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Research Article

Fixed Point Theorems for Almost Kannan on Complete ν-Generalized Metric Space

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Abstract: The ν-generalized metric space, first formulated by Branciari, broadens the traditional framework of fixed point theory. In this study, we derive several novel fixed point results concerning almost Kannan-type contraction defined on complete ν-generalized metric structures. It is proved that if $T : X \to CB(X)$ satisfies

$$H(Tx, Ty) \leq \delta[d(x, Tx) + d(y, Ty)] + Ld(y, Tx),$$

where $\delta \in \left[0, \frac{1}{2}\right)$ and $L \geq 0$, then T has at least one fixed point in X. The results are further extended to Mizoguchi-Takahashi and φ-functions. Illustrative examples are provided to verify the validity of the obtained results.

Keywords: ν-generalized metric space, Kannan contraction mapping, Mizoguchi-Takahashi function, fixed point

MSC: 46S40, 47H10, 54H25

1. Introduction

The origin of metric spaces dates back to Fréchet [1], representing a key development in mathematical analysis. Since a metric induces topological properties, it has found extensive applications across mathematics, with essential applications in the theory of fixed points. The natural progression of mathematics has led to generalizing these concepts, resulting in various Generalized Metric Spaces (GMSs) proposed by researchers over the years. In 2000, Branciari [2] later generalized the classical metric by relaxing the triangle inequality, introducing the ν-GMS structure, which is weaker than that of a metric space and stronger than that of a semimetric space. This new structure opened avenues for exploring topological properties and theorems on fixed points within ν-GMSs, as explored in various later studies (refer to [3–6]). Moreover, among the foundational results in this area is the Banach contraction theory [7] that has had a profound impact across mathematical disciplines. Additionally, the Banach principle [8] ensured the uniqueness of fixed points for contraction mappings, a cornerstone in nonlinear analysis. Later, in 1968, Kannan [9, 10] refined this principle, further enriching our understanding of contraction mapping and its applications.

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However, unlike previous studies that mainly considered single-valued or strictly contractive mappings, the present work introduces a new type of mapping—the almost Kannan contraction—derived from the multivalued weak contraction proposed by Berinde and Berinde [11]. This mapping is then extended to the framework of complete ν-generalized metric spaces, enriching fixed point theory by allowing weaker contractive conditions and broader applicability. Furthermore, several related results, including those involving the Mizoguchi-Takahashi function and the φ-function, are obtained as exceptional cases of the proposed framework, thus unifying and generalizing many known fixed-point theorems.

2. Preliminaries

We begin by revisiting some fundamental definitions of ν-GMS.

Definition 1 [2] Let X be a set, and let d be a mapping from $X \times X$ into $[0, \infty)$, where ν is an element in $\mathbb{N}$. The pair (X, d) is known as a ν-GMS if the following conditions hold:

(N1) $d(x, z) = 0$, iff x is equal z.

(N2) The function $d(x, z)$ is symmetric; meaning for any $x, z \in X$, $d(x, z) = d(z, x)$.

(N3) $d(x, z) \leq d(x, r_1) + d(r_1, r_2) + \cdots + d(r_N, z)$, $\forall\, x, r_1, r_2, \ldots, r_N, z \in X$, where the variables $x, r_1, r_2, \ldots, r_N, z$ are dissimilar.

When $\nu = 1$, conditions (N1)-(N3) coincide with those of a standard metric space. Moreover, any metric space is a ν-GMS; besides, the converse of this statement is not true (refer to [2], Example 1 and [12], Example 2).

We now outline the following essential topological properties of ν-GMS.

Definition 2 [13] Let (X, d) represent a ν-GMS. Then

• A sequence $\{x_n\}$ in X is referred to as a *Cauchy* sequence iff:

$$\limsup_{n\to\infty,\, m>n} d(x_m, x_n) = 0,$$

holds.

• A sequence $\{x_n\}$ within X is defined as a 2-*Cauchy* sequence iff:

$$\limsup_{n\to\infty} \{d(x_n, x_{n+1+2j}) : j = 0, 1, 2, \ldots\} = 0,$$

holds.

Definition 3 [13] In a ν-GMS (X, d), a sequence of points $\{x_n\}$ converges to x iff

$$\lim_{n\to\infty} d(x_n, x) = 0.$$

Definition 4 [13] A ν-GMS (X, d) is considered complete if every *Cauchy* sequence in X converges in X.

Proposition 1 [14] If (X, d) is a 2-GMS with d being continuous in its variable, then (X, d) is metrizable and every 3-GMS is metrizable. Also, for every $\nu \geq 4$, it is metrizable if every convergent sequence within the space is also a *Cauchy* sequence.

3. Main results

The current section gives proofs for several fixed-point results within ν-GMSs. We begin by collecting key concepts related to the theory of multivalued mappings in ν-GMSs.

Definition 5 Let $CB(X)$ denote the class of closed and bounded nonempty subsets of X. Define the function for $U, V \in CB(X)$ as $H : CB(X) \times CB(X) \to \mathbb{R}^+$, (where $\mathbb{R}^+$ denotes the set of all nonnegative real numbers) as follows:

$$H(U, V) = \max\{\delta(U, V), \delta(V, U)\},$$

where

$$\delta(U, V) = \sup\{d(u, V) : u \in U\}, \quad \text{and} \quad \delta(V, U) = \sup\{d(v, U) : v \in V\},$$

with

$$d(u, Z) = \inf\{d(u, y), y \in Z\}.$$

The function H, derived from the ν-GMS d, plays the role of a Hausdorff-type ν-GMS (see [15–17]).

Moreover, in the classical metric framework, Berinde and Berinde [11] proved an essential result concerning the almost contraction theorem; read the following theorem.

Theorem 1 Let (X, d) be a metric space, and let T be a mapping from X to $\mathscr{P}(X)$, a multi-valued weak contraction iff there are constants $\theta \in (0, 1)$ and $L \geq 0$ such that

$$H(Tx_1, Tx_2) \leq \theta.d(x_1, x_2) + L.d(x_2, Tx_1) \text{ holds for any } x_1, x_2 \in X.$$

First, by generalized Theorem 7 in Kannan contraction mapping, we introduce the almost Kannan contraction in ν-GMS.

Definition 6 Let (X, d) be a metric space. And $T : X \to CB(X)$ multi-valued operator. We call T an almost Kannan-type contraction mapping, if there exists a constant $\delta \in \left(0, \frac{1}{2}\right)$ and $L \geq 0$ such that

$$H(Tx_1, Tx_2) \leq \delta\,[d(x_1, Tx_1) + d(x_2, Tx_2)] + L d(x_2, Tx_1), \quad \forall\, x_1, x_2 \in X.$$

The following proposition contributes to the establishment of our results.

Proposition 2 [13] Let (X, d) be a ν-GMS. Let $\{x_n\}$ be a sequence of distinct points in X. Satisfying,

$$\sum_{n=1}^{\infty} d(x_n, x_{n+1}) < \infty \quad \text{and} \quad \lim_{n\to\infty} d(x_n, x_{n+2}) = 0.$$

Hence, $\{x_n\}$ is a *Cauchy* sequence in X.

Therefore, the study is well-prepared to state and prove the next theorem.

Theorem 2 Suppose (X, d) is a ν-GMS, and $T : X \to CB(X)$ is a multi-valued mapping. Then, T is called an almost Kannan contraction multi-valued mapping if there exists $\delta \in \left[0, \frac{1}{2}\right)$, and $L \geq 0$, such that

$$H(Tx, Ty) \leq \delta [d(x, Tx) + d(y, Ty)] + Ld(y, Tx), \quad \text{for any } x, y \in X. \tag{1}$$

Proof. We begin by assuming x_0, an arbitrary element in X; and given that $Tx_0 \neq \emptyset$, we can choose $x_1 \in Tx_0$. Now, if $x_1 = x_0$, the proof is complete. However, consider $x_0 \neq x_1$ and take $1 < \lambda$ such that $\lambda\delta < \frac{1}{2}$; there exists $x_2 \in Tx_1$ which

$$d(x_1, x_2) \leq \lambda H(Tx_0, Tx_1). \tag{2}$$

Applying inequality (1) to (2), we obtain the following expression

$$d(x_1, x_2) \leq \lambda [\delta d(x_0, Tx_0) + d(x_1, Tx_1)] + Ld(x_1, Tx_0),$$

$$d(x_1, x_2) \leq \lambda\delta [d(x_0, Tx_0) + d(x_1, Tx_1)] + Ld(x_1, x_1),$$

$$d(x_1, x_2) \leq \frac{\lambda\delta}{(1 - \lambda\delta)} d(x_0, x_1).$$

Let $h = \frac{\lambda\delta}{(1-\lambda\delta)} < 1$, observe that $x_1 \in Tx_0$, hence $d(x_1, Tx_0) = 0$. Moreover, there exists $x_3 \in Tx_2$, if $x_3 = x_2$ we are done, such that

$$d(x_2, x_3) \leq hd(x_1, x_2).$$

Repeating this construction produces a sequence $\{x_n\}_{n\in\mathbb{N}}$ that satisfies

$$d(x_n, x_{n+1}) \leq h^n d(x_0, x_1). \tag{3}$$

Moreover, the same trend on (3), from $n = 1 \to \infty$ establishes that

$$\sum_{n=1}^{\infty} d(x_n, x_{n+1}) < \infty.$$

In addition, the next shows that

$$\lim_{n\to\infty} d(x_n, x_{n+2}) = 0.$$

Again, by (1) we have

$$\begin{aligned} d(x_n, x_{n+2}) &\leq \lambda H(Tx_{n-1}, Tx_{n+1}), \\ &\leq \lambda\delta\left(d(x_{n-1}, Tx_{n-1}) + d(x_{n+1}, Tx_{n+1})\right) + \lambda L d(x_{n+1}, Tx_{n-1}), \\ &\leq \lambda\delta\left(d(x_{n-1}, x_n) + d(x_{n+1}, x_{n+2})\right) + \lambda L d(x_{n+1}, x_n), \end{aligned}$$

such that

$$\lim_{n\to\infty} d(x_n, x_{n+2}) = 0.$$

In this regard, for both sides, from Proposition 9, one brings to a conclusion that the sequence $\{x_n\}$ is *Cauchy* in a ν-GMS that is complete. The $\{x_n\}$ is convergent to some point $y \in X$. To verify the fixed point property of y, we continue as follows

$$\begin{aligned} d(y, Ty) &= \lim_{n\to\infty} d(x_{n+1}, Ty), \\ &\leq \lim_{n\to\infty} H(Tx_n, Ty), \\ &\leq \lim_{n\to\infty} \left[\delta(d(x_n, Tx_n) + d(y, Ty)) + Ld(y, Tx_n)\right], \\ d(y, Ty) &\leq \lim_{n\to\infty} \frac{\delta}{1-\delta} d(x_n, x_{n+1}) + Ld(y, x_{n+1}) = 0. \end{aligned}$$

Therefore, $d(y, Ty) = 0$, which thus shows that $y \in Ty$ is indeed a fixed point. □

To substantiate our main theorem, we study the following example.

Example 1 Consider $X = \mathbb{N}$, $\nu \in \mathbb{N}$, $\nu \geq 2$, and further define the metric $d : X \times X \to [0, \infty)$ as:

$$\begin{aligned} d(x, x) &= 0, \\ d(1, s) &= \nu + 1, \quad \text{if } s \in \mathbb{N} \setminus \{1, 2\}, \\ d(x, y) &= 1, \quad \text{elsewhere.} \end{aligned}$$

Then (X, d) is a ν-GMS where $\nu \geq 2$. Additionally, the following inequality holds:

$$H(Tx, Ty) \leq s\left[d(x, Tx) + d(y, Ty)\right] + Ld(y, Tx). \tag{4}$$

Now we define $T : X \to C(X)$ by:

$$Tx = \begin{cases} \{1, x\}, & x \in \mathbb{N} \setminus \{1, 2\}, \\ \\ \{1\}, & x \in \{1, 2\}. \end{cases}$$

This results in the cases:
Case 1: If $x, y \in \{1, 2\}$, then:

$$Tx = Ty = \{1\}.$$

If $x = 1$, and $y = 2$, then the following holds:

$$H(Tx, Ty) = \max\{\delta(Tx, Ty), \delta(Ty, Tx)\},$$

$$H(\{1\}, \{1\}) = \max\{\delta(\{1\}, \{1\}), \delta(\{1\}, \{1\})\} = \delta(\{1\}, \{1\}) = 0,$$

and

$$d(x, Tx) = 0,$$

$$d(y, Ty) = 1,$$

$$d(y, Tx) = 1.$$

Substituting these values into inequality (4) yields

$$0 \leq s[0 + 1] + L \times (1),$$

$$0 \leq s + L.$$

For $s \in \left(0, \frac{1}{2}\right]$ and $L \geq 0$, the inequality holds. Similarly, when $x = 2$ and $y = 1$, one gets

$$H(Tx, Ty) = 0, \quad d(x, Tx) = 1, \text{ and } d(y, Tx) = 0.$$

And the inequality (4) holds. Also, a trivial case is when $x = y$ is held.
Case 2: If $x, y \in \mathbb{N} \setminus \{1, 2\}$, then

$$Tx = \{1, x\}, \quad Ty = \{1, y\},$$

and

$$H(Tx, Ty) = H(\{1, x\}, \{1, y\}) = \max\{\delta(\{1, x\}, \{1, y\}), \delta(\{1, y\}, \{1, x\})\},$$

$$\delta(\{1, x\}, \{1, y\}) = \max\{d(1, \{1, y\}), d(x, \{1, y\})\},$$

$$d(1, \{1, y\}) = 0,$$

$$d(x, \{1, y\}) = 1,$$

then

$$\delta(\{1, x\}, \{1, y\}) = \max\{0, 1\} = 1.$$

Similarly,

$$\delta(\{1, y\}, \{1, x\}) = 1.$$

Then $H(Tx, Ty) = 1$. Also, one gets

$$d(x, Tx) = d(x, \{1, x\}) = 0,$$

$$d(y, Ty) = d(y, \{1, y\}) = 0,$$

$$d(y, Tx) = d(y, \{1, x\}) = 1.$$

Substituting these values into equation (4), one gets

$$1 \leq s[0+0] + L \times (1) \Rightarrow 1 \leq L,$$

where for $L \geq 1$, the relation expressed in (4) is satisfied.

Case 3: If $x \in \{1, 2\}$, $y \in \mathbb{N} \setminus \{1, 2\}$, then

$$Tx = \{1\}, \quad Ty = \{1, y\}.$$

Then

$$H(Tx, Ty) = H(\{1\}, \{1, y\}) = \max\{\delta(\{1\}, \{1, y\}), \delta(\{1, y\}, \{1\})\},$$

where

$$\delta(\{1\}, \{1, y\}) = \max\{d(1, \{1, y\})\} = 0.$$

Similarly, $\delta(\{1, y\}, \{1\}) = 0$, then $H(Tx, Ty) = 0$. Also, one gets

$$d(x, Tx) = d(x, \{1, 2\}) = 0,$$

$$d(y, Ty) = d(y, \{1, y\}) = 0,$$

$$d(y, Tx) = d(y, \{1\}) = \nu + 1.$$

Now, substitute these values into equation (4), one gets

$$0 \leq s[0+0] + L \times (\nu + 1) \Rightarrow 0 \leq L(\nu + 1),$$

which is true for all $L \geq 0$. We conclude that the inequality holds for all $s \in \left(0, \frac{1}{2}\right]$ and $L \geq 1$. If we choose $L = 1$, then the condition of almost Kannan is substituted in this example, and from Theorem 10, this multi-valued mapping has a fixed point, that is, when $x = 1$.

Furthermore, based on prior studies involving the Mizoguchi-Takahashi (M-T) framework, the following definition is adopted due to its relevance.

Definition 7 [18] A function $\varphi : [0, \infty) \to [0, 1]$ qualifies as an M-T function if it meets the M-T condition, which requires that $\limsup_{s \to t^+} \varphi(s) < 1$ for each t in the interval $[0, \infty)$.

Lemma 1 [18] If $\varphi : [0, \infty) \to [0, 1]$ is an M-T function, then $\kappa : [0, \infty) \to [0, 1]$, where $\kappa(t)$ is defined as $\kappa(t) = \frac{\varphi(t)+1}{2}$ is also an M-T function.

We now proceed to the next result utilizing the M-T function.

Theorem 3 If (X, d) is a complete ν-GMS, and suppose $T : X \to CB(X)$ to be a multi-valued map satisfying

$$H(Tx, Ty) \leq \kappa(d(x, y))[d(x, Tx) + d(y, Ty)] + Ld(y, Tx), \tag{5}$$

for $L \geq 0$, and $x, y \in X$. If κ is an M-T function, then T possesses a fixed point.

Proof. Consider an arbitrary $x_0 \in X$, and since $Tx_0 \neq \emptyset$, then there exists $x_1 \in Tx_0$. Now, when $x_1 = x_0$, we are done. Further, assume $x_1 \neq x_0$ and we take $1 < \lambda$ such that $\lambda\kappa < 1$ since $\kappa(t) \in [0, 1)$. Hence, there exists $x_2 \in Tx_1$ which

$$d(x_1, x_2) \leq \lambda H(Tx_0, Tx_1). \tag{6}$$

From inequality (5), we get

$$\begin{aligned} d(x_1, x_2) &\leq \lambda[\kappa d(x_0, Tx_0) + d(x_1, Tx_1)] + Ld(x_1, Tx_0), \\ &= \lambda\kappa d(x_0, Tx_0) + \lambda\kappa d(x_1, Tx_1) + \lambda L d(x_1, Tx_0), \\ &= \lambda\kappa d(x_0, x_1) + \lambda\kappa d(x_1, x_2) + \lambda L d(x_1, x_1), \\ d(x_1, x_2) &\leq \frac{\lambda\kappa}{1-\lambda\kappa} d(x_0, x_1). \end{aligned}$$

Let $h = \frac{\lambda\kappa}{1-\lambda\kappa}$. Since $\lambda\kappa < \frac{1}{2} \Rightarrow \frac{\lambda\kappa}{1-\lambda\kappa} < \frac{1}{2} \Rightarrow h < \frac{1}{2}$. Now assume $x_3 \in Tx_2$, when $x_3 = x_2$ the proof is complete. Let $x_2 \neq x_3$, upon which

$$d(x_2, x_3) \leq h d(x_1, x_2).$$

Consequently, a sequence $\{x_n\}_{n\in\mathbb{N}}$ is acquired upon repeating the same procedure that satisfies

$$d(x_n, x_{n+1}) \leq h d(x_{n-1}, x_n), \quad \forall\, n \in \mathbb{N}.$$

Therefore, applying mathematical induction yields

$$d(x_n, x_{n+1}) \leq h^n d(x_0, x_1). \tag{7}$$

Applying (7) as $n = 1 \to \infty$, one further proves that

$$\sum_{n=1}^{\infty} d(x_n, x_{n+1}) < \infty.$$

Furthermore, it is required to demonstrate that

$$\lim_{n\to\infty} d(x_n, x_{n+2}) = 0.$$

Through the use of (6), one obtains

$$d(x_n, x_{n+2}) \leq \lambda H(Tx_{n-1}, Tx_{n+1}),$$

$$\leq \lambda \left[\kappa (d(x_{n-1}, Tx_{n-1}) + d(x_{n+1}, Tx_{n+1})) + Ld(x_{n+1}, Tx_{n-1})\right],$$

$$= \lambda \kappa d(x_{n-1}, x_n) + \lambda \kappa d(x_{n+1}, x_{n+2}) + \lambda L d(x_{n+1}, x_n).$$

Thus, we find that

$$\lim_{n\to\infty} d(x_n, x_{n+2}) = 0.$$

Moreover, from Proposition 9, it is concluded that the sequence $\{x_n\}$ is *Cauchy* in the complete ν-GMS. In addition, the sequence $\{x_n\}$ is convergent to some point $y \in X$, where y is to be shown to be a fixed point as follows

$$d(y, Ty) = \lim_{n\to\infty} d(x_{n+1}, Ty),$$

$$\leq \lim_{n\to\infty} H(Tx_n, Ty),$$

$$\leq \lim_{n\to\infty} \kappa [d(x_n, Tx_n) + d(y, Ty)] + Ld(y, Tx_n),$$

$$d(y, Ty) \leq \lim_{n\to\infty} \frac{\kappa}{1-\kappa} d(x_n, x_{n+1}) + Ld(y, x_{n+1}) = 0.$$

Therefore, $d(y, Ty) = 0$, indicating that y is indeed a fixed point of Ty. □

Finally, to establish the next results, we need the following function.

Remark 1 [19] The two results defined by φ, the set of maps $\varphi : X \to [0, \infty)$ that fulfills the conditions as follows:
(N1) φ possesses the property of lower semi-continuity,
(N2) $\varphi(t) \leq t$ for every positive t.

Hence, we prove the following theorem by using the function φ.

Theorem 4 Suppose (X, d) represents a complete ν-GMS, let $\varphi : X \to [0, \infty)$ denotes a lower semi-continuous map, and $T : X \to CB(X)$ be a multi-valued function. Therefore, T is referred to as an almost Kannan contraction multi-valued function if there exists $\delta \in \left[0, \frac{1}{2}\right)$ with $L \geq 0$ satisfying the following inequality

$$H(Tx, Ty) + \varphi(Tx) + \varphi(Ty)$$

$$\leq \delta [d(x, Tx) + d(y, Ty) + \varphi(x) + \varphi(y) + \varphi(Tx) + \varphi(Ty)] + Ld(y, Tx). \tag{8}$$

For any $x, y \in X$, the function T possesses a fixed point $\varphi(x^*) = x^* \in X$.

Further, assume $x_1 \neq x_0$ and we take $0 < \lambda < \frac{1}{2}$ such that $\lambda\kappa < \frac{1}{2}$ since $\kappa(t) \in [0, 1)$. Hence, there exists $x_2 \in Tx_1$ upon which.

Proof. Consider an arbitrary $x_0 \in X$, and the fact that $T(x_0) \neq \emptyset$, we can choose $x_1 \in T(x_0)$. Next, given that $x_1 = x_0$, the result is established. Further, presume that $x_0 \neq x_1$ and take $\lambda\delta < \frac{1}{2}$, then it exists that $x_2 \in T(x_1)$ which

$$d(x_1, x_2) + \varphi(x_1) + \varphi(x_2) \leq \lambda[H(Tx_0, Tx_1) + \varphi(Tx_0) + \varphi(x_1)]. \tag{9}$$

Then, the deployment of the result in (8) on (9) yields

$$d(x_1, x_2) + \varphi(x_1) + \varphi(x_2) \leq \lambda\delta[d(x_0, Tx_0) + d(x_1, Tx_1) + \varphi(x_0) + \varphi(x_1)] + Ld(x_1, Tx_0).$$

This implies that

$$d(x_1, x_2) + \varphi(x_1) + \varphi(x_2) \leq \frac{\lambda\delta}{1-\lambda\delta}[d(x_0, x_1) + \varphi(x_0) + \varphi(x_1)].$$

Let $h = \frac{\lambda\delta}{1-\lambda\delta} < \frac{1}{2}$. Notice again that $x_1 \in Tx_0$, which eventually shows that $d(x_1, Tx_0) = 0$.
Moreover, repeating the procedure above gives a sequence $\{x_n\}_{n\in\mathbb{N}}$, and satisfying

$$d(x_n, x_{n+1}) + \varphi(x_{n+1}) + \varphi(x_n) \leq h^n[d(x_0, x_1) + \varphi(x_0) + \varphi(x_1)].$$

For all $n \geq 0$, the remaining part of the proof remains as in that of Theorem 10, where T possesses a fixed point x^*. Additionally, to show that we have $x^* \in X$, given that $x_n \to x^*$ as $n \to \infty$, one writes

$$\varphi(x_n) \leq h^n[d(x_0, x_1) + \varphi(x_0) + \varphi(x_1)], \quad \forall n \geq 0.$$

Assuming that $n \to +\infty$ in the lower inequality relation, one thus gets $\varphi(x_n) \to 0$ as $n \to \infty$. More so, the fact that φ is lower semi-continuous, it then reveals

$$0 \leq \varphi(x^*) \leq \liminf_{n\to\infty} \varphi(x_n) = \lim_{n\to\infty} \varphi(x_n) = 0.$$

Which is, $\varphi(x^*) = 0$. Thus, proving that x^* is a fixed point requires the use of (8) to obtain

$$H(x_{n+1}, Tx^*) + \varphi(x_{n+1}) + \varphi(Tx^*)$$

$$\leq \delta[d(x_n, x_{n+1}) + d(x^*, Tx^*) + \varphi(x_n) + \varphi(x_{n+1}) + \varphi(x^*) + \varphi(Tx^*)] + Ld(x^*, x_{n+1}),$$

$$H(x^*, Tx^*) + \varphi(Tx^*) \leq \delta[d(x^*, Tx^*) + \varphi(Tx^*)].$$

Since $0 < \delta \leq \frac{1}{2}$, one gets $H(x^*, Tx^*) = \varphi(Tx^*) = 0$, then $x^* \in Tx^*$. □

Remark 2 By setting $\varphi \equiv 0$ in Theorem 16, one recovers Theorem 10 as a special case.

In 2015, Suzuki and others studied the Kannan contraction single-valued mapping in ν-GMS.

Theorem 5 [13] Let (X, d) be a complete ν-GMS, and let T be a Kannan mapping on X, that is, there exists $\delta \in \left[0, \frac{1}{2}\right)$ such that for all $x, y \in X$, $d(Tx, Ty) \leq \delta[d(x, Tx) + d(y, Ty)]$, then T has a unique fixed point.

By applying the following lemma, we are prepared to prove the final theorem.

Lemma 2 [13] Consider (X, d) as a ν-GMS with T as a mapping defined on X. Presuming

$$\sum_{n=1}^{\infty} d(T^n u, T^{n+1} u) < \infty,$$

for some $u \in X$. Assume also either of the following

- v is odd,
- v is even, and

$$\lim_{n\to\infty} d(T^n u, T^{n+2} u) = 0,$$

holds.

Then $\{T^n u\}$ is a *Cauchy* sequence.

By using the function φ with a single-Kannan contraction mapping, we are ready to prove the next theorem.

Theorem 6 Consider (X, d) as a complete ν-GMS, let $\varphi : X \to [0, \infty)$ be a lower semi-continuous function, and a specified mapping $T : X \to X$. Assume further that there exists a constant $\delta \in \left[0, \frac{1}{2}\right)$ upon which $\forall\, x, y \in X$,

$$d(Tx, Ty) + \varphi(Tx) + \varphi(Ty) \leq \delta[d(x, Tx) + d(y, Ty) + \varphi(y) + \varphi(x) + \varphi(Tx) + \varphi(Ty)]. \tag{10}$$

Consequently, T possesses a unique fixed point $x^* \in X$, and additionally, $\varphi(x^*) = 0$.

Proof. Since

$$d(Tx, T^2x) + \varphi(Tx) + \varphi(T^2x) \leq \delta[d(x, Tx) + d(Tx, T^2x) + \varphi(x) + \varphi(Tx) + \varphi(T^2x)],$$

can rewrite this as

$$d(Tx, T^2x) + \varphi(Tx) + \varphi(T^2x) \leq \frac{\delta}{1-\delta}[d(x, Tx) + \varphi(x) + \varphi(Tx)].$$

So,

$$d(Tx, T^2x) + \varphi(Tx) + \varphi(T^2x) \leq r[d(x, Tx) + \varphi(x) + \varphi(Tx)],$$

for any $x \in X$ with $r = \dfrac{\delta}{1-\delta} \in [0, 1)$. Fixing $u \in X$, one then obtains

$$\sum_{n=1}^{\infty} d(T^n u, T^{n+1}u) + \varphi(T^n u) + \varphi(T^{n+1}u) \leq \sum_{n=1}^{\infty} r^n [d(u, Tu) + \varphi(u) + \varphi(Tu)] < \infty,$$

implying

$$\lim_{n\to\infty} d(T^n u, T^{n+1}u) = 0.$$

We also have

$$\lim_{n\to\infty} d(T^n u, T^{n+2}u) + \varphi(T^n u) + \varphi(T^{n+2}u)$$

$$\leq \lim_{n\to\infty} \left(\delta[d(T^{n-1}u, T^n u) + d(T^{n+1}u, T^{n+2}u)] + \varphi(T^{n-1}u) + \varphi(T^n u) + \varphi(T^{n+1}u) + \varphi(T^{n+2}u)\right) = 0.$$

For both sides, according to Lemma 19, the sequence $\{T^n u\}$ is *Cauchy* in the complete ν-GMS. In addition, $\{T^n u\}$ converges to a point $x^* \in X$. Moreover, to demonstrate that x^* is a fixed point, we proceed with

$$\varphi(T^n u) \leq r^n [d(u, Tu) + \varphi(u) + \varphi(Tu)].$$

$\forall n \geq 0$, and letting $n \to \infty$ in the lower inequality, we get $\varphi(T^n u) \to 0$ as $n \to \infty$, since φ is lower semi-continuous. This implies that

$$0 \leq \varphi(x^*) \leq \liminf_{n\to\infty} \varphi(T^n u) = \lim_{n\to\infty} \varphi(T^n u) = 0.$$

Which is, $\varphi(x^*) = 0$. Further, to show that x^* is a fixed point, one needs to demonstrate as follows

$$d(x^*, Tx^*) + \varphi(x^*) + \varphi(Tx^*) = \lim_{n\to\infty} d(T^n u, Tx^*) + \varphi(T^n u) + \varphi(Tx^*),$$

$$\leq \lim_{n\to\infty} \left(\delta[d(T^{n-1}u, T^n u) + d(x^*, Tx^*)] + \varphi(T^{n-1}u) + \varphi(x^*) + \varphi(T^n u) + \varphi(Tx^*)\right),$$

$$= \delta[d(x^*, Tx^*) + \varphi(x^*) + \varphi(Tx^*)],$$

entailing that $d(x^*, Tx^*) = 0$, hence x^* is a fixed point for T. Next, consider y^* to be a different fixed point of T.

$$d(x^*, y^*) + \varphi(x^*) + \varphi(y^*) = d(Tx^*, Ty^*) + \varphi(Tx^*) + \varphi(Ty^*),$$

$$\leq \delta[d(x^*, Tx^*) + d(y^*, Ty^*) + \varphi(x^*) + \varphi(Tx^*)$$

$$+ \varphi(y^*) + \varphi(Ty^*)] = 0.$$

Thus, $x^* = y^*$ is valid. Consequently, a unique fixed point x^* is attained. □

Remark 3 Taking $\varphi \equiv 0$ in Theorem 20, we obtain Theorem 18.

4. Conclusions

In summary, this research has introduced a refined framework for studying almost Kannan-type contractions in the setting of complete ν-generalized metric spaces. The developed theory enhances the analytical depth of fixed point results by relaxing classical contractive assumptions and establishing broader unifying conditions that capture both single-valued and multi-valued mappings. Beyond generalization, the results presented herein provide a cohesive structure that bridges several strands of fixed point theory, thereby contributing to a more flexible and inclusive understanding of contractive principles. Future investigations may focus on exploring the stability and uniqueness of fixed points, developing iterative algorithms based on the introduced mappings, and extending these results to nonlinear models and other abstract systems that arise in mathematical analysis and applied sciences.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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