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Research Article

Results for Fuzzy Generalized Contractive Mappings Tailored with Directed Graph

Usman Shehzad[1], Samina Batul[1], Dania Santina[2], Nabil Mlaiki[2*]

[1]Department of Mathematics, Capital University of Science and Technology, Islamabad 44000, Pakistan
[2]Department of Mathematics and Sciences, Prince Sultan University, Riyadh 11586, Saudi Arabia
E-mail: nmlaiki@psu.edu.sa

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Abstract: This paper introduces a novel class of graphic fuzzy generalized rational F-contractive mappings in controlled metric spaces endowed with a directed graph structure. Using these mappings, we establish the existence of fuzzy coincidence and fuzzy fixed points. Several significant consequences are derived for single-valued mappings in controlled metric spaces endowed with a directed graph structure. Similarly, important results are established for set-valued mappings in controlled metric spaces, tailored with a directed graph structure. To demonstrate the applicability of our results, we provide two examples in which all weights are explicitly computed for $n = 4$, confirming the existence of fuzzy coincidence and fixed points and the applicability of the proposed framework. Furthermore, we establish the existence of solutions for fractional differential equations and integral equations within the controlled metric space framework, highlighting the practical relevance of our approach. Our results generalize and extend several existing fixed point and fuzzy fixed point theorems in the literature, recovering many known cases as special instances.

***Keywords*:** fuzzy fixed point, fuzzy coincidence point, F-contraction, controlled metric space, directed graph, fractional differential equations, graphic contraction

MSC: 47H10, 54E40, 03E72

1. Introduction

The renowned Banach Contraction Principle [1] is a fundamental tool for establishing the existence of solutions to various linear and nonlinear functional equations. This principle provides sufficient criteria to ensure that successive approximations, starting from an initial guess, will converge to the exact solution.

The existence of fixed points in certain mappings defined on partially ordered metric spaces was explored in [2]. This direction of research has been further pursued in [3–7]. Jachymski and Jozwik [8] deviated from the traditional approach of utilizing order structures to establish fixed point results, instead opting to employ graph theory. Building upon this foundation, Bojor [9] integrated fixed point theory on metric spaces with graph theory to derive fixed point results for Reich-type contractions. The theory for specific types of mappings on metric spaces has been improved and generalized as a result of this innovative method [10].

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Abbas and Nazir [11] innovatively employed visual structure to establish the existence of fixed points for a pair of power graph contractions in a metric space context. Wardowski [12] subsequently introduced a generalization of the classical Banach Contraction Principle, yielding significant fixed point results by using the notion of F-contraction. In a recent development, Abbas et al. [13] derived fixed point results for set-valued mappings that satisfy specific graphic contraction conditions on sets tailored with directed graphs (also see [14]). Raja et al. [15] extended fixed point theory by using Clarke's subdifferential approach to study fractional differential systems with delay. Their results, supported by fixed point theorems, show the existence of solutions for systems of order $1 < r < 2$. Raja et al. [16] contributed to fixed point theory by studying a new class of impulsive nonlocal Caputo fractional dynamical differential inclusions of order $1 < r < 2$ in a Banach space, where the existence of mild solutions is established using fixed point techniques.

However, mathematical modeling of real-world phenomena is fraught with challenges, particularly the ambiguities inherent in categorizing events. Classical set theory is ill-tailored to handle such uncertainties, prompting the development of fuzzy set theory [17]. By replacing crisp sets with fuzzy sets, this framework enables the effective management of non-statistical uncertainty in information, garnering widespread recognition. The fundamental concepts of fuzzy sets have undergone significant refinements and enhancements, as evident in [18–21]. Heilpern's groundbreaking work in 1981 [22] introduced fuzzy set-valued maps, leading to a fuzzy analogue of Nadler's fixed point theorems [23]. This pioneering research sparked a surge of interest in the existence of fixed points in fuzzy set-valued maps, with notable contributions from Al-Mazrooei et al. [24, 25], Azam et al. [26, 27], Bose and Sahani [28], Qiu and Shu [29] and numerous other researchers in the field.

Using fuzzy sets, Dinesh [30] introduced the fuzzy incidence graph structures. Later, Meenakshi [31] explored the modular product of cubic fuzzy graph structures. Younis et al. [32] introduced graphical bipolar metric-type space, which combines the notions of graph theory with fixed point theory. In 2025, Shi et al. [33] introduced a new framework for fuzzy information management by combining fuzzy mixed graph and graph structures along with mentioning its properties.

This paper introduces the concept of graphic fuzzy generalized rational F-contractive mappings in controlled metric spaces, tailored with a directed graph G, leading to the establishment of common fuzzy coincidence and fixed point results for such contractions. Illustrative examples are presented to highlight the applicability of the theoretical findings. In the first example, we consider $n = 4$ elements and explicitly compute all the weights of the directed graph, demonstrating the existence of fuzzy coincidence and fixed points for the mapping. In the second example, with $n = 4$ elements, a similar computation is performed. Applications to fractional differential equations and integral equations are presented to highlight the applicability of the theoretical findings in controlled metric spaces.

Moreover, the obtained results generalize, extend, and unify several existing fixed point and fuzzy fixed point results available in the current literature. In particular, many known theorems for single-valued and set-valued mappings in metric, controlled metric, and graph-based spaces can be recovered as special cases of our main results by imposing suitable restrictions on the control functions, graph structure, and fuzzy mappings. Our framework integrates graph structure with fuzzy mapping to model both explicit and implicit relationships. The graph provides a discrete, binary representation of known connections, while the fuzzy mapping extends this by assigning continuous membership values, capturing a spectrum of associations.

Potential applications in the real world include transportation networks, social systems, image segmentation, traffic flow prediction, and biological networks. By integrating controlled metric spaces, graphs, and fuzzy logic, this framework models complex systems with uncertainty and supports the development of novel methodologies for diverse fields (see Table 1).

Table 1. Overview of recent works related to fuzzy and mixed graph structures

Year	Author(s)	Contribution
1965	Zadeh [17]	Fuzzy set
2015	Abbas et al. [13]	Directed graph
2016	Abbas et al. [14]	Directed graph
2020	Dinesh [30]	Fuzzy incidence graph structures
2022	Rafique et al. [34]	Directed graph endowed with fuzzy set
2023	Meenakshi [31]	Cubic fuzzy graph structures
2024	Younis et al. [32]	Graph structures
2025	Shi et al. [33]	Fuzzy mixed graph structures
–	This article	Directed graph endowed with fuzzy set

2. Preliminaries

In 2017, Kamran et al. [35] initiated the idea of an extended b-metric space as a generalization of a b-metric space, as follows:

Definition 1 [35] Assume $X \neq \emptyset$ and $\sigma : X \times X \to [1,\infty)$. A function $d_\sigma : X \times X \to [0,\infty)$ is called an extended b-metric if it satisfies the following conditions for all $x_1, x_2, x_3 \in X$:

(1): $d_\sigma(x_1,x_2) = 0 \Leftrightarrow x_1 = x_2$;

(2): $d_\sigma(x_1,x_2) = d_\sigma(x_2,x_1)$;

(3): $d_\sigma(x_1,x_3) \leq \sigma(x_1,x_3)\{d_\sigma(x_1,x_2) + d_\sigma(x_2,x_3)\}$.

Then the pair (X, d_σ) is said to be an extended b-metric space.

In 2018, Mlaiki et al. [36] introduced the concept of controlled metric spaces as follows:

Definition 2 Assume $X \neq \emptyset$ and $\sigma : X \times X \to [1,\infty)$. Then a function $d_\sigma : X \times X \to [0,\infty)$ is called a controlled metric if for all $x_1, x_2, x_3 \in X$, it satisfies the following properties:

(1): $d_\sigma(x_1,x_2) = 0 \Leftrightarrow x_1 = x_2$;

(2): $d_\sigma(x_1,x_2) = d_\sigma(x_2,x_1)$;

(3): $d_\sigma(x_1,x_3) \leq [\sigma(x_1,x_2)d_\sigma(x_1,x_2) + \sigma(x_2,x_3)d_\sigma(x_2,x_3)]$.

Then the pair (X, d_σ) is called a controlled metric space.

Definition 3 [12] A mapping $T : X \longrightarrow X$ is called F-contraction if there exists $\tau > 0$ such that for all $x_1, x_2 \in X$,

$$d_\sigma(T(x_1), T(x_2)) > 0 \Rightarrow \tau + F(d_\sigma(T(x_1), T(x_2)) \leq F(d_\sigma(x_1,x_2)),$$

where $F : \mathbb{R}_+ \to \mathbb{R}$ is a mapping satisfying:

(F_1): F is strictly increasing;

(F_2): for each sequence $\{\alpha_n\}$ of positive numbers

$$\lim_{n\longrightarrow\infty} \alpha_n = 0 \Leftrightarrow \lim_{n\longrightarrow\infty} F(\alpha_n) = -\infty;$$

(F_3): there exists $k \in (0,1)$ such that $\lim_{\alpha\longrightarrow 0^+} \alpha^k F(\alpha) = 0$.

In accordance with Jacehymski [10], let X be a nonempty set and let Δ represent the diagonal of $X \times X$, that is, $\Delta = \{(x,x)\}$. Let G be a directed graph whose set of vertices coincides with X, and its set of edges is denoted by $\Xi(G)$, which includes all loops, i.e., $\Delta \subseteq \Xi(G)$. Assume that G has no multiple $\Xi(G)$, allowing us to identify G with the pair $(\nu(G), \Xi(G))$. Furthermore, let d_σ be a controlled metric on X, the value $d_\sigma(x_1,x_2)$ represents the weight of the edge (x_1,x_2) in G.

The Hausdorff metric, also known as the Hausdorff distance or Hausdorff measure, is a way to measure the dissimilarity or similarity between two subsets of a metric space. It quantifies the maximum distance between points of one set to the closest point in the other set.

Definition 4 Assume $CB(X)$ is the family of closed and bounded subsets of a controlled metric space (X, d_σ). For $x_1 \in X$ and A, B are nonempty closed and bounded subsets of X, set

$$d_\sigma(x_1, B) = \inf_{b \in B}(d_\sigma(x_1, b)).$$

Define $H_\sigma : CB(X) \times CB(X) \longrightarrow [0, \infty)$ as

$$H_\sigma(A, B) = \max\left\{\sup_{a \in A} d_\sigma(a, B), \sup_{b \in B} d_\sigma(b, A)\right\},$$

then $(CB(X), H_\sigma)$ is called a Pompeiu-Hausdorff controlled metric space.

In accordance with Abbas et al. [14], we define $(CB(X), H_\sigma)$ as the Pompeiu-Hausdorff controlled metric space induced by X, with Δ representing the diagonal of $CB(X) \times CB(X)$. In this article, we presume that any A, B are nonempty closed and bounded subsets of X are connected by an edge, implying the existence of an edge between some $a_1 \in A$ and $a_2 \in B$, denoted by $(A, B) \in \Xi(G)$. We identify the directed graph G, referred to as a set of directed graph G, as the pair $(\nu(G), \Xi(G))$, where $\nu(G)$ coincides with $CB(X)$ and $\Xi(G)$ comprises all $\Xi(G)$, including loops $(\Delta \subseteq \Xi(G))$. We also assume that G has no multiple $\Xi(G)$. Furthermore, for every U, V are nonempty closed and bounded subsets of X, the value $H_\sigma(U, V)$ represents the weight of the edge (U, V) in the directed set graph G.

Abbas et al. [14] define property $\mathtt{P}^*$ for set of directed graph G as follows:

Definition 5 If, for each sequence of sets $\{X_n\} \in CB(X)$ with $X_n \longrightarrow X$ as n approaches ∞, there is an edge between X_{n+1} and X_n for $n \in \mathbb{N}$, it also implies that, for any $n \in \mathbb{N}$, there is a subsequence X_{n_k} of X_n with an edge between X and X_{n_k}.

The idea of fuzzy sets was first introduced by Zadeh [17] as follows:

Definition 6 [37] A fuzzy set in X is a function with domain in X and values in $[0, 1]$. If $\mathtt{F}$ is a fuzzy set in X, then the function value $\mathtt{F}(x)$ is called grade of membership of x in $\mathtt{F}$. The collection of all fuzzy sets in X is denoted by $\mathcal{F}(X)$. The mapping $\mathtt{f} : X \longrightarrow \mathcal{F}(X)$ is known as fuzzy mapping. For a number $\alpha \in (0, 1]$, an alpha-level set of a fuzzy set $\mathtt{F}$ is defined as

$$[\mathtt{F}]_\alpha = \left\{x \in X : \mathtt{F}(x) \geq \alpha\right\},$$

$$[\mathtt{F}]_0 = \overline{\left\{x \in X : \mathtt{F}(x) > 0\right\}},$$

where $\overline{\mathtt{F}}$ indicates the closure of the set $\mathtt{F}$.

In the further discussion, we will take $\alpha \in (0, 1]$, so, in the rest of the paper, it is not significant if α is not cited as part of $(0, 1]$.

Definition 7 Let $A, B \in \mathcal{F}(X)$

(1): For $[A]_\alpha$, $[B]_\alpha \subset X$ we say that $([A]_\alpha, [B]_\alpha) \subset \Xi(G)$ if there is an edge between $x_1 \in [A]_\alpha$ and $x_2 \in [B]_\alpha$.

(2): For $[A]_\alpha$, $[B]_\alpha \subset \mathcal{F}(X)$ we say that there is path between $[A]_\alpha$ and $[B]_\alpha$ if there is a path between $x_1 \in [A]_\alpha$ and $x_2 \in [B]_\alpha$.

Definition 8 [34] Define the set $\mathrm{I}_{FC}(X)$ by

$$\mathsf{I}_{FC}(X) = \{A \in \digamma(X) : [A]_\alpha \in CB(X)\}.$$

The interpretation of a relation R on $\mathsf{I}_{FC}(X)$ is as follows: If there is a path between $[A]_\alpha$ and $[B]_\alpha$, then for $[A]_\alpha$, $[B]_\alpha \in \mathsf{I}_{FC}(X)$ we say that $[A]_\alpha R [B]_\alpha$. It can easily be verified that R is transitive relation.

Definition 9 Consider the fuzzy set-valued mapping $T : CB(X) \longrightarrow \mathsf{I}_{FC}(X)$ then the set X_T is explained as

$$X_T = \{U \in CB(X) : (U, [T(U)]_\alpha) \subseteq \Xi(G)\}.$$

Definition 10 Consider the fuzzy set-valued mappings $S, T : CB(X) \longrightarrow \mathsf{I}_{FC}(X)$ then the $U \in CB(X)$ is a fuzzy coincidence point of S and T if $[T(U)]_\alpha = [S(U)]_\alpha$. Also a set $A \in CB(X)$ is a fuzzy fixed point of S if $[S(A)]_\alpha = A$.

Definition 11 [34] Consider the fuzzy set-valued mappings $S, T : CB(X) \longrightarrow \mathsf{I}_{FC}(X)$. These mappings are called weakly compatible if they commute at their coincidence point.

Definition 12 Let (X, d_σ) be a controlled metric space and $\varepsilon > 0$ we say (X, d_σ) is ε chainable if for given $x, \mathrm{y} \in X$ there exists $n \in \mathbb{N}$ a finite sequence $\{x_n\} \in X$ such that

$$x_0 = x, x_n = \mathrm{y} \text{ and } d_\sigma(x_{i-1}, x_i) < \varepsilon \text{ for } i = 1, 2, ..., n.$$

Lemma 1 Let (X, d_σ) be a controlled metric space. If $H_\sigma(A, B) < \varepsilon$ for all nonempty $A, B \in CB(X)$ then for every $a_1 \in A$ there exists $a_2 \in B$ such that $d_\sigma(a_1, a_2) < \varepsilon$.

Definition 13 A subset U of $CB(X)$ is called complete if for two fuzzy sets $A, B \in \mathsf{I}_{FC}(X)$ such that if $[A]_\alpha, [B]_\alpha \subseteq U$ there is an edge between $[A]_\alpha$ and $[B]_\alpha$.

3. Main results

Throughout this article, we represent the collection of all fuzzy coincidence points of S and T by $\tilde{\mathsf{C}}_F(S, T)$, property $\mathtt{P}^*$ by $\mathtt{P}^*$ and the collection of all fuzzy fixed points of S by $Fuz(S)$. In addition, we suppose that H_σ is continuous, while for nonempty sets $A, B \in CB(X)$ we define $M(A, B)$ and $E(A, B)$ as follows unless specified otherwise,

$$M(A,B) = \max \Bigg\{ H_\sigma([T(A)]_\alpha, [T(B)]_\alpha), H_\sigma([S(A)]_\alpha, [T(A)]_\alpha),$$

$$H_\sigma([S(B)]_\alpha, [T(B)]_\alpha), \frac{\{H_\sigma([T(A)]_\alpha, [S(A)]_\alpha) \times H_\sigma([S(B)]_\alpha, [T(B)]_\alpha)\}}{\tau(M(A,B)) + H_\sigma([S(A)]_\alpha, [S(B)]_\alpha))},$$

$$\frac{\{H_\sigma([T(A)]_\alpha, [S(A)]_\alpha) \times H_\sigma([S(B)]_\alpha, [T(B)]_\alpha)\}}{1 + H_\sigma([S(A)]_\alpha, [S(B)]_\alpha))} \Bigg\},$$

$$E(A,B) = \min \begin{Bmatrix} H_\sigma([S(A)]_\alpha, [T(A)]_\alpha), & H_\sigma([S(B)]_\alpha, [T(B)]_\alpha) \\ H_\sigma([S(A)]_\alpha, [T(B)]_\alpha), & H_\sigma([S(B)]_\alpha, [T(A)]_\alpha) \end{Bmatrix}.$$

Motivated by the idea of generalized graphic fuzzy F-contractive mappings, we have introduced the notion of graphic fuzzy generalized rational F contractive mappings in a controlled metric space as follows:

Definition 14 Let $S,T : CB(X) \longrightarrow \mathsf{I}_{FC}(X)$ be the set-valued maps, then the pair (S,T) is called graphic fuzzy generalized rational F contractive mappings if the following holds:

(i) For any $U \in CB(X)$, $([S(U)]_\alpha, U) \subset \Xi(G)$ and $(U,[T(U)]_\alpha) \subset \Xi(G)$.

(ii) There is a function $\tau : [0,\infty) \longrightarrow [0,\infty)$ having $\lim\limits_{\delta \longrightarrow \mathsf{q}^+} \inf \tau(\delta) > 0$ for all $\mathsf{q} \geq 0$ such that there exists an edge between A and B with $[S(A)]_\alpha \neq [S(B)]_\alpha$,

$$\tau(M(A,B)) + F(H_\sigma([S(A)]_\alpha, [S(B)]_\alpha)) \leq F(M(A,B)) + LE(A,B),$$

where $F \in \mathsf{F}$ *and* $L \geq 0$.

If (S,T) is a pair of set-valued mappings from $CB(X)$ to $\mathsf{I}_{FC}(X)$ that is graphic fuzzy generalized rational F-contractive on G, then (S,T) is also such a pair on G^{-1} and $\tilde{G}$. Moreover, (S,T) on G is also graphic fuzzy generalized rational F-contractive on the complete graph G_0 with $\Xi(G_0) = X \times X$.

Here, we establish some widely used fuzzy coincidence and fixed point results for graphic fuzzy generalized rational F contractive mappings with fuzzy set-valued maps on $\mathsf{I}_{FC}(X)$.

Theorem 1 Let (X,d_σ) be a complete controlled metric space tailored with a directed graph G such that $\nu(G) = X$, $\Xi(G) \supseteq \Delta$ together with the transitive relation R on $\mathsf{I}_{FC}(X)$. Let $S,T : CB(X) \to \mathsf{I}_{FC}(X)$ be a pair of graphic fuzzy generalized rational F contractive mappings such that the range of T contains the range of S and

$$\sup_{m\geq 0} \lim_{n \longrightarrow \infty} \frac{\sigma([T(A_{n+1})]_\alpha, [T(A_{n+2})]_\alpha)\sigma([T(A_{n+1})]_\alpha, [T(A_m)]_\alpha)}{\sigma([T(A_n)]_\alpha, [T(A_{n+1})]_\alpha)} < \frac{1}{\mathtt{L}} \text{ for all } \mathtt{L} \in (0,1),$$

then the following statements are true

(i) Given that G is weakly connected and satisfies $\mathtt{P}^*$, such that $[T(X)]_\alpha$ is a complete subspace of $\mathsf{I}_{FC}(X)$ and $\tilde{\mathsf{C}}_F(S,T) \neq \emptyset$ for some α.

(ii) If $\tilde{\mathsf{C}}_F(S,T)$ is complete, then the Hausdorff weight assigned to $[S(U)]_\alpha$ and $[S(V)]_\alpha$ is 0 and for all $U,V \in \tilde{\mathsf{C}}_F(S,T)$.

(iii) If $\tilde{\mathsf{C}}_F(S,T)$ is complete, and S,T are weakly compatible, then $Fuz(S) \cap Fuz(T)$ is a singleton.

(iv) $Fuz(S) \cap Fuz(T)$ is complete $\Leftrightarrow$ $Fuz(S) \cap Fuz(T)$ is a singleton.

Proof. (i) Let A_0 be an element in $CB(X)$. As the range of T contains the range of S, take $A_1 \in CB(X)$ such that $[S(A_0)]_\alpha = [T(A_1)]_\alpha$ take $A_2 \in CB(X)$ such that $[S(A_1)]_\alpha = [T(A_2)]_\alpha$. By the same above mentioned procedure, for $A_n \in CB(X)$ we obtain $A_{n+1} \in CB(X)$ such that $[S(A_n)]_\alpha = [T(A_{n+1})]_\alpha$ for all $n \in \mathbb{N}$.

By Definition 14 it is given that (S,T) is the pair of graphic fuzzy generalized rational F contractive mappings, so for $A_n, A_{n+1} \in CB(X)$ we have $(A_{n+1}, [T(A_{n+1})]_\alpha) \subseteq \Xi(G)$ and $([S(A_n)]_\alpha, A_n) \subseteq \Xi(G)$ as $[S(A_n)]_\alpha = [T(A_{n+1})]_\alpha$, therefore $([S(A_n)]_\alpha, A_n) = ([T(A_{n+1})]_\alpha, A_n) \subseteq \Xi(G)$.

By the property of transitivity,

$$(A_{n+1}, A_n) \subseteq \Xi(G).$$

If we take $A_n, A_{n+1} \in CB(X)$ for all $n \in \mathbb{N}$ then, $[S(A_n)]_\alpha \neq [S(A_{n+1})]_\alpha$, otherwise for some $k \in \mathbb{N}$ we have $[S(A_{2k})]_\alpha = [S(A_{2k+1})]_\alpha$, also as $[S(A_{2k})]_\alpha = [T(A_{2k+1})]_\alpha$. Then $[T(A_{2k+1})]_\alpha = [S(A_{2k+1})]_\alpha$ and hence $A_{2k+1} \in \tilde{\mathsf{C}}_F(S,T)$.

Since $(A_{n+1}, A_n) \subseteq \Xi(G)$ for all $n \in \mathbb{N}$, by using contractive condition,

$$\tau(M(A_n, A_{n+1})) + F(H_\sigma([T(A_{n+1})]_\alpha, [T(A_{n+2})]_\alpha))$$

$$= \tau(M(A_n,A_{n+1})) + F(H_\sigma([S(A_n)]_\alpha,[S(A_{n+1})]_\alpha))$$

$$\leq F(M(A_n,A_{n+1})) + LE(A_n,A_{n+1}) \leq F(M(A_n,A_{n+1})).$$

So $M(A_n,A_{n+1})$ and $E(A_n,A_{n+1})$ becomes

$$M(A_n,A_{n+1})$$

$$= \max\left\{ H_\sigma([T(A_n)]_\alpha,[T(A_{n+1})]_\alpha), H_\sigma([S(A_n)]_\alpha,[T(A_n)]_\alpha),\right.$$

$$H_\sigma([S(A_{n+1})]_\alpha,[T(A_{n+1})]_\alpha), \frac{\{H_\sigma([T(A_n)]_\alpha,[S(A_n)]_\alpha)\times H_\sigma([S(A_{n+1})]_\alpha,[T(A_{n+1})]_\alpha)\}}{\tau(M(A_n,A_{n+1})) + H_\sigma([S(A_n)]_\alpha,[S(A_{n+1})]_\alpha))},$$

$$\left.\frac{\{H_\sigma([T(A_n)]_\alpha,[S(A_n)]_\alpha)\times H_\sigma([S(A_{n+1})]_\alpha,[T(A_{n+1})]_\alpha)\}}{1 + H_\sigma([S(A_n)]_\alpha,[S(A_{n+1})]_\alpha))}\right\},$$

$$= \max\left\{ H_\sigma([T(A_n)]_\alpha,[T(A_{n+1})]_\alpha), H_\sigma([T(A_{n+1})]_\alpha,[T(A_n)]_\alpha),\right.$$

$$H_\sigma([T(A_{n+1})]_\alpha,[T(A_{n+2})]_\alpha), \frac{\{H_\sigma([T(A_n)]_\alpha,[T(A_{n+1})]_\alpha)\times H_\sigma([T(A_{n+2})]_\alpha,[T(A_{n+1})]_\alpha)\}}{\tau(M(A_n,A_{n+1})) + H_\sigma([T(A_{n+1})]_\alpha,[T(A_{n+2})]_\alpha))},$$

$$\left.\frac{\{H_\sigma([T(A_n)]_\alpha,[T(A_{n+1})]_\alpha)\times H_\sigma([T(A_{n+2})]_\alpha,[T(A_{n+1})]_\alpha)\}}{1 + H_\sigma([T(A_{n+1})]_\alpha,[T(A_{n+2})]_\alpha))}\right\},$$

$$\leq \max\left\{ H_\sigma([T(A_n)]_\alpha,[T(A_{n+1})]_\alpha), H_\sigma([T(A_{n+1})]_\alpha,[T(A_{n+2})]_\alpha),\right.$$

$$\frac{\{H_\sigma([T(A_n)]_\alpha,[T(A_{n+1})]_\alpha)\times H_\sigma([T(A_{n+2})]_\alpha,[T(A_{n+1})]_\alpha)\}}{\tau(M(A_n,A_{n+1})) + H_\sigma([T(A_{n+1})]_\alpha,[T(A_{n+2})]_\alpha))},$$

$$\left.\frac{\{H_\sigma([T(A_n)]_\alpha,[T(A_{n+1})]_\alpha)\times H_\sigma([T(A_{n+2})]_\alpha,[T(A_{n+1})]_\alpha)\}}{1 + H_\sigma([T(A_{n+1})]_\alpha,[T(A_{n+2})]_\alpha))}\right\},$$

$$\leq \max\left\{ H_\sigma([T(A_n)]_\alpha,[T(A_{n+1})]_\alpha), H_\sigma([T(A_{n+1})]_\alpha,[T(A_{n+2})]_\alpha)\right\},$$

and

$$E(A_n,A_{n+1}) = \min\begin{Bmatrix} H_\sigma([S(A_n)]_\alpha,[T(A_n)]_\alpha), & H_\sigma([S(A_{n+1})]_\alpha,[T(A_{n+1})]_\alpha) \\ H_\sigma([S(A_n)]_\alpha,[T(A_{n+1})]_\alpha), & H_\sigma([S(A_{n+1})]_\alpha,[T(A_n)]_\alpha) \end{Bmatrix}$$

$$\leq \min\begin{Bmatrix} H_\sigma([T(A_{n+1})]_\alpha,[T(A_n)]_\alpha), & H_\sigma([T(A_{n+2})]_\alpha,[T(A_{n+1})]_\alpha) \\ H_\sigma([T(A_{n+1})]_\alpha,[T(A_{n+1})]_\alpha), & H_\sigma([T(A_{n+2})]_\alpha,[T(A_n)]_\alpha) \end{Bmatrix} = 0.$$

Consequently we obtain

$$\begin{aligned} &\tau(M(A_n,A_{n+1})) + F(H_\sigma([T(A_{n+1})]_\alpha,[T(A_{n+2})]_\alpha)) \\ &\leq F\left[\max\left\{H_\sigma([T(A_n)]_\alpha,[T(A_{n+1})]_\alpha), H_\sigma([T(A_{n+1})]_\alpha,[T(A_{n+2})]_\alpha)\right\}\right]. \end{aligned} \tag{1}$$

If we take

$$\begin{aligned} &\max\left\{H_\sigma([T(A_n)]_\alpha,[T(A_{n+1})]_\alpha), H_\sigma([T(A_{n+1})]_\alpha,[T(A_{n+2})]_\alpha)\right\} \\ &= H_\sigma([T(A_{n+1})]_\alpha,[T(A_{n+2})]_\alpha). \end{aligned}$$

Then,

$$\begin{aligned} &\tau(H_\sigma([T(A_{n+1})]_\alpha,[T(A_{n+2})]_\alpha)) + F(H_\sigma([T(A_{n+1})]_\alpha,[T(A_{n+2})]_\alpha)) \\ &\leq F(H_\sigma([T(A_{n+1})]_\alpha,[T(A_{n+2})]_\alpha)), \end{aligned}$$

which further implies that

$$\begin{aligned} &F(H_\sigma([T(A_{n+1})]_\alpha,[T(A_{n+2})]_\alpha)) \\ &\leq F(H_\sigma([T(A_{n+1})]_\alpha,[T(A_{n+2})]_\alpha)) - \tau(H_\sigma([T(A_{n+1})]_\alpha,[T(A_{n+2})]_\alpha)) \\ &< F(H_\sigma([T(A_{n+1})]_\alpha,[T(A_{n+2})]_\alpha)). \end{aligned}$$

As F is strictly increasing,

$$H_\sigma([T(A_{n+1})]_\alpha,[T(A_{n+2})]_\alpha) < H_\sigma([T(A_{n+1})]_\alpha,[T(A_{n+2})]_\alpha),$$

a contradiction.

Hence

$$\max\left\{H_\sigma([T(A_n)]_\alpha,[T(A_{n+1})]_\alpha),H_\sigma([T(A_{n+1})]_\alpha,[T(A_{n+2})]_\alpha)\right\}$$

$$= H_\sigma([T(A_n)]_\alpha,[T(A_{n+1})]_\alpha).$$

Then, from (1)

$$\tau(M(A_n,A_{n+1}))+F(H_\sigma([T(A_{n+1})]_\alpha,[T(A_{n+2})]_\alpha))\leq F(H_\sigma([T(A_n)]_\alpha,[T(A_{n+1})]_\alpha)).$$

That is

$$F(H_\sigma([T(A_{n+1})]_\alpha,[T(A_{n+2})]_\alpha))\leq F(H_\sigma([T(A_n)]_\alpha,[T(A_{n+1})]_\alpha)).$$

Thus $H_\sigma([T(A_n)]_\alpha,[T(A_{n+1})]_\alpha)$ is a decreasing sequence.
Next we will prove

$$\lim_{n\longrightarrow\infty} H_\sigma([T(A_n)]_\alpha,[T(A_{n+1})]_\alpha)=0.$$

By definition of τ, for all $n_0\in\mathbb{N}$ there exists $\mathscr{C}>0$ such that $\tau(M(A_n,A_{n+1}))>\mathscr{C}$ for all $n\geq n_0$.

$$F(H_\sigma([T(A_n)]_\alpha,[T(A_{n+1})]_\alpha))$$

$$\leq F(H_\sigma([T(A_{n-1})]_\alpha,[T(A_n)]_\alpha))-\tau(M(A_{n-1},A_n))$$

$$\leq F(H_\sigma([T(A_{n-2})]_\alpha,[T(A_{n-1})]_\alpha))-\tau(M(A_{n-1},A_n))-\tau(M(A_{n-2},A_{n-1}))$$

$$\leq\cdots\leq F(H_\sigma([T(A_0)]_\alpha,[T(A)]_\alpha))-\left[\tau(M(A_{n-1},A_n))+\tau(M(A_{n-2},A_{n_1}))+\cdots+\tau(M(A_0,A_1))\right]$$

$$\leq F(H_\sigma([T(A_0)]_\alpha,[T(A_1)]_\alpha))-n\mathscr{C}.$$

Letting $n\longrightarrow\infty$, in above inequality we get

$$\lim_{n\longrightarrow\infty} F(H_\sigma([T(A_n)]_\alpha,[T(A_{n+1})]_\alpha))=-\infty.$$

From condition F_2

$$\lim_{n\longrightarrow 0} H_\sigma([T(A_n)]_\alpha,[T(A_{n+1})]_\alpha)=0.$$

From F_3 there exists $k\in(0,1)$ such that,

$$\lim_{n\longrightarrow\infty} H_\sigma([T(A_n)]_\alpha,[T(A_{n+1})]_\alpha)^k F(H_\sigma([T(A_n)]_\alpha,[T(A_{n+1})]_\alpha))=0.$$

Consider the following

$$\begin{aligned}
&H_\sigma([T(A_n)]_\alpha,[T(A_{n+1})]_\alpha)^k F(H_\sigma([T(A_n)]_\alpha,[T(A_{n+1})]_\alpha))\\
&-H_\sigma([T(A_n)]_\alpha,[T(A_{n+1})]_\alpha)^k F(H_\sigma([T(A_0)]_\alpha,[T(A_1)]_\alpha))\\
&\leq\left[H_\sigma([T(A_n)]_\alpha,[T(A_{n+1})]_\alpha)^k F(H_\sigma([T(A_0)]_\alpha,[T(A_1)]_\alpha))-n\mathscr{C}\right]\\
&-\left[H_\sigma([T(A_n)]_\alpha,[T(A_{n+1})]_\alpha)^k F(H_\sigma([T(A_0)]_\alpha,[T(A_1)]_\alpha))\right]\\
&\leq -n\mathscr{C}[H_\sigma([T(A_n)]_\alpha,[T(A_{n+1})]_\alpha)^k]\leq 0.
\end{aligned}$$

Taking limit n tends to ∞, in above

$$\lim_{n\longrightarrow\infty} n[H_\sigma([T(A_n)]_\alpha,[T(A_{n+1})]_\alpha)^k]=0.$$

There exists $n_1\in\mathbb{N}$ such that, for all $n\geq n_0,\ n[H_\sigma([T(A_n)]_\alpha,[T(A_{n+1})]_\alpha)^k]\leq 1$.
Thus, we get

$$\lim_{n\longrightarrow\infty}[H_\sigma([T(A_n)]_\alpha,[T(A_{n+1})]_\alpha)]\leq\frac{1}{n^{\frac{1}{k}}}. \tag{2}$$

By triangular inequality, and for all $m,n\geq n_0$

$$\begin{aligned}
&H_\sigma([T(A_n)]_\alpha,[T(A_m)]_\alpha)\\
&\leq\sigma([T(A_n)]_\alpha,[T(A_{n+1})]_\alpha)H_\sigma([T(A_n)]_\alpha,[T(A_{n+1})]_\alpha)
\end{aligned}$$

$$
\begin{aligned}
&+\sigma([T(A_{n+1})]_\alpha,[T(A_m)]_\alpha)H_\sigma([T(A_{n+1})]_\alpha,[T(A_m)]_\alpha)\\
&\le \sigma([T(A_n)]_\alpha,[T(A_{n+1})]_\alpha)H_\sigma([T(A_n)]_\alpha,[T(A_{n+1})]_\alpha)\\
&\quad+\sigma([T(A_{n+1})]_\alpha,[T(A_m)]_\alpha)\Big[\sigma([T(A_{n+1})]_\alpha,[T(A_{n+2})]_\alpha)H_\sigma([T(A_{n+1})]_\alpha,[T(A_{n+2})]_\alpha)\\
&\quad+\sigma([T(A_{n+2})]_\alpha,[T(A_m)]_\alpha)H_\sigma([T(A_{n+2})]_\alpha,[T(A_m)]_\alpha)\Big]\\
&= \sigma([T(A_n)]_\alpha,[T(A_{n+1})]_\alpha)H_\sigma([T(A_n)]_\alpha,[T(A_{n+1})]_\alpha)\\
&\quad+\sigma([T(A_{n+1})]_\alpha,[T(A_m)]_\alpha)\sigma([T(A_{n+1})]_\alpha,[T(A_{n+2})]_\alpha)H_\sigma([T(A_{n+1})]_\alpha,[T(A_{n+2})]_\alpha)\\
&\quad+\sigma([T(A_{n+1})]_\alpha,[T(A_m)]_\alpha)\sigma([T(A_{n+2})]_\alpha,[T(A_m)]_\alpha)H_\sigma([T(A_{n+2})]_\alpha,[T(A_m)]_\alpha)\\
&\quad\vdots\\
&\le \sigma([T(A_n)]_\alpha,[T(A_{n+1})]_\alpha)H_\sigma([T(A_n)]_\alpha,[T(A_{n+1})]_\alpha)\\
&\quad+\sum_{i=1}^{m-2}\Big(\prod_{\mathrm{j}=n+1}^{\mathrm{i}}\sigma([T(A_j)]_\alpha,[T(A_m)]_\alpha)\Big)\sigma([T(A_i)]_\alpha,[T(A_{i+1})]_\alpha)H_\sigma([T(A_i)]_\alpha,[T(A_{i+1})]_\alpha)\\
&\quad+\Big(\prod_{\mathrm{i}=n+1}^{\mathrm{m}-1}\sigma([T(A_i)]_\alpha,[T(A_m)]_\alpha)\Big)\sigma([T(A_{m-1})]_\alpha,[T(A_m)]_\alpha)H_\sigma([T(A_{m-1})]_\alpha,[T(A_m)]_\alpha)\\
&= \sigma([T(A_n)]_\alpha,[T(A_{n+1})]_\alpha)H_\sigma([T(A_n)]_\alpha,[T(A_{n+1})]_\alpha)\\
&\quad+\sum_{i=n+1}^{m-1}\Big(\prod_{\mathrm{j}=n+1}^{\mathrm{i}}\sigma([T(A_j)]_\alpha,[T(A_m)]_\alpha)\Big)\sigma([T(A_i)]_\alpha,[T(A_{i+1})]_\alpha)H_\sigma([T(A_i)]_\alpha,[T(A_{i+1})]_\alpha)\\
&\le \sigma([T(A_n)]_\alpha,[T(A_{n+1})]_\alpha)H_\sigma([T(A_n)]_\alpha,[T(A_{n+1})]_\alpha)\\
&\quad+\sum_{i=n+1}^{m-1}\Big(\prod_{\mathrm{j}=0}^{\mathrm{i}}\sigma([T(A_j)]_\alpha,[T(A_m)]_\alpha)\Big)\sigma([T(A_i)]_\alpha,[T(A_{i+1})]_\alpha)H_\sigma([T(A_i)]_\alpha,[T(A_{i+1})]_\alpha).
\end{aligned}
$$

Therefore,

$$\Rightarrow H_\sigma([T(A_n)]_\alpha,[T(A_m)]_\alpha)$$

$$\leq \sigma([T(A_n)]_\alpha,[T(A_{n+1})]_\alpha)H_\sigma([T(A_n)]_\alpha,[T(A_{n+1})]_\alpha)$$

$$+\sum_{i=n+1}^{m-1}\left(\prod_{\mathrm{j}=0}^{\mathrm{i}}\sigma([T(A_j)]_\alpha,[T(A_m)]_\alpha)\right)\sigma([T(A_i)]_\alpha,[T(A_{i+1})]_\alpha)\frac{1}{i^{\frac{1}{k}}}. \tag{3}$$

Now

$$\sum_{i=n+1}^{m-1}\left(\prod_{\mathrm{j}=0}^{\mathrm{i}}\sigma([T(A_j)]_\alpha,[T(A_m)]_\alpha)\right)\sigma([T(A_i)]_\alpha,[T(A_{i+1})]_\alpha)\frac{1}{i^{\frac{1}{k}}}$$

$$\leq \sum_{i=n+1}^{\infty}\frac{1}{i^{\frac{1}{k}}}\left(\prod_{\mathrm{j}=0}^{\mathrm{i}}\sigma([T(A_j)]_\alpha,[T(A_m)]_\alpha)\right)\sigma([T(A_i)]_\alpha,[T(A_{i+1})]_\alpha) \tag{4}$$

$$=\sum_{i=n+1}^{\infty}\mathscr{U}_i\mathscr{V}_i,$$

where $\mathscr{U}_i=\frac{1}{i^{\frac{1}{k}}}$ and $\mathscr{V}_i=\left(\prod_{\mathrm{j}=0}^{\mathrm{i}}\sigma([T(A_j)]_\alpha,[T(A_m)]_\alpha)\right)\sigma([T(A_i)]_\alpha,[T(A_{i+1})]_\alpha)$.

Since $\frac{1}{k}>0$, therefore $\sum_{i=n+1}^{\infty}\frac{1}{i^{\frac{1}{k}}}$ converges.

Also, $\{\mathscr{V}_i\}_i$ is increasing and bounded above, so $\lim_{i\to\infty}\{\mathscr{V}_i\}$ (which is nonzero) exists.

Consider the partial sum

$$\mathsf{R}_p=\sum_{i=1}^{p}\left(\prod_{\mathrm{j}=0}^{\mathrm{i}}\sigma([T(A_j)]_\alpha,[T(A_m)]_\alpha)\right)\sigma([T(A_i)]_\alpha,[T(A_{i+1})]_\alpha)\frac{1}{i^{\frac{1}{k}}}.$$

Hence, for every $m>n$, we have

$$H_\sigma([T(A_n)]_\alpha,[T(A_m)]_\alpha)\leq[\sigma([T(A_n)]_\alpha,[T(A_{n+1})]_\alpha)H_\sigma([T(A_n)]_\alpha,[T(A_{n+1})]_\alpha)+\mathsf{R}_{\mathrm{m}-1}-\mathsf{R}_{\mathrm{n}}].$$

By using a similar procedure to that in Theorem 8 of [38], we obtain

$$\lim_{n\to\infty}H_\sigma([T(A_n)]_\alpha,[T(A_m)]_\alpha)=0.$$

Thus, $\{[T(A_n)]_\alpha\}_{n\geq 0}$ is a Cauchy sequence in $[T(X)]_\alpha$. The completeness of $([T(X)]_\alpha,d_\sigma)$ in $\mathsf{I}_{FC}(X)$ implies that $[T(A_n)]_\alpha\longrightarrow V$ as $n\longrightarrow\infty$ for some $V\in\mathsf{I}_{FC}(X)$. Also $U\in CB(X)$ can be found such that $[T(U)]_\alpha=V$.

Next suppose that $[T(U)]_\alpha = [S(U)]_\alpha$. Otherwise $([T(A_{n+1})]_\alpha, [T(A_n)]_\alpha) \subseteq \Xi(G)$, and by P^* there exists a subsequence $[T(A_{n_k+1})]_\alpha$ of $[T(A_{n+1})]_\alpha$ such that $([S(U)]_\alpha, [T(A_{n_k+1})]_\alpha) \subseteq \Xi(G)$. Since, $(U, [T(U)]_\alpha) \subseteq \Xi(G)$ and $([T(A_{n_k+1})]_\alpha, A_{n_k}) = ([S(A_{n_k})]_\alpha, A_{n_k}) \subseteq \Xi(G)$ implies $(U, A_{n_k}) \subseteq \Xi(G)$. It is given that (S,T) is the pair of graphic fuzzy generalized rational F contractive mappings, so we get

$$\tau(M(U,A_{n_k})) + F(H_\sigma([S(U)]_\alpha, [T(A_{n_k+1})]_\alpha)) = \tau(M(U,A_{n_k})) + F(H_\sigma([S(U)]_\alpha, [S(A_{n_k})]_\alpha))$$

$$\leq F(M(U,A_{n_k}) + LE(U,A_{n_k}),$$

where

$$M(U,A_{n_k}) = \max\Bigg\{ H_\sigma([T(U)]_\alpha, [T(A_{n_k})]_\alpha), H_\sigma([S(U)]_\alpha, [T(U)]_\alpha),$$

$$H_\sigma([S(A_{n_k})]_\alpha, [T(A_{n_k})]_\alpha), \frac{\{H_\sigma([T(U)]_\alpha, [S(U)]_\alpha) \times H_\sigma([S(A_{n_k})]_\alpha, [T(A_{n_k})]_\alpha)\}}{\tau(M(U,A_{n_k})) + H_\sigma([S(U)]_\alpha, [S(A_{n_k})]_\alpha))},$$

$$\frac{\{H_\sigma([T(U)]_\alpha, [S(U)]_\alpha) \times H_\sigma([S(A_{n_k})]_\alpha, [T(A_{n_k})]_\alpha)\}}{1 + H_\sigma([S(U)]_\alpha, [S(A_{n_k})]_\alpha))} \Bigg\},$$

$$= \max\Bigg\{ H_\sigma([T(U)]_\alpha, [T(A_{n_k})]_\alpha), H_\sigma([S(U)]_\alpha, [T(U)]_\alpha),$$

$$H_\sigma([S(A_{n_k})]_\alpha, [T(A_{n_k})]_\alpha), \frac{\{H_\sigma([T(U)]_\alpha, [S(U)]_\alpha) \times H_\sigma([T(A_{n_k+1})]_\alpha, [T(A_{n_k})]_\alpha)\}}{\tau(M(U,A_{n_k})) + H_\sigma([S(U)]_\alpha, [T(A_{n_k+1})]_\alpha))},$$

$$\frac{\{H_\sigma([T(U)]_\alpha, [S(U)]_\alpha) \times H_\sigma([T(A_{n_k+1})]_\alpha, [T(A_{n_k})]_\alpha)\}}{1 + H_\sigma([S(U)]_\alpha, [T(A_{n_k+1})]_\alpha))} \Bigg\},$$

and

$$E(U,A_{n_k}) = \min\Bigg\{ H_\sigma([S(U)]_\alpha, [T(U)]_\alpha), H_\sigma([S(A_{n_k})]_\alpha, [T(A_{n_k})]_\alpha)$$

$$H_\sigma([S(U)]_\alpha, [T(A_{n_k})]_\alpha), H_\sigma([S(A_{n_k})]_\alpha, [T(U)]_\alpha) \Bigg\}$$

$$\leq \min\Bigg\{ H_\sigma([S(U)]_\alpha, [T(U)]_\alpha), H_\sigma([T(A_{n_k+1})]_\alpha, [T(A_{n_k})]_\alpha)$$

$$H_\sigma([S(U)]_\alpha,[T(A_{n_k})]_\alpha),H_\sigma([T(A_{n_k+1})]_\alpha,[T(U)]_\alpha)\Big\}.$$

For $M(U,A_{n_k})$ the following five cases arises:

Case 1:

If $M(U,A_{n_k})=H_\sigma([T(U)]_\alpha,[T(A_{n_k})]_\alpha)$, then

$$F(H_\sigma([S(U)]_\alpha,[T(A_{n_k+1})]_\alpha))$$

$$=F(H_\sigma([T(U)]_\alpha,[T(A_{n_k})]_\alpha))-\tau(H_\sigma([T(U)]_\alpha,[T(A_{n_k})]_\alpha)).$$

Taking k tends to ∞ implies

$$F(H_\sigma(H_\sigma([S(U)]_\alpha,[T(U)]_\alpha))<F(H_\sigma([T(U)]_\alpha,[T(U)]_\alpha)),$$

a contradiction.

Case 2:

If $M(U,A_{n_k})=H_\sigma([S(U)]_\alpha,[T(U)]_\alpha)$, then we have

$$F(H_\sigma([S(U)]_\alpha,[T(U)]_\alpha))=F(H_\sigma([S(U)]_\alpha,[T(U)]_\alpha))-\tau(H_\sigma([S(U)]_\alpha,[T(U)]_\alpha)),$$

a contradiction.

Case 3:

If $M(U,A_{n_k})=H_\sigma([S(A_{n_k})]_\alpha,[T(A_{n_k})]_\alpha)$, then we have

$$F(H_\sigma([S(U)]_\alpha,[T(A_{n_k+1})]_\alpha))$$

$$=F(H_\sigma([T(A_{n_k+1})]_\alpha,[T(A_{n_k})]_\alpha))-\tau(H_\sigma([T(A_{n_k+1})]_\alpha,[T(A_{n_k})]_\alpha)),$$

by taking the limit as k tends to ∞

$$F(H_\sigma([S(U)]_\alpha,[T(U)]_\alpha))<F(H_\sigma([T(U)]_\alpha,[T(U)]_\alpha))-\tau(H_\sigma([S(U)]_\alpha,[T(U)]_\alpha)),$$

a contradiction.

Case 4:

If

$$M(U,A_{n_k})=\frac{\{H_\sigma([T(U)]_\alpha,[S(U)]_\alpha)\times H_\sigma([T(A_{n_k+1})]_\alpha,[T(A_{n_k})]_\alpha)\}}{\tau(M(U,A_{n_k}))+H_\sigma([S(U)]_\alpha,[T(A_{n_k+1})]_\alpha))},$$

then we have

$$F(H_\sigma([S(U)]_\alpha,[T(A_{n_k+1})]_\alpha))$$

$$= F(\frac{\{H_\sigma([T(U)]_\alpha,[S(U)]_\alpha)\times H_\sigma([T(A_{n_k+1})]_\alpha,[T(A_{n_k})]_\alpha)\}}{\tau(M(U,A_{n_k}))+H_\sigma([S(U)]_\alpha,[T(A_{n_k+1})]_\alpha))})$$

$$-\tau(\frac{\{H_\sigma([T(U)]_\alpha,[S(U)]_\alpha)\times H_\sigma([T(A_{n_k+1})]_\alpha,[T(A_{n_k})]_\alpha)\}}{\tau(M(U,A_{n_k}))+H_\sigma([S(U)]_\alpha,[T(A_{n_k+1})]_\alpha))}),$$

by taking the limit as k tends to ∞

$$F(H_\sigma([S(U)]_\alpha,[T(U)]_\alpha)) = F(\frac{\{H_\sigma([T(U)]_\alpha,[S(U)]_\alpha)\times H_\sigma([T(U)]_\alpha,[T(U)]_\alpha)\}}{\tau(M(U,U))+H_\sigma([S(U)]_\alpha,[T(U)]_\alpha))})$$

$$-\tau(\frac{\{H_\sigma([T(U)]_\alpha,[S(U)]_\alpha)\times H_\sigma([T(U)]_\alpha,[T(U)]_\alpha)\}}{\tau(M(U,U))+H_\sigma([S(U)]_\alpha,[T(U)]_\alpha))})$$

$$< F(0)$$

a contradiction.

Case 5:

If

$$M(U,A_{n_k}) = \frac{\{H_\sigma([T(U)]_\alpha,[S(U)]_\alpha)\times H_\sigma([T(A_{n_k+1})]_\alpha,[T(A_{n_k})]_\alpha)\}}{1+H_\sigma([S(U)]_\alpha,[T(A_{n_k+1})]_\alpha))},$$

then we have

$$\Rightarrow F(H_\sigma([S(U)]_\alpha,[T(A_{n_k+1})]_\alpha))$$

$$= F(\frac{\{H_\sigma([T(U)]_\alpha,[S(U)]_\alpha)\times H_\sigma([T(A_{n_k+1})]_\alpha,[T(A_{n_k})]_\alpha)\}}{1+H_\sigma([S(U)]_\alpha,[T(A_{n_k+1})]_\alpha))})$$

$$-\tau(\frac{\{H_\sigma([T(U)]_\alpha,[S(U)]_\alpha)\times H_\sigma([T(A_{n_k+1})]_\alpha,[T(A_{n_k})]_\alpha)\}}{1+H_\sigma([S(U)]_\alpha,[T(A_{n_k+1})]_\alpha))}),$$

by taking the limit as k tends to ∞

$$F(H_\sigma([S(U)]_\alpha,[T(U)]_\alpha)) = F(\frac{\{H_\sigma([T(U)]_\alpha,[S(U)]_\alpha)\times H_\sigma([T(U)]_\alpha,[T(U)]_\alpha)\}}{1+H_\sigma([S(U)]_\alpha,[T(U)]_\alpha))})$$

$$- \tau(\frac{\{H_\sigma([T(U)]_\alpha,[S(U)]_\alpha) \times H_\sigma([T(U)]_\alpha,[T(U)]_\alpha)\}}{1+H_\sigma([S(U)]_\alpha,[T(U)]_\alpha))})$$

$$< F(0)$$

a contradiction.

For $E(U,A_{n_k})$ the following four cases arises:

Case 1:

If $E(U,A_{n_k}) = H_\sigma([S(U)]_\alpha,[T(U)]_\alpha)$ then proceeding same as in **Case** 2 of $M(U,A_{n_k})$ we obtain a contradiction.

Case 2:

If $E(U,A_{n_k}) = H_\sigma([S(A_{n_k})]_\alpha,[T(A_{n_k})]_\alpha)$ then proceeding same as in **Case** 3 of $M(U,A_{n_k})$ we obtain a contradiction.

Case 3:

If $E(U,A_{n_k}) = H_\sigma([S(U)]_\alpha,[T(A_{n_k})]_\alpha)$ then

$$F(H_\sigma([S(U)]_\alpha,[T(A_{n_k})]_\alpha)) = F(H_\sigma([S(U)]_\alpha,[T(A_{n_k})]_\alpha)) - \tau(H_\sigma([S(U)]_\alpha,[T(A_{n_k})]_\alpha)),$$

by taking the limit as k tends to ∞

$$F(H_\sigma([S(U)]_\alpha,[T(U)]_\alpha)) = F(H_\sigma([S(U)]_\alpha,[T(U)]_\alpha)) - \tau(H_\sigma([S(U)]_\alpha,[T(U)]_\alpha)),$$

a contradiction.

Case 4:

If $E(U,A_{n_k}) = H_\sigma([S(A_{n_k})]_\alpha,[T(U)]_\alpha)$ then

$$F(H_\sigma([S(A_{n_k})]_\alpha,[T(U)]_\alpha)) = F(H_\sigma([S(A_{n_k})]_\alpha,[T(U)]_\alpha)) - \tau(H_\sigma([S(A_{n_k})]_\alpha,[T(U)]_\alpha)),$$

by taking the limit as k tends to ∞

$$F(H_\sigma([S(U)]_\alpha,[T(U)]_\alpha)) = F(H_\sigma([S(U)]_\alpha,[T(U)]_\alpha)) - \tau(H_\sigma([S(U)]_\alpha,[T(U)]_\alpha)),$$

a contradiction.

From above we conclude that $[S(U)]_\alpha = [T(U)]_\alpha$ that is $U \in \tilde{\mathsf{C}}_F(S,T)$.

For (ii) Take $U,V \in \tilde{\mathsf{C}}_F(S,T)$, also assume on contrary a non-zero Pompeiu-Hausdorff weight for $[S(U)]_\alpha$ and $[S(V)]_\alpha$. It is given that (S,T) is the pair of graphic fuzzy generalized rational F contractive mappings, so we get

$$\tau(M(U,V)) + F(H_\sigma([S(U)]_\alpha,[S(V)]_\alpha)) \leq F(M(U,V) + LE(U,V)$$

$$\leq F(M(U,V),$$

where

$$M(U,V) = \max\Bigg\{ H_\sigma([T(U)]_\alpha,[T(V)]_\alpha), H_\sigma([S(U)]_\alpha,[T(U)]_\alpha), H_\sigma([S(V)]_\alpha,[S(V)]_\alpha),$$

$$\frac{\{H_\sigma([T(U)]_\alpha,[S(U)]_\alpha)\times H_\sigma([S(V)]_\alpha,[T(V)]_\alpha)\}}{\tau(M(U,V))+H_\sigma([S(U)]_\alpha,[S(V)]_\alpha))},$$

$$\frac{\{H_\sigma([T(U)]_\alpha,[S(U)]_\alpha)\times H_\sigma([S(V)]_\alpha,[T(V)]_\alpha)\}}{1+H_\sigma([S(U)]_\alpha,[S(V)]_\alpha))}\Bigg\},$$

$$= \max\Bigg\{ H_\sigma([S(U)]_\alpha,[S(V)]_\alpha), H_\sigma([S(U)]_\alpha,[T(U)]_\alpha), H_\sigma([S(V)]_\alpha,[S(V)]_\alpha),$$

$$\frac{\{H_\sigma([S(U)]_\alpha,[S(U)]_\alpha)\times H_\sigma([S(V)]_\alpha,[S(V)]_\alpha)\}}{\tau(M(U,V))+H_\sigma([S(U)]_\alpha,[S(V)]_\alpha))},$$

$$\frac{\{H_\sigma([S(U)]_\alpha,[S(U)]_\alpha)\times H_\sigma([S(V)]_\alpha,[S(V)]_\alpha)\}}{1+H_\sigma([S(U)]_\alpha,[S(V)]_\alpha))}\Bigg\}$$

$$= H_\sigma([S(U)]_\alpha,[S(V)]_\alpha),$$

and

$$E(U,V) = \min\Bigg\{ H_\sigma([S(U)]_\alpha,[T(U)]_\alpha), H_\sigma([S(V)]_\alpha,[T(V)]_\alpha)$$

$$H_\sigma([S(U)]_\alpha,[T(V)]_\alpha), H_\sigma([S(V)]_\alpha,[T(U)]_\alpha)\Bigg\}$$

$$= \min\Bigg\{ H_\sigma([S(U)]_\alpha,[S(U)]_\alpha), H_\sigma([S(V)]_\alpha,[S(V)]_\alpha)$$

$$H_\sigma([S(U)]_\alpha,[S(V)]_\alpha), H_\sigma([S(V)]_\alpha,[S(U)]_\alpha)\Bigg\}.$$

Thus,

$$F(H_\sigma([S(U)]_\alpha,[S(V)]_\alpha)) \le F(H_\sigma([S(U)]_\alpha,[S(V)]_\alpha)) - \tau(M(U,V))$$

$$< F(H_\sigma([S(U)]_\alpha, [S(V)]_\alpha)),$$

again a contradiction.

(iii) First we will prove $Fuz(S) \cap Fuz(T) \neq 0$. By taking $\mathtt{K} = [\mathtt{J}(U)]_\alpha = [\mathtt{P}(U)]_\alpha$ then we obtain $[T(\mathtt{K})]_\alpha = [T([S(U)]_\alpha)]_\alpha = [S([T(U)]_\alpha)]_\alpha = [\mathtt{J}(\mathtt{K})]_\alpha$ implies $\mathtt{K} \in \tilde{\mathtt{C}}_F(S,T)$. So the Pompeiu-Hausdorff weights for $[S(U)]_\alpha$ and $[\mathtt{J}(\mathtt{K})]_\alpha$ are zero by (ii). Thus, $[S(\mathtt{K})]_\alpha = [T(\mathtt{K})]_\alpha = \mathtt{K}$ that is $\mathtt{K} \in Fuz(S) \cap Fuz(T)$. Since $\tilde{\mathtt{C}}_F(S,T)$ is singleton set, then $Fuz(S) \cap Fuz(T)$ also singleton.

(iv) Assume on contrary that the set $Fuz(S) \cap Fuz(T)$ is complete. We have to prove that $Fuz(S) \cap Fuz(T)$ is singleton. Let on contrary U, V are nonempty closed and bounded subsets of X so that there exists $U, V \in Fuz(S) \cap Fuz(T)$ and $U \neq V$. Since $Fuz(S) \cap Fuz(T)$ is complete there exists an edge between U and V. Since it is given that (S,T) is the pair of graphic fuzzy generalized rational F contractive mappings, so we get

$$\tau(M(U,V)) + F(H_\sigma(U,V)) = \tau(M(U,V)) + F(H_\sigma([S(U)]_\alpha, [S(V)]_\alpha))$$

$$\leq F(M(U,V)) + LE(U,V)$$

$$< F(M(U,V)),$$

where

$$M(U,V) = \max\left\{ H_\sigma([T(U)]_\alpha, [T(V)]_\alpha), H_\sigma([S(U)]_\alpha, [T(U)]_\alpha), \right.$$

$$H_\sigma([S(V)]_\alpha, [T(V)]_\alpha), \frac{\{H_\sigma([T(U)]_\alpha, [S(U)]_\alpha) \times H_\sigma([S(V)]_\alpha, [T(V)]_\alpha)\}}{\tau(M(U,V)) + H_\sigma([S(U)]_\alpha, [S(V)]_\alpha))},$$

$$\left. \frac{\{H_\sigma([T(U)]_\alpha, [S(U)]_\alpha) \times H_\sigma([S(V)]_\alpha, [T(V)]_\alpha)\}}{1 + H_\sigma([S(U)]_\alpha, [S(V)]_\alpha))} \right\},$$

$$= \max\left\{ H_\sigma(U,V), H_\sigma(U,U), H_\sigma(V,V), \frac{\{H_\sigma(U,U) \times H_\sigma(V,V)\}}{\tau(M(U,V)) + H_\sigma(U,V)}, \frac{\{H_\sigma(U,U) \times H_\sigma(V,V)\}}{1 + H_\sigma(U,V)} \right\}$$

$$= H_\sigma(U,V),$$

and

$$E(U,V) = \left\{ H_\sigma([S(U)]_\alpha, [T(U)]_\alpha), H_\sigma([S(V)]_\alpha, [T(V)]_\alpha) H_\sigma([S(U)]_\alpha, [T(V)]_\alpha), H_\sigma([S(V)]_\alpha, [T(U)]_\alpha) \right\}$$

$$= \left\{ H_\sigma(U,U), H_\sigma(V,V), H_\sigma(U,V)), H_\sigma(V,U) \right\}$$

$$= 0.$$

Thus,

$$F(H_\sigma(U,V)) \leq F(H_\sigma(U,V)) - \tau(M(U,V)),$$

a contradiction. So, $U = V$. Conversely, if $Fuz(S) \cap Fuz(T)$ is singleton, then $\Xi(X) \supseteq \Delta$ shows $Fuz(S) \cap Fuz(T)$ is a singleton set. □

In order to validate Theorem 1 consider the following examples in the framework of controlled metric spaces:

Example 1 Let $X = \{3n : n \in \{1,2,3,4\}\} = \nu(G)$, $\Xi(G) = \{(p,q) \in X \times X : p < q\}$. Define $d_\sigma : \nu(G) \times \nu(G) \longrightarrow [0,\infty)$ by

$$d_\sigma(x_1,x_2) = \begin{cases} 0, & \text{if } x_1 = x_2, \\ \dfrac{1}{3n}, & \text{if } x_1 \in \{3,6\} \text{ with } x_1 \neq x_2, \\ \dfrac{3n}{3n+1}, & \text{otherwise.} \end{cases}$$

then (X, d_σ) is a complete controlled metric space.

Moreover, the Pompeiu-Hausdorff controlled metric is given by

$$H_\sigma(A,B) = \begin{cases} \dfrac{1}{3n}, & \text{if } A,B \subseteq \{3,6\} \text{ with } A \neq B, \\ \dfrac{3n}{3n+1}, & \text{if } A \text{ or } B \text{ (or both) } \subsetneq \{3,6\} \text{ with } A \neq B, \\ 0, & \text{if } B = A. \end{cases}$$

The Pompeiu-Hausdorff weights for ($n = 4$) assigned to $A, B \in CB(X)$ are shown in Figure 1.

For the case $n = 4$, the vertex set is $\nu(G) = \{3,6,9,12\}$. The Pompeiu-Hausdorff metric H_σ assigns different weights depending on whether the subsets lie entirely in $\{3,6\}$ or involve other elements. Specifically, the distance between $\{3\}$ and $\{6\}$ is $\frac{1}{12}$, reflecting the special role of the elements 3 and 6 in the definition of d_σ. All other pairs of distinct nodes, such as $(3,9), (3,12), (6,9), (6,12)$, and $(9,12)$, involve at least one element outside $\{3,6\}$ and therefore receive the larger weight $\frac{12}{13}$. Consequently, the figure illustrates how the metric structure distinguishes between the pair $(3,6)$, which is assigned a smaller distance, and all other connections, which are uniformly assigned the larger distance. This weighted representation provides a clear visualization of the Pompeiu-Hausdorff geometry induced by d_σ when $n = 4$.

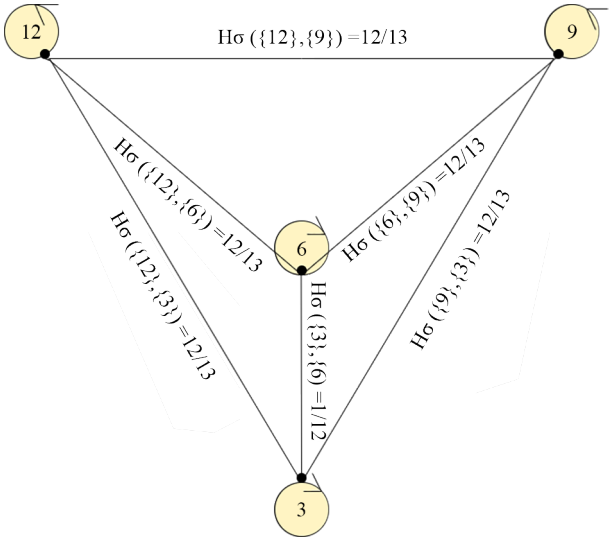


Figure 1. The Pompeiu-Hausdorff weights ($n = 4$)

Define $S, T : CB(X) \longrightarrow \mathrm{l}_{FC}(X)$ by, for $U \subseteq \{3,6\}$

$$S(U)(V) = \begin{cases} 1, & \text{if } V = \{3\}, \\ \frac{1}{3}, & \text{if } V \neq \{3\}, \\ 0, & \text{elsewhere.} \end{cases}$$

For $U \subsetneq \{3,6\}$

$$S(U)(V) = \begin{cases} 1, & \text{if } V = \{3,6\}, \\ \frac{1}{3}, & \text{if } V \neq \{3,6\}, \\ 0, & \text{elsewhere.} \end{cases}$$

Now

$$[S(U)]_1 = \{V : S(U)(V) = 1\} = \begin{cases} \{3\}, & \text{if } U \subset \{3,6\}, \\ \{3,6\}, & \text{if } U \subsetneq \{3,6\}. \end{cases}$$

For $U = \{3\}$

$$T(U)(V) = \begin{cases} 1, & \text{if } V = \{3\}, \\ \frac{1}{3}, & \text{if } V \neq \{3\}, \\ 0, & \text{elsewhere.} \end{cases}$$

For $U \subseteq \{6,9\}$

$$T(U)(V) = \begin{cases} 1, & \text{if } V = \{3,6,9\}, \\ \frac{1}{3}, & \text{if } V \neq \{3,6,9\}, \\ 0, & \text{elsewhere.} \end{cases}$$

For $U \subsetneq \{3,6,9\}$

$$T(U)(V) = \begin{cases} 1, & \text{if } V = \{3,6,9,\ldots,3n\}, \\ \frac{1}{3}, & \text{if } V \neq \{3,6,9,\ldots,3n\}, \\ 0, & \text{elsewhere.} \end{cases}$$

Now

$$[T(U)]_1 = \{V : T(U)(V) = 1\} = \begin{cases} \{3\}, & \text{if } U = \{3\}, \\ \{3,6,9\}, & \text{if } U \subset \{6,9\}, \\ \{3,6,9,\ldots,3n\}, & \text{if } U \subsetneq \{3,6,9\}. \end{cases}$$

Note that for all $V \in CB(X), (V,[T(V)]_\alpha) \subset \Xi(G)$ and $(V,[S(V)]_\alpha) \subset \Xi(G)$.

Take $F(p) = \ln(p) + p$ and $\tau(\mathsf{r}) = \begin{cases} \frac{12}{13}, & \text{if } \mathsf{r} = [0,1], \\ \ln(1+3\mathsf{r}), & \text{if } \mathsf{r} > 1. \end{cases}$

Now for nonempty $V_1, V_2 \in CB(X)$ with $[S(V_1)]_\alpha \neq [S(V_2)]_\alpha$. We have to verify

$$\tau(M(V_1,V_2)) + F(H_\sigma([S(V_1)]_\alpha,[S(V_2)]_\alpha)) \leq F(M(V_1,V_2)) + LE(V_1,V_2).$$

The following cases arises:

Case 1:

For $\alpha = 1$ and $V_1 \subseteq \{3,6\}$ and $V_2 = \{9\}$ with $V_1, V_2 \in \Xi(G)$, we have

$$
\begin{aligned}
& H_\sigma([S(V_1)]_1,[S(V_2)]_1)e^{H_\sigma([S(V_1)]_1,[S(V_2)]_1)-M(V_1,V_2)-LE(V_1,V_2)} \\
& \leq H_\sigma([S(V_1)]_1,[S(V_2)]_1)e^{H_\sigma([S(V_1)]_1,[S(V_2)]_1)-H_\sigma([S(V_2)]_1,[T(V_2)]_1)} \\
& \leq H_\sigma(\{3\},\{3,6\})e^{H_\sigma(\{3\},\{3,6\})-H_\sigma(\{3\},\{3,6,9\})} \\
& < \frac{1}{3n}e^{\frac{1}{3n}-\frac{3n}{3n+1}} \\
& < \frac{3n}{3n+1}e^{\frac{-12}{13}} \\
& = e^{\frac{-12}{13}}H_\sigma(\{3,6\},\{3,6,9\}) \\
& = e^{\frac{-12}{13}}H_\sigma([S(V_2)]_1,[T(V_2)]_1) \\
& \leq e^{-\tau(M(V_1,V_2))}M(V_1,V_2).
\end{aligned}
$$

Case 2:
For $\alpha = 1$ and $V_1 \subseteq \{3,6\}$ and $V_2 = \{3,6,9\}$ with $V_1, V_2 \in \Xi(G)$, we have

$$
\begin{aligned}
& H_\sigma([S(V_1)]_1,[S(V_2)]_1)e^{H_\sigma([S(V_1)]_1,[S(V_2)]_1)-M(V_1,V_2)-LE(V_1,V_2)} \\
& \leq H_\sigma([S(V_1)]_1,[S(V_2)]_1)e^{H_\sigma([S(V_1)]_1,[S(V_2)]_1)-H_\sigma([S(V_2)]_1,[T(V_2)]_1)} \\
& \leq H_\sigma(\{3\},\{3,6\})e^{H_\sigma(\{3\},\{3,6\})-H_\sigma(\{3,6\},\{3,6,9,\ldots,3n\})} \\
& < \frac{1}{3n}e^{\frac{1}{3n}-\frac{3n}{3n+1}} \\
& < \frac{3n}{3n+1}e^{\frac{-12}{13}} \\
& = e^{\frac{-12}{13}}H_\sigma(\{3,6\},\{3,6,9,\ldots,3n\}) \\
& = e^{\frac{-12}{13}}H_\sigma([S(V_2)]_1,[T(V_2)]_1) \\
& \leq e^{-\tau(M(V_1,V_2))}M(V_1,V_2).
\end{aligned}
$$

Case 3: For $\alpha = 1$ and $V_1 = \{9\}$ and $V_2 \subseteq \{3,6\}$ with $V_1, V_2 \in \Xi(G)$, we have

$$H_\sigma([S(V_1)]_1, [S(V_2)]_1) e^{H_\sigma([S(V_1)]_1,[S(V_2)]_1) - M(V_1,V_2) - LE(V_1,V_2)}$$

$$\leq H_\sigma([S(V_1)]_1, [S(V_2)]_1) e^{H_\sigma([S(V_1)]_1,[S(V_2)]_1) - H_\sigma([S(V_1)]_1,[T(V_2)]_1)}$$

$$\leq H_\sigma(\{3,6\}, \{3\}) e^{H_\sigma(\{3,6\},\{3\}) - H_\sigma(\{3,6\},\{3,6,9\})}$$

$$< \frac{1}{3n} e^{\frac{1}{3n} - \frac{3n}{3n+1}}$$

$$< \frac{3n}{3n+1} e^{\frac{-12}{13}}$$

$$= e^{\frac{-12}{13}} H_\sigma(\{3,6\}, \{3,6,9\})$$

$$= e^{\frac{-12}{13}} H_\sigma([S(V_1)]_1, [T(V_2)]_1)$$

$$\leq e^{-\tau(M(V_1,V_2))} M(V_1, V_2).$$

Case 4:
For $\alpha = 1$ and $V_1 \subsetneq \{3,6,9\}$ and $V_2 \subseteq \{3,6\}$ with $V_1, V_2 \in \Xi(G)$, we have

$$H_\sigma([S(V_1)]_1, [S(V_2)]_1) e^{H_\sigma([S(V_1)]_1,[S(V_2)]_1) - M(V_1,V_2) - LE(V_1,V_2)}$$

$$\leq H_\sigma([S(V_1)]_1, [S(V_2)]_1) e^{H_\sigma([S(V_1)]_1,[S(V_2)]_1) - H_\sigma([S(V_1)]_1,[T(V_2)]_1)}$$

$$\leq H_\sigma(\{3\}, \{3,6\}) e^{H_\sigma(\{3\},\{3,6\}) - H_\sigma(\{3\},\{3,6,9\})}$$

$$< \frac{1}{3n} e^{\frac{1}{3n} - \frac{3n}{3n+1}}$$

$$< \frac{3n}{3n+1} e^{\frac{-12}{13}}$$

$$= e^{\frac{-12}{13}} H_\sigma(\{3,6\}, \{3,6,9,...,3n\})$$

$$= e^{\frac{-12}{13}} H_\sigma([S(V_1)]_1, [T(V_2)]_1)$$

$$\leq e^{-\tau(M(V_1,V_2))}M(V_1,V_2).$$

For any V_1, V_2 are nonempty closed and bounded subsets of X with a connecting edge, the conditions are met. Thus, all requirements of Theorem 1 are satisfied. Additionally, $\{3\}$ is common fuzzy fixed point.

Example 2 Let $X = \{3n : n \in \{1,2,3,...,e\}\} = \nu(G),\ e \geq 3$.

$\Xi(G) = \{(3,3),(6,6),(9,9),...,(3n,3n),(3,6),(3,9),...,(3,3n)\}$.

Define $d_\sigma : \nu(G) \times \nu(G) \longrightarrow [0,\infty)$ by $d_\sigma(x_1,x_2) = \begin{cases} 0, & \text{if } x_1 = x_2, \\ \frac{1}{3n}, & \text{if } x_1 \in \{3\} \text{ with } x_1 \neq x_2, \\ \frac{3n}{3n+1}, & \text{otherwise,} \end{cases}$

then (X, d_σ) is a complete controlled metric space.

Moreover, the Pompeiu-Hausdorff controlled metric is given by

$$H_\sigma(A,B) = \begin{cases} \dfrac{1}{3n}, & \text{if } A,B \subseteq \{3\} \text{ with } A \neq B, \\ \dfrac{3n}{3n+1}, & \text{if } A \text{ or } B \text{ (or both) } \subsetneq \{3,6\} \text{ with } A \neq B, \\ 0, & \text{if } B = A. \end{cases}$$

The Pompeiu-Hausdorff weights for $(n = 4)$ assigned to $U, V \in CB(X)$ are shown in Figure 2.

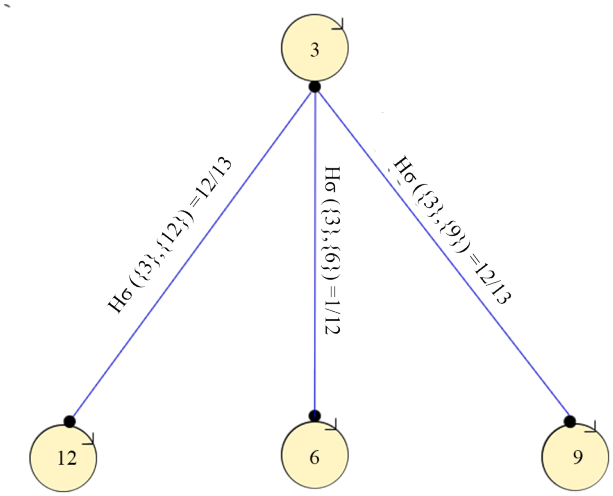


Figure 2. The Pompeiu-Hausdorff weights $(n = 4)$

Define $S, T : CB(X) \longrightarrow I_{FC}(X)$ by,

For $U = \{3\}$

$$S(U)(V) = \begin{cases} 1, & \text{if } V = \{3\}, \\ \dfrac{1}{3}, & \text{if } V \neq \{3\}, \\ 0, & \text{elsewhere.} \end{cases}$$

For $U \neq \{3\}$

$$S(U)(V) = \begin{cases} 1, & \text{if } V = \{3,6\}, \\ \frac{1}{3}, & \text{if } V \neq \{3,6\}, \\ 0, & \text{elsewhere.} \end{cases}$$

Now $[S(U)]_1 = \{V : S(U)(V) = 1\} = \begin{cases} \{3\}, & \text{if} \quad U = \{3\}, \\ \{3,6\}, & \text{if } U \neq \{3\}. \end{cases}$

For $U = \{3\}$

$$T(U)(V) = \begin{cases} 1, & \text{if } V = \{3\}, \\ \frac{1}{3}, & \text{if } V \neq \{3\}, \\ 0, & \text{elsewhere.} \end{cases}$$

For $U \neq \{3\}$

$$T(U)(V) = \begin{cases} 1, & \text{if } V = \{3,6,9,\ldots,3n\}, \\ \frac{1}{3}, & \text{if } V \neq \{3,6,9,\ldots,3n\}, \\ 0, & \text{elsewhere.} \end{cases}$$

Now $[T(U)]_1 = \{V : T(U)(V) = 1\} = \begin{cases} \{3\}, & \text{if} \quad U = \{3\}, \\ \{3,6,9,...,3n\}, & \text{if} \quad U \neq \{3\}. \end{cases}$

Note that for all $V \in CB(X), (V, [T(V)]_\alpha) \subset \Xi(G)$ and $(V, [S(V)]_\alpha) \subset \Xi(G)$.

Take $F(p) = \ln(p) + p$ and $\tau(\mathsf{r}) = \begin{cases} \frac{7}{8}, & \text{if} \quad \mathsf{r} = [0,1], \\ \ln(1+3\mathsf{r}^2), & \text{if} \quad \mathsf{r} > 1. \end{cases}$

We have to verify

$$\tau(M(V_1,V_2)) + F(H_\sigma([S(V_1)]_\alpha, [S(V_2)]_\alpha)) \leq F(M(V_1,V_2)) + LE(V_1,V_2).$$

Now for all V_1, V_2 are nonempty closed and bounded subsets of X with $[S(V_1)]_\alpha \neq [S(V_2)]_\alpha$ with following cases:

Case 1:

For $\alpha = 1$ and $V_1 = \{3\}$ and $V_2 \neq \{3\}$ with $(V_1, V_2) \in \Xi(G)$, we have

$$H_\sigma([S(V_1)]_1, [S(V_2)]_1) e^{H_\sigma([S(V_1)]_1, [S(V_2)]_1) - M(V_1,V_2) - LE(V_1,V_2)}$$

$$\leq H_\sigma([S(V_1)]_1,[S(V_2)]_1)e^{H_\sigma([S(V_1)]_1,[S(V_2)]_1)-H_\sigma([S(V_2)]_1,[T(V_2)]_1)}$$

$$< \frac{1}{3n}e^{\frac{1}{3n}-\frac{3n}{3n+1}}$$

$$< \frac{3n}{3n+1}e^{\frac{-7}{8}}$$

$$= e^{\frac{-7}{8}}H_\sigma([S(V_2)]_1,[T(V_2)]_1)$$

$$\leq e^{-\tau(M(V_1,V_2))}M(V_1,V_2).$$

Case 2:
For $\alpha = 1$ and $V_1 \neq \{3\}$ and $V_2 = \{3\}$ with $(V_1,V_2) \in \Xi(G)$, we have

$$H_\sigma([S(V_1)]_1,[S(V_2)]_1)e^{H_\sigma([S(V_1)]_1,[S(V_2)]_1)-M(V_1,V_2)-LE(V_1,V_2)}$$

$$\leq H_\sigma([S(V_1)]_1,[S(V_2)]_1)e^{H_\sigma([S(V_1)]_1,[S(V_2)]_1)-H_\sigma([S(V_2)]_1,[T(V_2)]_1)}$$

$$< \frac{1}{3n}e^{\frac{1}{3n}-\frac{3n}{3n+1}}$$

$$< \frac{3n}{3n+1}e^{\frac{-7}{8}}$$

$$= e^{\frac{-7}{8}}H_\sigma([S(V_2)]_1,[T(V_2)]_1)$$

$$\leq e^{-\tau(M(V_1,V_2))}M(V_1,V_2).$$

Similarly, one can verify that for any V_1, V_2, which are nonempty closed and bounded subsets of X having an edge between V_1 and V_2, the condition is fulfilled. Hence, for the pairs $(3,6)$, $(3,9)$, and $(3,12)$, all the requirements of Theorem 1 are fulfilled. Furthermore $\{3\}$ is the common fuzzy fixed point of T and S.

By taking $E = 0$ in Definition (14), we obtain the following:

Theorem 2 Let (X,d_σ) be a complete controlled metric space tailored with a directed graph G with $\nu(G) = X$, $\Xi(G) \supseteq \Delta$ and the relation R on $\mathsf{I}_{FC}(X)$ is transitive. Let $S,T : CB(X) \to \mathsf{I}_{FC}(X)$ are the pair of graphic fuzzy generalized rational F contractive mappings such that range of T contains the range of S and

$$\sup_{m\geq 0}\lim_{n\longrightarrow\infty}\frac{\sigma([T(A_{n+1})]\alpha,[T(A_{n+2})]\alpha)\sigma([T(A_{n+1})]\alpha,[T(A_m)]\alpha)}{\sigma([T(A_n)]\alpha,[T(A_{n+1})]\alpha)} < \frac{1}{\mathtt{L}} \quad \text{for all } \mathtt{L} \in (0,1),$$

then, the following statements are true.

(i) Given that G is weakly connected, satisfies $\mathtt{P}^*$, such that $[T(X)]_{\alpha_{T(X)}}$ is a complete subspace of $\mathsf{I}_{FC}(X)$, and $\tilde{\mathsf{C}}_F(S,T) \neq \emptyset$ for some α.

(ii) If $\tilde{\mathsf{C}}_F(S,T)$ is complete, then Hausdorff weight assigned to $[S(U)]_\alpha$ and $[S(V)]_\alpha$ is 0 and for all $U,V \in \tilde{\mathsf{C}}_F(S,T)$.

(iii) If $\tilde{\mathsf{C}}_F(S,T)$ is complete, and S,T are weakly compatible, then $Fuz(S) \cap Fuz(T)$ is singleton.

(iv) $Fuz(S) \cap Fuz(T)$ is complete $\Leftrightarrow$ $Fuz(S) \cap Fuz(T)$ is singleton. where $M(A,B)$ is defined same as in Theorem 1.

If we take $S = T$ in Definition (14) and

$$M(A,B) = \max \Bigg\{ H_\sigma([S(A)]_\alpha, [S(B)]_\alpha), H_\sigma([S(A)]_\alpha, [S(A)]_\alpha),$$

$$H_\sigma([S(B)]_\alpha, [S(B)]_\alpha), \frac{\{H_\sigma([S(A)]_\alpha, [S(A)]_\alpha) \times H_\sigma([S(B)]_\alpha, [S(B)]_\alpha)\}}{\tau(M(A,B)) + H_\sigma([S(A)]_\alpha, [S(B)]_\alpha))},$$

$$\frac{\{H_\sigma([S(A)]_\alpha, [S(A)]_\alpha) \times H_\sigma([S(B)]_\alpha, [S(B)]_\alpha)\}}{1 + H_\sigma([S(A)]_\alpha, [S(B)]_\alpha))} \Bigg\},$$

$$E(A,B) = \min \begin{Bmatrix} H_\sigma([S(A)]_\alpha, [S(A)]_\alpha), & H_\sigma([S(B)]_\alpha, [S(B)]_\alpha) \\ H_\sigma([S(A)]_\alpha, [S(B)]_\alpha), & H_\sigma([S(B)]_\alpha, [S(A)]_\alpha) \end{Bmatrix},$$

we obtain the following:

Corollary 1 Let (X, d_σ) be a complete controlled metric space tailored with a directed graph G with $\nu(G) = X$, $\Xi(G) \supseteq \Delta$ and the relation R on $\mathsf{I}_{FC}(X)$ is transitive. Assume that $S : CB(X) \to \mathsf{I}_{FC}(X)$ satisfies the following conditions:

(1) For every $U \in CB(X)$, $(U, [S(U)]) \subset \Xi(G)$.

(2) There exists $\tau : [0,\infty) \longrightarrow [0,\infty)$ such that

$$\lim_{p \longrightarrow \mathsf{q}^+} \inf \tau(p) > 0, \text{ for all } \mathsf{q} > 0$$

and there is a edge between $A, B \in CB(X)$ with $[S(A)]_\alpha \neq [S(B)]_\alpha$ such that $\tau(M(A,B)) + F(H_\sigma([S(A)]_\alpha, [S(B)]_\alpha)) \leq F(M(A,B)) + LE(A,B)$, for all $L \geq 0$.

(3) $\sup_{m \geq 0} \lim_{n \longrightarrow \infty} \dfrac{\sigma([S(A_{n+1})]_\alpha, [S(A_{n+2})]_\alpha)\sigma([S(A_{n+1})]_\alpha, [S(A_m)]_\alpha)}{\sigma([S(A_n)]_\alpha, [S(A_{n+1})]_\alpha)} < \dfrac{1}{\mathtt{L}}$ for all $\mathtt{L} \in (0,1)$, then, the following statements are true

(i) $Fuz(S)$ is complete, then the Hausdorff weights assigned to the $U, V \in Fuz(S)$ is 0.

(ii) If the weakly connected graph G satisfies $\mathtt{P}^*$, then S has fuzzy fixed point.

(iii) $Fuz(S)$ is complete $\Leftrightarrow$ $Fuz(S)$ is singleton.

If we take $L = 0$ in Corollary3, we obtain the following:

Corollary 2 Let (X, d_σ) be a complete controlled metric space tailored with a directed graph G with $\nu(G) = X$, $\Xi(G) \supseteq \Delta$ and the relation R on $\mathsf{I}_{FC}(X)$ is transitive. Assume that $S : CB(X) \to \mathsf{I}_{FC}(X)$ satisfies the following conditions:

(1) For every $U \in CB(X)$, $(U, [S(U)]) \subset \Xi(G)$.

(2) There exists $\tau : [0,\infty) \longrightarrow [0,\infty)$ such that

$$\lim_{p \longrightarrow \mathsf{q}^+} \inf \tau(p) > 0, \text{ for all } \mathsf{q} > 0,$$

there is a edge between $A,B \in CB(X)$ with $[S(A)]_\alpha \neq [S(B)]_\alpha$ such that $\Rightarrow \tau(M(A,B)) + F(H_\sigma([S(A)]_\alpha,[S(B)]_\alpha)) \leq F(M(A,B))$.

(3) $\sup_{m\geq 0}\lim_{n\longrightarrow\infty} \dfrac{\sigma([S(A_{n+1})]_\alpha,[S(A_{n+2})]_\alpha)\sigma([S(A_{n+1})]_\alpha,[S(A_m)]_\alpha)}{\sigma([S(A_n)]_\alpha,[S(A_{n+1})]_\alpha)} < \dfrac{1}{\mathtt{L}}$ for all $\mathtt{L} \in (0,1)$, then the following statements are fulfilled:

(i) $Fuz(S)$ is complete, then Hausdorff weights assigned to the $U,V \in Fuz(S)$ is 0.

(ii) If the weakly connected graph G satisfies $\mathtt{P}^*$, then S has fuzzy fixed point.

(iii) $Fuz(S)$ is complete $\Leftrightarrow$ $Fuz(S)$ is singleton.

Theorem 3 Let (X,d_σ) be a ε chainable complete controlled metric space for $\varepsilon > 0$ and $S,T : CB(X) \to \mathfrak{l}_{FC}(X)$ be the fuzzy set valued mappings. Assume that for all $A,B \in CB(X)$ such that $0 < H_\sigma([T(A)]_\alpha,[S(B)]_\alpha) < \varepsilon$. If

(i) $\sup_{m\geq 0}\lim_{n\longrightarrow\infty} \dfrac{\sigma([T(A_{n+1})]_\alpha,[T(A_{n+2})]_\alpha)\sigma([T(A_{n+1})]_\alpha,[T(A_m)]_\alpha)}{\sigma([T(A_n)]_\alpha,[T(A_{n+1})]_\alpha)} < \dfrac{1}{\mathtt{L}}$ for all $\mathtt{L} \in (0,1)$.

(ii) There exists $\tau : [0,\infty) \longrightarrow [0,\infty)$ such that

$$\lim_{p\longrightarrow \mathtt{q}^+} \inf \tau(p) > 0, \text{ for all } \mathtt{q} > 0.$$

(iii) $\tau(M(A,B)) + F(H_\sigma([S(A)]_\alpha,[S(B)]_\alpha)) \leq F(M(A,B)) + LE(A,B),$ for all $L \geq 0$.

Then T and S possess common fuzzy fixed point provided that T and S are weakly compatible.

Proof. We are provided $S,T : CB(X) \to \mathfrak{l}_{FC}(X)$ be the fuzzy set valued mappings and $0 < H_\sigma([S(A)]_\alpha,[S(B)]_\alpha) < \varepsilon$, we have $H_\sigma(A,B) < \varepsilon$ then by Lemma 1 for every $a_1 \in A$, we have an element $a_2 \in B$ such that $d_\sigma(a_1,a_2) < \varepsilon$. Take a graph G with $\nu(G) = X$ and $\Xi(G) = \{(a_1,a_2) \in X \times X : 0 < d_\sigma(a_1,a_2) < \varepsilon\}$.

Then by the ε chainability of (X,d_σ) means that G is connected. This implies that for A,B are nonempty closed and bounded subsets of X so we have $(A,B) \subset \Xi(G)$. Therefore by Theorem 1 $\tau(M(A,B)) + F(H_\sigma([S(A)]_\alpha,[S(B)]_\alpha)) \leq F(M(A,B)) + LE(A,B)$, where

$$M(A,B) = \max\Bigg\{H_\sigma([T(A)]_\alpha,[T(B)]_\alpha), H_\sigma([S(A)]_\alpha,[T(A)]_\alpha),$$

$$H_\sigma([S(B)]_\alpha,[T(B)]_\alpha), \frac{\{H_\sigma([T(A)]_\alpha,[S(A)]_\alpha) \times H_\sigma([S(B)]_\alpha,[T(B)]_\alpha)\}}{\tau(M(A,B)) + H_\sigma([S(A)]_\alpha,[S(B)]_\alpha))},$$

$$\frac{\{H_\sigma([T(A)]_\alpha,[S(A)]_\alpha) \times H_\sigma([S(B)]_\alpha,[T(B)]_\alpha)\}}{1 + H_\sigma([S(A)]_\alpha,[S(B)]_\alpha))}\Bigg\},$$

and $E(A,B) = \min\begin{Bmatrix} H_\sigma([S(A)]_\alpha,[T(A)]_\alpha), & H_\sigma([S(B)]_\alpha,[T(B)]_\alpha) \\ H_\sigma([S(A)]_\alpha,[T(B)]_\alpha), & H_\sigma([S(B)]_\alpha,[T(A)]_\alpha) \end{Bmatrix}$.

Then the pair (S,T) are graphic fuzzy generalized rational F contractive mappings. Also G possess $\mathtt{P}^*$. Indeed if $\{X_n\} \in CB(X)$ with $X_n \longrightarrow X$ as $n \longrightarrow \infty$ and $(X_n,X) \in \Xi(G)$ for $n \in \mathbb{N}$ means that there is subsequence of $\{X_{n_k}\}$ of $\{X_n\}$ such that $(X_{n_k},X) \subset \Xi(G)$. So by Theorem 1 S and T have a common fuzzy fixed point. □

Next, we deduce some consequences and comparative results of our main theorem with in the framework of both single-valued and set-valued mappings. Now take $M(C,\mathtt{D})$ and $E(C,\mathtt{D})$ as follows:

$$M(C,\mathsf{D}) = \max\Bigg\{H_\sigma(T(C),T(\mathsf{D})), H_\sigma(S(C),T(C)), H_\sigma(S(\mathsf{D}),T(\mathsf{D})),$$

$$\frac{\{H_\sigma(T(C),S(C))\times H_\sigma(S(\mathsf{D}),T(\mathsf{D}))\}}{\tau(M(C,\mathsf{D}))+H_\sigma(S(C),S(\mathsf{D})))}, \frac{\{H_\sigma(T(C),S(C))\times H_\sigma(S(\mathsf{D}),T(\mathsf{D}))\}}{1+H_\sigma(S(C),S(\mathsf{D})))}\Bigg\},$$

$$E(C,\mathsf{D}) = \min\begin{Bmatrix} H_\sigma(S(C),T(C)), & H_\sigma(S(\mathsf{D}),T(\mathsf{D})) \\ H_\sigma(S(C),T(\mathsf{D})), & H_\sigma(S(\mathsf{D}),T(C)) \end{Bmatrix}.$$

Corollary 3 Let (X,d_σ) be a complete controlled metric space tailored with a directed graph G with $\nu(G)=X$, $\Xi(G)\supseteq\Delta$ and the relation R on $\mathsf{I}_{FC}(X)$ is transitive. Let $S,T: CB(X)\to CB(X)$ be the set valued mappings with $S(X)\supseteq T(X)$ satisfies the following:

(1) $(S(P),P)\subset\Xi(G)$ and $(P,T(P))\subset\Xi(G)$ for any $P\in CB(X)$.

(2) There exist $\tau:[0,\infty)\longrightarrow[0,\infty)$ such that

$$\lim_{p\longrightarrow \mathsf{q}^+}\inf\tau(p)>0, \text{for all } \mathsf{q}>0$$

such that for all $F\in\mathsf{F}$ there is an edge between C and D with $S(C)\neq S(\mathsf{D})$ such that $\tau(M(C,\mathsf{D}))+F(H_\sigma(S(C),S(\mathsf{D})))\leq F(M(C,\mathsf{D}))+LE(C,\mathsf{D})$.

(3) $\sup\limits_{m\geq 0}\lim\limits_{n\longrightarrow\infty}\dfrac{\sigma([T(\mathsf{D}_{n+1})]_\alpha,[T(\mathsf{D}_{n+2})]_\alpha)\sigma([T(\mathsf{D}_{n+1})]_\alpha,[T(\mathsf{D}_m)]_\alpha)}{\sigma([T(\mathsf{D}_n)]_\alpha,[T(\mathsf{D}_{n+1})]_\alpha)}<\dfrac{1}{\mathtt{L}}$ for all $\mathtt{L}\in(0,1)$, then the following statements are true.

(i) Given that G is weakly connected which satisfy $\mathtt{P}^*$ and $T(X)$ is a complete subspace of $CB(X)$ and $\mathfrak{CP}(S,T)\neq\emptyset$.

(ii) If $\mathfrak{CP}(S,T)$ is complete, then H_σ assigned to $S(\mathsf{P})$ and $T(U)$ is 0 for all $\mathsf{P},U\in\mathfrak{CP}(S,T)$.

(iii) If $\mathfrak{CP}(S,T)$ is complete, and S,T are weakly compatible, then $Fix(S)\cap Fix(T)$ is singleton.

(iv) $Fix(S)\cap Fix(T)$ is complete $\Leftrightarrow$ $Fix(S)\cap Fix(T)$ is singleton.

Proof. Assume the mappings $\phi,\psi: X\longrightarrow(0,1]$ and define the pair of mappings $S,T: CB(X)\to\mathsf{I}_{FC}(X)$ by $S(A)(\mathsf{D})=\begin{cases}\phi(x), & \text{if } \mathsf{D}=\{S(C)\},\\ 0, & \text{if } \mathsf{D}\neq\{S(C)\},\end{cases}$ and $T(B)(\mathsf{E})=\begin{cases}\psi(y), & \text{if } \mathsf{E}=\{T(\mathsf{D})\},\\ 0, & \text{if } \mathsf{E}\neq\{T(\mathsf{D})\}.\end{cases}$

Now for $\alpha(x)=\phi(x)\in(0,1]$ for all $x\in X$, we have the following

$$[S(A)]_\alpha=\Big\{\mathsf{D}\in CB(X): S(A)(\mathsf{D})\geq\alpha(x)\Big\}=\{S(C)\}.$$

Now for $\alpha(y)=\psi(y)\in(0,1]$ for all $y\in X$, we have the following

$$[T(B)]_\alpha=\Big\{\mathsf{E}\in CB(X): T(B)(\mathsf{E})\geq\alpha(y)\Big\}=\{T(\mathsf{D})\}.$$

By Theorem 1 admits a common fixed point. □

Now consider $\mathsf{f},\mathsf{g}: X \longrightarrow X$ be self mappings and while $\mathfrak{Cp}(\mathsf{f},\mathsf{g})$ we mean the set of coincidence point of f and g. By taking by taking $M(x,y)$ and $E(x,y)$ as follows:

$$M(x,y) = \max\left\{ d_\sigma(\mathsf{g}(x),\mathsf{g}(y)), d_\sigma(\mathsf{f}(x),\mathsf{g}(x)), d_\sigma(\mathsf{f}(y),\mathsf{g}(y)), \frac{\{d_\sigma(\mathsf{g}(x),\mathsf{f}(x))\times d_\sigma(\mathsf{f}(y),\mathsf{g}(y))\}}{\tau(M(x,y))+d_\sigma(\mathsf{f}(x),\mathsf{f}(y)))}, \right.$$

$$\left. \frac{\{d_\sigma(\mathsf{g}(x),\mathsf{f}(x))\times d_\sigma(\mathsf{f}(y),\mathsf{g}(y))\}}{1+d_\sigma(\mathsf{f}(x),\mathsf{f}(y)))} \right\},$$

$$E(x,y) = \min\begin{Bmatrix} d_\sigma(\mathsf{f}(x),\mathsf{g}(x)), & d_\sigma(\mathsf{f}(y),\mathsf{g}(y)) \\ d_\sigma(\mathsf{f}(x),\mathsf{g}(y)), & d_\sigma(\mathsf{f}(y),\mathsf{g}(x)) \end{Bmatrix},$$

we obtain the following:

Corollary 4 Let (X,d_σ) be a complete controlled metric space tailored with a directed graph G with $\nu(G)=X$, $\Xi(G)\supseteq\Delta$ and the relation R on X is transitive. Let $\mathsf{f},\mathsf{g}: X\to X$ be the self maps satisfy the following conditions.

(1) $(\mathsf{f}(x_1),x_1)\subset\Xi(G)$ and $(x_1,\mathsf{g}(x_1))\subset\Xi(G)$ for any $x_1\in X$.

(2) $\exists\ \tau:[0,\infty)\longrightarrow[0,\infty)$ such that

$$\lim_{p\longrightarrow \mathsf{q}^+}\inf\tau(p)>0,\ \text{ for all } \mathsf{q}>0$$

such that for all $F\in\mathsf{F}$ there is an $\Xi(G)$ between x and y with $\mathsf{f}(x)\neq\mathsf{f}(y)$ such that $\tau(M(x,y))+F(d_\sigma(\mathsf{f}(x),\mathsf{f}(y)))\leq F(M(x,y))+LE(x,y)$.

(3) $\sup\limits_{m\geq 0}\lim\limits_{n\longrightarrow\infty}\dfrac{\sigma(\mathsf{g}[(x_{n+1})]_\alpha,[\mathsf{g}(x_{n+2})]_\alpha)\sigma([\mathsf{g}(x_{n+1})]_\alpha,[\mathsf{g}(x_m)]_\alpha)}{\sigma(\mathsf{g}[(x_n)]_\alpha,[\mathsf{g}(x_{n+1})]_\alpha)}<\dfrac{1}{\mathtt{L}}$ for all $\mathtt{L}\in(0,1)$, then the following statements are holds.

(i) Given that G is weakly connected satisfy $\mathtt{P}^*$ such that $\mathsf{g}(X)$ is a complete subspace of X and $\mathfrak{Cp}(\mathsf{f},\mathsf{g})\neq\emptyset$.

(ii) If $\mathfrak{Cp}(\mathsf{f},\mathsf{g})$ is complete, then $\mathsf{f}(\mathsf{p})=\mathsf{f}(\mathsf{q})$ for all $\mathsf{p},\mathsf{q}\in\mathfrak{CP}(\mathsf{f},\mathsf{g})$.

(iii) If f,g are weakly compatible, and $\mathfrak{Cp}(\mathsf{f},\mathsf{g})$ is complete, then $Fix(\mathsf{f})\cap Fix(\mathsf{g})$ is singleton.

(iv) $Fix(\mathsf{f})\cap Fix(\mathsf{g})$ is complete $\Leftrightarrow$ $Fix(\mathsf{f})\cap Fix(\mathsf{g})$ is singleton.

Proof. Assume the mappings $\phi,\psi: X\longrightarrow(0,1]$ and define the pair of mappings $S,T: CB(X)\to \mathsf{I}_{FC}(X)$ by $S(A)(C)=\begin{cases}\phi(x), & \text{if } C=\{\mathsf{f}(x)\},\\ 0, & \text{if } C\neq\{\mathsf{f}(x)\},\end{cases}$ and $T(B)(\mathsf{E})=\begin{cases}\psi(y), & \text{if } \mathsf{E}=\{\mathsf{g}(y)\},\\ 0, & \text{if } \mathsf{E}\neq\{\mathsf{g}(y)\}.\end{cases}$

Now for $\alpha(x)=\phi(x)\in(0,1]$ for all $x\in X$, we have the following

$$[S(A)]_\alpha=\Big\{C\in CB(X): S(A)(C)\geq\alpha(x)\Big\}=\{\mathsf{f}(x)\}.$$

Now for $\alpha(y)=\psi(y)\in(0,1]$ for all $y\in X$, we have the following

$$[T(B)]_\alpha=\Big\{\mathsf{E}\in CB(X): T(B)(\mathsf{E})\geq\alpha(y)\Big\}=\{\mathsf{g}(y)\}.$$

Clearly $\{f(x)\},\{g(y)\} \in CB(X)$ for all $x,y \in X$. By Theorem 1 there exists a common fixed point of f and g. □

Remark 1 To highlight the advancement beyond existing results, we emphasize three key innovations in our work:

(i) If we take $\sigma(x_1,x_2) = \sigma(x_2,x_3) = \sigma(x_1,x_3)$ in Definition 2, then controlled metric space becomes an extended b-metric space and all the results proved above becomes corollaries in an extended b-metric spaces. Therefore our results are more general.

(ii) If we take $\sigma(x_1,x_3) = s$ in Definition 2, set $L = 0$, and suitably modify $M(x_1,x_2)$, then the controlled metric space reduces to a b-metric space. Consequently, the results of Rafique et al. [34] appear as special cases of our more general framework.

(iii) If we take $\sigma(x_1,x_3) = s = 1$ in Definition 2, set $L = 0$, and suitably modify $M(x_1,x_2)$, then the controlled metric space reduces to a metric space. Consequently, the results of Abbas et al. [14] appear as special cases of our more general framework.

4. Application to fractional differential equations

The concept of fractional differentials was first introduced by Pierre-Simon Lacroix in 1819, providing a foundational framework that has since been extensively developed by various researchers. By employing generalized contractions and applying them to Fractional Differential Equations, scholars have successfully established numerous fixed point theorems across diverse metric spaces (refer to key studies [29, 39] for examples). This section will explore the existence of one such model within the context of controlled metric spaces.

Let $X = [0,1]$ and $\mathsf{C}(X,\mathbb{R})$ be the set of continuous functions defined on X. Define the metric $d_\sigma : \mathsf{C}(X,\mathbb{R}) \times \mathsf{C}(X,\mathbb{R}) \to [0,\infty)$ by

$$d_\sigma(k,j) = |k-j|_\infty^4 = \max_{x\in[0,1]} |k(x)-j(x)|^4,$$

for all $k,j \in \mathsf{C}(X,\mathbb{R})$. Then $(\mathsf{C}[0,1],d_\sigma)$ is a complete controlled metric space, where $\sigma : X \times X \longrightarrow (0,1]$ be defined by

$$\sigma(k,j) = \begin{cases} 1, & \text{if } j,k \in [0,0.5), \\ k+j+2, & \text{otherwise.} \end{cases}$$

Let $\mathtt{K}_1,\mathtt{K}_2 : X \times \mathbb{R} \longrightarrow \mathbb{R}$ be the two mappings such that $\mathtt{K}_1(x,k(\mathsf{r})),\mathtt{K}_2(x,j(\mathsf{r})) \geq 0$ for all $k,j \in \mathsf{C}(X,\mathbb{R})$ and $x \in X$. We will look into the following system of Fractional Differential equations:

$$^{\mathsf{C}}\mathrm{D}^{\mathtt{v}}k(x) = \mathtt{K}_1(x,k(x)); \quad k \in \mathsf{C}(X,\mathbb{R}) \tag{5}$$

$$^{\mathsf{C}}\mathrm{D}^{\mathtt{v}}j(x) = \mathtt{K}_2(x,j(x)); \quad j \in \mathsf{C}(X,\mathbb{R}), \tag{6}$$

with boundary conditions $k(0) = 0, Xk(1) = k^{'}(0), j(0) = 0, Xj(1) = j^{'}(0)$.

Here, $^{\mathsf{C}}\mathrm{D}^{\mathtt{v}}$ represent the Caputo-Fabrizio derivative of order $\mathtt{v}$ given as

$$^{\mathsf{C}}\mathrm{D}^{\mathtt{v}}k(x) = \frac{1}{\Gamma(k-\mathtt{v})}\int_0^x (x-\mathsf{r})^{k-\mathtt{v}-1}k(\mathsf{r})d\mathsf{r} \quad (k-1<\mathtt{v}<k, k=[n]+1),$$

and $X^{\mathtt{v}}k$ is defined by:

$$X^{\mathtt{v}}k(x)=\frac{1}{\Gamma(\mathtt{v})}\int_0^x(x-\mathsf{r})^{\mathtt{v}-1}k(\mathsf{r})d\mathsf{r}.$$

Then equations (5) and (6) can be modified into

$$k(x)=\frac{1}{\Gamma(\mathtt{v})}\int_0^x(x-\mathsf{r})^{\mathtt{v}-1}\mathtt{K}_1(\mathsf{r},k(\mathsf{r}))d\mathsf{r}+\frac{2x}{\Gamma(\mathtt{v})}\int_0^B\int_0^{\mathsf{r}}(\mathsf{r}-\mathsf{e})^{\mathtt{v}-1}\mathtt{K}_1(\mathsf{e},k(\mathsf{e}))d\mathsf{r}d\mathsf{e},$$

$$j(x)=\frac{1}{\Gamma(\mathtt{v})}\int_0^x(x-\mathsf{r})^{\mathtt{v}-1}\mathtt{K}_2(\mathsf{r},j(\mathsf{r}))d\mathsf{r}+\frac{2x}{\Gamma(\mathtt{v})}\int_0^B\int_0^{\mathsf{r}}(\mathsf{r}-\mathsf{e})^{\mathtt{v}-1}\mathtt{K}_2(\mathsf{e},j(\mathsf{e}))d\mathsf{r}d\mathsf{e}.$$

Suppose that,
(i) there exists $\tau(M(x,y))>0$ such that

$$|\mathtt{K}_1(x,k(\mathsf{r}))-\mathtt{K}_2(x,j(\mathsf{r}))|\leq\frac{e^{\frac{-\tau(M(x,y))}{4}}\Gamma(\mathtt{v}+1)}{6}|k(\mathsf{r})-j(\mathsf{r})|,\quad\text{for all }\mathsf{r}\in X;$$

(ii) $\mathsf{f}_0\in\mathsf{C}(X,\mathbb{R})$ so that for any $x\in X$,

$$k_0(x)\leq\frac{1}{\Gamma(\mathtt{v})}\int_0^x(x-\mathsf{r})^{\mathtt{v}-1}\mathtt{K}_1(\mathsf{r},k_0(\mathsf{r}))d\mathsf{r}+\frac{2x}{\Gamma(\mathtt{v})}\int_0^B\int_0^{\mathsf{r}}(\mathsf{r}-\mathsf{e})^{\mathtt{v}-1}\mathtt{K}_1(\mathsf{e},k_0(\mathsf{e}))d\mathsf{r}d\mathsf{e},$$

$$j_0(x)\leq\frac{1}{\Gamma(\mathtt{v})}\int_0^x(x-\mathsf{r})^{\mathtt{v}-1}\mathtt{K}_2(\mathsf{r},j_0(\mathsf{r}))d\mathsf{r}+\frac{2x}{\Gamma(\mathtt{v})}\int_0^B\int_0^{\mathsf{r}}(\mathsf{r}-j)^{\mathtt{v}-1}\mathtt{K}_2(\mathsf{e},j_0(\mathsf{e}))d\mathsf{r}d\mathsf{e};$$

(iii) Let $X=\{u\in\mathsf{C}(X,\mathbb{R}):u(x)\geq0\}$ for all $x\in X$, and define the operator $\mathsf{f},\mathsf{g}:X\longrightarrow X$ by :

$$(\mathsf{f}k)(x)=\frac{1}{\Gamma(\mathtt{v})}\int_0^x(x-\mathsf{r})^{\mathtt{v}-1}\mathtt{K}_1(\mathsf{r},k(\mathsf{r}))d\mathsf{r}+\frac{2x}{\Gamma(\mathtt{v})}\int_0^B\int_0^{\mathsf{r}}(\mathsf{r}-\mathsf{e})^{\mathtt{v}-1}\mathtt{K}_1(\mathsf{e},k(\mathsf{e}))d\mathsf{r}d\mathsf{e},$$

$$(\mathsf{g}j)(x)=\frac{1}{\Gamma(\mathtt{v})}\int_0^x(x-\mathsf{r})^{\mathtt{v}-1}\mathtt{K}_2(\mathsf{r},j(\mathsf{r}))d\mathsf{r}+\frac{2x}{\Gamma(\mathtt{v})}\int_0^B\int_0^{\mathsf{r}}(\mathsf{r}-\mathsf{e})^{\mathtt{v}-1}\mathtt{K}_2(\mathsf{e},j(\mathsf{e}))d\mathsf{r}d\mathsf{e},$$

satisfying: $F(M(x,y))+LE(x,y)\leq|(\mathsf{f}k)(x)-(\mathsf{g}j)(x)|$, where

$$M(x,y)=\max\Big\{d_\sigma(\mathsf{g}(x),\mathsf{g}(y)),d_\sigma(\mathsf{f}(x),\mathsf{g}(x)),d_\sigma(\mathsf{f}(y),\mathsf{g}(y)),\ \frac{\{d_\sigma(\mathsf{g}(x),\mathsf{f}(x))\times d_\sigma(\mathsf{f}(y),\mathsf{g}(y))\}}{\tau(M(x,y))+d_\sigma(\mathsf{f}(x),\mathsf{f}(y))},$$

$$\frac{\{d_\sigma(\mathsf{g}(x),\mathsf{f}(x))\times d_\sigma(\mathsf{f}(y),\mathsf{g}(y))\}}{1+d_\sigma(\mathsf{f}(x),\mathsf{f}(y))}\Big\},$$

and $E(x,y)=\min\begin{Bmatrix} d_\sigma(\mathsf{f}(x),\mathsf{g}(x)), & d_\sigma(\mathsf{f}(y),\mathsf{g}(y)) \\ d_\sigma(\mathsf{f}(x),\mathsf{g}(y)), & d_\sigma(\mathsf{f}(y),\mathsf{g}(x)) \end{Bmatrix}$.

Theorem 4 If the conditions (i)–(iii) are fulfilled then a common solution in $\mathsf{C}(X,\mathbb{R})$ for equations (5) and (6) exists.

Proof. Consider,

$$|(\mathsf{f}k)(x)-(\mathsf{g}j)(x)| = |\frac{1}{\Gamma(\mathsf{v})}\int_0^x (x-\mathsf{r})^{\mathsf{v}-1}\mathtt{K}_1(\mathsf{r},k(\mathsf{r}))d\mathsf{r} - \frac{1}{\Gamma(\mathsf{v})}\int_0^x (x-\mathsf{r})^{\mathsf{v}-1}\mathtt{K}_2(\mathsf{r},j(\mathsf{r}))d\mathsf{r}+$$

$$\frac{2x}{\Gamma(\mathsf{v})}\int_0^B\int_0^{\mathsf{e}}(\mathsf{e}-\mathsf{h})^{\mathsf{v}-1}\mathtt{K}_1(\mathsf{h},k(\mathsf{h}))d\mathsf{e}d\mathsf{h} - \frac{2x}{\Gamma(\mathsf{v})}\int_0^B\int_0^{\mathsf{e}}(\mathsf{e}-\mathsf{h})^{\mathsf{v}-1}\mathtt{K}_2(\mathsf{h},j(\mathsf{h}))d\mathsf{e}d\mathsf{h}|,$$

$$|(\mathsf{f}k)(x)-(\mathsf{g}j)(x)|$$

$$= |\frac{1}{\Gamma(\mathsf{v})}\int_0^x (x-\mathsf{r})^{\mathsf{v}-1}\mathtt{K}_1(\mathsf{r},k(\mathsf{r}))d\mathsf{r} - \frac{1}{\Gamma(\mathsf{v})}\int_0^x (x-\mathsf{r})^{\mathsf{v}-1}\mathtt{K}_2(\mathsf{r},j(\mathsf{r}))d\mathsf{r}|+$$

$$|\frac{2x}{\Gamma(\mathsf{v})}\int_0^B\int_0^{\mathsf{e}}(\mathsf{e}-\mathsf{h})^{\mathsf{v}-1}\mathtt{K}_1(\mathsf{h},k(\mathsf{h}))d\mathsf{e}d\mathsf{h} - \frac{2x}{\Gamma(\mathsf{v})}\int_0^B\int_0^{\mathsf{e}}(\mathsf{e}-\mathsf{h})^{\mathsf{v}-1}\mathtt{K}_2(\mathsf{h},j(\mathsf{h}))d\mathsf{e}d\mathsf{h}|$$

$$\leq \frac{1}{\Gamma(\mathsf{v})}\frac{e^{\frac{-\tau(M(x,y))}{4}}\Gamma(\mathsf{v}+1)}{6}\int_0^x (x-\mathsf{r})^{\mathsf{v}-1}|k(\mathsf{r})-j(\mathsf{r})|d\mathsf{r}$$

$$+\frac{2}{\Gamma(\mathsf{v})}\frac{e^{\frac{-\tau(M(x,y))}{4}}\Gamma(\mathsf{v}+1)}{6}\int_0^B\int_0^{\mathsf{e}}\frac{2x}{\Gamma(\mathsf{v})}(\mathsf{e}-\mathsf{h})^{\mathsf{v}-1}|k(\mathsf{h})-j(\mathsf{h})|d\mathsf{e}d\mathsf{h}$$

$$\leq \frac{1}{\Gamma(\mathsf{v})}\frac{e^{\frac{-\tau(M(x,y))}{4}}\Gamma(\mathsf{v}+1)}{6}\int_0^x (x-\mathsf{r})^{\mathsf{v}-1}|k(\mathsf{r})-j(\mathsf{r})|d\mathsf{r}$$

$$+\frac{2}{\Gamma(\mathsf{v})}\frac{e^{\frac{-\tau(M(x,y))}{4}}\Gamma(\mathsf{v}+1)}{6}\int_0^B\int_0^{\mathsf{e}}\frac{2x}{\Gamma(\mathsf{v})}(\mathsf{e}-\mathsf{h})^{\mathsf{v}-1}|k(\mathsf{h})-j(\mathsf{h})|d\mathsf{e}d\mathsf{h}$$

$$\leq \frac{1}{\Gamma(\mathsf{v})}\frac{e^{\frac{-\tau(M(x,y))}{4}}\Gamma(\mathsf{v}+1)}{6}|k(\mathsf{r})-j(\mathsf{r})|\int_0^x (x-\mathsf{r})^{\mathsf{v}-1}d\mathsf{r}$$

$$+\frac{2}{\Gamma(\mathsf{v})}\frac{e^{\frac{-\tau(M(x,y))}{4}}\Gamma(\mathsf{v}+1)\Gamma(\mathsf{v})}{6\Gamma(\mathsf{v})\Gamma(\mathsf{v}+1)}|k(\mathsf{h})-j(\mathsf{h})|\int_0^B\int_0^{\mathsf{e}}\frac{2x}{\Gamma(\mathsf{v})}(\mathsf{e}-\mathsf{h})^{\mathsf{v}-1}|k(\mathsf{h})-j(\mathsf{h})|d\mathsf{e}d\mathsf{h}$$

$$\leq \frac{e^{\frac{-\tau(M(x,y))}{4}}\Gamma(\mathsf{v}+1)\Gamma(\mathsf{v})}{6\Gamma(\mathsf{v})\Gamma(\mathsf{v}+1)}|k(\mathsf{r}))-j(\mathsf{r})|+2e^{\frac{-\tau(M(x,y))}{4}}\mathtt{B}(\mathsf{v}+1,1)\frac{\Gamma(\mathsf{v}+1)\Gamma(\mathsf{v})}{6\Gamma(\mathsf{v})\Gamma(\mathsf{v}+1)}|k(\mathsf{r})-j(\mathsf{r})|$$

$$\leq \frac{e^{\frac{-\tau(M(x,y))}{4}}}{6}|k(\mathsf{r})-j(\mathsf{r})|+\frac{e^{\frac{-\tau(M(x,y))}{4}}}{3}|k(\mathsf{r})-j(\mathsf{r})|$$

$$\leq \frac{e^{\frac{-\tau(M(x,y))}{4}}}{3}|k(\mathsf{r})-j(\mathsf{r})|,$$

where B is beta function.

Which further implies that

$$|(\mathsf{f}k)(x)-(\mathsf{g}j)(x)| \leq \frac{e^{\frac{-\tau(M(x,y))}{4}}}{3}|k(\mathsf{r})-j(\mathsf{r})|,$$

by taking power four on both sides we get

$$|(\mathsf{f}k)(x)-(\mathsf{g}j)(x)|^4 \leq \frac{e^{-\tau(M(x,y))}}{81}|k(\mathsf{r})-j(\mathsf{r})|^4 \leq e^{-\tau(M(x,y))}|k(\mathsf{r})-j(\mathsf{r})|^4.$$

By assumption (iii) the above inequality written as

$$F(d_\sigma(\mathsf{f}(x),\mathsf{f}(y))) \leq F(M(x,y))+LE(x,y) \leq e^{-\tau(M(x,y))}|(\mathsf{f}k)(x)-(\mathsf{g}j)(x)|^4. \tag{7}$$

Define a function $F:\mathbb{R}^+ \longrightarrow \mathbb{R}^+$ by $F(u)=\ln(u)$ for all $u \in \mathbb{R}^+$. Also $\exists\ \tau:[0,\infty) \longrightarrow [0,\infty)$ such that

$$\lim_{p \longrightarrow \mathsf{q}^+} \inf \tau(p) > 0, \quad \text{for all } \mathsf{q} > 0.$$

So, (7) becomes

$$\ln e^{\tau(M(x,y))}+\ln\left\{F(M(x,y))+LE(x,y)\right\} \leq \ln d_\sigma(x,y)$$

$$\tau(M(x,y))+\ln\left\{F(M(x,y))+LE(x,y)\right\} \leq \ln d_\sigma(x,y).$$

So all the conditions of Corollary 4 are satisfied. Hence, the equations (5) and (6) admits a common solution in $\mathsf{C}(X,\mathbb{R})$. □

5. Application to integral equations

In this section, we investigate the existence of solutions to a nonlinear integral equation.

Let $X = (\mathsf{C}[0,1], \mathbb{R}_+)$ be the set of continuous functions in $[0,1]$ endowed with the metric $\mathsf{d}_\sigma : X \times X \to [0,\infty)$ defined by

$$\mathsf{d}_\sigma(\mathtt{f}_1, \mathtt{f}_2) = \max_{x \in [0,1]} |\mathtt{f}_1(x) - \mathtt{f}_2(x)|^4,$$

for all $\mathtt{f}_1, \mathtt{f}_2 \in \mathsf{C}[0,1]$. Then $(\mathsf{C}[0,1], \mathsf{d}_\sigma)$ is a complete controlled metric space, where $\sigma : X \times X \longrightarrow (0,1]$ be defined by $\sigma(x_1, x_2) = \begin{cases} 1, & \text{if } x_1, x_2 \in [0, 0.5), \\ x_1 + x_2 + 2, & \text{otherwise.} \end{cases}$

Consider the following integral equation:

$$\mathtt{f}(x) = Q(x) + \int_0^x \mathtt{K}(x, \mathsf{p}, \mathtt{f}(\mathsf{p}))\, d\mathsf{p} \tag{8}$$

where $\mathtt{K} : [0,1] \times [0,1] \times X \longrightarrow \mathbb{R}$ and $Q : [0,1] \longrightarrow \mathbb{R}$ are continuous. We have to prove the existence of a solution of the equation (8) by applying Theorem 1.

Theorem 5 Let $X = (\mathsf{C}[0,1], \mathbb{R})$ and $S, T : X \longrightarrow X$ be the integral operators defined as follows:

$$S(\mathtt{f}_1(x)) = Q(x) + \int_0^x \mathtt{K}_1(x, \mathsf{p}, \mathtt{f}_1(\mathsf{p}))\, d\mathsf{p} \quad \text{and}$$

$$T(\mathtt{f}_2(x)) = Q(x) + \int_0^x \mathtt{K}_2(x, \mathsf{p}, \mathtt{f}_2(\mathsf{p}))\, d\mathsf{p} \text{ where } \mathtt{f}_1, \mathtt{f}_2 \in \mathsf{C}[0,1].$$

Assume the following hold:

(i) There exists a continuous function $\mathfrak{u} : [0,1] \longrightarrow \mathbb{R}_+$ such that $\int_0^x [\mathfrak{u}(\mathsf{p})] d\mathsf{p} \leq e^{\frac{-\tau(\mathsf{M}(\mathtt{f}_1, \mathtt{f}_2))}{4}}$ and satisfying, for all $\mathtt{f}_1, \mathtt{f}_2 \in X$ so that $\mathtt{f}_1(x) \leq \mathtt{f}_2(x)$

$$|\mathtt{K}_1(x, (\mathsf{p}), \mathtt{f}_1(\mathsf{p})) - \mathtt{K}_2(x, (\mathsf{p}), \mathtt{f}_2(\mathsf{p}))| \leq [\mathfrak{u}(\mathsf{p})] \max_{x \in [0,1]} |\mathtt{f}_1(\mathsf{p}) - \mathtt{f}_2(\mathsf{p})|;$$

(ii) for all $\mathtt{f}_1, \mathtt{f}_2 \in X$ such that $\mathtt{f}_1(x) \leq \mathtt{f}_2(x)$, we have

$$F(M(\mathtt{f}_1, \mathtt{f}_2)) + LE(\mathtt{f}_1, \mathtt{f}_2) \leq |\mathtt{K}_1(x, (\mathsf{p}), \mathtt{f}_1(\mathsf{p})) - \mathtt{K}_2(x, (\mathsf{p}), \mathtt{f}_2(\mathsf{p}))|, \quad \text{for all } \tau > 0,$$

where

$$M(\mathtt{f}_1, \mathtt{f}_2) = \max \Big\{ d_\sigma(T(\mathtt{f}_1), T(\mathtt{f}_2)), d_\sigma(S(\mathtt{f}_1), T(\mathtt{f}_1)), d_\sigma(S(\mathtt{f}_2), T(\mathtt{f}_2)),$$

$$\frac{\{d_\sigma(T(\mathtt{f}_1),S(\mathtt{f}_1))\times d_\sigma(S(\mathtt{f}_2),[T(\mathtt{f}_2)]_\alpha)\}}{\tau(M(\mathtt{f}_1,\mathtt{f}_2))+d_\sigma(S(\mathtt{f}_1),S(\mathtt{f}_2))},\frac{\{d_\sigma(T(\mathtt{f}_1),S(\mathtt{f}_1))\times d_\sigma(S(\mathtt{f}_2),T(\mathtt{f}_2))\}}{1+d_\sigma(S(\mathtt{f}_1),S(\mathtt{f}_2))}\Bigg\},$$

and $E(\mathtt{f}_1,\mathtt{f}_2)=\min\begin{Bmatrix} d_\sigma(S(\mathtt{f}_1),T(\mathtt{f}_1)), & d_\sigma(S(\mathtt{f}_2),T(\mathtt{f}_2))\\ d_\sigma([S(\mathtt{f}_1),T(\mathtt{f}_2)), & d_\sigma(S(\mathtt{f}_2),T(\mathtt{f}_1))\end{Bmatrix}$.

Then integral equation (8) has a solution in $\mathsf{C}[0,1]$.

Proof. Let $\mathtt{f}_1,\mathtt{f}_2\in X=(\mathsf{C}[0,1],\mathbb{R}_+)$ and

$$|S(\mathtt{f}_1(x))-T(\mathtt{f}_2(x))|^4=\left|\left(Q(x)+\int_0^x \mathtt{K}_1(x,\mathsf{p},\mathtt{f}_1(\mathsf{p}))\,d\mathsf{p}\right)-\left(Q(x)+\int_0^x \mathtt{K}_2(x,\mathsf{p},\mathtt{f}_2(\mathsf{p}))\,d\mathsf{p}\right)\right|^4$$

$$\leq\left|\int_0^x|\mathtt{K}_1(x,(\mathsf{p}),\mathtt{f}_1(\mathsf{p}))-\mathtt{K}_2(x,(\mathsf{p}),\mathtt{f}_2(\mathsf{p}))\right|^4 d\mathsf{p}$$

$$\leq\int_0^x[\mathfrak{u}(\mathsf{p})]^4\max_{x\in[0,1]}|\mathtt{f}_1(\mathsf{p})-\mathtt{f}_2(\mathsf{p})|^4 d\mathsf{p}$$

$$\leq e^{-\tau(\mathsf{M}(\mathtt{f}_1,\mathtt{f}_2))}\mathsf{d}_\sigma(\mathtt{f}_1,\mathtt{f}_2).$$

Thus, by (ii), we have

$$F(d_\sigma(\mathsf{f}(x),\mathsf{f}(y)))\leq F(M(\mathtt{f}_1,\mathtt{f}_2))+LE(\mathtt{f}_1,\mathtt{f}_2)\leq e^{-\tau(\mathsf{M}(\mathtt{f}_1,\mathtt{f}_2))}\mathsf{d}_\sigma(\mathtt{f}_1,\mathtt{f}_2)$$

$$e^{\tau(\mathsf{M}(\mathtt{f}_1,\mathtt{f}_2))}F(d_\sigma(\mathsf{f}(x),\mathsf{f}(y)))\leq F(M(\mathtt{f}_1,\mathtt{f}_2))+LE(\mathtt{f}_1,\mathtt{f}_2)\leq\mathsf{d}_\sigma(\mathtt{f}_1,\mathtt{f}_2).$$

Define a function $\mathtt{F}:\mathbb{R}^+\longrightarrow\mathbb{R}^+$ by $\mathtt{F}(\mathsf{t})=\ln(\mathsf{t})$ for all $\mathsf{t}\in\mathbb{R}^+$ implies

$$\ln e^{\tau(\mathsf{M}(\mathtt{f}_1,\mathtt{f}_2))}+\ln(F(d_\sigma(\mathsf{f}(x),\mathsf{f}(y))))\leq\ln(M(\mathtt{f}_1,\mathtt{f}_2))+LE(\mathtt{f}_1,\mathtt{f}_2)\leq\ln\mathsf{d}_\sigma(\mathtt{f}_1,\mathtt{f}_2).$$

That is,

$$\tau(\mathsf{M}(\mathtt{f}_1,\mathtt{f}_2))+\ln(d_\sigma(\mathsf{f}(x),\mathsf{f}(y)))\leq\ln(M(\mathtt{f}_1,\mathtt{f}_2))+LE(\mathtt{f}_1,\mathtt{f}_2)\leq\ln\mathsf{d}_\sigma(\mathtt{f}_1,\mathtt{f}_2).$$

Hence, the conditions of Corollary 4 are satisfied. That is, the operators S and T have a common fixed point. Thus, the integral equation (8) has a solution. □

6. Conclusion

This study's findings have significantly expanded the boundaries of fuzzy fixed point theory and multi-valued mapping fixed points, introducing graphic fuzzy generalized rational F contractive mappings. The fundamental concepts, presented in the context of controlled metric spaces, provide a solid foundation for future developments. These findings not only broaden the scope of fuzzy fixed point theory and multi-valued mappings but also establish a solid framework for further research. In future directions, these ideas may be adapted to broader metric frameworks, including double controlled metric, generalized metric, F-metric, and other generalized metric spaces, where further refinements and generalizations can be achieved. Moreover, the fuzzy set-valued map component can be extended to encompass various advanced mapping concepts, such as intuitionistic fuzzy, soft set-valued, and other related mappings, potentially leading to new breakthroughs and deeper insights.

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Conflict of interest

The authors declare no competing financial interest.

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