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Research Article

Extensions and Improvements of Jensen's and Majorization Type Inequalities for *p*-harmonic Convex Functions

Faiza Azhar[1], **Muhammad Imran Asjad**[1,2], **Majeed Ahmad Yousif**[3], **Imran Abbas Baloch**[4,5], **Dumitru Baleanu**[6], **Pshtiwan Othman Mohammed**[7,8,9*]

[1] Department of Mathematics, University of Management and Technology, Lahore, 54770, Pakistan
[2] Jadara Research Center, Jadara University, Irbid, 21110, Jordan
[3] Department of Mathematics, College of Education, University of Zakho, Duhok, 42001, Iraq
[4] Department of Mathematics, Abdus Salam School of Mathematical Sciences, Government College University, Lahore, 54600, Pakistan
[5] Higher Education Department, Government Graduate College for Boys Gulberg, Lahore, 54600, Pakistan
[6] Department of Computer Science and Mathematics, Lebanese American University, Beirut, 11022801, Lebanon
[7] Department of Mathematics, College of Education, University of Sulaimani, Sulaymaniyah, 46001, Iraq
[8] Department of Computer Engineering, Biruni University, Istanbul, 34010, Turkey
[9] International Telematic University Uninettuno, Rome, 00186, Italy
E-mail: pshtiwansangawi@gmail.com

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Abstract: In this paper some integral inequalities of Jensen's type for generalised non-convex functions defined on real-line intervals and some results on Majorization type inequalities are established. We introduce a novel category of harmonic convex functions, referred to as p-harmonic convex functions. Additionally, new findings are also presented, with generalizations of previously established results. Further validation of the obtained solutions with the existing literature is also discussed.

Keywords: convex functions, harmonic convex functions, p-harmonic convex functions, Jensen's inequality, majorization inequality

MSC: 26D10, 26D15, 39B62

1. Introduction

Integral inequalities are essential and have important role in both applied and theoretical mathematics fields. Over the past two decades, mathematical inequalities and their applications have made significant contributions to engineering [1], statistics [2], approximation theory [3–5], information theory [6], operator theory [7], and other fields of mathematics [8–11].

The field of convex analysis has a long history started with the pioneers, Werner Fenchel and Hermann Minkowski, investigated the geometric properties of functions and sets in convexity. Since the systematic study of convex analysis was initiated in the 1970s by Rockafellar et al. [12], this field of study has attracted a lot of attention because of its numerous applications in estimation theory, signal processing, control systems, economics, data analysis, and other

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fields. However, the classical convexity has been a helpful tool in engineering applications, it does not always solve every problem and requires the development of several generalizations of convexity. Some important generalisations are m-convexity [13], p-convexity [14], h-convexity [15], η-convexity [16], ϕ-convexity [17], k-convexity [18], cr-Log-h-Convexity [19], and s-convexity [20], among others [21, 22]. During the study of convexity, researchers have been captivated by the generalisation of new classes of convex functions. They have attempted to identify functions that are similar to these classes but not necessarily convex. These classes include harmonic (α, m) [23], harmonic (s, m) [24, 25], simple harmonic [26], and harmonic $(p, (s, m))$ [27] convex functions. Harmonic convex functions and their variations play a significant role in inequality theory, applications, and other fields of mathematics. Researchers have focused on harmonic convex functions due to its potential significance for further exploration.

Jensen, Fejer, Hermite-Hadamard, and fractional integral inequalities are the key findings in convexity. These inequalities have wide applications across various branches of mathematics, including optimisation, probability theory, control theory, and harmonic analysis. Jensen's inequality is the direct cause of several inequalities, such as the Hermite-Hadamard, Young, and Hölder inequalities, and many more. In 2003, Mercer et al. [28] introduced a Jensen's inequality variant, significantly impacting the theory of inequality. A novel version of the discrete Jensen-type inequality for harmonic convex functions was introduced by Baloch et al. [29], which was first introduced by Dragomir [30] for harmonic convex functions.

Second part of the paper deals with the concept of majorization. This mathematical concept provides a framework for comparing two vectors or sequences in a specific order according to certain criteria [31, 32]. Fundamentally, majorization examines the distribution or arrangement of elements within a vector in comparison to another, assessing whether one vector is more dispersed than the other based on a specific criterion [33]. Many mathematicians have explored majorization inequalities, employing different methods to derive related results.

The research gap lies in the absence of detailed exploration regarding the discrete Jensen-type and Majorization-type inequalities for harmonic convex functions. The present study is a more generalized version of discrete Jensen-type and Majorization-type inequalities for p-harmonic convex functions.

The main aim of this paper is to introduce p-harmonic convex functions and establish a novel variant of discrete-Jensen's inequality for this class. Specially, to introduce the new concept of convexity by unifying p-harmonic convex functions. This new class broadens the definition of convex functions and provides a versatile framework for studying convexity. These inequalities can be applied in several fields of mathematics, such as optimisation, analysis, and probability theory, as they extend the classical results of convexity. The main contributions highlight the need of investigating new ideas about convexity and their potential benefits.

The paper is organised as follows: Section 2 presents some preliminaries, which also covers the fundamental characteristics of the novel idea of convexity that is presented in the paper. Section 3, 4 and 5 contain the major results for p-harmonic convex functions. These results include Jensen-type, discrete variant of Jensen-type, and Majorization type inequalities which establish the usefulness of the novel concept of convexity in extending classical results.

2. Preliminaries and some basic results

The following section, we introduced several novel categories of convex functions.

2.1 *Classical & harmonic convexity*

Definition 1 [34] A function $\mathscr{G}$ defined over an interval I is said to be convex if it satisfies the following inequality

$$\mathscr{G}(\eta\phi_1+(1-\eta)\phi_2)\leq\eta\mathscr{G}(\phi_1)+(1-\eta)\mathscr{G}(\phi_2),\quad\forall\phi_1,\,\phi_2\in I,\quad\eta\in[0,\,1].$$

Definition 2 [35] Let $I=[a,\,b]\subseteq\mathbb{R}\setminus\{0\}$. The interval I is termed a *harmonic convex set* if it satisfies the condition

$$\frac{\vartheta\chi}{\kappa\vartheta+(1-\kappa)\chi} \in I, \quad \forall \vartheta, \chi \in I, \quad \kappa \in [0, 1].$$

Definition 3 [36] Let $\mathscr{G}\colon I = [a, b] \subseteq \mathbb{R}\setminus\{0\} \to \mathbb{R}$ be a function. Then, $\mathscr{G}$ is termed *harmonic convex* if and only if it satisfies the condition

$$\mathscr{G}\left(\frac{\vartheta\chi}{\kappa\vartheta+(1-\kappa)\chi}\right) \leq (1-\kappa)\mathscr{G}(\vartheta)+\kappa\mathscr{G}(\chi), \quad \forall \vartheta, \chi \in I, \quad \kappa \in [0, 1].$$

Conversely, if the inequality is reversed, the function $\mathscr{G}$ is termed *harmonically concave*.

The following fundamental yet impactful results, independently derived by Dragomir and further extended by Baloch et al. in [37, 38], are formulated as follows.

Fundamental Criteria:

• Let $\mathscr{G}\colon [a, b] \subset I \subset (0, \infty)$ be a function, and define an associated function $\mathscr{H}\colon \left[\frac{1}{b}, \frac{1}{a}\right] \to \mathbb{R}$ as $\mathscr{H}(\kappa) = \mathscr{G}\left(\frac{1}{\kappa}\right)$. Then, $\mathscr{G}$ is harmonic convex over $[a, b]$ if and only if $\mathscr{H}$ is convex in the conventional sense over $\left[\frac{1}{b}, \frac{1}{a}\right]$.

• Consider two functions $\mathscr{G}$, $\mathscr{Q}\colon [a, b] \subset (0, \infty) \to \mathbb{R}$, where $\mathscr{Q}(\kappa) = \kappa\mathscr{G}(\kappa)$ for all $\kappa \in [a, b]$. Then, $\mathscr{G}$ exhibits harmonic convexity over $[a, b]$ if and only if $\mathscr{Q}$ maintains convexity over the same domain.

Dragomir [30] established a highly significant and influential result concerning the Jensen-type inequality for harmonic convex functions. Baloch et al. subsequently leveraged this inequality and its variations to present a concise and elegant proof of the weighted Harmonic Geometric Arithmetic (HGA) inequality in [39].

Theorem 1 [30] Let $I \subseteq (0, \infty)$ be an interval, and consider a function $\mathscr{G}\colon I \to \mathbb{R}$ that exhibits harmonic convexity. The Jensen-type inequality for such functions is expressed as follows

$$\mathscr{G}\left(\frac{1}{\sum_{i=1}^{n}\frac{\vartheta_i}{\varphi_i}}\right) \leq \sum_{i=1}^{n}\vartheta_i\mathscr{G}(\varphi_i),$$

which holds for all $\varphi_1, \varphi_2, \ldots, \varphi_n \in I$ and weights $\vartheta_1, \vartheta_2, \ldots, \vartheta_n \geq 0$ satisfying the normalization constraint

$$\sum_{i=1}^{n}\vartheta_i = 1.$$

Lemma 1 [34] Let $I \subseteq \mathbb{R}$ and $\mathscr{G}\colon I \to \mathbb{R}$ be a function of real valued. The function $\mathscr{G}$ is classified as convex on I if and only if the associated bivariate function

$$\hbar_{\mathscr{G}}(\phi, \psi) = \frac{\mathscr{G}(\psi)-\mathscr{G}(\phi)}{\psi-\phi}$$

is increasing with respect to the variables ϕ and ψ.

Remark 1 [36] Suppose I is an interval in $\mathbb{R}$ that does not include zero, and $\mathscr{G}\colon I \to \mathbb{R}$ is a mapping. So,

• If $I \subseteq (0, \infty)$ and the function $\mathscr{G}$ is both convex and non-decreasing, then $\mathscr{G}$ necessarily satisfies the conditions of harmonic convexity.

• If $I \subseteq (0, \infty)$ and the function $\mathscr{G}$ is both harmonic convex and non-increasing, then $\mathscr{G}$ must also be convex.

• If $I \subseteq (-\infty, 0)$ and $\mathscr{G}$ is both harmonic convex and non-decreasing, then it follows that $\mathscr{G}$ is convex.

• If $I \subseteq (-\infty, 0)$ and the function $\mathscr{G}$ is both convex and non-increasing behavior, then $\mathscr{G}$ must also be harmonic convex as well.

Lemma 2 [40] Let Φ: $I \to \mathbb{R}$ be a differentiable function for convex defined on $I \subset \mathbb{R}$. Suppose that $n, \mathscr{N} \in \mathbb{R}$ satisfy $n < \mathscr{N}$ and that $[n, \mathscr{N}] \subset I^0$, where I^0 denotes the interior of I. If a function $\mathscr{G}$: $\omega \to \mathbb{R}$ is υ-measurable and satisfies the bounds

$$-\infty < n \leq \mathscr{G}(\phi) \leq \mathscr{N} < \infty \quad \text{for} \quad \upsilon\text{-almost everywhere } \phi \in \omega, \tag{1}$$

and further, if $\mathscr{G}, \Psi \circ \mathscr{G}, \Psi' \circ \mathscr{G}, (\Psi' \circ \mathscr{G})\mathscr{G} \in L_w(\omega, \upsilon)$, where $w \geq 0$ is υ-integrable with $\int_\omega w d\upsilon = 1$, then the following inequality holds

$$0 \leq \int_\omega (\Psi \circ \mathscr{G}) w d\upsilon - \Psi\left(\int_\omega \mathscr{G} w d\upsilon\right) \tag{2}$$

$$\leq \frac{1}{4}[\Psi'(\mathscr{N}) - \Psi'(n)](\mathscr{N} - n).$$

Jensen's inequality for harmonic convex functions yield the following results.

Theorem 2 [40] Let $\hbar$: $I \subset (0, \infty) \to \mathbb{R}$ be a differentiable function of harmonic convex, then suppose that $[l, \mathscr{L}] \subset I$. Assume that a function ϕ: $\omega \to \mathbb{R}$ is υ-measurable and satisfies the bounds

$$0 < l \leq \phi(\eta) \leq \mathscr{L} < \infty \quad \text{for} \quad \upsilon\text{-almost everywhere } \eta \in \omega. \tag{3}$$

If $w \geq 0$ is υ-measurable on ω with $\int_\omega w d\upsilon = 1$, then

(i) If $\hbar \circ \phi, \frac{1}{\phi}, \frac{1}{\phi^2}, (\hbar' \circ \phi)\phi, (\hbar' \circ \phi)\phi^2 \in L_w(\omega, \upsilon)$, then

$$0 \leq \int_\omega (\hbar \circ \phi) w d\upsilon - \hbar\left(\frac{1}{\int_\omega \frac{w}{\phi} d\upsilon}\right) \leq \frac{1}{4\mathscr{L}l}[\mathscr{L}^2\hbar'(\mathscr{L}) - l^2\hbar'(l)](\mathscr{L} - l).$$

(ii) If $\hbar \circ \phi, (\hbar \circ \phi)\phi, (\hbar' \circ \phi)\phi, (\hbar' \circ \phi)\phi^2, \phi, \phi^2 \in L_w(\omega, \upsilon)$, then

$$0 \leq \int_\omega (\hbar \circ \phi)\phi w d\upsilon - \Psi\left(\int_\omega \phi w d\upsilon\right)\int_\omega \phi w d\upsilon$$

$$\leq \frac{1}{4}\left[\hbar(\mathscr{L}) + \mathscr{L}\hbar'(\mathscr{L}) - \hbar(l) - l\hbar'(l)\right](\mathscr{L} - l).$$

Theorem 3 [40] Let $\hbar$: $I \subset (0, \infty) \to \mathbb{R}$ be a function of harmonic convex, and let $[l, \mathscr{L}] \subset I$. Assume that ϕ: $\omega \to \mathbb{R}$ is υ-measurable, satisfies the bounds (1), and that $w \geq 0$ is υ-measurable on ω with $\int_\omega w d\upsilon = 1$. Then

(i) Define the function $\Delta_\hbar(., l, \mathscr{L})$: $(\frac{1}{\mathscr{L}}, \frac{1}{l}) \to \mathbb{R}$ as

$$\Delta_{\hbar}(\eta, l, \mathscr{L}) = \frac{\hbar(l) - \hbar(\frac{1}{\eta})}{\frac{1}{l} - \eta} - \frac{\hbar(\frac{1}{\eta}) - \hbar(\mathscr{L})}{\eta - \frac{1}{\mathscr{L}}},$$

If $(\hbar \circ \phi)$, $\frac{1}{\phi} \in L_w(\omega, \upsilon)$, then

$$0 \leq \int_{\omega} (\hbar \circ \phi) w d\upsilon - \hbar \left(\frac{1}{\int_{\omega} \frac{w}{\phi} d\upsilon} \right)$$

$$\leq \frac{1}{4\mathscr{L} l} \left[\mathscr{L}^2 \hbar^{'}(\mathscr{L}) - l^2 \hbar^{'}(l) \right] (\mathscr{L} - l).$$

(ii) Define the function $\Upsilon_{\hbar}(., l, \mathscr{L})$: $(l, \mathscr{L}) \to \mathbb{R}$ as

$$\Upsilon_{\hbar}(., l, \mathscr{L}) = \frac{\mathscr{L}\hbar(\mathscr{L}) - \eta\hbar(\eta)}{\mathscr{L} - \eta} - \frac{\eta\hbar(\eta) - l\hbar(l)}{\eta - l},$$

If $(\hbar \circ \phi)\phi$, $\phi \in L_w(\omega, \upsilon)$, then

$$0 \leq \int_{\omega} (\hbar \circ \phi) \phi w d\upsilon - \Psi \left(\int_{\omega} \phi w d\upsilon \right) \int_{\omega} \phi w d\upsilon$$

$$\leq \frac{1}{4} \left[\hbar(\mathscr{L}) + \mathscr{L}\hbar^{'} - (\mathscr{L}) - \hbar(l) - l\hbar^{'} + (l) \right] (\mathscr{L} - l).$$

2.2 *p-harmonic convexity*

Definition 4 [41] Let I be a subset of $(0, \infty)$, and let p be an element of $\mathbb{R} \setminus \{0\}$. A function $\mathscr{G}: I \to \mathbb{R}$ is termed a p-convex function if it satisfies the inequality

$$\mathscr{G} \left(\left[\vartheta \varphi^p + (1 - \vartheta) \psi^p \right]^{\frac{1}{p}} \right) \leq \vartheta \mathscr{G}(\varphi) + (1 - \vartheta) \mathscr{G}(\psi),$$

for all φ, $\psi \in I$ and $\vartheta \in [0, 1]$.

Theorem 4 [42] Let $\mathscr{G}: I \subset (0, \infty) \to \mathbb{R}$ be a p-convex function with $p \in \mathbb{R} \setminus \{0\}$, and let ϑ_1, $\vartheta_2 \in I$ satisfy $\vartheta_1 < \vartheta_2$. If $\mathscr{G} \in L[\vartheta_1, \vartheta_2]$, then the following inequality holds

$$\mathscr{G} \left(\left[\frac{\vartheta_1^p + \vartheta_2^p}{2} \right]^{\frac{1}{p}} \right) \leq \frac{p}{\vartheta_2^p - \vartheta_1^p} \int_{\vartheta_1}^{\vartheta_2} \frac{\mathscr{G}(\xi)}{\xi^{1-p}} d\xi \leq \mathscr{G} \left(\frac{\vartheta_1 + \vartheta_2}{2} \right).$$

Definition 5 [43] A subset $[a, b] \subset \mathbb{R} \setminus \{0\}$ is called p-harmonic convex if it satisfies the condition

$$\left[\frac{\varphi^p \psi^p}{\eta \varphi^p+(1-\eta)\psi^p}\right]^{\frac{1}{p}} \in I, \quad \forall \varphi,\, \psi \in I,\ \eta \in [0,\, 1], \quad p \neq 0.$$

Definition 6 [43] Let $I=[a,\, b] \subset \mathbb{R} \setminus \{0\}$ be p-harmonic convex set. A function $\mathscr{G}\colon I=[a,\, b] \subseteq \mathbb{R}\setminus\{0\} \to \mathbb{R}$ is said to p-harmonic convex if

$$\mathscr{G}\left(\left[\frac{\phi^p \psi^p}{\eta \phi^p+(1-\eta)\psi^p}\right]^{\frac{1}{p}}\right) \leq (1-\eta)\mathscr{G}(\phi)+\eta\mathscr{G}(\psi), \qquad \forall \phi,\, \psi \in I,\, \eta \in [0,\, 1].$$

Setting $\eta=\frac{1}{2}$ yields

$$\mathscr{G}\left(\left[\frac{2\phi^p \psi^p}{\phi^p+\psi^p}\right]^{\frac{1}{p}}\right) \leq \frac{\mathscr{G}(\phi)+\mathscr{G}(\psi)}{2}, \qquad \forall \phi,\, \psi \in I,$$

which is referred to as the Jensen-type p-harmonic convex function.

Remark 2 Suppose that $[a,\, b] \subset \mathbb{R}\setminus\{0\}$ is a p-harmonic convex set. If $[a,\, b] \subset (0,\, \infty)$ and $p \in \mathbb{R}\setminus\{0\}$, then Definition 6 is equivalent to Definition 2.1 in [27].

Lemma 3 [37] Assuming a nontrivial interval $I \subseteq (0,\, \infty)$, we examine a real-valued mapping $\mathscr{G}\colon I \to \mathbb{R}$. In the domain I, the function $\mathscr{G}$ is *harmonically p-convex* over I iff the composition $\phi^p\mathscr{G}(\phi)$ satisfies the traditional convexity condition.

Theorem 5 [43] Let $\mathscr{G}\colon I=[a,\, b] \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be a function that satisfies p-harmonic convexity over the domain $[a,\, b]$. Then, the following inequalities hold

$$\mathscr{G}\left(\left[\frac{2a^p b^p}{a^p+b^p}\right]^{\frac{1}{p}}\right) \leq \frac{1}{2}\left[\mathscr{G}\left(\left[\frac{4a^p b^p}{a^p+3b^p}\right]^{\frac{1}{p}}\right)+\mathscr{G}\left(\left[\frac{4a^p b^p}{3a^p+b^p}\right]^{\frac{1}{p}}\right)\right]$$

$$\leq \frac{\mathscr{G}(a)+\mathscr{G}(\vartheta)}{2}.$$

Remark 3 Let I is an interval in $\mathbb{R}$ that does not include zero, $p \in \mathbb{R}\setminus\{0\}$ and $\mathscr{G}\colon I \to \mathbb{R}$ is a mapping. So,

- If $p \geq 1$ and the function $\mathscr{G}$ is both harmonic convex and non-decreasing, then $\mathscr{G}$ necessarily satisfies the conditions of p-harmonic convexity.
- If $p \geq 1$ and the function $\mathscr{G}$ is both p-harmonic convex and non-increasing, then $\mathscr{G}$ must also be harmonic convex.
- If $p \geq 1$ and the function $\mathscr{G}$ is both p-harmonic concave and non-decreasing, then $\mathscr{G}$ is harmonic concave as well.
- If $p \geq 1$ and the function $\mathscr{G}$ is both harmonic concave and non-increasing, then $\mathscr{G}$ is p-harmonic concave.
- If $p \leq 1$ and the function $\mathscr{G}$ is both p-harmonic convex and non-decreasing, then $\mathscr{G}$ is harmonic convex.
- If $p \leq 1$ and the function $\mathscr{G}$ is both harmonic convex and non-increasing, then $\mathscr{G}$ is p-harmonic convex.
- If $p \leq 1$ and the function $\mathscr{G}$ is both harmonic concave and non-decreasing, then $\mathscr{G}$ must also be p-harmonic concave.
- If $p \leq 1$ and the function $\mathscr{G}$ is both p-harmonic concave and non-increasing, then $\mathscr{G}$ is harmonic concave.

Example 1 • The function $\mathscr{G}(\phi)=e^{\phi}$ is both convex and non-decreasing, which implies by Remarks 1 and 3 that $\mathscr{G}$ is p-harmonic convex.

• If $\mathscr{G}(\phi) = \phi^p$ for $\phi \geq 0$, $p \geq 0$, is both convex and non-decreasing function, then by Remarks 1 and 3, $\mathscr{G}$ is p-harmonic convex function.

• If $\mathscr{G}(\phi) = \sqrt{\phi}$ for $\phi \geq 0$, is convex and non-decreasing function, then by Remarks 1 and 3, $\mathscr{G}$ is p-harmonic convex.

• The function $\mathscr{G}(\phi) = \ln \phi$ for $\phi > 0$ is convex and non-decreasing, then by Remarks 1 and 3, $\mathscr{G}$ is p-harmonic convex.

• The function $\mathscr{G}(\phi) = -\ln \phi$ for $\phi > 0$ is convex and non-increasing, neither harmonic nor p-harmonic convex.

• The function $\mathscr{G}(\phi) = \frac{1}{\phi}$ for $\phi > 0$ is convex and non-increasing, neither harmonic nor p-harmonic convex.

• The function $\mathscr{G}(\phi) = e^{-\phi}$ is convex and non-increasing, neither harmonic nor p-harmonic convex.

Remark 4 For $p = 1$ and $p = -1$, the notion of p-harmonic convexity reduces to standard convexity and harmonic convexity for functions defined on $I \subset (0, \infty)$, respectively. By selecting different values of p, we obtain various classes of convex functions, demonstrating that p-harmonic convex functions constitute a broader class that includes both harmonic convex and classical convex functions as particular cases.

2.3 *Majorization-type inequality*

Definition 7 Let $m \geq 2$, and consider two m-tuples $\gamma = (\gamma_1, \ldots, \gamma_m)$ and $\delta = (\delta_1, \ldots, \delta_m)$. We say that δ is majorized by γ, or equivalently, γ majorizes δ (denoted as $\gamma \succ \delta$), if the following conditions hold

$$\sum_{i=1}^{l} \gamma_i \geq \sum_{i=1}^{l} \delta_i,$$

for all $l = 1, 2, \ldots, m-1$, and

$$\sum_{i=1}^{m} \gamma_i = \sum_{i=1}^{m} \delta_i,$$

where γ_i represents the ith largest element in the sequence.

The fundamental theorem related to majorization is presented below.

Theorem 6 [44] Let $\gamma = (\gamma_1, \ldots, \gamma_m)$ and $\delta = (\delta_1, \ldots, \delta_m)$ be two m-tuples belonging to a set $I \subseteq \mathbb{R} \setminus \{0\}$, such that γ majorizes δ (denoted by $\gamma \succ \delta$). If $\mathscr{G} \colon I \to \mathbb{R}$ is a function of harmonic convex, so

$$\sum_{i=1}^{m} \gamma_i \mathscr{G}(\gamma_i) \geq \sum_{i=1}^{m} \delta_i \mathscr{G}(\delta_i).$$

The weighted variant of this result is as follows.

Theorem 7 [44] Let $\gamma = (\gamma_1, \ldots, \gamma_m)$ and $\delta = (\delta_1, \ldots, \delta_m)$ be two m-tuples belonging to a set $I \subseteq \mathbb{R} \setminus \{0\}$, such that γ majorizes δ. Additionally, consider a weight $\acute{w} = (\acute{w}_1, \acute{w}_2, \ldots, \acute{w}_m) \in (0, \infty)^m$, representing strictly positive coefficients. If the function $\mathscr{G} \colon I \to \mathbb{R}$ is harmonic convex, consequently, the following inequality

$$\sum_{i=1}^{m} \acute{w}_i \gamma_i \mathscr{G}(\gamma_i) \geq \sum_{i=1}^{m} \acute{w}_i \delta_i \mathscr{G}(\delta_i), \tag{4}$$

holds.

3. Main results

3.1 *Jensen type inequalities*

Theorem 8 Let the hypotheses of Theorem 2 be fulfilled for a p-harmonic convex function. Then, the following assertions hold:

(i) If $\hbar\circ\phi^p$, $\frac{1}{\phi^p}$, $\frac{1}{\phi^{1+p}}$, $(\hbar'\circ\phi^p)\phi^p$, $(\hbar'\circ\phi^p)\phi^{1+p}\in L_w(\omega,\,\upsilon)$, then the subsequent inequalities are satisfied

$$0\leq\int_\omega(\hbar\circ\phi)wd\upsilon-\hbar\left(\frac{1}{\left(\int_\omega\frac{w}{\phi^p}d\upsilon\right)^{\frac{1}{p}}}\right) \tag{5}$$

$$\leq\frac{1}{p}\int_\omega(\hbar'\circ\phi)\phi^{p+1}wd\upsilon\int_\omega\frac{1}{\phi^p}wd\upsilon-\frac{1}{p}\int_\omega(\hbar'\circ\phi)\phi wd\upsilon$$

$$\leq\frac{1}{2p}\left[p\vartheta^{(1+\frac{1}{p})}\hbar'(\vartheta^{\frac{1}{p}})-p\vartheta^{(1+\frac{1}{p})}\hbar'(\vartheta^{\frac{1}{p}})\right]\int_\omega\left|\frac{1}{\phi^p}-\int_\omega\frac{w}{\phi^p}d\upsilon\right|wd\upsilon$$

$$\leq\frac{1}{2p}\left[p\vartheta^{(1+\frac{1}{p})}\hbar'(\vartheta^{\frac{1}{p}})-p\vartheta^{(1+\frac{1}{p})}\hbar'(\vartheta^{\frac{1}{p}})\right]\left[\int_\omega\left(\frac{1}{\phi^p}\right)^2wd\upsilon-\left(\int_\omega\frac{w}{\phi^p}d\upsilon\right)^2\right]^{\frac{1}{2}}$$

$$\leq\frac{1}{4\mathscr{M}\vartheta}\left[\vartheta^{(1+\frac{1}{p})}\hbar'(\vartheta^{\frac{1}{p}})-\vartheta^{(1+\frac{1}{p})}\hbar'(\vartheta^{\frac{1}{p}})\right](\mathscr{M}-\vartheta).$$

(ii) If $\hbar\circ\phi^p$, $(\hbar'\circ\phi^p)\phi^p$, $(\hbar\circ\phi^p)\phi^p$, $(\hbar'\circ\phi^p)\phi^{1+p}$, ϕ^p, $\phi^{p+1}\in L_w(\omega,\,\upsilon)$, then the following inequalities hold

$$0\leq\int_\omega(\hbar\circ\phi^p)\phi^pwd\upsilon-\Psi\left(\int_\omega\phi^pwd\upsilon\right)\int_\omega\phi^pwd\upsilon \tag{6}$$

$$\leq\frac{1}{p}\int_\omega(\hbar'\circ\phi)\phi^{p+1}wd\upsilon\int_\omega\frac{1}{\phi^p}wd\upsilon-\frac{1}{p}\int_\omega(\hbar'\circ\phi)\phi wd\upsilon$$

$$\leq\frac{1}{2}\left[p\mathscr{M}^{(p-1)}\hbar(\mathscr{M})+\mathscr{M}^{(p)}\hbar'(\mathscr{M})-p(\vartheta)^{p-1}\hbar(\vartheta)+(\vartheta)^p\hbar'(\vartheta)\right]\int_\omega\left|\phi^p-\int_\omega w\phi^pd\upsilon\right|wd\upsilon$$

$$\leq\frac{1}{2}\left[p\mathscr{M}^{(p-1)}\hbar(\mathscr{M})+\mathscr{M}^{(p)}\hbar'(\mathscr{M})-p(\vartheta)^{p-1}\hbar(\vartheta)+(\vartheta)^p\hbar'(\vartheta)\right]$$

$$\times\left[\int_\omega(\phi^p)^2wd\upsilon-\left(\int_\omega w\phi^pd\upsilon\right)^2\right]^{\frac{1}{2}}$$

$$\leq\frac{1}{4}\left[p\mathscr{M}^{(p-1)}\hbar(\mathscr{M})+\mathscr{M}^{(p)}\hbar'(\mathscr{M})-p(\vartheta)^{p-1}\hbar(\vartheta)+(\vartheta)^p\hbar'(\vartheta)\right](\mathscr{M}-\vartheta).$$

Proof.

(i) Given that the function $\hbar$: $[\vartheta, \mathscr{M}] \subset I \to \mathbb{R}$ is differentiable and p-harmonic convex, it follows from Fundamental Criterion 1 that the transformation Θ: $[\frac{1}{\mathscr{M}}, \frac{1}{\vartheta}] \to \mathbb{R}$, defined as $\Theta(z) = \hbar\left(\frac{1}{z^{\frac{1}{p}}}\right)$, is differentiable and convex. Consequently, we obtain

$$\Theta^{'}(z) = -\frac{1}{pz^{(1+\frac{1}{p})}}\hbar^{'}\left(\frac{1}{z^{\frac{1}{p}}}\right), \quad z \in \left[\frac{1}{\mathscr{M}}, \frac{1}{\vartheta}\right].$$

Now, define $\lambda = \frac{1}{\mathscr{M}}$ and $\Lambda = \frac{1}{\vartheta}$, and introduce the function $\mathscr{G}(\eta) = \frac{1}{\phi^p}, \ \eta \in \omega$. From inequality (3), we derive

$$\lambda \leq \mathscr{G}(\eta) \leq \Lambda < \infty, \quad \text{for almost everywhere } \eta \in \omega$$

with respect to measure υ.

Furthermore, we establish

$$\Theta^{'}(\lambda) = \Theta^{'}\left(\frac{1}{\mathscr{M}}\right) = -p\mathscr{M}^{\left(1+\frac{1}{p}\right)}\hbar^{'}\left(\mathscr{M}^{\frac{1}{p}}\right),$$

and similarly,

$$\Theta^{'}(\Lambda) = \Theta^{'}\left(\frac{1}{\vartheta}\right) = -p\vartheta^{\left(1+\frac{1}{p}\right)}\hbar^{'}\left(\vartheta^{\frac{1}{p}}\right).$$

□

Using inequality (3) with the above choices, we derive the desired result (5).

(ii) Given that the function $\hbar$: $[\vartheta, \mathscr{M}] \subset I \to \mathbb{R}$ is differentiable and p-harmonic convex, Fundamental Criterion 1 implies that the function Θ: $[\vartheta, \mathscr{M}] \to \mathbb{R}$, defined as

$$\Theta(z) = z^p \hbar(z),$$

is differentiable and convex. Differentiating $\Theta^{'}$, we obtain

$$\Theta^{'}(z) = pz^{p-1}\hbar(z) + z^p\hbar^{'}(z), \quad z \in [\vartheta, \mathscr{M}].$$

Applying inequality (2) to this function leads to the required result (6). □

The following result establishes a reverse form of Jensen's inequality [45, 46].

Lemma 4 Consider a mapping Ψ: $I \to \mathbb{R}$, where I is a given interval $I \subset \mathbb{R}$. Suppose that $\lambda, \Lambda \in \mathbb{R}$ satisfy $\lambda < \Lambda$ and that $[\lambda, \Lambda] \subset I$. If $\mathscr{G}$: $\omega \to \mathbb{R}$ is υ-measurable and satisfies the constraints given in (1), and if $\mathscr{G}, \Psi \circ \mathscr{G} \in L_w(\omega, \upsilon)$, where $w \geq 0$ is υ-almost everywhere on ω with $\int_\omega w d\upsilon = 1$, then

Proof.

$$0 \leq \int_{\omega} (\Psi \circ \mathscr{G}) w \, d\upsilon - \Psi(\overline{\mathscr{G}}_{\omega, w}), \tag{7}$$

$$\leq \frac{(\Lambda - \overline{\mathscr{G}}_{\omega, w})(\overline{\mathscr{G}}_{\omega, w} - \lambda)}{\Lambda - \lambda} sup_{\zeta \in (\lambda, \Lambda)} \Psi_{\Phi}(\zeta; \lambda, \Lambda),$$

$$\leq (\Lambda - \overline{\mathscr{G}}_{\omega, w})(\overline{\mathscr{G}}_{\omega, w} - \lambda) \frac{\Phi'_{-}(\Lambda) - \Phi'_{+}(\lambda)}{\Lambda - \lambda},$$

$$\leq \frac{1}{4}(\Lambda - \lambda)[\Psi'_{-}(\Lambda) - \Psi'_{+}(\lambda)],$$

where

$$\overline{\mathscr{G}}_{\omega, w} := \int_{\omega} w(\phi) \mathscr{G}(\phi) \, d\upsilon(\phi) \in [\lambda, \Lambda],$$

and the function Φ_{Ψ}: $(\lambda, \Lambda) \to \mathbb{R}$ is given by

$$\Phi_{\Psi}(\eta; \lambda, \Lambda) = \frac{\Psi(\Lambda) - \Psi(\eta)}{\Lambda - \eta} - \frac{\Psi(\eta) - \Psi(\lambda)}{\eta - \lambda}.$$

Moreover, we obtain the inequality

$$0 \leq \int_{\omega} (\Psi \circ \mathscr{G}) w \, d\theta - \Psi(\overline{\mathscr{G}}_{\omega, w}) \tag{8}$$

$$\leq \frac{1}{4}(\Lambda - \lambda) \Phi_{\Psi}(\overline{\mathscr{G}}_{\omega, w}; \lambda, \Lambda)$$

$$\leq \frac{1}{4}(\Lambda - \lambda)[\Psi'_{-}(\Lambda) - \Psi'_{+}(\lambda)],$$

provided that $\overline{\mathscr{G}}_{\omega, w} \in (\lambda, \Lambda)$.

Consider a function $\hbar$: $[\vartheta, \mathscr{M}] \subset I \to \mathbb{R}$, which is assumed to be p-harmonic convex. Then, defining Ψ: $[\frac{1}{\mathscr{M}}, \frac{1}{\vartheta}] \to \mathbb{R}$ by

$$\Psi(z) = \hbar \left(\frac{1}{z^{\frac{1}{p}}} \right),$$

it follows from Criterion 1 that Ψ is p-convex. Setting $\lambda = \frac{1}{\mathscr{M}}$ and $\Lambda = \frac{1}{\vartheta}$, we obtain

$$\Phi_{\Psi}(\eta;\lambda,\Lambda)=\frac{\hbar(\vartheta^{\frac{1}{p}})-\hbar\left(\frac{1}{\eta^{\frac{1}{p}}}\right)}{\frac{1}{\vartheta}-\eta}-\frac{\hbar\left(\frac{1}{\eta^{\frac{1}{p}}}\right)-\hbar(\vartheta^{\frac{1}{p}})}{\eta-\frac{1}{\vartheta}},\quad \forall\eta\in\left(\frac{1}{\mathscr{M}},\frac{1}{\vartheta}\right).$$

Similarly, for a p-harmonic convex function $\hbar$: $[\vartheta,\mathscr{M}]\subset I\to\mathbb{R}$, consider the function Ψ: $[\vartheta,\mathscr{M}]\to\mathbb{R}$ defined by

$$\Psi(z)=z^p\hbar(z),$$

which is p-convex by Fundamental Criterion. Thus, for $\lambda=\vartheta$ and $\Lambda=\mathscr{M}$, we have

$$\Phi_{\Psi}(\eta;\lambda,\Lambda)=\frac{\hbar(\mathscr{M}^p)\hbar(\mathscr{M})-\eta^p\hbar(\eta)}{\mathscr{M}-\eta}-\frac{\eta^p\hbar(\eta)-\vartheta^p\hbar(\vartheta)}{\eta-\vartheta},\quad \forall\eta\in(\vartheta,\mathscr{M}).$$

□

Using Lemma 4, we may also derive the following result.

Theorem 9 Let the hypotheses of Theorem 2 hold for the class of p-harmonic convex functions. Then,

(i). Introduce the function $\Xi_{\hbar}(.,\vartheta,\mathscr{M})$: $\left(\frac{1}{\mathscr{M}},\frac{1}{\vartheta}\right)\to\mathbb{R}$ given by

$$\Xi_{\hbar}(\eta,\vartheta,\mathscr{M})=\frac{\hbar(\vartheta^{\frac{1}{p}})-\hbar\left(\frac{1}{\eta^{\frac{1}{p}}}\right)}{\frac{1}{\vartheta}-\eta}-\frac{\hbar\left(\frac{1}{\eta^{\frac{1}{p}}}\right)-\hbar(\vartheta^{\frac{1}{p}})}{\eta-\frac{1}{\vartheta}}. \tag{9}$$

Proof. If $\hbar\circ\phi,\ \frac{1}{\phi^p}\in L_w(\omega,\nu)$, then

$$0\leq\int_{\omega}(\hbar\circ\phi)\omega d\nu-\hbar\left(\frac{1}{\int_{\omega}\frac{\omega}{\phi^p}d\nu}\right) \tag{10}$$

$$\leq\vartheta\mathscr{M}\frac{\left(\frac{1}{\vartheta}-\int_{\omega}\frac{\omega}{\phi^p}d\nu\right)\left(\int_{\omega}\frac{\omega}{\phi^p}d\nu-\frac{1}{\vartheta}\right)}{\mathscr{M}-\vartheta}\sup_{\eta\in\left(\frac{1}{\mathscr{M}},\frac{1}{\vartheta}\right)}\Xi_{\hbar}(\eta,\vartheta,\mathscr{M})$$

$$\leq\mathscr{M}\vartheta\left(\frac{1}{\vartheta}-\int_{\omega}\frac{\omega}{\phi^p}d\nu\right)\left(\int_{\omega}\frac{\omega}{\phi^p}d\nu-\frac{1}{\vartheta}\right)\frac{p\mathscr{M}^{\left(1+\frac{1}{p}\right)}\hbar'_{-}(\mathscr{M}^{\frac{1}{p}})-p\vartheta^{\left(1+\frac{1}{p}\right)}\hbar'_{+}(\vartheta^{\frac{1}{p}})}{p(\mathscr{M}-\vartheta)}$$

$$\leq\frac{1}{4p\mathscr{M}\vartheta}(\mathscr{M}-\vartheta)\left[p\mathscr{M}^{\left(1+\frac{1}{p}\right)}\hbar'_{-}(\mathscr{M}^{\frac{1}{p}})-p\vartheta^{\left(1+\frac{1}{p}\right)}\hbar'_{+}(\vartheta^{\frac{1}{p}})\right].$$

Moreover, we deduce the following inequality

$$0 \leq \int_{\omega} (\hbar \circ \phi)\omega d\nu - \hbar\left(\frac{1}{\int_{\omega} \frac{\omega}{\phi^p} d\nu}\right) \leq \frac{1}{4p\mathscr{M}\vartheta}(\mathscr{M}-\vartheta)\Xi_{\hbar}\left(\int_{\omega} \frac{\omega}{\phi^p} d\nu,\, \vartheta,\, \mathscr{M}\right) \tag{11}$$

$$\leq \frac{1}{4p\mathscr{M}\vartheta}(\mathscr{M}-\vartheta)\left[p\mathscr{M}^{\left(1+\frac{1}{p}\right)}\hbar'_{-}(\mathscr{M}^{\frac{1}{p}}) - p\vartheta^{\left(1+\frac{1}{p}\right)}\hbar'_{+}(\vartheta^{\frac{1}{p}})\right].$$

Given that $\int_{\omega} \frac{w}{\phi^p} d\upsilon \in \left(\frac{1}{\mathscr{M}},\, \frac{1}{\vartheta}\right)$.
(ii) Define the function $\Upsilon_{\hbar}(.,\, \vartheta,\, \mathscr{M})$: $(\vartheta,\, \mathscr{M}) \to \mathbb{R}$ as follows

$$\Upsilon_{\hbar}(\eta,\, \vartheta,\, \mathscr{M}) = \frac{\mathscr{M}^p\hbar(\mathscr{M}) - \eta^p\hbar(\eta)}{\mathscr{M}-\eta} - \frac{\eta^p\hbar(\eta) - \vartheta^p\hbar(\vartheta)}{\eta-\vartheta}. \tag{12}$$

If $(\hbar \circ \phi^p)\phi^p,\, \phi^p \in L_w(\omega,\, \upsilon)$, then we obtain the following inequality

$$0 \leq \int_{\omega} (\hbar \circ \phi^p)\phi^p w d\upsilon - \Psi\left(\int_{\omega} \phi^p w d\upsilon\right)\int_{\omega} \phi^p w d\upsilon \tag{13}$$

$$\leq \frac{(\mathscr{M} - \int_{\omega} \phi^p w d\upsilon)(\int_{\omega} \phi^p w d\upsilon - \vartheta)}{\mathscr{M}-\vartheta} \sup_{\eta \in (\vartheta,\, \mathscr{M})} \Upsilon_{\hbar}(\eta,\, \vartheta,\, \mathscr{M})$$

$$\leq (\mathscr{M} - \int_{\omega} \phi^p w d\upsilon)(\int_{\omega} \phi^p w d\upsilon - \vartheta)$$

$$\times \frac{[p\mathscr{M}^{(p-1)}\hbar(\mathscr{M}) + \mathscr{M}^p\hbar'(\mathscr{M}) - p\vartheta^{p-1}\hbar(\vartheta) - \vartheta^p\hbar'(\vartheta)]}{\mathscr{M}-\vartheta}$$

$$\leq \frac{1}{4}(\mathscr{M}-\vartheta)[p\mathscr{M}^{(p-1)}\hbar(\mathscr{M}) + \mathscr{M}^p\hbar'(\mathscr{M}) - p\vartheta^{p-1}\hbar(\vartheta) - \vartheta^p\hbar'(\vartheta)].$$

Similarly, the following inequality holds

$$0 \leq \int_{\omega} (\hbar \circ \phi^p)\phi^p w d\upsilon - \Psi\left(\int_{\omega} \phi^p w d\upsilon\right)\int_{\omega} \phi^p w d\upsilon \tag{14}$$

$$\leq \frac{1}{4}(\mathscr{M}-\vartheta)\Upsilon_{\hbar}\left(\int_{\omega} \phi^p w d\upsilon,\, \vartheta,\, \mathscr{M}\right)$$

$$\leq \frac{1}{4}(\mathscr{M}-\vartheta)[p\mathscr{M}^{(p-1)}\hbar(\mathscr{M}) + \mathscr{M}^p\hbar'(\mathscr{M}) - p\vartheta^{p-1}\hbar(\vartheta) - \vartheta^p\hbar'(\vartheta)].$$

Given that $\int_{\omega} \phi^p w d\upsilon \in (\vartheta,\, \mathscr{M})$. □

Furthermore, the reverse of Jensen's inequality also holds [45, 46].

Lemma 5 Assuming Lemma 4 holds, we derive the following inequalities

$$0 \leq \int_{\omega} (\Psi \circ \mathscr{G}) w d\upsilon(\phi) - \Psi(\overline{\mathscr{G}}_{\omega, w}) \tag{15}$$

$$\leq 2 \max \left\{ \frac{\Lambda - \overline{\mathscr{G}}_{\omega, w}}{\Lambda - \lambda}, \frac{\overline{\mathscr{G}}_{\omega, w} - \lambda}{\Lambda - \lambda} \right\} \left[\frac{\Phi(\lambda) + \Phi(\Lambda)}{2} - \Phi\left(\frac{\lambda + \Lambda}{2}\right) \right]$$

$$\leq 2 \left[\frac{\Psi(\lambda) + \Psi(\Lambda)}{2} - \Psi\left(\frac{\lambda + \Lambda}{2}\right) \right].$$

Applying Lemma 5, we establish the following theorem

Theorem 10 Let the conditions of Theorem 2 be satisfied for a p-harmonic convex function. Then the following inequalities hold

Proof. (i). If $\hbar \circ \phi$, $\frac{1}{\phi^p} \in L_w(\omega, \upsilon)$, then we have

$$0 \leq \int_{\omega} (\hbar \circ \phi) w d\upsilon - \hbar\left(\frac{1}{\int_{\omega} \frac{w}{\phi^p} d\upsilon} \right) \tag{16}$$

$$\leq 2 \left[\frac{\hbar(\vartheta) + \hbar(\mathscr{M})}{2} - \hbar\left(\frac{2\mathscr{M}\vartheta}{\mathscr{M} + \vartheta} \right) \right].$$

(ii). If $(\hbar \circ \phi^p)\phi^p$, $\phi^p \in L_w(\omega, \upsilon)$, then the following inequality holds

$$0 \leq \int_{\omega} (\hbar \circ \phi^p)\phi^p w d\upsilon - \hbar\left(\frac{1}{\int_{\omega} \phi^p w d\upsilon} \right) \int_{\omega} \phi^p w d\upsilon \tag{17}$$

$$\leq 2 \left[\frac{\vartheta\hbar(\vartheta) + \mathscr{M}\hbar(\mathscr{M})}{2} - \hbar\left(\frac{\mathscr{M} + \vartheta}{2} \right) \frac{\mathscr{M} + \vartheta}{2} \right].$$

□

3.2 *Discrete inequalities*

Let $s = (s_1, \ldots, s_n)$ be a probability distribution satisfying $s_i \geq 0$ for all $i \in \{1, \ldots, n\}$ and

$$\sum_{i=1}^{n} s_i = 1.$$

By applying the inequalities from Theorem 8 to the discrete measure, we derive the following discrete inequalities.

Assume that $\phi_i \in [\vartheta, \mathscr{M}] \subset (0, \infty)$ for all $i \in \{1, \ldots, n\}$, and let $s = (s_1, \ldots, s_n)$ be a probability distribution. If the function $\hbar$: $[\vartheta, \mathscr{M}] \to \mathbb{R}$ is p-harmonic convex and differentiable, then by using inequality (5), we obtain

$$0 \leq \sum_{i=1}^{n} s_i \hbar(\phi_i) - \hbar\left(\frac{1}{\left(\sum_{i=1}^{n} \frac{s_i}{\phi_i^p}\right)^{\frac{1}{p}}}\right) \tag{18}$$

$$\leq \frac{1}{p}\sum_{i=1}^{n} s_i \hbar'(\phi_i)\phi_i^{(1+p)} \sum_{i=1}^{n} \frac{s_i}{\phi_i^p} - \sum_{i=1}^{n} s_i \hbar'(\phi_i)\phi_i$$

$$\leq \frac{1}{2p}\left[p\mathscr{M}^{(1+\frac{1}{p})}\hbar'(\mathscr{M}^{\frac{1}{p}}) - p\vartheta^{(1+\frac{1}{p})}\hbar'(\vartheta^{\frac{1}{p}})\right] \sum_{i=1}^{n} s_i \left|\frac{1}{\phi_i^p} - \sum_{j=1}^{n} \frac{s_j}{\phi_j^p}\right|$$

$$\leq \frac{1}{2p}\left[p\mathscr{M}^{(1+\frac{1}{p})}\hbar'(\mathscr{M}^{\frac{1}{p}}) - p\vartheta^{(1+\frac{1}{p})}\vartheta'(l^{\frac{1}{p}})\right] \left[\sum_{i=1}^{n} s_i \frac{s_i}{(\phi_i^p)^2} - \left(\sum_{i=1}^{n} \frac{s_i}{\phi_j^p}\right)^2\right]^{\frac{1}{2}}$$

$$\leq \frac{1}{4p\mathscr{M}\vartheta}\left[p\mathscr{M}^{(1+\frac{1}{p})}\hbar'(\mathscr{M}^{\frac{1}{p}}) - p\vartheta^{(1+\frac{1}{p})}\hbar'(\vartheta^{\frac{1}{p}})\right](\mathscr{M} - \vartheta).$$

Similarly, applying inequality (6) gives

$$0 \leq \sum_{i=1}^{n} s_i \hbar(\phi_i^p)\phi_i^{p^2} - \hbar\left(\sum_{i=1}^{n} s_i \phi_i^p\right) \sum_{i=1}^{n} s_i \phi^p \tag{19}$$

$$\leq \frac{1}{p}\sum_{i=1}^{n} s_i \left[(\hbar' \circ \phi_i^p) + \hbar'(\phi_i^p)\right] \phi_i^{p+1} - \frac{1}{p}\sum_{i=1}^{n} s_i \left[\hbar(\phi_i) + (\hbar' \circ \phi_i)\phi_i\right] \sum_{i=1}^{n} s_i \phi_i$$

$$\leq \frac{1}{2}\left[p\mathscr{M}^{(p-1)}\hbar(\mathscr{M}) + \mathscr{M}^p \hbar'(\mathscr{M}) - p\vartheta^{p-1}\hbar(\vartheta) + \vartheta^p \hbar'(\vartheta)\right] \sum_{i=1}^{n} s_i \left|\phi^p - \sum_{j=1}^{n} s_j \phi_j^p\right|$$

$$\leq \frac{1}{2}\left[p\mathscr{M}^{(p-1)}\hbar(\mathscr{M}) + \mathscr{M}^p \hbar'(\mathscr{M}) - p\vartheta^{p-1}\hbar(\vartheta) + \vartheta^p \hbar'(\vartheta)\right] \left[\sum_{i=1}^{n} s_i (\phi_i^p)^2 - \left(\sum_{j=1}^{n} s_j \phi_j^p\right)^2\right]^{\frac{1}{2}}$$

$$\leq \frac{1}{4}\left[p\mathscr{M}^{(p-1)}\hbar(\mathscr{M}) + \mathscr{M}^p \hbar'(\mathscr{M}) - p\vartheta^{p-1}\hbar(\vartheta) + \vartheta^p \hbar'(\vartheta)\right](\mathscr{M} - \vartheta).$$

Supposed that mapping $\Xi_{\hbar}(.,\vartheta,\mathscr{M})\colon \left(\frac{1}{\mathscr{M}}, \frac{1}{\vartheta}\right) \to \mathbb{R}$, which is defined by Equation (9). If $\phi_i \in [\vartheta, \mathscr{M}] \subset (0, \infty)$ for $i \in 1, \ldots, n$ and $s = (s_1, \ldots, s_n)$ is a probability distribution, then for $\hbar\colon [\vartheta, \mathscr{M}] \to \mathbb{R}$ being p-harmonic convex, from inequality (10), we have

$$0 \leq \sum_{i=1}^{n} s_i \hbar(\phi_i) - \hbar\left(\frac{1}{\sum_{i=1}^{n} \frac{s_i}{\phi_i^p}}\right) \tag{20}$$

$$\leq \vartheta \mathscr{M} \frac{\left(\frac{1}{\vartheta} - \sum_{i=1}^{n} \frac{s_i}{\phi_i^p}\right)\left(\sum_{i=1}^{n} \frac{s_i}{\phi_i^p} - \frac{1}{\mathscr{M}}\right)}{\mathscr{M} - \vartheta} \sup_{\eta \in \left(\frac{1}{\mathscr{M}},\, \frac{1}{\vartheta}\right)} \Xi_\hbar(\eta,\, \vartheta,\, \mathscr{M})$$

$$\leq \frac{1}{4p\mathscr{M}\vartheta}(\mathscr{M} - \vartheta)\left[p\mathscr{M}^{\left(1+\frac{1}{p}\right)}\hbar'_{-}(\mathscr{M}^{\frac{1}{p}}) - p\vartheta^{\left(1+\frac{1}{p}\right)}\hbar'_{+}(\vartheta^{\frac{1}{p}})\right].$$

Similarly, from inequality (11), we obtain

$$0 \leq \sum_{i=1}^{n} s_i \hbar(\phi_i) - \hbar\left(\frac{1}{\sum_{i=1}^{n} \frac{s_i}{\phi_i^p}}\right) \tag{21}$$

$$\leq \frac{1}{4p\mathscr{M}\vartheta}(\mathscr{M} - \vartheta)\Xi_\hbar\left(\sum_{i=1}^{n} \frac{s_i}{\phi_i^p},\, \vartheta,\, \mathscr{M}\right)$$

$$\leq \frac{1}{4p\mathscr{M}\vartheta}(\mathscr{M} - \vartheta)\left[p\mathscr{M}^{\left(1+\frac{1}{p}\right)}\hbar'_{-}(\mathscr{M}^{\frac{1}{p}}) - p\vartheta^{\left(1+\frac{1}{p}\right)}\hbar'_{+}(\vartheta^{\frac{1}{p}})\right],$$

provided that $\sum_{i=1}^{n} \frac{s_i}{\phi_i^p} \in \left(\frac{1}{\mathscr{M}},\, \frac{1}{\vartheta}\right)$.

Now, consider the function $\Upsilon_\hbar(.,\, \vartheta,\, \mathscr{M})$: $[\vartheta,\, \mathscr{M}] \to \mathbb{R}$, as defined by inequality (12). Let $\phi_i^p \in [\vartheta,\, \mathscr{M}] \subset (0,\, \infty)$ for $i \in 1,\, \ldots,\, n$ and $s = (s_1,\, \ldots,\, s_n)$ be a probability distribution.

If $\hbar$: $[\vartheta,\, \mathscr{M}] \to \mathbb{R}$ is p-harmonic convex, then from inequality (13)

$$0 \leq \sum_{i=1}^{n} s_i \hbar(\phi_i^p)\phi_i^p - \hbar\left(\sum_{i=1}^{n} s_i \phi_i^p\right)\sum_{i=1}^{n} s_i \phi_i^p \tag{22}$$

$$\leq \frac{(\mathscr{M} - \sum_{i=1}^{n} s_i \phi_i^p)(\sum_{i=1}^{n} s_i \phi_i^p - \vartheta)}{\mathscr{M} - \vartheta} \sup_{\eta \in (\vartheta,\, \mathscr{M})} \Upsilon_\hbar(\eta,\, \vartheta,\, \mathscr{M})$$

$$\leq \frac{1}{4}(\mathscr{M} - \vartheta)[p\mathscr{M}^{(p-1)}\hbar(\mathscr{M}) + \mathscr{M}^p \hbar'_{-}(\mathscr{M}) - p\vartheta^{p-1}\hbar(\vartheta) - \vartheta^p \hbar'_{+}(\vartheta)].$$

From inequality (14), we obtain

$$0 \leq \sum_{i=1}^{n} s_i \hbar(\phi_i^p)\phi_i^p - \hbar\left(\sum_{i=1}^{n} s_i \phi_i^p\right)\sum_{i=1}^{n} s_i \phi_i^p \tag{23}$$

$$\leq \frac{1}{4}(\mathscr{M}-\vartheta)\Xi_\hbar\left(\sum_{i=1}^{n} s_i\phi_i^p,\, \vartheta,\, \mathscr{M}\right)$$

$$\leq \frac{1}{4}(\mathscr{M}-\vartheta)\left[p\mathscr{M}^{(p-1)}\hbar(\mathscr{M})+\mathscr{M}^p\hbar'_-(\mathscr{M})-p(\vartheta)^{p-1}\hbar(\vartheta)-\vartheta^p\hbar'_+(\vartheta)\right].$$

This holds under the condition that $\sum_{i=1}^{n} s_i\phi_i^p \in (\vartheta,\, \mathscr{M})$.
Furthermore, applying inequality (16), we derive

$$0 \leq \sum_{i=1}^{n} s_i\hbar(\phi_i)-\hbar\left(\frac{1}{\sum_{i=1}^{n}\frac{s_i}{\phi_i^p}}\right) \tag{24}$$

$$\leq \frac{2\mathscr{M}\vartheta \max}{\mathscr{M}-\vartheta}\left\{\frac{1}{\vartheta}-\sum_{i=1}^{n}\frac{s_i}{\phi_i^p},\, \sum_{i=1}^{n}\frac{s_i}{\phi_i^p}-\frac{1}{\mathscr{M}}\right\}$$

$$\times\left[\frac{\hbar(\vartheta)+\hbar(\mathscr{M})}{2}-\hbar\left(\frac{2\mathscr{M}\vartheta}{\mathscr{M}+\vartheta}\right)\right]$$

$$\leq 2\left[\frac{\hbar(\vartheta)+\hbar(\mathscr{M})}{2}-\hbar\left(\frac{2\mathscr{M}\vartheta}{\mathscr{M}+\vartheta}\right)\right].$$

Likewise, using inequality (17), we obtain

$$0 \leq \sum_{i=1}^{n} s_i\hbar(\phi_i^p)-\hbar\left(\frac{1}{\sum_{i=1}^{n}\frac{s_i}{\phi_i^p}}\right) \tag{25}$$

$$\leq \frac{2\mathscr{M}\vartheta \max}{\mathscr{M}-\vartheta}\left\{\mathscr{M}-\sum_{i=1}^{n}\frac{s_i}{\phi_i^p},\, \sum_{i=1}^{n}\frac{s_i}{\phi_i^p}-\vartheta\right\}$$

$$\times\left[\frac{\vartheta\hbar(\vartheta)+\mathscr{M}\hbar(\mathscr{M})}{2}-\hbar\left(\frac{\mathscr{M}+\vartheta}{2}\right)\right]$$

$$\leq 2\left[\frac{\vartheta\hbar(\vartheta)+\mathscr{M}\hbar(\mathscr{M})}{2}-\hbar\left(\frac{\mathscr{M}+\vartheta}{2}\right)\frac{\mathscr{M}+\vartheta}{2}\right].$$

Remark 5 • Classical and Harmonic convexities are retrieved from p-harmonic convexity as a limiting case.
• On letting $p \to 1$, Theorem 8, 9 and 10 refines and coincides with Theorem 5, 6 and 7 of [30].
• On letting $p \to 1$, Our main results of subsection 3.2 (Discrete inequalities) refines as well as coincides with Dragomir's Discrete inequalities [30].

3.3 *Majorization inequalities*

Theorem 11 Let all the assumptions of Theorem 6 be fulfilled for a p-harmonic convex function. Then the following inequality

$$\sum_{\tilde{i}=1}^{\tilde{m}}(\gamma_{\tilde{i}})^p\mathfrak{f}(\gamma_{\tilde{i}})\geq\sum_{\tilde{i}=1}^{\tilde{m}}(\delta_{\tilde{i}})^p\mathfrak{f}(\delta_{\tilde{i}}), \tag{26}$$

holds.

Proof. Define the function $\chi(\mathfrak{z})=\mathfrak{z}^p\mathscr{G}(\mathfrak{z})$, and let

$$\lambda_{\tilde{i}}=\frac{g(\delta_{\tilde{i}})-g(\gamma_{\tilde{i}})}{\delta_{\tilde{i}}-\gamma_{\tilde{i}}}.$$

By utilizing Lemmas 3 and 1, along with the p-harmonic convexity of $\mathscr{G}$, we conclude that g is convex on I. Consequently, the sequence $\{\lambda_{\tilde{i}}\}_{\tilde{i}=1}^{\tilde{m}}$ is decreasing.

Define the sums

$$E_{\tilde{l}}=\sum_{\tilde{i}=1}^{\tilde{l}}\gamma_{\tilde{i}},\quad F_{\tilde{l}}=\sum_{\tilde{i}=1}^{\tilde{l}}\delta_{\tilde{i}},\quad E_0=F_0.$$

Since $\gamma\succ\delta$, we have $E_m=F_m$, leading to

$$\begin{aligned}\sum_{\tilde{i}=1}^{\tilde{m}}(\gamma_{\tilde{i}})^p\mathscr{G}(\gamma_{\tilde{i}})-\sum_{\tilde{i}=1}^{\tilde{m}}(\delta_{\tilde{i}})^p\mathscr{G}(\delta_{\tilde{i}})&=\sum_{\tilde{i}=1}^{\tilde{m}}\chi(\gamma_{\tilde{i}})-\sum_{\tilde{i}=1}^{\tilde{m}}\chi(\delta_{\tilde{i}})\\
&=\sum_{\tilde{i}=1}^{\tilde{m}}\lambda_{\tilde{i}}(\gamma_{\tilde{i}}-\delta_{\tilde{i}})\\
&=\sum_{\tilde{i}=1}^{\tilde{m}}\lambda_{\tilde{i}}(E_{\tilde{i}}-E_{\tilde{i}-1}-F_{\tilde{i}}+F_{\tilde{i}-1})\\
&=\sum_{\tilde{i}=1}^{\tilde{m}}\lambda_{\tilde{i}}(E_{\tilde{i}}-F_{\tilde{i}})-\sum_{\tilde{i}=1}^{\tilde{m}}\lambda_{\tilde{i}}(E_{\tilde{i}-1}-F_{\tilde{i}-1})\\
&=\sum_{\tilde{i}=1}^{\tilde{m}-1}\lambda_{\tilde{i}}(E_{\tilde{i}}-F_{\tilde{i}})-\sum_{\tilde{i}=1}^{\tilde{m}-1}\lambda_{\tilde{i}+1}(E_{\tilde{i}}-F_{\tilde{i}})\\
&=\sum_{\tilde{i}=1}^{\tilde{m}-1}(\lambda_{\tilde{i}}-\lambda_{\tilde{i}+1})(E_{\tilde{i}}-F_{\tilde{i}}).\end{aligned}$$

Thus, inequality (26) follows from the monotonicity of the sequence $\{\lambda_{\tilde{i}}\}_{\tilde{i}=1}^{\tilde{m}-1}$ and the condition $E_{\tilde{i}} \geq F_{\tilde{i}}$ for all $\tilde{i} = 1, 2, \ldots, \tilde{m}-1$, together with the previously established identity. □

The weighted version of the previous theorem is stated as follows.

Theorem 12 Let all the assumptions of Theorem 7 be fulfilled for a p-harmonic convex function. Then the following inequality

$$\sum_{\tilde{i}=1}^{\tilde{m}} \acute{w}_{\tilde{i}} \gamma_{\tilde{i}}^{p} \mathscr{G}(\gamma_{\tilde{i}}) \geq \sum_{\tilde{i}=1}^{\tilde{m}} \acute{w}_{\tilde{i}} \delta_{\tilde{i}}^{p} \mathscr{G}(\delta_{\tilde{i}}), \tag{27}$$

holds.

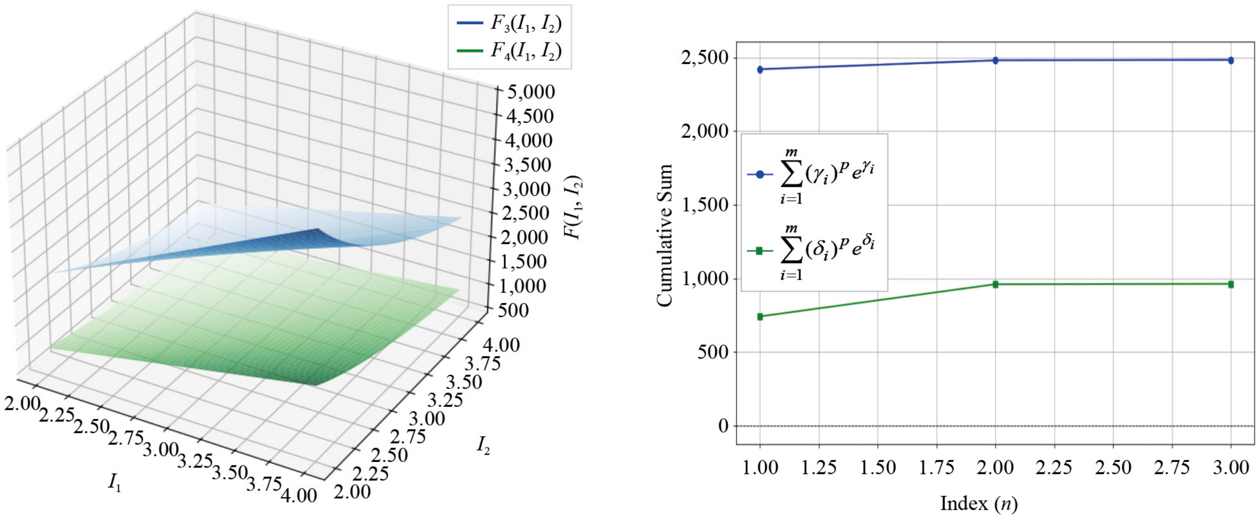
Proof. Applying the method of Theorem 11 and inserting the weight factors $\acute{w}_{\tilde{i}}$ yields the claimed result. □

Corollary 1 Suppose the conditions of Theorem 11 hold, and let $\mathscr{G}\colon I \to \mathbb{R}$ be a p-harmonic convex function for $p \geq 1$. Then,

$$\sum_{\tilde{i}=1}^{\tilde{m}} (\gamma_{\tilde{i}})^{p} e^{\gamma_{\tilde{i}}} \geq \sum_{\tilde{i}=1}^{\tilde{m}} (\delta_{\tilde{i}})^{p} e^{\delta_{\tilde{i}}}.$$

Table 1. Comparison of Corollary 1 for different values of $p = 0, 1, 2, 3, 4$, and three-tuples $\gamma = (6, 3, 1)$ and $\delta = (5, 4, 1)$

p	$\sum(\gamma_i)^p e^{\gamma_i}$	$\sum(\delta_i)^p e^{\delta_i}$	Observation
0	$e^6 + e^3 + e^1 \approx 436.63$	$e^5 + e^4 + e^1 \approx 190.23$	Holds (since e^x is convex)
1	$6e^6 + 3e^3 + 1e^1 \approx 2{,}483.55$	$5e^5 + 4e^4 + 1e^1 \approx 963.18$	Holds
2	$6^2e^6 + 3^2e^3 + 1^2e^1 \approx 14{,}790.4$	$5^2e^5 + 4^2e^4 + 1^2e^1 \approx 4{,}974.38$	Holds
3	$6^3e^6 + 3^3e^3 + 1^3e^1 \approx 88{,}682.4$	$5^3e^5 + 4^3e^4 + 1^3e^1 \approx 25{,}829.4$	Holds strongly
4	$6^4e^6 + 3^4e^3 + 1^4e^1 \approx 532{,}164.9$	$5^4e^5 + 4^4e^4 + 1^4e^1 \approx 133{,}076.8$	Holds strongly



(a) 3D plot illustrating the majorization of a three-tuple

(b) 2D plot illustrating the majorization of a three-tuple

Figure 1. Analysis of Corollary 1 for $p = 1$

Proof. From Theorem 11, choosing $\mathscr{G}(\phi) = e^{\phi}$ ensures that $\mathscr{G}(\phi)$ is both a convex function as a harmonic and a convex function as p-harmonic. Thus, for $p \geq 1$, the required result follows after simplifications. □

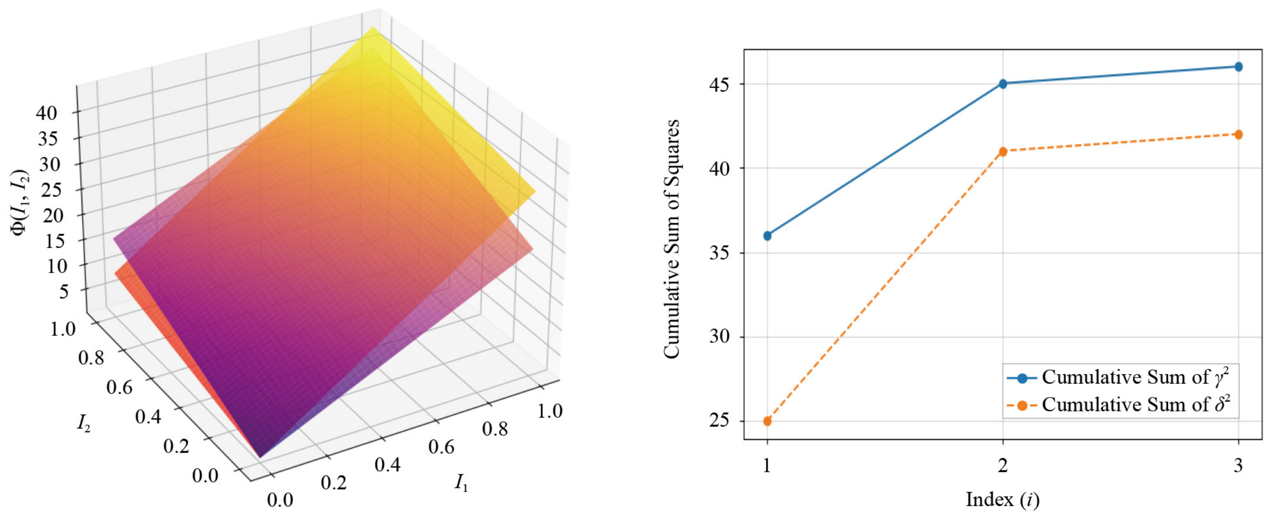
Corollary 2 Under the assumptions of Theorem 11, let $\mathscr{G}\colon I \to \mathbb{R}$ be a p-harmonic convex function. Consequently, the following inequality is true:

$$\sum_{\tilde{i}=1}^{\tilde{m}} \gamma_{\tilde{i}}^{2p} \geq \sum_{\tilde{i}=1}^{\tilde{m}} \delta_{\tilde{i}}^{2p}. \tag{28}$$

Proof. Since $\mathscr{G}(\phi) = \phi$ is a function of harmonic convex and $\mathscr{G}(\phi) = \phi^p$ is p-harmonic convex, applying Theorem 11 with $\mathscr{G}(\phi) = \phi^p$ yields the desired result for $p \geq 1$ after simplifications. □

Table 2. Comparison of Corollary 2 for $p = 0, 1, 2, 3, 4$ and three-tuples $\gamma = (6, 3, 1)$ and $\delta = (5, 4, 1)$

p	$\sum_{i=1}^{3} \gamma_i^{2p}$	$\sum_{i=1}^{3} \delta_i^{2p}$	Inequality Result	Observation
0	$6^0 + 3^0 + 1^0 = 3$	$5^0 + 4^0 + 1^0 = 3$	$=$	Equal (trivial case)
1	$6^2 + 3^2 + 1^2 = 46$	$5^2 + 4^2 + 1^2 = 42$	$>$	Holds
2	$6^4 + 3^4 + 1^4 = 1{,}378$	$5^4 + 4^4 + 1^4 = 882$	$>$	Holds strongly
3	$6^6 + 3^6 + 1^6 = 47{,}386$	$5^6 + 4^6 + 1^6 = 19{,}722$	$>$	Holds strongly
4	$6^8 + 3^8 + 1^8 = 1{,}686{,}178$	$5^8 + 4^8 + 1^8 = 456{,}162$	$>$	Holds very strongly



(a) 3D plot illustrating majorization of a three-tuple for $p = 1$

(b) 2D plot illustrating majorization of a three-tuple for $p = 1$

Figure 2. Analysis of Corollary 2 for $p = 1$

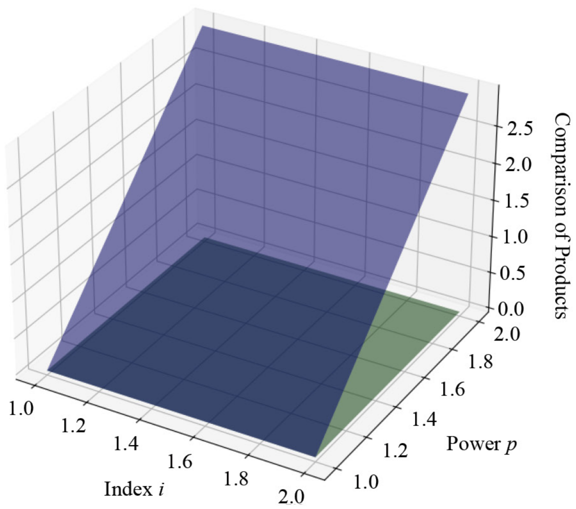
Corollary 3 Suppose the conditions of Theorem 11 hold, and let $\mathscr{G}\colon I \to \mathbb{R}$ called a function of p-harmonic convex. Consequently, the following inequality is true:

$$\prod_{\tilde{i}=1}^{\tilde{m}} (\gamma_{\tilde{i}})^{\gamma_{\tilde{i}}^{p}} \geq \prod_{\tilde{i}=1}^{\tilde{m}} (\delta_{\tilde{i}})^{\delta_{\tilde{i}}^{p}}.$$

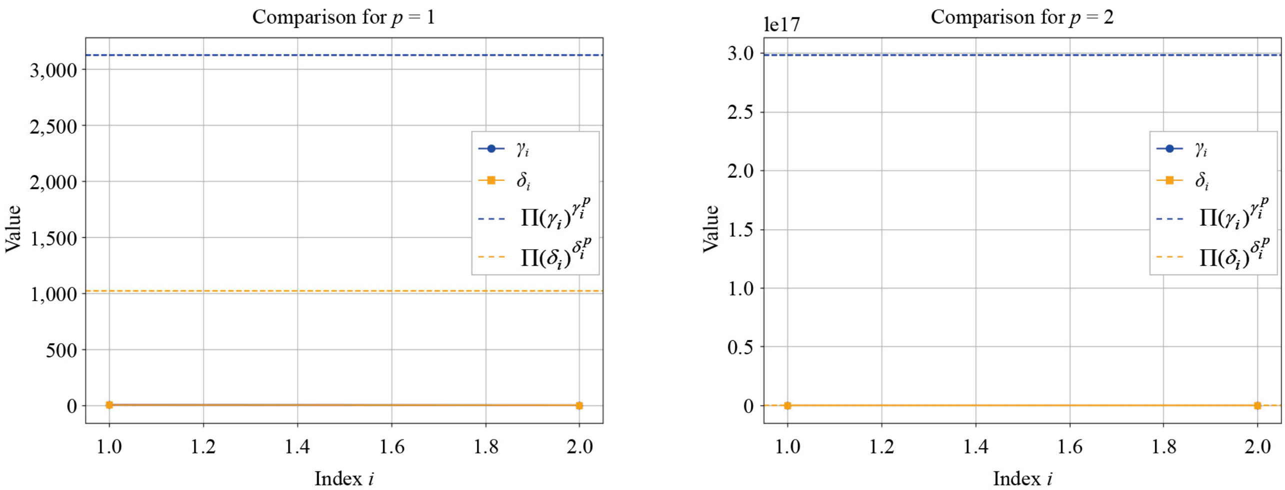
Proof. Since $\mathscr{G}(\phi) = \ln\phi$ is both harmonic convex and non-decreasing, it follows that $\mathscr{G}(\phi) = \ln\phi$ is also p-harmonic convex. By applying Theorem 11 with $\mathscr{G}(\phi) = \ln\phi$, we obtain the required result for $p \geq 1$ after simplifications. □

Table 3. Comparison of Corollary 3 for two-tuples $\gamma = (5, 1)$, $\delta = (4, 2)$ and $p \in [1, 2]$

p	$\gamma_1^{\gamma_1^p}$	$\gamma_2^{\gamma_2^p}$	LHS product	$\delta_1^{\delta_1^p}$	$\delta_2^{\delta_2^p}$	RHS product
1.0	$5^5 = 3.12 \times 10^3$	$1^1 = 1$	3.12×10^3	$4^4 = 256$	$2^2 = 4$	1.02×10^3
1.2	$5^{5^{1.2}} \approx 1.04 \times 10^5$	1	1.04×10^5	$4^{4^{1.2}} \approx 1.21 \times 10^4$	$2^{2^{1.2}} \approx 4.6$	5.57×10^4
1.4	$5^{5^{1.4}} \approx 2.19 \times 10^6$	1	2.19×10^6	$4^{4^{1.4}} \approx 1.01 \times 10^5$	$2^{2^{1.4}} \approx 7.5$	7.58×10^5
1.6	$5^{5^{1.6}} \approx 2.16 \times 10^7$	1	2.16×10^7	$4^{4^{1.6}} \approx 5.72 \times 10^5$	$2^{2^{1.6}} \approx 12.0$	6.86×10^6
1.8	$5^{5^{1.8}} \approx 1.44 \times 10^9$	1	1.44×10^9	$4^{4^{1.8}} \approx 2.25 \times 10^6$	$2^{2^{1.8}} \approx 18.9$	4.26×10^7
2.0	$5^{5^2} \approx 2.98 \times 10^{17}$	H1	2.98×10^{17}	$4^{16} \approx 4.29 \times 10^9$	$2^4 = 16$	6.87×10^{10}



(a) 3D comparison of sequences corresponding to two tuples



(b) 2D comparison of sequences corresponding to two tuples

Figure 3. Analysis of Corollary 3 for $p = 1$ & 2

Remark 6 • Setting $p = 1$ and considering the three-tuples $\gamma = (6, 3, 1)$ and $\delta = (5, 4, 1)$ in Corollaries 1 and 2, we observe that γ is majorized by δ.

• Setting $p = 2$ and considering the four-tuples $\gamma = (8, 3, 2, 1)$ and $\delta = (7, 4, 2, 1)$ in Corollaries 1 and 2, we conclude that γ is majorized by δ.

• Setting $p = 3$ and considering the five-tuples $\gamma = (10, 6, 3, 2, 1)$ and $\delta = (8, 7, 3, 3, 1)$ in Corollaries 1 and 2, we again infer that γ is majorized by δ.

• Setting $p \in [1, 2]$ and considering the two-tuples $\gamma = (5, 1)$ and $\delta = (4, 2)$ in Corollary 3, we conclude that γ is majorized by δ.

Remark 7 As $p \to 1$, Theorem 11 and Theorem 12 not only refine but also coincide with Theorem 6 and Theorem 7, respectively.

Theorem 13 Let I_1 and I_2 be two intervals in $\mathbb{R}$, and let $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_{\tilde{m}})$ and $\delta = (\delta_1, \delta_2, \ldots, \delta_{\tilde{m}})$ be two $\tilde{m}$-tuples such that $\gamma_{\tilde{i}}, \delta_{\tilde{i}} \in I_1$ for all $\tilde{i} = 1, 2, \ldots, \tilde{m}$.

Similarly,

If $\gamma \succ \delta$ and $\alpha \succ \beta$, then

$$\sum_{\tilde{i}=1}^{\tilde{m}} \sum_{\tilde{j}=1}^{\tilde{n}} \gamma_{\tilde{i}}^p \alpha_{\tilde{j}}^p \mathfrak{f}(\gamma_{\tilde{i}}, \alpha_{\tilde{j}}) \geq \sum_{\tilde{i}=1}^{\tilde{m}} \sum_{\tilde{j}=1}^{\tilde{n}} \delta_{\tilde{i}}^p \beta_{\tilde{j}}^p \mathfrak{f}(\delta_{\tilde{i}}, \beta_{\tilde{j}}) \tag{29}$$

for every p-harmonic convex function $\varphi(\phi, \psi) = (\phi\psi)^p \mathscr{G}(\phi, \psi)$, where $\varphi \colon I_1 \times I_2 \to \mathbb{R}$.

Proof. Assume that the sequences $\gamma, \delta, \alpha,$ and β are arranged in descending order, where $\gamma_{\tilde{i}} \neq \delta_{\tilde{i}}$ and $\alpha_{\tilde{j}} \neq \beta_{\tilde{j}}$ for all $\tilde{i}$ and $\tilde{j}$. Define the cumulative sums:

$$E_l = \sum_{\tilde{i}=1}^{\tilde{l}} \gamma_{\tilde{i}}, \quad F_l = \sum_{\tilde{i}=1}^{\tilde{l}} \delta_{\tilde{i}} \quad (\tilde{l} = 1, 2, \ldots., \tilde{m}),$$

$$G_\ell = \sum_{\tilde{j}=1}^{\tilde{\ell}} \alpha_{\tilde{j}}, \quad H_\ell = \sum_{\tilde{j}=1}^{\tilde{\ell}} \beta_{\tilde{j}} \quad (\tilde{\ell} = 1, 2, \ldots., \tilde{n}),$$

$$E_0 = F_0 = G_0 = H_0 = 0$$

By the definition of majorization, it follows that

$$E_{\tilde{m}} = F_{\tilde{m}}, \quad G_{\tilde{n}} = H_{\tilde{n}}.$$

Define the terms $\eta_{\tilde{i}, \tilde{j}}$ and $r_{\tilde{i}, \tilde{j}}$ as

$$\eta_{\tilde{i}, \tilde{j}} := \nabla \varphi(\gamma_{\tilde{i}}, \delta_{\tilde{i}}; \alpha_{\tilde{j}}) = \frac{\gamma_{\tilde{i}}^p \alpha_{\tilde{j}}^p \mathfrak{f}(\gamma_{\tilde{i}}, \alpha_{\tilde{j}}) - \delta_{\tilde{i}}^p \alpha_{\tilde{j}}^p \mathfrak{f}(\delta_{\tilde{i}}, \alpha_{\tilde{j}})}{\gamma_{\tilde{i}} - \delta_{\tilde{i}}}$$

$$r_{\tilde{i}, \tilde{j}} := \nabla \varphi(\delta_{\tilde{i}}; \alpha_{\tilde{j}}, \beta_{\tilde{j}}) = \frac{\delta_{\tilde{i}}^p \alpha_{\tilde{j}}^p \mathscr{G}(\delta_{\tilde{i}}, \alpha_{\tilde{j}}) - \delta_{\tilde{i}}^p \beta_{\tilde{j}}^p \mathscr{G}(\delta_{\tilde{i}}, \beta_{\tilde{j}})}{\alpha_{\tilde{j}} - \beta_{\tilde{j}}}$$

Thus, it is evident that

$$\gamma_{\tilde{i}}^{p}\alpha_{\tilde{j}}^{p}\mathscr{G}(\gamma_{\tilde{i}},\,\alpha_{\tilde{j}})-\delta_{\tilde{i}}^{p}\beta_{\tilde{j}}^{p}\mathscr{G}(\delta_{\tilde{i}},\,\beta_{\tilde{j}}) =\gamma_{\tilde{i}}^{p}\alpha_{\tilde{j}}^{p}\mathscr{G}(\gamma_{\tilde{i}},\,\alpha_{\tilde{j}})-\delta_{\tilde{i}}^{p}\alpha_{\tilde{j}}^{p}\mathscr{G}(\delta_{\tilde{i}},\,\alpha_{\tilde{j}})+\delta_{\tilde{i}}^{p}\alpha_{\tilde{j}}^{p}\mathscr{G}(\delta_{\tilde{i}},\,\alpha_{\tilde{j}})-\delta_{\tilde{i}}^{p}\beta_{\tilde{j}}^{p}\mathscr{G}(\delta_{\tilde{i}},\,\beta_{\tilde{j}})$$

$$=\frac{\gamma_{\tilde{i}}^{p}\alpha_{\tilde{j}}^{p}\mathscr{G}(\gamma_{\tilde{i}},\,\alpha_{\tilde{j}})-\delta_{\tilde{i}}^{p}\alpha_{\tilde{j}}^{p}\mathscr{G}(\delta_{\tilde{i}},\,\alpha_{\tilde{j}})}{\gamma_{\tilde{i}}-\delta_{\tilde{i}}}(\gamma_{\tilde{i}}-\delta_{\tilde{i}})$$

$$+\frac{\delta_{\tilde{i}}^{p}\alpha_{\tilde{j}}^{p}\mathscr{G}(\delta_{\tilde{i}},\,\alpha_{\tilde{j}})-\delta_{\tilde{i}}^{p}\beta_{\tilde{j}}^{p}\mathscr{G}(\delta_{\tilde{i}},\,\beta_{\tilde{j}})}{\alpha_{\tilde{j}}-\beta_{\tilde{j}}}(\alpha_{\tilde{j}}-\beta_{\tilde{j}})$$

$$=\eta_{\tilde{i},\,\tilde{j}}(E_{\tilde{i}}-E_{\tilde{i}-1}-F_{\tilde{i}}+F_{\tilde{i}-1})+r_{\tilde{i},\,\tilde{j}}(G_{\tilde{j}}-G_{\tilde{j}-1}-H_{\tilde{j}}+H_{\tilde{j}-1}).$$

By summing across all $\tilde{i}$ and $\tilde{j}$, we derive

$$\sum_{\tilde{i}=1}^{\tilde{m}}\sum_{\tilde{j}=1}^{\tilde{n}}\gamma_{\tilde{i}}^{p}\alpha_{\tilde{j}}^{p}\mathscr{G}(\gamma_{\tilde{i}},\,\alpha_{\tilde{j}})-\sum_{\tilde{i}=1}^{\tilde{m}}\sum_{\tilde{j}=1}^{\tilde{n}}\delta_{\tilde{i}}^{p}\beta_{\tilde{j}}^{p}\mathscr{G}(\delta_{\tilde{i}},\,\beta_{\tilde{j}})$$

$$=\sum_{\tilde{i}=1}^{\tilde{j}=1}\eta_{\tilde{i},\,\tilde{j}}(E_{\tilde{i}}-E_{\tilde{i}-1}-F_{\tilde{i}}+F_{\tilde{i}-1})+\sum_{\tilde{i}=1}^{\tilde{m}}\sum_{\tilde{j}=1}^{\tilde{n}}r_{\tilde{i},\,\tilde{j}}(G_{\tilde{j}}-G_{\tilde{j}-1}-H_{\tilde{j}}+H_{\tilde{j}-1})$$

$$=\sum_{\tilde{j}=1}^{\tilde{n}}\left[\sum_{\tilde{i}=1}^{\tilde{m}}\eta_{\tilde{i},\,\tilde{j}}(E_{\tilde{i}}-F_{\tilde{i}})-\sum_{\tilde{i}=1}^{\tilde{m}}\eta_{\tilde{i},\,\tilde{j}}(E_{\tilde{i}-1}-F_{\tilde{i}-1})\right]$$

$$+\sum_{\tilde{i}=1}^{\tilde{m}}\left[\sum_{\tilde{j}=1}^{\tilde{n}}r_{\tilde{i},\,\tilde{j}}(G_{\tilde{j}}-H_{\tilde{j}})-\sum_{\tilde{j}=1}^{\tilde{n}}r_{\tilde{i},\,\tilde{j}}(G_{\tilde{j}-1}-H_{\tilde{j}-1})\right]$$

$$=\sum_{\tilde{j}=1}^{\tilde{n}}\left[\sum_{\tilde{i}=1}^{\tilde{m}-1}\eta_{\tilde{i},\,\tilde{j}}(E_{\tilde{i}}-F_{\tilde{i}})-\sum_{\tilde{i}=2}^{\tilde{m}}\eta_{\tilde{i},\,\tilde{j}}(E_{\tilde{i}-1}-F_{\tilde{i}-1})\right]$$

$$+\sum_{\tilde{i}=1}^{\tilde{m}}\left[\sum_{\tilde{j}=1}^{\tilde{n}-1}r_{\tilde{i},\,\tilde{j}}(G_{\tilde{j}}-H_{\tilde{j}})-\sum_{\tilde{j}=2}^{\tilde{n}}r_{\tilde{i},\,\tilde{j}}(G_{\tilde{j}-1}-H_{\tilde{j}-1})\right]$$

$$=\sum_{\tilde{j}=1}^{\tilde{n}}\left[\sum_{\tilde{i}=1}^{\tilde{m}-1}\eta_{\tilde{i},\,\tilde{j}}(E_{\tilde{i}}-F_{\tilde{i}})-\sum_{\tilde{i}=1}^{\tilde{m}-1}\eta_{\tilde{i}+1,\,\tilde{j}}(E_{\tilde{i}}-F_{\tilde{i}})\right]$$

$$+\sum_{\tilde{i}=1}^{\tilde{m}}\left[\sum_{\tilde{j}=1}^{\tilde{n}-1}r_{\tilde{i},\,\tilde{j}}(G_{\tilde{j}}-H_{\tilde{j}})-\sum_{\tilde{j}=1}^{\tilde{n}-1}r_{\tilde{i},\,\tilde{j}+1}(G_{\tilde{j}}-H_{\tilde{j}})\right]$$

$$= \sum_{\tilde{j}=1}^{\tilde{n}} \left[\sum_{\tilde{i}=1}^{\tilde{m}-1} (\eta_{\tilde{i},\tilde{j}} - \eta_{\tilde{i}+1,\tilde{j}})(E_{\tilde{i}} - F_{\tilde{i}}) \right]$$

$$+ \sum_{\tilde{i}=1}^{\tilde{m}} \left[\sum_{\tilde{j}=1}^{\tilde{n}-1} (r_{\tilde{i},\tilde{j}} - r_{\tilde{i},\tilde{j}+1})(G_{\tilde{j}} - H_{\tilde{j}}) \right]. \tag{30}$$

As φ is a p-harmonic convex function on $I_1 \times I_2$, it consequently serves as a coordinate convex function on $I_1 \times I_2$ as well. Thus, $\eta_{\tilde{i},\tilde{j}}$ decreases as $\tilde{i}$ increases for each fixed $\tilde{j}$, and $r_{\tilde{i},\tilde{j}}$ decreases as $\tilde{j}$ increases for each fixed $\tilde{i}$. Consequently, we have $\eta_{\tilde{i},\tilde{j}} - \eta_{\tilde{i}+1,\tilde{j}} \geq 0,\ \forall\, \tilde{i} \in \{1, 2, \ldots, \tilde{m}-1\}$, and $r_{\tilde{i},\tilde{j}} - r_{\tilde{i},\tilde{j}+1} \geq 0,\ \forall\, \tilde{j} \in \{1, 2, \ldots, \tilde{n}-1\}$. From the definition of majorization, we derive that $E_{\tilde{i}} - F_{\tilde{i}} \geq 0$ for every $\tilde{i} \in \{1, 2, \ldots, \tilde{m}-1\}$ and $G_{\tilde{j}} - H_{\tilde{j}} \geq 0$ for every $\tilde{j} \in \{1, 2, \ldots, \tilde{n}-1\}$. Thus, the right-hand side of inequality (30) is non-negative, which leads to the conclusion that

$$\sum_{\tilde{i}=1}^{\tilde{m}} \sum_{\tilde{j}=1}^{\tilde{n}} \gamma_{\tilde{i}}^p \alpha_{\tilde{j}}^p \mathscr{G}(\gamma_{\tilde{i}}, \alpha_{\tilde{j}}) - \sum_{\tilde{i}=1}^{\tilde{m}} \sum_{\tilde{j}=1}^{\tilde{n}} \delta_{\tilde{i}}^p \beta_{\tilde{j}}^p \mathscr{G}(\delta_{\tilde{i}}, \beta_{\tilde{j}}) \geq 0,$$

which is equivalent to inequality (29). □

The following theorem establishes a general inequality for the convex functions defined on rectangles. This result encompasses the majorization inequality for specific tuples.

Theorem 14 Let I_1 and I_2 be two intervals in $\mathbb{R}$, and consider two $\tilde{m}$-tuples $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_{\tilde{m}})$ and $\delta = (\delta_1, \delta_2, \ldots, \delta_{\tilde{m}})$ such that $\gamma_{\tilde{i}}, \delta_{\tilde{i}} \in I_1$ for all $\tilde{i} = 1, 2, \ldots, \tilde{m}$. Similarly, let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{\tilde{n}})$ and $\beta = (\beta_1, \beta_2, \ldots, \beta_{\tilde{n}})$ be two $\tilde{n}$-tuples satisfying $\alpha_{\tilde{j}}, \beta_{\tilde{j}} \in I_2$ for all $\tilde{j} = 1, 2, \ldots, \tilde{n}$. Additionally, let $v = (v_1, v_2, \ldots, v_{\tilde{m}})$ and $\acute{w} = (\acute{w}_1, \acute{w}_2, \ldots, \acute{w}_{\tilde{n}})$ be two positive real $\tilde{m}$- and $\tilde{n}$-tuples, respectively. If φ: $I_1 \times I_2 \to \mathbb{R}$ is a p-harmonic convex function, then the following inequality holds:

$$\sum_{\tilde{i}=1}^{\tilde{m}} \sum_{\tilde{j}=1}^{\tilde{n}} (\gamma_{\tilde{i}}\alpha_{\tilde{j}})^p v_{\tilde{i}} \acute{w}_{\tilde{j}} \mathscr{G}(\gamma_{\tilde{i}}, \alpha_{\tilde{j}}) - \sum_{\tilde{i}=1}^{\tilde{m}} \sum_{\tilde{j}=1}^{\tilde{n}} (\delta_{\tilde{i}}\beta_{\tilde{j}})^p v_{\tilde{i}} \acute{w}_{\tilde{j}} \mathscr{G}(\delta_{\tilde{i}}, \beta_{\tilde{j}})$$

$$\geq \sum_{\tilde{i}=1}^{\tilde{m}} \sum_{\tilde{j}=1}^{\tilde{n}} p\mathscr{G}(\delta_{\tilde{i}}, \beta_{\tilde{j}}) v_{\tilde{i}} \acute{w}_{\tilde{j}} \left\{ \delta_{\tilde{i}}^{p-1} \beta_{\tilde{j}}^p (\gamma_{\tilde{i}} - \delta_{\tilde{i}}) + \delta_{\tilde{i}}^p \beta_{\tilde{j}}^{p-1} (\alpha_{\tilde{i}} - \beta_{\tilde{j}}) \right\}$$

$$+ \sum_{\tilde{i}=1}^{\tilde{m}} \sum_{\tilde{j}=1}^{\tilde{n}} (\delta_{\tilde{i}}\beta_{\tilde{j}})^p v_{\tilde{i}} \acute{w}_{\tilde{j}} \left\{ \eta_{\tilde{i}}(\gamma_{\tilde{i}} - \delta_{\tilde{i}}) + r_{\tilde{j}}(\alpha_{\tilde{j}} - \beta_{\tilde{j}}) \right\} \tag{31}$$

Here, $\eta_{\tilde{i}}$ denotes the non-negative partial derivative of φ with respect to its first variable evaluated at $\delta_{\tilde{i}} (\tilde{i} = 1, 2, ..., \tilde{m})$. Likewise $r_{\tilde{j}}$ represents the non-negative partial derivative of φ with respect to its second variable evaluated at $\beta_{\tilde{j}}$ for $(\tilde{j} = 1, 2, ..., \tilde{n})$.

Proof. The convexity of p-harmonic function

$$\varphi(\phi, \psi) = (\phi\psi)^p \mathscr{G}(\phi, \psi): I_1 \times I_2 \to \mathbb{R}$$

implies the following inequality

$$\varphi(\phi,\,\psi)-\varphi(\acute{w},\,\mathfrak{z})\geq\langle\nabla\varphi(\acute{w},\,\mathfrak{z}),\,(\phi-\acute{w},\,\psi-\mathfrak{z})\rangle,$$

for all $(\phi,\,\psi),\,(\acute{w},\,\mathfrak{z})\in I_1\times I_2$. This expands to:

$$(\phi\psi)^p\mathscr{G}(\phi,\,\psi)-(\acute{w}\mathfrak{z})^p\mathscr{G}(\acute{w},\,\mathfrak{z})\geq p\mathscr{G}(\acute{w},\,\mathfrak{z})\left\{\acute{w}^{p-1}\mathfrak{z}^p(\phi-\acute{w})+\acute{w}^p\mathfrak{z}^{p-1}(\psi-\mathfrak{z})\right\}$$

$$+(\acute{w}\mathfrak{z})^p\left\{\frac{\partial\mathscr{G}}{\partial\acute{w}}(\acute{w},\,\mathfrak{z})(\phi-\acute{w})+\frac{\partial\mathscr{G}}{\partial\mathfrak{z}}(\acute{w},\,\mathfrak{z})(\psi-\mathfrak{z})\right\}. \tag{32}$$

Substituting $\phi\to\gamma_{\tilde{i}}$, $\psi\to\alpha_{\tilde{i}}$, $\acute{w}\to\delta_{\tilde{i}}$, and $\mathfrak{z}\to\beta_{\tilde{j}}$ in (32), we obtain

$$(\gamma_{\tilde{i}}\alpha_{\tilde{j}})^p\mathscr{G}(\gamma_{\tilde{i}},\,\alpha_{\tilde{j}})-(\delta_{\tilde{i}}\beta_{\tilde{j}})^p\mathscr{G}(\delta_{\tilde{i}},\,\beta_{\tilde{j}})\geq p\mathscr{G}(\delta_{\tilde{i}},\,\beta_{\tilde{j}})\left\{\delta_{\tilde{i}}^{p-1}\beta_{\tilde{j}}^{p}(\gamma_{\tilde{i}}-\delta_{\tilde{i}})+\delta_{\tilde{i}}^{p}\beta_{\tilde{j}}^{p-1}(\alpha_{\tilde{i}}-\beta_{\tilde{j}})\right\}$$

$$+(\delta_{\tilde{i}}\beta_{\tilde{j}})^p\left\{\eta_{\tilde{i}}(\gamma_{\tilde{i}}-\delta_{\tilde{i}})+r_{\tilde{j}}(\alpha_{\tilde{j}}-\beta_{\tilde{j}})\right\}. \tag{33}$$

Multiplying both sides of (33) by $v_{\tilde{i}}\acute{w}_{\tilde{j}}$ gives

$$\sum_{\tilde{i}=1}^{\tilde{m}}\sum_{\tilde{j}=1}^{\tilde{n}}(\gamma_{\tilde{i}}\alpha_{\tilde{j}})^p v_{\tilde{i}}\acute{w}_{\tilde{j}}\mathscr{G}(\gamma_{\tilde{i}},\,\alpha_{\tilde{j}})-\sum_{\tilde{i}=1}^{\tilde{m}}\sum_{\tilde{j}=1}^{\tilde{n}}(\delta_{\tilde{i}}\beta_{\tilde{j}})^p v_{\tilde{i}}\acute{w}_{\tilde{j}}\mathscr{G}(\delta_{\tilde{i}},\,\beta_{\tilde{j}})$$

$$\geq\sum_{\tilde{i}=1}^{\tilde{m}}\sum_{\tilde{j}=1}^{\tilde{n}}p\mathscr{G}(\delta_{\tilde{i}},\,\beta_{\tilde{j}})v_{\tilde{i}}\acute{w}_{\tilde{j}}\left[\delta_{\tilde{i}}^{p-1}\beta_{\tilde{j}}^{p}(\gamma_{\tilde{i}}-\delta_{\tilde{i}})+\delta_{\tilde{i}}^{p}\beta_{\tilde{j}}^{p-1}(\alpha_{\tilde{i}}-\beta_{\tilde{j}})\right]$$

$$+\sum_{\tilde{i}=1}^{\tilde{m}}\sum_{\tilde{j}=1}^{\tilde{n}}(\delta_{\tilde{i}}\beta_{\tilde{j}})^p v_{\tilde{i}}\acute{w}_{\tilde{j}}\left[\eta_{\tilde{i}}(\gamma_{\tilde{i}}-\delta_{\tilde{i}})+r_{\tilde{j}}(\alpha_{\tilde{j}}-\beta_{\tilde{j}})\right].$$

□

Theorem 15 Let I_1 and I_2 be two intervals in $\mathbb{R}$, and let φ: $I_1\times I_2\to\mathbb{R}$ be a p-harmonic convex function. Consider two $\tilde{m}$-tuples $\gamma=(\gamma_1,\,\gamma_2,\,\dots,\,\gamma_{\tilde{m}})$ and $\delta=(\delta_1,\,\delta_2,\,\dots,\,\delta_{\tilde{m}})$, where $\gamma_{\tilde{i}},\,\delta_{\tilde{i}}\in I_1$ for all $\tilde{i}=1,\,2,\,\dots,\,\tilde{m}$. Additionally, let $\alpha=(\alpha_1,\,\alpha_2,\,\dots,\,\alpha_{\tilde{n}})$ and $\beta=(\beta_1,\,\beta_2,\,\dots,\,\beta_{\tilde{n}})$ be two $\tilde{n}$-tuples with $\alpha_{\tilde{j}},\,\beta_{\tilde{j}}\in I_2$ for all $\tilde{j}=1,\,2,\,\dots,\,\tilde{n}$. Let $c=(c_1,\,c_2,\,\dots,\,c_{\tilde{m}})$ and $\acute{w}=(\acute{w}_1,\,\acute{w}_2,\,\dots,\,\acute{w}_{\tilde{n}})$ be tuples of real positive numbers. Consequently, the following inequality is true

$$\sum_{\tilde{i}=1}^{\tilde{l}}\delta_{\tilde{i}}v_{\tilde{i}}\leq\sum_{\tilde{i}=1}^{\tilde{l}}\gamma_{\tilde{i}}v_{\tilde{i}},\quad(\tilde{l}=1,\,2,\,\dots,\,\tilde{m}-1), \tag{34}$$

$$\sum_{\tilde{j}=1}^{\tilde{l}}\beta_{\tilde{j}}\acute{w}_{\tilde{j}}\leq\sum_{\tilde{j}=1}^{\tilde{l}}\alpha_{\tilde{j}}\acute{w}_{\tilde{j}},\quad(\tilde{l}=1,\,2,\,\dots,\,\tilde{n}-1), \tag{35}$$

$$\sum_{\tilde{i}=1}^{\tilde{m}} \delta_{\tilde{i}} v_{\tilde{i}} = \sum_{\tilde{i}=1}^{\tilde{m}} \gamma_{\tilde{i}} v_{\tilde{i}}, \tag{36}$$

$$\sum_{\tilde{j}=1}^{\tilde{n}} \beta_{\tilde{j}} \acute{w}_{\tilde{j}} = \sum_{\tilde{j}=1}^{\tilde{n}} \alpha_{\tilde{j}} \acute{w}_{\tilde{j}}. \tag{37}$$

The statements below are true:
(i) If δ and β are decreasing sequences, then

$$\sum_{\tilde{i}=1}^{\tilde{m}} \sum_{\tilde{j}=1}^{\tilde{n}} v_{\tilde{i}} \acute{w}_{\tilde{j}} (\delta_{\tilde{i}} \beta_{\tilde{j}})^p \mathscr{G}(\delta_{\tilde{i}}, \beta_{\tilde{j}}) \leq \sum_{\tilde{i}=1}^{\tilde{m}} \sum_{\tilde{j}=1}^{\tilde{n}} v_{\tilde{i}} \acute{w}_{\tilde{j}} (\gamma_{\tilde{i}} \alpha_{\tilde{j}})^p \mathscr{G}(\gamma_{\tilde{i}}, \alpha_{\tilde{j}}). \tag{38}$$

(ii) If γ and α are increasing sequences, then

$$\sum_{\tilde{i}=1}^{\tilde{m}} \sum_{\tilde{j}=1}^{\tilde{n}} v_{\tilde{i}} \acute{w}_{\tilde{j}} (\gamma_{\tilde{i}} \alpha_{\tilde{j}})^p \mathscr{G}(\gamma_{\tilde{i}}, \alpha_{\tilde{j}}) \leq \sum_{\tilde{i}=1}^{\tilde{m}} \sum_{\tilde{j}=1}^{\tilde{n}} v_{\tilde{i}} \acute{w}_{\tilde{j}} (\delta_{\tilde{i}} \beta_{\tilde{j}})^p \mathscr{G}(\delta_{\tilde{i}}, \beta_{\tilde{j}}). \tag{39}$$

Proof. To establish part (i), we apply Abel's transformation. Define the following summations:

$$E_0 = F_0 = 0, \quad E_l = \sum_{\tilde{i}=1}^{\tilde{l}} v_{\tilde{i}} \gamma_{\tilde{i}}, \quad F_k = \sum_{\tilde{i}=1}^{\tilde{l}} v_{\tilde{i}} \delta_{\tilde{i}}, \quad (\tilde{l} = 1, 2, \ldots, \tilde{m})$$

and

$$G_0 = H_0 = 0, \quad G_l = \sum_{\tilde{j}=1}^{\tilde{l}} \acute{w}_{\tilde{j}} \alpha_{\tilde{j}}, \quad H_l = \sum_{\tilde{j}=1}^{\tilde{l}} \acute{w}_{\tilde{j}} \beta_{\tilde{j}}, \quad (\tilde{l} = 1, 2, \ldots, \tilde{n}).$$

By utilizing inequalities (36) and (37), we derive

$$E_{\tilde{m}} = F_{\tilde{m}}, \quad G_{\tilde{n}} = H_{\tilde{n}}.$$

Since φ is a p-harmonic convex function on $I_1 \times I_2$, it follows that φ is also coordinate-wise p-harmonic convex on $I_1 \times I_2$. If δ and β are decreasing $\tilde{m}$- and $\tilde{n}$-tuples, then $(\eta_1, \eta_2, ..., \eta_{\tilde{m}})$ and $(r_1, r_2, ..., r_{\tilde{n}})$ are also decreasing $\tilde{m}$- and $\tilde{n}$-tuples, respectively. Here $\eta_{\tilde{i}}$ is the non-negative partial derivative of φ with respect to its first variable, evaluated at $\delta_{\tilde{i}}$ $(\tilde{i} = 1, 2, ..., \tilde{m})$ and $r_{\tilde{j}}$ is the non-negative partial derivative of φ with respect to its second variable, evaluated at $\beta_{\tilde{j}}$ $(\tilde{j} = 1, 2, ..., \tilde{n})$. Using inequality (31), we obtain

$$\sum_{\tilde{i}=1}^{\tilde{m}} \sum_{\tilde{j}=1}^{\tilde{n}} (\gamma_{\tilde{i}} \alpha_{\tilde{j}})^p v_{\tilde{i}} \acute{w}_{\tilde{j}} \mathscr{G}(\gamma_{\tilde{i}}, \alpha_{\tilde{j}}) - \sum_{\tilde{i}=1}^{\tilde{m}} \sum_{\tilde{j}=1}^{\tilde{n}} (\delta_{\tilde{i}} \beta_{\tilde{j}})^p v_{\tilde{i}} \acute{w}_{\tilde{j}} \mathscr{G}(\delta_{\tilde{i}}, {}_{\tilde{j}})$$

$$\geq \sum_{\tilde{i}=1}^{\tilde{m}} \sum_{\tilde{j}=1}^{\tilde{n}} p\mathscr{G}(\delta_{\tilde{i}}, \beta_{\tilde{j}}) v_{\tilde{i}} \acute{w}_{\tilde{j}} \left\{ \delta_{\tilde{i}}^{p-1} \beta_{\tilde{j}}^{p} (\gamma_{\tilde{i}} - \delta_{\tilde{i}}) + \delta_{\tilde{i}}^{p} \beta_{\tilde{j}}^{p-1} (\alpha_{\tilde{i}} - \beta_{\tilde{j}}) \right\}$$

$$+ \sum_{\tilde{i}=1}^{\tilde{m}} \sum_{\tilde{j}=1}^{\tilde{n}} (\delta_{\tilde{i}} \beta_{\tilde{j}})^{p} v_{\tilde{i}} \acute{w}_{\tilde{j}} \left\{ \eta_{\tilde{i}} (\gamma_{\tilde{i}} - \delta_{\tilde{i}}) + r_{\tilde{j}} (\alpha_{\tilde{j}} - \beta_{\tilde{j}}) \right\}$$

$$= \sum_{\tilde{i}=1}^{\tilde{m}} \sum_{\tilde{j}=1}^{\tilde{n}} p\mathscr{G}(\delta_{\tilde{i}}, \beta_{\tilde{j}}) v_{\tilde{i}} \acute{w}_{\tilde{j}} \left\{ \delta_{\tilde{i}}^{p-1} \beta_{\tilde{j}}^{p} (\gamma_{\tilde{i}} - \delta_{\tilde{i}}) \right\} + \sum_{\tilde{i}=1}^{\tilde{m}} \sum_{\tilde{j}=1}^{\tilde{n}} p\mathscr{G}(\delta_{\tilde{i}}, \beta_{\tilde{j}}) v_{\tilde{i}} \acute{w}_{\tilde{j}} \left\{ \delta_{\tilde{i}}^{p} \beta_{\tilde{j}}^{p-1} (\alpha_{\tilde{i}} - \beta_{\tilde{j}}) \right\}$$

$$+ \sum_{\tilde{i}=1}^{\tilde{m}} \sum_{\tilde{j}=1}^{\tilde{n}} (\delta_{\tilde{i}} \beta_{\tilde{j}})^{p} v_{\tilde{i}} \acute{w}_{\tilde{j}} \{ \eta_{\tilde{i}} (\gamma_{\tilde{i}} - \delta_{\tilde{i}}) \} + \sum_{\tilde{i}=1}^{\tilde{m}} \sum_{\tilde{j}=1}^{\tilde{n}} (\delta_{\tilde{i}} \beta_{\tilde{j}})^{p} v_{\tilde{i}} \acute{w}_{\tilde{j}} \left\{ r_{\tilde{j}} (\alpha_{\tilde{j}} - \beta_{\tilde{j}}) \right\}$$

$$= \sum_{\tilde{j}=1}^{\tilde{n}} p \acute{w}_{\tilde{j}} \beta_{\tilde{j}}^{p} \left[\sum_{\tilde{i}=1}^{\tilde{m}} \mathscr{G}(\delta_{\tilde{i}}, \beta_{\tilde{j}}) \left\{ \delta_{\tilde{i}}^{p-1} (v_{\tilde{i}} \gamma_{\tilde{i}} - v_{\tilde{i}} \delta_{\tilde{i}}) \right\} \right]$$

$$+ \sum_{\tilde{i}=1}^{\tilde{m}} p v_{\tilde{i}} \delta_{\tilde{i}}^{p} \left[\sum_{\tilde{j}=1}^{\tilde{n}} \mathscr{G}(\delta_{\tilde{i}}, \beta_{\tilde{j}}) \left\{ \beta_{\tilde{j}}^{p-1} (\acute{w}_{\tilde{j}} \alpha_{\tilde{i}} - \acute{w}_{\tilde{j}} \beta_{\tilde{j}}) \right\} \right]$$

$$+ \sum_{\tilde{j}=1}^{\tilde{n}} \beta_{\tilde{j}}^{p} \acute{w}_{\tilde{j}} \left[\sum_{\tilde{i}=1}^{\tilde{m}} \delta_{\tilde{i}}^{p} \eta_{\tilde{i}} (v_{\tilde{i}} \gamma_{\tilde{i}} - v_{\tilde{i}} \delta_{\tilde{i}}) \right] + \sum_{\tilde{i}=1}^{\tilde{m}} \delta_{\tilde{i}}^{p} v_{\tilde{i}} \left[\sum_{\tilde{j}=1}^{\tilde{n}} \beta_{\tilde{j}}^{p} r_{\tilde{j}} (\acute{w}_{\tilde{j}} \alpha_{\tilde{j}} - \acute{w}_{\tilde{j}} \beta_{\tilde{j}}) \right]$$

$$= \sum_{\tilde{j}=1}^{\tilde{n}} p \acute{w}_{\tilde{j}} \beta_{\tilde{j}}^{p} \left[\sum_{\tilde{i}+1}^{\tilde{m}} \mathscr{G}(\delta_{\tilde{i}}, \beta_{\tilde{j}}) \left\{ \delta_{\tilde{i}}^{p-1} (E_{\tilde{i}} - E_{\tilde{i}-1} - F_{\tilde{i}} + F_{\tilde{i}-1}) \right\} \right] + \sum_{\tilde{i}=1}^{\tilde{m}} p v_{\tilde{i}} \delta_{\tilde{i}}^{p}$$

$$\times \left[\sum_{\tilde{j}=1}^{\tilde{n}} \mathscr{G}(\delta_{\tilde{i}}, \beta_{\tilde{j}}) \left\{ \beta_{\tilde{j}}^{p-1} (G_{\tilde{i}} - G_{\tilde{i}-1} - F_{\tilde{i}} + F_{\tilde{i}-1}) \right\} \right]$$

$$+ \sum_{\tilde{j}=1}^{\tilde{n}} \beta_{\tilde{j}}^{p} \acute{w}_{\tilde{j}} \left[\sum_{\tilde{i}=1}^{\tilde{m}} \delta_{\tilde{i}}^{p} \eta_{\tilde{i}} (E_{\tilde{i}} - E_{\tilde{i}-1} - F_{\tilde{i}} + F_{\tilde{i}-1}) \right] + \sum_{\tilde{i}=1}^{\tilde{m}} \delta_{\tilde{i}}^{p} v_{\tilde{i}} \left[\sum_{\tilde{j}=1}^{\tilde{n}} \beta_{\tilde{j}}^{p} r_{\tilde{j}} (G_{\tilde{i}} - G_{\tilde{i}-1} - H_{\tilde{i}} + H_{\tilde{i}-1}) \right]$$

$$= \sum_{\tilde{j}=1}^{\tilde{n}} p \acute{w}_{\tilde{j}} \beta_{\tilde{j}}^{p} \left[\sum_{\tilde{i}=1}^{\tilde{m}} \mathscr{G}(\delta_{\tilde{i}}, \beta_{\tilde{j}}) \delta_{\tilde{i}}^{p-1} (E_{\tilde{i}} - F_{\tilde{i}}) - \sum_{\tilde{i}=1}^{\tilde{m}} \mathscr{G}(\delta_{\tilde{i}}, \beta_{\tilde{j}}) \delta_{\tilde{i}}^{p-1} (E_{\tilde{i}-1} - F_{\tilde{i}-1}) \right] + \sum_{\tilde{i}=1}^{\tilde{m}} p v_{\tilde{i}} \delta_{\tilde{i}}^{p}$$

$$\times \left[\sum_{\tilde{j}=1}^{\tilde{n}} \mathscr{G}(\delta_{\tilde{i}}, \beta_{\tilde{j}}) \beta_{\tilde{j}}^{p-1} (G_{\tilde{i}} - H_{\tilde{i}}) - \sum_{\tilde{j}=1}^{\tilde{n}} \beta_{\tilde{j}}^{p-1} (G_{\tilde{i}-1} - H_{\tilde{i}-1}) \right]$$

$$+\sum_{\tilde{j}=1}^{\tilde{n}}\beta_{\tilde{j}}^{p}\acute{w}_{\tilde{j}}\left[\sum_{\tilde{i}=1}^{\tilde{m}}\delta_{\tilde{i}}^{p}\eta_{\tilde{i}}(E_{\tilde{i}}-F_{\tilde{i}})-\sum_{\tilde{i}=1}^{\tilde{m}}\delta_{\tilde{i}}^{p}\eta_{\tilde{i}}(E_{\tilde{i}-1}-F_{\tilde{i}-1})\right]$$

$$+\sum_{\tilde{i}=1}^{\tilde{m}}\delta_{\tilde{i}}^{p}v_{\tilde{i}}\left[\sum_{\tilde{j}=1}^{\tilde{n}}\beta_{\tilde{j}}^{p}r_{\tilde{j}}(G_{\tilde{i}}-H_{\tilde{i}})-\sum_{\tilde{j}=1}^{\tilde{n}}\beta_{\tilde{j}}^{p}r_{\tilde{j}}(G_{\tilde{i}-1}-H_{\tilde{i}-1})\right]$$

$$=\sum_{\tilde{j}=1}^{\tilde{n}}p\acute{w}_{\tilde{j}}\beta_{\tilde{j}}^{p}\left[\sum_{\tilde{i}=1}^{\tilde{m}-1}\mathscr{G}(\delta_{\tilde{i}},\beta_{\tilde{j}})(\delta_{\tilde{i}}^{p-1}-\delta_{\tilde{i}+1}^{p-1})(E_{\tilde{i}}-F_{\tilde{i}})\right]$$

$$+\sum_{\tilde{i}=1}^{\tilde{m}}pv_{\tilde{i}}\delta_{\tilde{i}}^{p}\left[\sum_{\tilde{j}=1}^{\tilde{n}-1}\mathscr{G}(\delta_{\tilde{i}},\beta_{\tilde{j}})(\beta_{\tilde{j}}^{p-1}-\beta_{\tilde{j}+1})(G_{\tilde{i}}-H_{\tilde{i}})\right]$$

$$+\sum_{\tilde{j}=1}^{\tilde{n}}\beta_{\tilde{j}}^{p}\acute{w}_{\tilde{j}}\left[\sum_{\tilde{i}=1}^{\tilde{m}-1}\delta_{\tilde{i}}^{p}(\eta_{\tilde{i}}-\eta_{\tilde{i}+1})(E_{\tilde{i}}-F_{\tilde{i}})\right]+\sum_{\tilde{i}=1}^{\tilde{m}}\delta_{\tilde{i}}^{p}v_{\tilde{i}}\left[\sum_{\tilde{j}=1}^{\tilde{n}-1}\beta_{\tilde{j}}^{p}(r_{\tilde{j}}-r_{\tilde{j}+1})(G_{\tilde{i}}-H_{\tilde{i}})\right]. \tag{40}$$

Since $(\delta_1, \delta_2, \ldots, \delta_{\tilde{m}})$, $(\beta_1, \beta_2, \ldots, \beta_{\tilde{n}})$, $(\eta_1, \eta_2, \ldots, \eta_{\tilde{m}})$, and $(r_1, r_2, \ldots, r_{\tilde{n}})$ are decreasing $\tilde{m}$- and $\tilde{n}$-tuples, respectively, it follows that $(\delta_{\tilde{i}}^{p-1}-\delta_{\tilde{i}+1}^{p-1})\geq 0$ and $\eta_{\tilde{i}}-\eta_{\tilde{i}+1}\geq 0$ for $\tilde{i}=1, 2, \ldots, \tilde{m}-1$, and $(\beta_{\tilde{j}}^{p-1}-\beta_{\tilde{j}+1}^{p-1})\geq 0$ and $r_{\tilde{j}}-r_{\tilde{j}+1}\geq 0$ for $\tilde{j}=1, 2, \ldots, \tilde{n}-1$. Additionally, from the assumed inequalities (34) and (35), we know that $E_{\tilde{i}}-F_{\tilde{i}}\leq 0$ for $\tilde{i}=1, 2, \ldots, \tilde{m}-1$, and $G_{\tilde{j}}-H_{\tilde{j}}\leq 0$ for $\tilde{j}=1, 2, \ldots, \tilde{n}-1$. Thus, we obtain

$$\sum_{\tilde{j}=1}^{\tilde{n}}p\acute{w}_{\tilde{j}}\beta_{\tilde{j}}^{p}\left[\sum_{\tilde{i}=1}^{\tilde{m}-1}\mathscr{G}(\delta_{\tilde{i}},\beta_{\tilde{j}})(\delta_{\tilde{i}}^{p-1}-\delta_{\tilde{i}+1}^{p-1})(E_{\tilde{i}}-F_{\tilde{i}})\right]+\sum_{\tilde{i}=1}^{\tilde{m}}pv_{\tilde{i}}\delta_{\tilde{i}}^{p}\left[\sum_{\tilde{j}=1}^{\tilde{n}-1}\mathscr{G}(\delta_{\tilde{i}},\beta_{\tilde{j}})(\beta_{\tilde{j}}^{p-1}-\beta_{\tilde{j}+1}^{p-1})(G_{\tilde{i}}-H_{\tilde{i}})\right]$$

$$+\sum_{\tilde{j}=1}^{\tilde{n}}\beta_{\tilde{j}}^{p}\acute{w}_{\tilde{j}}\left[\sum_{\tilde{i}=1}^{\tilde{m}-1}\delta_{\tilde{i}}^{p}(\eta_{\tilde{i}}-\eta_{\tilde{i}+1})(E_{\tilde{i}}-F_{\tilde{i}})\right]+\sum_{\tilde{i}=1}^{\tilde{m}}\delta_{\tilde{i}}^{p}v_{\tilde{i}}\left[\sum_{\tilde{j}=1}^{\tilde{n}-1}\beta_{\tilde{j}}^{p}(r_{\tilde{j}}-r_{\tilde{j}+1})(G_{\tilde{i}}-H_{\tilde{i}})\right]\geq 0. \tag{41}$$

Substituting inequality (41) into (40), we obtain

$$\sum_{\tilde{i}=1}^{\tilde{m}}\sum_{\tilde{j}=1}^{\tilde{n}}v_{\tilde{i}}\acute{w}_{\tilde{j}}(\gamma_{\tilde{i}}\alpha_{\tilde{j}})^{p}\mathscr{G}(\gamma_{\tilde{i}},\alpha_{\tilde{j}})-\sum_{\tilde{i}=1}^{\tilde{m}}\sum_{\tilde{j}=1}^{\tilde{n}}v_{\tilde{i}}\acute{w}_{\tilde{j}}(\delta_{\tilde{i}}\beta_{\tilde{j}})^{p}\mathscr{G}(\delta_{\tilde{i}},\beta_{\tilde{j}})\geq 0.$$

This is equivalent to inequality (38). Similarly, the remaining cases can be proved. □

4. Discussion of results

The numerical comparisons reported in Table 1 confirm the validity of Corollary 1 for different values of the parameter p, where the majorization inequality holds in all cases and becomes stronger as p increases. This clearly supports the theoretical predictions derived for p-harmonic convex functions.

The graphical illustration in Figure 1 further reinforces these findings by visually demonstrating the separation between the cumulative sums of the majorizing and majorized sequences, providing an intuitive verification of the analytical result.

Similarly, Table 2 validates Corollary 2, showing that the inequality is strict for $p \geq 1$, except for the trivial case. The corresponding plots in Figure 2 highlight this behavior and emphasize the strengthening effect of increasing p.

Finally, the results presented in Table 3 and Figure 3 confirm Corollary 3 for the multiplicative form of the inequality, illustrating a pronounced dominance of the majorizing sequence and demonstrating the consistency and robustness of the proposed framework.

5. Conclusion

We introduced refinement of Jensen's integral inequalities. Further, we generalized Jensen's Mercer type inequalities for a class which is not harmonic convex and next we gave results for functions of this new class which is p-harmonic convex functions. Additionally, we generalized results for Majorization inequalities and produced a new variety of discrete Jensen-type results. Our methods and findings are novel in mathematics, particularly for the class of p-harmonic convex functions, and we anticipate that they will serve as inspiration for future studies. The results we have obtained represent improvements and generalization of several previously established results.

Author contributions

F.A.: Methodology, Software, Validation, Investigation, Writing-original draft preparation, Visualization. D.B.: Conceptualization, Validation, Investigation, Funding acquisition, Writing-original draft preparation. M.I.A.: Methodology, Writing-original draft preparation, Supervision. M.A.Y.: Methodology, Formal analysis, Validation, Visualization, Writing-review and editing. I.A.B.: Conceptualization, Investigation, Resources, Visualization, Writing-review and editing. P.O.M.: Conceptualization, Formal analysis, Supervision, Writing-original draft preparation. All authors read and approved the manuscript.

Conflicts of interest

The authors declare no competing financial interest.

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