Contemporary Mathematics
https://ojs.wiserpub.com/index.php/CM/

Research Article

Percentage Points for Testing Equality of Two Covariance Matrices Under Intraclass Correlation Structure

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Received: 25 September 2025; **Revised:** 24 November 2025; **Accepted:** 1 December 2025

Abstract: By employing the inverse Mellin transform and the calculus of residues, we derive the exact null distribution of the likelihood ratio statistic for testing the homogeneity of covariance matrices of two p-variate Gaussian populations having compound symmetry. As a practical component of this work, critical values (percentage points) have been computed for $p = 3(1)9$.

***Keywords*:** test criterion, intraclass correlation, null moments, inverse Mellin transform

MSC: 62E10, 62E15, 62H10

1. Introduction

The term intraclass correlation was originally coined by Fisher [1] to describe a multivariate population characterized by equal variances and equal covariances. This specific structure is commonly encountered in repeated measures designs which are not time-dependent [2]. When the assumption of interchangeability of order of responses is assumed, the covariances are said to be exchangeable, a term that is synonymous with compound symmetry. Gaussian models featuring compound symmetry are frequently applied to study symmetries in animals and plants and have also demonstrated significant utility in applied fields like medical research and psychometrics.

When the dispersion matrix Σ of a p-variate distribution exhibits compound symmetry, it can be expressed as

$$\Sigma_{vc} = \sigma^2[(1-\rho)I_p + \rho J],$$

where I_p denotes an identity matrix of order p, J is a $p \times p$ matrix having each element equals to unity; σ^2 and ρ are known scalars, $\sigma^2 \in (0, \infty)$ and $\rho \in (-1/(p-1), 1)$.

The problem of testing $H_{vc} : \Sigma = \Sigma_{vc}$, for the multivariate Gaussian distribution, was first considered by Wilks [3] who obtained the likelihood ratio statistic Λ_{vc} and computed the distribution $p = 2$ and $p = 3$. A Statistical Analysis System (SAS) program that computes Λ_{vc} for testing the compound symmetry was developed by Khattree and Naik [4] p.158.

DOI: https://doi.org/10.37256/cm.7120268725

Nagar et al. [5] have tabulated the exact percentage points for $p = 4(1)8$. The problem of testing H_{vc} has important applications in areas such as medical research and psychometrics. Furthermore, such models also emerge in the analysis of familial data [6]. For more general structures of the covariance matrices, the reader is referred to Votaw [7], Votaw et al. [8], Szatrowski [9], Olkin and Press [10], Olkin [11], Quereshi [12], Coelho and Marques [13]. For some recent work, reference may be made to Coelho and Roy [14], Jurková et al. [15], Tsukada [16], and Zhao et al. [17].

Let $\Pi_1, \ldots, \Pi_q$ be q independent p-variate normal populations with mean vectors $\mu_1, \ldots, \mu_q$ and positive definite covariance matrices $\Sigma_1, \ldots, \Sigma_q$, respectively. Let samples of sizes $N_1, \ldots, N_q$ be available from these q populations.

Assume that Σ_g has intra-class correlation structure. That is

$$\Sigma_g = \Sigma_{g,\, vc} = \sigma_g^2[(1-\rho_g)I_p + \rho_g J], \quad g = 1, \ldots, q,$$

where, for $g = 1, 2, \ldots, q$, $\sigma_g^2 \in (0, \infty)$ and $\rho_g \in (-1/(p-1), 1)$ are unknown scalars. Consider the hypothesis

$$H_q(VC|vc) : \Sigma_{1,\, vc} = \cdots = \Sigma_{q,\, vc} = \Sigma_{vc} \tag{1}$$

against the alternative K which states that $H_q(VC|vc)$ is not true. The modified likelihood ratio statistic for testing $H_q(VC|vc)$ can be stated as

$$\Lambda_q^*(VC|vc) = \frac{n_0^{pn_0/2}}{\prod_{g=1}^q n_g^{pn_g/2}} \frac{\prod_{g=1}^q \left[[\mathrm{tr}\,((pI_p - J)A_g)]^{p-1}\,\mathrm{tr}\,(JA_g)\right]^{n_g/2}}{\left[[\mathrm{tr}\,((pI_p - J)A)]^{p-1}\,\mathrm{tr}(JA)\right]^{n_0/2}},$$

where A_g/n_g is the sample variance covariance matrix formed from the g-th sample, $A = \sum_{g=1}^q A_g$, $n_g = N_g - 1$, $g = 1, \ldots, q$, and $n_0 = \sum_{g=1}^q ng$. The h-th null moment of the modified likelihood ratio statistic $\Lambda_q^*(VC|vc)$, derived by Han [18], can be expressed as

$$E(\Lambda_q^{*h}(VC|vc)) = \frac{n_0^{n_0 ph/2}}{\prod_{g=1}^q n_g^{n_g ph/2}} \frac{\Gamma(n_0/2)\,\Gamma[n_0(p-1)/2]}{\Gamma[n_0(1+h)/2]\,\Gamma[n_0(p-1)(1+h)/2]} \prod_{g=1}^q \frac{\Gamma[n_g(1+h)/2]\,\Gamma[n_g(p-1)(1+h)/2]}{\Gamma(n_g/2)\,\Gamma[n_g(p-1)/2]},$$

where $n_g > 0$, $g = 1, \ldots, q$ and $\mathrm{Re}(h) > -m/2$. When $n_1 = \cdots = n_q = n$, the h-th null moment of $V = [\Lambda_q^*(VC|vc)]^{2/n}$ simplifies to

$$E(V^h) = q^{qph} \frac{\Gamma(nq/2)\Gamma[nq(p-1)/2]}{\Gamma[q(n/2+h)]\Gamma[q(p-1)(n/2+h)]} \frac{\Gamma^q(n/2+h)\Gamma^q[(p-1)(n/2+h)]}{\Gamma^q(n/2)\Gamma^q[n(p-1)/2]}. \tag{2}$$

For the univariate case ($p = 1$), the hypothesis in (1) simplifies to the standard Neyman-Pearson hypothesis for testing the homogeneity of variances of Gaussian models. Han [18] has shown that the test based on the modified Likelihood Ratio Criterion (LRC) is better than the test derived by using Roy's union intersection procedure (see [19, 20]). Gupta and Nagar [21] derived the asymptotic nonnull distribution of a constant multiple of $-2\ln\Lambda_q^*(VC|vc)$. For $p = 2$ and $p = 3$, the exact distribution and percentage points of $[\Lambda_q^*(VC|vc)]^{2/n}$, for $n_1 = \cdots = n_q = n$, are obtained in Gupta and Nagar [21] and Gupta et al. [22].

In this article, we consider the case of two multivariate normal populations. We derive the exact distribution of $[\Lambda_2^*(VC|vc)]^{2/n}$ for testing $H_2(VC|vc)$ by using the inverse Mellin transform and the residue theorem [5, 23–26]. By using properties and results on gamma, psi and Riemann zeta functions [27, 28], we give the density for p-even and p-odd in series form. By using distributional results derived in this article and suitable software such as Mathematica, we obtain the significance points for the test statistic $[\Lambda_2^*(VC|vc)]^{2/n}$.

2. The density of V

Substituting $q = 2$ in (2) and using Gauss-Legendre multiplication formula for gamma function, namely,

$$\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right),$$

and simplifying, the h-th moment of V is

$$E(V^h) = \frac{\Gamma(n/2+h)\Gamma[(n+1)/2]}{\Gamma(n/2)\Gamma[(n+1)/2+h]}\prod_{k=0}^{p-2}\frac{\Gamma[n/2+k/(p-1)+h]\Gamma[n/2+1/(2(p-1))+k/(p-1)]}{\Gamma[n/2+k/(p-1)]\Gamma[n/2+1/(2(p-1))+k/(p-1)+h]}.$$

Now, using the inverse Mellin transform and the above moment expression, the density of V is obtained as

$$f(v) = K(n,\, p)(2\pi\imath)^{-1}\int_C \frac{\Gamma(n/2+h)}{\Gamma[(n+1)/2+h]}\prod_{k=0}^{p-2}\frac{\Gamma[n/2+k/(p-1)+h]}{\Gamma[n/2+1/(2(p-1))+k/(p-1)+h]}v^{-1-h}dh, \tag{3}$$

where $0 < v < 1$, $\imath = \sqrt{-1}$, C is a suitable contour containing poles of the integrand and

$$K(n,\, p) = \frac{\Gamma[(n+1)/2]}{\Gamma(n/2)}\prod_{k=0}^{p-2}\frac{\Gamma[n/2+1/(2(p-1))+k/(p-1)]}{\Gamma[n/2+k/(p-1)]}.$$

Substituting $n/2+h = t$ and simplifying, the density (3) is restated as

$$f(v) = K(n,\, p)(2\pi\imath)^{-1}v^{(n-2)/2}\int_{C_1}\Delta(t)v^{-t}dt, \quad 0 < v < 1,$$

where the contour C_1 encloses the poles of the integrand (for the existence of such a contour the reader is referred to Luke [29] p.143). For p even

$$\Delta(t) = \frac{\Gamma^2(t)\prod_{k=1}^{p-2}\Gamma[t+k/(p-1)]}{\Gamma(t+1/2)\prod_{k=0}^{p-2}\Gamma[t+1/(2(p-1))+k/(p-1)]}$$

and for p odd

$$\Delta(t)=\frac{\Gamma^2(t)\prod_{k(\neq(p-1)/2)=1}^{p-2}\Gamma[t+k/(p-1)]}{\prod_{k=0}^{p-2}\Gamma[t+1/(2(p-1))+k/(p-1)]}.$$

The poles of the integrand, for p even, are available by equating to zero each factor of $\prod_{i=0}^{\infty}\prod_{j=0}^{p-2}(t+i+j/(p-1))^{a_{ij}}$ where a_{ij} gives the order of the pole at $t=-i-j/(p-1)$. The order a_{ij} is given by $a_{i0}=2$ and $a_{ij}=1$ for $j\geq 1$.

The poles of the integrand, for p odd, are available by equating to zero each factor of $\prod_{i=0}^{\infty}\prod_{j(\neq(p-1)/2)=0}^{p-2}(t+i+j/(p-1))^{a_{ij}}$ where a_{ij} gives the order of the pole at $t=-i-j/(p-1)$. The order a_{ij} is given by $a_{i0}=2$ and $a_{ij}=1$ for $j\geq 1$.

Hence, by the residue theorem, the density for p even, is given by

$$f(v)=K(n,\,p)^{(n-2)/2}\sum_{i=0}^{\infty}\sum_{j=0}^{p-2}R_{ij},\quad 0<v<1, \tag{4}$$

where R_{ij} is the residue at $t=-i-j/(p-1)$. From the calculus of residues, the residue at $t=-i-j/(p-1)$, $j\geq 1$, is derived as

$$\begin{aligned}R_{ij}&=\lim_{t\to -i-j/(p-1)}\left[\left(t+i+\frac{j}{p-1}\right)\Delta(t)v^{-t}\right]\\&=\lim_{t\to -i-j/(p-1)}\left[A_{ij}v^{-t}\right],\end{aligned} \tag{5}$$

where

$$\begin{aligned}A_{ij}&=\frac{(t+i+j/(p-1))\Gamma^2(t)\prod_{k=1}^{p-2}\Gamma[t+k/(p-1)]}{\Gamma(t+1/2)\prod_{k=0}^{p-2}\Gamma[t+1/(2(p-1))+k/(p-1)]}\\&=\frac{\Gamma[t+i+j/(p-1)+1]\Gamma^2(t)\prod_{k(\neq j)=1}^{p-2}\Gamma[t+k/(p-1)]}{\Gamma(t+1/2)\prod_{\ell=0}^{i-1}(t+j/(p-1)+\ell)\prod_{k=0}^{p-2}\Gamma[t+1/(2(p-1))+k/(p-1)]}.\end{aligned}$$

Now, by taking $t\to -i-j/(p-1)$, the residue R_{ij} is evaluated as

$$R_{ij}=A_{ij0}v^{i+j/(p-1)}, \tag{6}$$

where

$$A_{ij0}=\frac{(-1)^i\Gamma^2[-i-j/(p-1)]\prod_{k(\neq j)=1}^{p-2}\Gamma[-i-(j-k)/(p-1)]}{i!\Gamma[-i-j/(p-1)+1/2]\prod_{k=0}^{p-2}\Gamma[-i-(j-k)/(p-1)+1/(2(p-1))]}.$$

The residue R_{i0} is obtained as

$$R_{i0} = \lim_{t \to -i} \left[\frac{\partial}{\partial t} \left(A_{i0} v^{-t} \right) \right] = \lim_{t \to -i} \left[\frac{\partial}{\partial t} (g(t)) \right], \tag{7}$$

where

$$A_{i0} = \frac{(t+i)^2 \Gamma^2(t) \prod_{k=1}^{p-2} \Gamma[t + k/(p-1)]}{\Gamma(t+1/2) \prod_{k=0}^{p-2} \Gamma[t + 1/(2(p-1)) + k/(p-1)]}.$$

Now, writing $(t+i)^2 \Gamma^2(t) = \Gamma^2(t+i+1)/\prod_{\ell=0}^{i-1}(t+\ell)^2$ above, we get:

$$= \frac{\Gamma^2(t+i+1) \prod_{k=1}^{p-2} \Gamma[t + k/(p-1)]}{\Gamma(t+1/2) \prod_{\ell=0}^{i-1}(t+\ell)^2 \prod_{k=0}^{p-2} \Gamma[t + 1/(2(p-1)) + k/(p-1)]}$$

and $g(t) = A_{i0} v^{-t}$. Taking logarithm of $g(t)$ and differentiating the resulting expression with respect to t, one gets

$$\frac{\partial \ln g(t)}{\partial t} = \frac{\partial}{\partial t} (\ln A_{i0} - t \ln v)$$

$$= B_{i0} - \ln v.$$

Further, noting the $\dfrac{\partial \ln g(t)}{\partial t} = \dfrac{1}{g(t)} \dfrac{\partial g(t)}{\partial t}$, we get

$$\frac{\partial g(t)}{\partial t} = (B_{i0} - \ln v)\, g(t) = (B_{i0} - \ln v) A_{i0} v^{-t},$$

where $\psi(\cdot)$ is the digamma function [29, 30]. Now, taking limit as $t \to -i$ in (7), we obtain

$$R_{i0} = [B_{i00} - \ln v] A_{i00} v^{i} \tag{8}$$

where

$$A_{i00} = \frac{\prod_{k=1}^{p-2} \Gamma[-i + k/(p-1)]}{\Gamma(-i+1/2)(i!)^2 \prod_{k=0}^{p-2} \Gamma[-i + 1/(2(p-1)) + k/(p-1)]}$$

and

$$B_{i00} = 2\psi(i+1) + \sum_{k=1}^{p-2} \psi\left(-i + \frac{k}{p-1}\right) - \psi\left(-i + \frac{1}{2}\right) - \sum_{k=0}^{p-2} \psi\left(-i + \frac{k}{p-1} + \frac{1}{2(p-1)}\right).$$

Finally, substituting (6) and (8) in (4), we obtain

$$f(v) = K(n,\, p)v^{(n-2)/2}\left[\sum_{i=0}^{\infty}\sum_{j=1}^{p-2} A_{ij0}v^{i+j/(p-1)} + \sum_{i=0}^{\infty}[B_{i00} - \ln v]A_{i00}v^{i}\right], \tag{9}$$

where $0 < v < 1$. Similarly, the density for p odd is given by

$$f(v) = K(n,\, p)v^{(n-2)/2}\left[\sum_{i=0}^{\infty}\sum_{\substack{j=1\\ \neq(p-1)/2}}^{p-2} A_{ij0}v^{i+j/(p-1)} + \sum_{i=0}^{\infty}[B_{i00} - \ln v]A_{i00}v^{i}\right], \tag{10}$$

where

$$A_{ij0} = \frac{(-1)^i\Gamma^2[-i-j/(p-1)]\prod_{k=1,\, k\notin\{j,\,(p-1)/2\}}^{p-2}\Gamma[-i-(j-k)/(p-1)]}{i!\prod_{k=0}^{p-2}\Gamma[-i-(j-k)/(p-1)+1/(2(p-1))]},$$

$$A_{i00} = \frac{\prod_{k(\neq(p-1)/2)=1}^{p-2}\Gamma[-i+k/(p-1)]}{(i!)^2\prod_{k=0}^{p-2}\Gamma[-i+1/(2(p-1))+k/(p-1)]}$$

and

$$B_{i00} = 2\psi(i+1) + \sum_{\substack{k=1\\ \neq(p-1)/2}}^{p-2} \psi\left(-i + \frac{k}{p-1}\right) - \sum_{k=0}^{p-2} \psi\left(-i + \frac{k}{p-1} + \frac{1}{2(p-1)}\right).$$

Substituting $p = 4$ in (9), the density of V simplifies to

$$f(v) = K(n,\, 4)v^{(n-2)/2}\left[\sum_{i=0}^{\infty} A_{i10}v^{i+1/3} + \sum_{i=0}^{\infty} A_{i20}v^{i+2/3} + \sum_{i=0}^{\infty}[B_{i00} - \ln(v)]A_{i00}v^{i}\right], \tag{11}$$

where

$$K(n,\, 4) = \frac{\Gamma(n/2+1/6)\Gamma^2(n/2+1/2)\Gamma(n/2+5/6)}{\Gamma(n/2+1/3)\Gamma(n/2+2/3)\Gamma^2(n/2)},$$

$$A_{i10} = \frac{(-1)^i \Gamma^2(-i-1/3)\Gamma(1/3-i)}{i!\Gamma(-i-1/6)\Gamma^2(1/6-i)\Gamma(1/2-i)},$$

$$A_{i20} = \frac{(-1)^i \Gamma^2(-i-2/3)\Gamma(-i-1/3)}{i!\Gamma(-i-1/2)\Gamma^2(-i-1/6)\Gamma(1/6-i)},$$

$$A_{i00} = \frac{\Gamma(1/3-i)\Gamma(2/3-i)}{(i!)^2\Gamma(1/6-i)\Gamma^2(1/2-i)\Gamma(5/6-i)},$$

and

$$B_{i00} = 2\psi(i+1) - 2\psi\left(\frac{1}{2}-i\right) + \psi\left(\frac{1}{3}-i\right) + \psi\left(\frac{2}{3}-i\right) - \psi\left(\frac{5}{6}-i\right).$$

Further simplification of A_{i10}, A_{i20}, A_{i00}, and B_{i00} can be achieved by rewriting gamma and digamma functions with the help of conversion formulas

$$\Gamma(\beta - j) = \frac{\Gamma(\beta)\Gamma(1-\beta)}{\Gamma(1-\beta+j)}$$

and

$$\psi(\beta - j) = \psi(\beta) - \psi(1-\beta) + \psi(1-\beta+j).$$

In continuation, we present a few graphs (Figure 1) of the density function defined by the expression (put the number of the equation of the density) for $n = 5$, 10, 15, and 20. By visual observation of shapes that emerge for different values of n one can appreciate efficient computation of the infinite series.

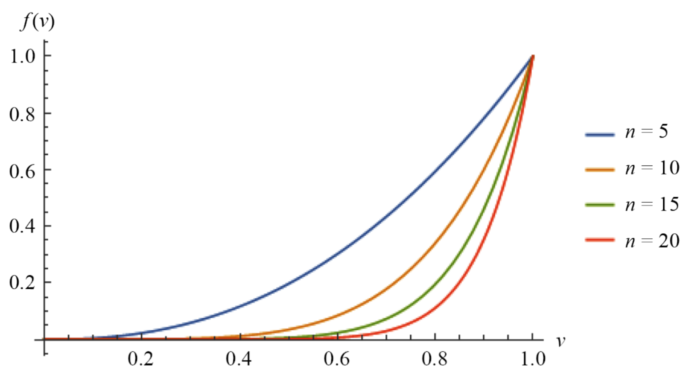

Figure 1. Graphs of $f(v)$ for $p = 4$ and different values of n

3. Computation

The computation of the exact percentage points has been carried out by using $F(v,\ p) = \int_0^v f(t)\,dt$ where $f(t)$ is given by (9) and (10). First, $f(t)$ is simplified for $p = 4(1)9$ using results on gamma and digamma functions. Then, the Cumulative Distribution Function (CDF) $F(v,\ p)$ for $p = 4(1)9$ is obtained by integrating term by term these simplified density functions. For each p, $F(v,\ p)$ is computed for various values of v. It is checked for monotonicity and for conditions $F(v,\ p) \to 0$ as $v \to 0$ and $F(v,\ p) \to 1$ as $v \to 1$. Then, v is computed for various values of p and $F(v,\ p)$. These are given in Tables 1-3. We have used MATHEMATICA 11.0 to carry out these computations. To compute v for a given value of $\alpha = F(v,\ p)$, we have used FindRoot which searches for a numerical solution to the given equation using Newton's method or a variant of the secant method. A six-place accuracy has been kept throughout. For higher values of p it is seen that the accuracy is being lost. Hence the tables are given for $p = 4(1)9$.

Table 1. Percentage points of V for $p = 4$ and $p = 5$

$p=4$					$p=5$				
n	$\alpha=0.01$	$\alpha=0.025$	$\alpha=0.05$	$\alpha=0.1$	n	$\alpha=0.01$	$\alpha=0.025$	$\alpha=0.05$	$\alpha=0.1$
2	0.0055	0.0150	0.0325	0.0707	2	0.0057	0.0155	0.0334	0.0723
3	0.0346	0.0669	0.1106	0.1832	3	0.0352	0.0679	0.1120	0.1851
4	0.0841	0.1371	0.1986	0.2882	4	0.0850	0.1383	0.2001	0.2898
5	0.1414	0.2083	0.2794	0.3749	5	0.1424	0.2095	0.2807	0.3763
6	0.1988	0.2738	0.3491	0.4451	6	0.1998	0.2750	0.3502	0.4462
7	0.2527	0.3320	0.4083	0.5022	7	0.2536	0.3330	0.4093	0.5031
8	0.3020	0.3830	0.4586	0.5491	8	0.3028	0.3839	0.4595	0.5500
9	0.3465	0.4277	0.5016	0.5883	9	0.3472	0.4285	0.5023	0.5890
10	0.3864	0.4668	0.5386	0.6214	10	0.3871	0.4675	0.5392	0.6220
11	0.4223	0.5012	0.5706	0.6497	11	0.4230	0.5019	0.5712	0.6502
12	0.4546	0.5317	0.5987	0.6741	12	0.4552	0.5323	0.5992	0.6746
13	0.4837	0.5589	0.6234	0.6954	13	0.4842	0.5594	0.6238	0.6958
14	0.5101	0.5831	0.6453	0.7141	14	0.5105	0.5836	0.6457	0.7144
15	0.5340	0.6049	0.6648	0.7307	15	0.5344	0.6053	0.6652	0.7310
16	0.5557	0.6246	0.6824	0.7454	16	0.5562	0.6250	0.6827	0.7457
17	0.5757	0.6425	0.6982	0.7587	17	0.5760	0.6428	0.6985	0.7589
18	0.5939	0.6588	0.7125	0.7706	18	0.5943	0.6591	0.7128	0.7708
19	0.6107	0.6736	0.7255	0.7814	19	0.6110	0.6739	0.7258	0.7816
20	0.6262	0.6873	0.7375	0.7913	20	0.6265	0.6876	0.7377	0.7915
21	0.6405	0.6999	0.7484	0.8003	21	0.6408	0.7001	0.7486	0.8005
22	0.6538	0.7115	0.7585	0.8086	22	0.6541	0.7117	0.7587	0.8087
23	0.6662	0.7222	0.7678	0.8162	23	0.6664	0.7225	0.7680	0.8163
24	0.6777	0.7322	0.7764	0.8232	24	0.6779	0.7324	0.7766	0.8233
25	0.6885	0.7415	0.7844	0.8297	25	0.6887	0.7417	0.7846	0.8299
26	0.6986	0.7502	0.7919	0.8358	26	0.6988	0.7504	0.7920	0.8359
27	0.7080	0.7584	0.7988	0.8414	27	0.7082	0.7585	0.7989	0.8415
28	0.7169	0.7660	0.8053	0.8467	28	0.7171	0.7661	0.8054	0.8468
29	0.7253	0.7731	0.8114	0.8516	29	0.7254	0.7733	0.8115	0.8517
30	0.7332	0.7799	0.8172	0.8562	30	0.7333	0.7800	0.8173	0.8563

While the methodology presented here can be extended to cases where $p \geq 10$, it comes with significant practical limitations. The series expansions obtained for these higher dimensions become extremely lengthy, posing a major challenge for computing the percentage points. For this reason, we recommend using an alternative approach, such as the one described by Coelho and Marques [13], when calculating significance points for higher values of p.

Table 2. Percentage points of V for $p = 6$ and $p = 7$

$p = 6$					$p = 7$				
n	$\alpha = 0.01$	$\alpha = 0.025$	$\alpha = 0.05$	$\alpha = 0.1$	n	$\alpha = 0.01$	$\alpha = 0.025$	$\alpha = 0.05$	$\alpha = 0.1$
2	0.0058	0.0158	0.0340	0.0733	2	0.0059	0.0161	0.0343	0.0739
3	0.0356	0.0686	0.1129	0.1863	3	0.0359	0.0690	0.1135	0.1870
4	0.0855	0.1390	0.2010	0.2909	4	0.0859	0.1395	0.2016	0.2916
5	0.1430	0.2102	0.2815	0.3772	5	0.1434	0.2107	0.2821	0.3777
6	0.2004	0.2756	0.3509	0.4470	6	0.2008	0.2761	0.3514	0.4474
7	0.2542	0.3336	0.4099	0.5037	7	0.2546	0.3340	0.4103	0.5041
8	0.3034	0.3845	0.4600	0.5504	8	0.3037	0.3848	0.4604	0.5508
9	0.3477	0.4289	0.5028	0.5894	9	0.3480	0.4292	0.5031	0.5897
10	0.3876	0.4679	0.5396	0.6224	10	0.3878	0.4682	0.5399	0.6226
11	0.4233	0.5022	0.5716	0.6505	11	0.4236	0.5025	0.5718	0.6507
12	0.4555	0.5326	0.5995	0.6748	12	0.4558	0.5328	0.5997	0.6750
13	0.4846	0.5597	0.6241	0.6960	13	0.4848	0.5599	0.6243	0.6962
14	0.5108	0.5838	0.6459	0.7147	14	0.5110	0.5840	0.6461	0.7148
15	0.5347	0.6056	0.6654	0.7312	15	0.5349	0.6057	0.6656	0.7313
16	0.5564	0.6252	0.6829	0.7459	16	0.5566	0.6254	0.6830	0.7460
17	0.5763	0.6430	0.6986	0.7591	17	0.5764	0.6432	0.6988	0.7592
18	0.5945	0.6593	0.7129	0.7710	18	0.5946	0.6594	0.7130	0.7711
19	0.6112	0.6741	0.7259	0.7818	19	0.6113	0.6742	0.7260	0.7819
20	0.6267	0.6877	0.7378	0.7916	20	0.6268	0.6878	0.7379	0.7917
21	0.6410	0.7003	0.7487	0.8006	21	0.6411	0.7004	0.7488	0.8006
22	0.6542	0.7118	0.7588	0.8088	22	0.6543	0.7119	0.7589	0.8089
23	0.6666	0.7226	0.7681	0.8164	23	0.6667	0.7227	0.7681	0.8165
24	0.6781	0.7325	0.7767	0.8234	24	0.6782	0.7326	0.7767	0.8235
25	0.6888	0.7418	0.7846	0.8299	25	0.6889	0.7419	0.7847	0.8300
26	0.6989	0.7505	0.7921	0.8360	26	0.6989	0.7506	0.7921	0.8360
27	0.7083	0.7586	0.7990	0.8416	27	0.7084	0.7587	0.7991	0.8416
28	0.7172	0.7662	0.8055	0.8469	28	0.7172	0.7663	0.8056	0.8469
29	0.7255	0.7734	0.8116	0.8518	29	0.7256	0.7734	0.8117	0.8518
30	0.7334	0.7801	0.8173	0.8564	30	0.7335	0.7801	0.8174	0.8564

Table 3. Percentage points of V for $p = 8$ and $p = 9$

	$p = 8$					$p = 9$			
n	$\alpha = 0.01$	$\alpha = 0.025$	$\alpha = 0.05$	$\alpha = 0.1$	n	$\alpha = 0.01$	$\alpha = 0.025$	$\alpha = 0.05$	$\alpha = 0.1$
2	0.0060	0.0162	0.0346	0.0744	2	0.0060	0.0163	0.0348	0.0748
3	0.0361	0.0693	0.1139	0.1876	3	0.0362	0.0696	0.1142	0.1880
4	0.0862	0.1399	0.2020	0.2920	4	0.0864	0.1402	0.2023	0.2924
5	0.1437	0.2111	0.2824	0.3781	5	0.1439	0.2113	0.2827	0.3784
6	0.2011	0.2764	0.3518	0.4478	6	0.2013	0.2766	0.3520	0.4480
7	0.2548	0.3343	0.4106	0.5044	7	0.2551	0.3345	0.4108	0.5046
8	0.3040	0.3851	0.4606	0.5510	8	0.3041	0.3853	0.4608	0.5512
9	0.3483	0.4295	0.5033	0.5899	9	0.3484	0.4296	0.5035	0.5901
10	0.3881	0.4684	0.5401	0.6228	10	0.3882	0.4685	0.5402	0.6229
11	0.4238	0.5027	0.5720	0.6509	11	0.4239	0.5028	0.5721	0.6510
12	0.4559	0.5330	0.5999	0.6751	12	0.4561	0.5331	0.5999	0.6752
13	0.4849	0.5600	0.6244	0.6963	13	0.4850	0.5601	0.6245	0.6964
14	0.5112	0.5841	0.6462	0.7149	14	0.5113	0.5842	0.6463	0.7150
15	0.5350	0.6059	0.6657	0.7314	15	0.5351	0.6059	0.6657	0.7314
16	0.5567	0.6255	0.6831	0.7461	16	0.5568	0.6255	0.6832	0.7461
17	0.5765	0.6433	0.6988	0.7592	17	0.5766	0.6433	0.6989	0.7593
18	0.5947	0.6595	0.7131	0.7711	18	0.5948	0.6595	0.7132	0.7712
19	0.6114	0.6743	0.7261	0.7819	19	0.6115	0.6744	0.7262	0.7820
20	0.6269	0.6879	0.7380	0.7917	20	0.6269	0.6880	0.7380	0.7918
21	0.6412	0.7004	0.7489	0.8007	21	0.6412	0.7005	0.7489	0.8007
22	0.6544	0.7120	0.7589	0.8089	22	0.6545	0.7120	0.7590	0.8090
23	0.6667	0.7227	0.7682	0.8165	23	0.6668	0.7228	0.7682	0.8165
24	0.6782	0.7327	0.7768	0.8235	24	0.6783	0.7327	0.7768	0.8236
25	0.6890	0.7420	0.7848	0.8300	25	0.6890	0.7420	0.7848	0.8300
26	0.6990	0.7506	0.7922	0.8361	26	0.6990	0.7507	0.7922	0.8361
27	0.7084	0.7587	0.7991	0.8417	27	0.7085	0.7588	0.7991	0.8417
28	0.7173	0.7663	0.8056	0.8469	28	0.7173	0.7663	0.8056	0.8469
29	0.7256	0.7734	0.8117	0.8518	29	0.7257	0.7735	0.8117	0.8519
30	0.7335	0.7802	0.8174	0.8564	30	0.7335	0.7802	0.8174	0.8565

Acknowledgments

This research work was supported by the Sistema Universitario de Investigación, Universidad de Antioquia [project no. 2023-66291].

Conflict of interest

The authors declare no competing financial interest.

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