigg)\|v^*-v\|+L\Omega_{r+\gamma}\|{}^{C}D_b^{\theta;\,\psi}v^*-{}^{C}D_b^{\theta;\,\psi}v\|,
\end{aligned}
\tag{51}
$$

and

$$
\begin{aligned}
&|{}^{C}D_b^{\theta;\,\psi}v^*(t)-{}^{C}D_b^{\theta;\,\psi}v(t)|\\
\le&|{}^{C}D_b^{\theta;\,\psi}v^*(t)-{}^{C}D_b^{\theta;\,\psi}\omega(t)|+|{}^{C}D_b^{\theta;\,\psi}\omega(t)-{}^{C}D_b^{\theta;\,\psi}v(t)|\\
\le&\tilde{\rho}L_\psi\chi(t)\varepsilon+I_b^{r+\gamma-\theta;\,\psi}|\phi(t,\,v^*(t),\,I_b^{\delta;\,\psi}v^*(t),\,{}^{C}D_b^{\theta;\,\psi}v^*(t))-\phi(t,\,v(t),\,I_b^{\delta;\,\psi}v(t),\,{}^{C}D_b^{\theta;\,\psi}v(t))|
\end{aligned}
$$

$$
\begin{aligned}
&+\kappa I_b^{\gamma-\theta;\,\psi}|v^*(t)-v(t)|+\frac{K^{\psi}_{\gamma-\theta,\,\vartheta}(t)}{|\Xi|}\Big[\rho\kappa I_b^{\gamma+q_1;\,\psi}|v^*(\eta)-v(\eta)|+\xi\kappa I_b^{\gamma-q_2;\,\psi}|v^*(T)-v(T)|\\
&+\rho I_b^{r+\gamma+q_1;\,\psi}|\phi(\eta,\,v^*(\eta),\,I_b^{\delta;\,\psi}\eta^*(T),\,{}^{C}D_b^{\theta;\,\psi}\eta^*(T))-\phi(\eta,\,v(\eta),\,I_b^{\delta;\,\psi}v(\eta),\,{}^{C}D_b^{\theta;\,\psi}v(\eta))|\\
&+\xi I_b^{r+\gamma-q_2;\,\psi}|\phi(T,\,v^*(T),\,I_b^{\delta;\,\psi}v^*(T),\,{}^{C}D_b^{\theta;\,\psi}v^*(T))-\varphi(T,\,v(T),\,I_b^{\delta;\,\psi}v(T),\,{}^{C}D_b^{\theta;\,\psi}v(T))|\\
\le&\tilde{\rho}L_{\psi}\chi(t)\varepsilon+(I_b^{r+\gamma-\theta;\,\psi})(L(\|v^*-v\|+K^{\psi}_{\delta;\,1}(T)\|v^*-v\|+\|{}^{C}D_b^{\theta;\,\psi}v^*-{}^{C}D_b^{\theta;\,\psi}v\|))+\kappa(I_b^{\gamma-\theta;\,\psi})\|v^*-v\|\\
&+\frac{K^{\psi}_{\gamma-\theta,\,\vartheta}(t)}{|\Xi|}\Big[\rho(I_b^{r+\gamma+q_1;\,\psi})((L\|v^*-v\|+K^{\psi}_{\delta,\,1}(T)\|v^*-v\|+\|{}^{C}D_b^{\theta;\,\psi}v^*-{}^{C}D_b^{\theta;\,\psi}v\|))\\
&+\xi(I_b^{r+\gamma-q_2;\,\psi})(L(\|v^*-v\|+K^{\psi}_{\delta,\,1}(T)\|v^*-v\|+\|{}^{C}D_b^{\theta;\,\psi}v^*-{}^{C}D_b^{\theta;\,\psi}v\|))\\
&+\rho\kappa(I_b^{\gamma+q_1;\,\psi})\|v^*-v\|+\xi\kappa(I_b^{\gamma-q_2;\,\psi})\|v^*-v\|\Big]\\
\le&\tilde{\rho}L_{\psi}\chi(t)\varepsilon+\Bigg\{L\Lambda(I_b^{r+\gamma;\,\psi})[1]+\kappa(I_b^{\gamma;\,\psi})[1]+\frac{K^{\psi}_{\gamma-\theta,\,\vartheta}(T)}{|\Xi|}\Big[L\Lambda\rho(I_b^{r+\gamma+q_1;\,\psi})[1]+L\Lambda\xi(I_b^{r+\gamma-q_2,\,\psi})[1]\\
&+\rho\kappa(I_b^{\gamma+q_1;\,\psi})[1]+\xi\kappa(I_b^{\gamma-q_2;\,\psi})[1]\Big]\Bigg\}\|u-v\|\\
&+\Bigg\{L(I_b^{r+\gamma;\,\psi})[1]+\frac{K^{\psi}_{\gamma,\,\vartheta}(T)}{|\Xi|}\Big[\rho L(I_b^{r+\gamma+q_1;\,\psi})[1]+\xi L(I_b^{r+\gamma-q_2;\,\psi})[1]\Big]\Bigg\}\|{}^{C}D_b^{\theta;\,\psi}v^*-{}^{C}D_b^{\theta;\,\psi}v\|\\
\le&\tilde{\rho}L_{\psi}\chi(t)\varepsilon+\Bigg(L\Lambda\tilde{\Omega}_{r+\gamma}+\kappa\tilde{\Omega}_{\gamma}\Bigg)\|v^*-v\|+L\tilde{\Omega}_{r+\gamma}\|{}^{C}D_b^{\theta;\,\psi}v^*-{}^{C}D_b^{\theta;\,\psi}v\|.
\end{aligned}\tag{52}
$$

From (51) and (52), we get

$$
\begin{aligned}
\|v^*-v\|_{\mathscr{V}}\le&(\rho+\tilde{\rho})L_{\psi}\chi(t)\varepsilon+\Bigg(L\Lambda(\Omega_{r+\gamma}+\tilde{\Omega}_{r+\gamma})+\kappa(\Omega_{\gamma}+\tilde{\Omega}_{\gamma})\Bigg)\|v^*-v\|+L(\Omega_{r+\gamma}+\tilde{\Omega}_{r+\gamma})\|{}^{C}D_b^{\theta;\,\psi}v^*-{}^{C}D_b^{\theta;\,\psi}v\|\\
\le&(\rho+\tilde{\rho})L_{\psi}\chi(t)\varepsilon+\Bigg(L\Lambda(\Omega_{r+\gamma}+\tilde{\Omega}_{r+\gamma})+\kappa(\Omega_{\gamma}+\tilde{\Omega}_{\gamma})\Bigg)(\|v^*-v\|_{\infty}+\|{}^{C}D_b^{\theta;\,\psi}v^*-{}^{C}D_b^{\theta;\,\psi}v\|_{\infty})
\end{aligned}
$$

$$\leq (\rho+\tilde{\rho})L_{\psi}\chi(t)\varepsilon+\Delta\|v^{*}-v\|_{\mathscr{V}}. \tag{53}$$

According to (53), we find

$$\|v^{*}-v\|\leq \tilde{C}_{\phi}\chi(t)\varepsilon,$$

where $\tilde{C}_{\phi}=\dfrac{(\rho+\tilde{\rho})L_{\psi}}{1-\Delta}$, and ρ, $\tilde{\rho}$ are given in (10) and (11) respectively.

Therefore, we conclude that problem (1)-(2) is UHRS. □

4. Example

Consider the following NLFDE:

$$\begin{cases} {}^{H}D_{0}^{\frac{3}{2},\,\frac{9}{10};\,\psi(t)}\left({}^{C}D_{0}^{\frac{8}{10};\,\psi(t)}+\dfrac{1}{10}\right)v(t)=\phi\left(t,\,v(t),\,I_{0}^{\frac{7}{10};\,\psi(t)}v(t),\,{}^{C}D_{0}^{\frac{6}{10};\,\psi(t)}v(t)\right),\;\; t\in[0,\,1], & (54)\\[2ex] v(0)=0,\;\; \dfrac{3}{7}I_{0}^{\frac{9}{10};\,\psi(t)}v(\eta)+\dfrac{4}{9}{}^{C}D_{0}^{\frac{4}{10};\,\psi(t)}x(T)=0. & (55)\end{cases}$$

For $t\in[0,\,1]$, we have $b=0$, $T=1$, $\eta=\frac{1}{2}$, $r=\frac{3}{2}$, $\beta=\frac{9}{10}$, $\gamma=\frac{8}{10}$, $\theta=\frac{6}{10}$, $q_1=\frac{9}{10}$, $q_2=\frac{4}{10}$, $\delta=\frac{7}{10}$, $\kappa=\frac{1}{10}$, $\rho=\frac{3}{7}$, $\xi=\frac{4}{9}$, $\vartheta=r+\beta(1-r)=\frac{21}{20}$. Assume that the nonlinear function ϕ is given by:

$$\phi\left(t,\,v(t),\,I_{0}^{\frac{7}{10};\,\psi(t)}v(t),\,{}^{C}D_{0}^{\frac{6}{10};\,\psi(t)}v(t)\right)=\frac{\sin(t)}{\exp(t^2)+99}\left(\frac{|v(t)|}{|v(t)|+1}+\frac{|I_{0}^{\frac{7}{10};\,\psi(t)}v(t)|}{1+|I_{0}^{\frac{7}{10};\,\psi(t)}v(t)|}+\frac{|{}^{C}D_{0}^{\frac{6}{10};\,\psi(t)}v(t)|}{1+|{}^{C}D_{0}^{\frac{6}{10};\,\psi(t)}v(t)|}\right),$$

It is clear that ϕ is continuous. For any u, v, w, $\overline{u}$, $\overline{v}$ and $\overline{w}\in\mathbb{R}$, we can conclude that

$$|\phi(t,\,u,\,v,\,w)-\phi(t,\,\overline{u},\,\overline{v},\,\overline{w})|\leq L(|u-\overline{u}|+|v-\overline{v}|+|w-\overline{w}|$$

$$=\left|\frac{\sin(t)}{\exp(t^2)+99}\left(\left(\frac{u}{1+u}-\frac{\overline{u}}{1+\overline{u}}\right)+\left(\frac{v}{1+v}-\frac{\overline{v}}{1+\overline{v}}\right)+\left(\frac{w}{1+w}-\frac{\overline{w}}{1+\overline{w}}\right)\right)\right|$$

$$\leq\frac{1}{\exp(t^2)+49}\Big((|u-\overline{u}|+|v-\overline{v}|+|w-\overline{w}|\Big)$$

$$\leq\frac{1}{100}\Big(|u-\overline{u}|+|v-\overline{v}|+|w-\overline{w}|\Big).$$

Thus, $L = \dfrac{1}{100}$.

In this numerical example, we consider four different choices for the function ψ, namely:

$$\psi(t) = t,\ \ \psi(t) = t^2 + 1,\ \ \psi(t) = \ln(t+1) \text{ and } \psi(t) = e^t.$$

For $\psi(t) = t$, with a simple calculation, we obtain

$$\Delta \colon = 0.476001 < 1.$$

Since all the hypotheses of **Theorem 3** are satisfied, the problem (54)-(55) admits a unique solution on the interval [0, 1].

To verify condition (C3), we have for any real number v, $\frac{|v|}{1+|v|} \leq 1$.

Thus,

$$0 \leq \frac{|v(t)|}{|v(t)|+1} \leq 1, \quad 0 \leq \frac{|I_0^{7/10;\,\psi} v(t)|}{1+|I_0^{7/10;\,\psi} v(t)|} \leq 1, \quad 0 \leq \frac{|{}^C D_0^{6/10;\,\psi} v(t)|}{1+|{}^C D_0^{6/10;\,\psi} v(t)|} \leq 1.$$

Hence,

$$|\phi(t, u, v, w)| \leq \frac{3}{\exp(b^2)+99} = \sigma(t) \in C([b, T], \mathbb{R}_+), \quad \forall (t, u, v, w) \in [b, T] \times \mathbb{R}^3.$$

Moreover, we have $\kappa\Omega_\gamma \colon = 0.182182 < 1$. Since all hypotheses of **Theorem 4** are satisfied, therefore, the problem (54)-(55) has at least one solution on [0, 1].

Consequently $\exists\, C_\omega \colon = 9.363418 > 0$, by a direct corollary of **Theorem 5**, the problem (54)-(55) is UHS.

Setting $\varphi_2(t) = (\psi(t) - \psi(0))^{\frac{1}{2}} \in C([0, 1], \mathbb{R})$, we have

$$\begin{aligned} I_0^{\alpha,\,\psi} \varphi_2(t) =& \frac{\Gamma(\frac{3}{2})}{\Gamma(\alpha+\frac{3}{2})} (\psi(t) - \psi(0))^{\alpha+\frac{1}{2}} \\ \leq& \frac{\Gamma(\frac{3}{2})}{\Gamma(\alpha+\frac{3}{2})} (\psi(T) - \psi(0))^{\alpha} (\psi(t) - \psi(0))^{\frac{1}{2}} \\ \leq& L_\psi (\psi(t) - \psi(0))^{\frac{1}{2}} \\ \leq& L_\psi \varphi_2(t),\ \ t \in [b, T], \end{aligned}$$

where $L_\psi = \max_{\alpha \in S} \left\{ \dfrac{\Gamma(\frac{3}{2})}{\Gamma(\alpha+\frac{3}{2})} (\psi(T) - \psi(0))^{\alpha} \right\},\ S = \{r+\gamma+q_1,\ r+\gamma,\ r+\gamma-q_2\}$.

Therefore, (C4) satisfies $L_\psi = 0.365610 > 0$. Since $\exists\, \tilde{C}_\phi := 2.050946 > 0$, the problem (54)-(55) is UHRS according to **Theorem 6**.

Finally, we summarize the comparison for all cases of ψ as in Table 1 (which computed using MATLAB (version R2016a(9.0.0.341360))).

Table 1. Numerical values for different parameters

$\psi(t)$	Δ	$\kappa\Omega_\gamma$	L_ψ	C_ϕ (UHS)	$\tilde{C}_\phi$ (UHRS)
t	0.476001	0.182182	0.365610	9.363418	2.050946
t^2+1	0.453332	0.175014	0.365610	8.276482	1.905704
$\ln(t+1)$	0.433469	0.140728	0.196075	11.113318	1.023827
e^t	0.596785	0.257712	0.917654	12.330105	6.662727

5. Conclusion

This work has contributed to the advancement of fractional calculus by investigating the existence, uniqueness, and stability of NLFDE involving a hybrid combination of ψ-Hilfer and ψ-Caputo fractional derivatives. This formulation leverages the distinctive features of both operators under mixed nonlocal boundary conditions. Two main theorems were established to ensure the well-posedness of the problem: the first guarantees the uniqueness of the solution via Banach's contraction principle, while the second confirms the existence of at least one solution using Krasnoselskii's fixed-point theorem. Moreover, the stability analysis was conducted in the sense of UHS and UHRS, highlighting the robustness of the obtained solutions against small perturbations. Finally, a numerical illustration was presented to validate the theoretical findings. The results, summarized in Table 1, compare the numerical constants associated with existence, uniqueness, and stability for different choices of the function $\psi(t)$.

The present analysis may be extended by reformulating the problem using new fractional operators with non-singular kernels, such as the Atangana-Baleanu or Prabhakar-type derivatives. In addition, future investigations could consider more general boundary conditions or incorporate time-delay effects to broaden the applicability of the proposed framework.

Data availability

This study was not supported by actual data.

Confilict of interest

The authors declare no competing financial interest.

References

[1] Podlubny I. *Fractional Differential Equations*. Academic Press; 1999.
[2] Metzler R, Jeon JH, Cherstvy AG, Barkai E. Anomalous diffusion models and their properties: non-stationarity, non-ergodicity, and ageing at the centenary of single particle tracking. *Physical Chemistry Chemical Physics*. 2014; 16(44): 24128-24164. Available from: https://doi.org/10.1039/C4CP03465A.
[3] Mouaouine A, Boukhouima A, Hattaf K, Yousfi N. A fractional order SIR epidemic model with nonlinear incidence rate. *Advances in Difference Equations*. 2018; 160. Available from: https://doi.org/10.1186/s13662-018-1613-z.

[4] Bachraoui M, Hattaf K, Yousfi N. Dynamics of a fractional order HBV infection model with capsids and CTL immune response. *Communications in Mathematical Biology and Neuroscience*. 2019; 2019: 6. Available from: https://doi.org/10.28919/cmbn/3835.

[5] Ghanbari B, Djilali S. Mathematical and numerical analysis of a three-species predator-prey model with herd behavior and time fractional-order derivative. *Mathematical Methods in the Applied Sciences*. 2020; 43(4): 1736-1752. Available from: https://doi.org/10.1002/mma.5999.

[6] Acay B, Bas E, Abdeljawad T. Fractional economic models based on market equilibrium in the frame of different type kernels. *Chaos, Solitons & Fractals*. 2020; 130: 109438. Available from: https://doi.org/10.1016/j.chaos.2019.109438.

[7] Derdar N. Nonlinear implicit Caputo-Hadamard fractional differential equation with fractional boundary conditions. *Jordan Journal of Mathematics and Statistics*. 2022; 15(4B): 999-1014. Available from: https://doi.org/10.47013/15.4.14.

[8] Kilbas AA, Srivastava HM, Trujillo JJ. *Theory and Applications of Fractional Differential Equations*. Elsevier; 2006.

[9] Samko SG, Kilbas AA, Mariche OI. *Fractional Integrals and Derivatives*. Gordon and Breach; 1993.

[10] Almeida R. A Caputo fractional derivative of a function with respect to another function. *Communications in Nonlinear Science and Numerical Simulation*. 2017; 44: 460-481. Available from: https://doi.org/10.1016/j.cnsns.2016.09.006.

[11] Almeida R, Malinowska A, Monteiro M. Fractional differential equations with a Caputo derivative with respect to a kernel function and their applications. *Mathematical Methods in the Applied Sciences*. 2018; 41(1): 336-352. Available from: https://doi.org/10.1002/mma.4617.

[12] Langevin P. On the theory of Brownian motion. *Comptes Rendus*. 1908; 146: 530-533.

[13] Bouchaud JP, Cont RA. Langevin approach to stock market fluctuations and crashes. *The European Physical Journal B-Condensed Matter and Complex Systems*. 1998; 6(4): 543-550. Available from: https://doi.org/10.1007/s100510050582.

[14] Kosinski RA, Grabowski A. Langevin equations for modeling evacuation processes. *Acta Physica Polonica B Proceedings Supplement*. 2010; 3(2): 365-376.

[15] Hinch EJ. Application of the Langevin equation to fluid suspensions. *Journal of Fluid Mechanics*. 1975; 72(3): 499-511. Available from: https://doi.org/10.1017/S0022112075003102.

[16] Fraaije J, Zvelindovsky AV, Sevink GJA, Maurits NM. Modulated self-organization in complex amphiphilic systems. *Molecular Simulation*. 2000; 25(3-4): 131-144.

[17] Wodkiewicz K, Zubairy KMS. Exact solution of a nonlinear Langevin equation with applications to photoelectron counting and noise-induced instability. *Journal of Mathematical Physics*. 1983; 24: 1401-1404. Available from: https://doi.org/10.1063/1.525874.

[18] Schluttig J, Alamanova JJD, Helms V, Schwarz US. Dynamics of proteinprotein encounter: a Langevin equation approach with reaction patches. *The Journal of Chemical Physics*. 2008; 129(15): 155106. Available from: https://doi.org/10.1063/1.2996082.

[19] Lim SC, Li M, Teo LP. Langevin equation with two fractional orders. *Physics Letters A*. 2008; 372: 6309-6320. Available from: https://doi.org/10.1016/j.physleta.2008.08.045.

[20] Belhamiti MM, Dahmani Z, Sarikaya MZ. Two order nonlinear fractional sequential Langevin equation with new chaotic dynamics. *Communications Faculty of Sciences University of Ankara Series A1: Mathematics and Statistics*. 2023; 72(3): 663-685. Available from: https://doi.org/10.31801/cfsuasmas.1126025.

[21] Bezziou M, Dahmani Z, Ndiaye A. Langevin differential equation of fractional order in non compactness Banach space. *Journal of Interdisciplinary Mathematics*. 2020; 23(4): 857-876. Available from: https://doi.org/10.1080/09720502.2020.1730515.

[22] Lozinski A, Owens RG, Phillips TN. The langevin and fokker-planck equations in polymer rheology. *Handbook of Numerical Analysis*. 2011; 16: 211-303. Available from: https://doi.org/10.1016/B978-0-444-53047-9.00002-2.

[23] Uranagas M, Munakata T. Generalized Langevin equation revisited: mechanical random force and self-consistent structure. *Journal of Physics A: Mathematical and Theoretical*. 2010; 43(45): 455003. Available from: https://doi.org/10.1088/1751-8113/43/45/455003.

[24] Derdar N. Nonlinear Langevin fractional differential equation with nonlocal mixed boundary conditions involving a Caputo-exponential. *TWMS Journal of Applied and Engineering Mathematics*. 2025; 15(8): 1901-1919.

[25] Almaghamsi I, Alghamdi A, Ghanmi A. Existence of solution for a Langevin equation involving the ψ-Hilfer fractional derivative: a variational approach. *AIMS Mathematics*. 2025; 10(1): 534-550.

[26] Boutiara A, Abdo MS, Alqudah MA, Abdeljawad T. On a class of Langevin equations in the frame of Caputo function-dependent-kernel fractional derivatives with antiperiodic boundary conditions. *AIMS Mathematics*. 2021; 6(6): 5518-5534. Available from: https://doi.org/10.3934/math.2021327.

[27] Ali SM, Abdo MS. Qualitative analysis for multiterm Langevin systems with generalized Caputo fractional operators of different orders. *Mathematical Problems in Engineering*. 2022; 2022(1): 1879152. Available from: https://doi.org/10.1155/2022/1879152.

[28] Ulam SM. *A Collection of Mathematical Problems*. Interscience Publishers; 1960.

[29] Bouzida H, Benalia A, Salim B, Erhan IC. Existence and Ulam stability results of hybrid Langevin pantograph ψ-fractional coupled systems. *Filomat*. 2025; 39(10): 3401-3424. Available from: https://doi.org/10.2298/FIL2510401B.

[30] Chen G, Jinbo N, Fu X. Existence and Ulam's types stability of higher-order fractional Langevin equations on a star graph. *AIMS Mathematics*. 2024; 9(5): 11877-11909. Available from: https://doi.org/10.3934/math.2024581.

[31] Cheng H, Naila, Zada A, Popa IL, Kallekh A. (k, ϕ)-Hilfer fractional Langevin differential equation having multipoint boundary conditions. *Boundary Value Problems*. 2024; 113: 1-25. Available from: https://doi.org/10.1186/s13661-024-01918-3.

[32] Rassias TM. On the stability of linear mappings in Banach spaces. *Proceedings of the American Mathematical Society*. 1978; 72: 297-300.

[33] Ullah Khan R, Samreen M, Ali G, Popa IL. New Results on the Stability and Existence of Langevin Fractional Differential Equations with Boundary Conditions. *Fractal and Fractional*. 2025; 9(2): 127. Available from: https://doi.org/10.3390/fractalfract9020127.

[34] Dhaniya S, Kumar A, Khan A, Abdeljawad T. Stability analysis of a class of Langevin equations in the frame of generalized Caputo fractional operator with nonlocal boundary conditions. *Boundary Value Problems*. 2025; 2025: 6309-6320. Available from: https://doi.org/10.1186/s13661-025-02024-8.

[35] Sousa JVC, De Oliveira EC. On the ψ-Hilfer fractional derivative. *Communications in Nonlinear Science and Numerical Simulation*. 2018; 60: 72-91.

[36] Granas A, Dugundji J. *Fixed Point Theory*. Springer; 2003.

[37] Krasnoselsky MA. Two remarks on the method of successive approximation. *Russian Mathematical Surveys*. 1955; 10(1): 123-127.