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Research Article

Properties of Tangent-Fibonacci Polynomials via Golden F-Calculus

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Abstract: In this paper, we introduce and systematically study a new class of two-variable Tangent-Fibonacci polynomials together with their corresponding numbers within the framework of Golden F-Calculus. By employing suitable generating functions, we establish several fundamental properties of these polynomials, including summation formulas, recurrence relations, symmetry identities, and F-derivative representations. We further explore their structural connections with the Stirling-Fibonacci numbers of the second kind and derive various convolution-type identities and explicit summation expressions. In addition, we propose new parametric extensions characterized by trigonometric-type generating functions, and investigate their analytical behavior using the F-differential operator and functional equation methods. Moreover, we obtain an explicit matrix representation that clarifies the relationship between the associated polynomial matrix and a generalized Pascal matrix via Fibonomial coefficients of the first kind. The developed framework not only extends the theory of Fibonacci-based special polynomials in a coherent and unified manner, but also provides a foundation for further applications in combinatorics, number theory, approximation theory, and matrix analysis.

Keywords: Golden calculus, Tangent polynomials, Tangent-Fibonacci polynomials, Stirling-Fibonacci numbers of the second kind

MSC: 11B39, 15A16, 68T07

1. Introduction

Special polynomials and numbers exhibit a profound and highly structured nature that has captivated mathematicians for centuries. Their importance is not confined to theoretical investigations; rather, they play a significant role in physics, engineering, and various other scientific fields. Over time, numerous analytical techniques have been developed to examine their structural, algebraic, and functional properties [1–10]. Among these methodologies, approaches rooted in Fibonacci theory have proven particularly effective in revealing deeper structural patterns and generating new classes of polynomial families.

Generating functions constitute one of the most powerful tools in the study of integer sequences and special polynomials. They offer a coherent and unified framework that enables infinite sequences to be analyzed through algebraic and analytic methods [11–17]. The introduction of additional parameters and multiple variables into generating functions has further

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expanded their versatility, making it possible to investigate more sophisticated combinatorial structures and a wider spectrum of polynomial families [18–22].

For $w \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, the Stirling numbers of the first kind are defined by [18]

$$(x)_w = \sum_{k=0}^{w} S_1(w,\, k) x^k, \tag{1}$$

where $(x)_0 = 1$, and $(x)_w = x(x-1)\cdots(x-w+1)$ for $(w \geq 1)$. From (1), we obtain

$$\frac{1}{r!}(\log(1+t))^r = \sum_{w=r}^{\infty} S_1(w,\, r)\frac{t^w}{w!}, \qquad (r \geq 0).$$

For $j \in \mathbb{N}_0$, the Stirling numbers of the second kind are defined by [23]

$$x^w = \sum_{q=0}^{w} S_2(w,\, q)(x)_q. \tag{2}$$

From (2), it follows that

$$\frac{1}{k!}(e^t - 1)^k = \sum_{w=k}^{\infty} S_2(w,\, k)\frac{t^w}{w!}. \tag{3}$$

For any nonnegative integer r, the r-Stirling numbers $S_{2,\, r}(j,\, k)$ of the second kind are defined by [24]

$$\frac{1}{k!}e^{rt}(e^t - 1)^k = \sum_{w=k}^{\infty} S_{2,\, r}(w+r,\, k+r)\frac{t^w}{w!}.$$

On the other hand, the tangent function is given by

$$s(t) = \frac{2}{e^{2t}+1},$$

where $t \in \mathbb{C}$. The tangent function has been extensively studied due to its wide range of applications [25]. In particular, multi-parameter extensions of the tangent function have been employed in modeling hybrid tangent networks. Considerable attention has also been devoted to investigating the zeros of tangent polynomials, their generalized versions within different analytical frameworks, and their connections with other well-known polynomial families. The generating function of the tangent polynomials is defined by [26, 27]

$$\frac{2}{e^{2t}+1}e^{xt} = \sum_{w=0}^{\infty} \mathbb{T}_w(x)\frac{t^w}{w!}. \tag{4}$$

For $x = 0$, the values $\mathbb{T}_w(0)$ reduce to the tangent numbers. In [23], Khan and Kızılateş introduced Frobenius-Sigmoid polynomials as a generalized form of Sigmoid polynomials and investigated several of their properties through generating-function techniques.

We now recall several basic definitions related to Golden Calculus, also known as F-calculus. The Fibonacci sequence is defined recursively by

$$F_w = F_{w-1} + F_{w-2}, \qquad w \geq 2,$$

with initial values $F_0 = 0$ and $F_1 = 1$. An explicit formula for Fibonacci numbers is

$$F_w = \frac{\alpha^w - \beta^w}{\alpha - \beta}, \qquad w \in \mathbb{N}_0,$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ (the Golden ratio) and $\beta = \frac{1-\sqrt{5}}{2}$. The Golden ratio appears naturally in many areas of mathematics, science, architecture, and art. The formal development of F-calculus was extensively studied by Pashaev and Nalci [28]. For further background, we refer the reader to Krot [29], Özvatan [30], Pashaev [31], and Kuş et al. [32].

For $w \in \mathbb{N}$, the F-factorial derived from Fibonacci numbers is defined as

$$F_1 F_2 F_3 \ldots F_w = F_w!,$$

with $F_0! = 1$.

The Golden binomial theorem (or F-binomial theorem) is expressed as

$$(a+b)_F^w = \sum_{s=0}^{w} (-1)^{\binom{s}{2}} \binom{w}{s}_F a^{w-s} b^s, \tag{5}$$

or equivalently,

$$(a +_F b)^w = \sum_{s=0}^{w} \binom{w}{s}_F a^{w-s} b^s, \tag{6}$$

where the Golden binomial coefficients (Fibonomials) are defined by

$$\binom{u}{s}_F = \frac{F_u!}{F_{u-s}! F_s!},$$

for nonnegative integers $u \geq s$.

The Golden derivative (F-derivative) is defined by

$$\frac{d_F}{d_F x}(g(x)) = \frac{g(\alpha x) - g(\beta x)}{(\alpha - \beta)x}. \tag{7}$$

The first and second types of Golden exponential functions are given by

$$e_F^x = \sum_{w=0}^{\infty} \frac{(x)_F^w}{F_w!}, \tag{8}$$

and

$$E_F^x = \sum_{w=0}^{\infty} (-1)^{\binom{w}{2}} \frac{(x)_F^w}{F_w!}, \tag{9}$$

respectively. For simplicity, we use the equivalent notations

$$e_F^x = \sum_{w=0}^{\infty} \frac{x^w}{F_w!}, \tag{10}$$

and

$$E_F^x = \sum_{w=0}^{\infty} (-1)^{\binom{w}{2}} \frac{x^w}{F_w!}. \tag{11}$$

These functions satisfy the identities [28, 29]

$$e_F^x e_F^y = e_F^{(x +_F y)}, \tag{12}$$

$$e_F^x E_F^y = e_F^{(x+y)_F}. \tag{13}$$

The Golden trigonometric functions are defined by

$$\cos_F(x) = \sum_{w=0}^{\infty} (-1)^w \frac{x^{2w}}{F_{2w}!}, \tag{14}$$

and

$$\sin_F(x) = \sum_{w=0}^{\infty} (-1)^w \frac{x^{2w+1}}{F_{2w+1}!}. \tag{15}$$

For an arbitrary constant ϕ, their F-derivatives are given by

$$\frac{d_F}{d_F x}\left(e_F^{\phi x}\right) = \phi e_F^{\phi x}, \tag{16}$$

$$\frac{d_F}{d_F x}\left(E_F^{\phi x}\right) = \phi E_F^{-\phi x}, \tag{17}$$

$$\frac{d_F}{d_F x}\left(\cos_F(\phi x)\right) = -\phi \sin_F(\phi x), \tag{18}$$

and

$$\frac{d_F}{d_F x}\left(\sin_F(\phi x)\right) = \phi \cos_F(\phi x). \tag{19}$$

The remainder of this paper is organized as follows. Section 2 introduces a new family of two-variable polynomials defined by Eq. (20) within the framework of Golden Calculus. In this section, we develop the concept of two-variable Tangent-Fibonacci polynomials and their associated numbers, establishing key properties such as recurrence relations, summation formulas, and derivative identities derived from generating functions and functional equations. Section 3 extends these results to parametric bivariate forms of Tangent-Fibonacci polynomials, where additional identities are obtained using F-calculus techniques. Finally, Section 4 introduces a new Tangent-Fibonacci matrix and presents a factorization of this matrix, highlighting its structural properties.

2. Two variable Tangent-Fibonacci polynomials and their properties

In this section, we introduce two-variables extension of Tangent-Fibonacci polynomials utilizing the Golden Calculus. Additionally, we derive several relations for these polynomials through various identities. We commence with the following definition:

Definition 1 Let w be a non-negative integer. The two-variable Tangent-Fibonacci polynomials $\mathbb{T}_{w,F}(x, y)$ are defined by the following generating function:

$$\frac{2}{e_F^{2t}+1} e_F^{xt} E_F^{yt} = \sum_{w=0}^{\infty} \mathbb{T}_{w,F}(x, y) \frac{t^w}{F_w!}. \tag{20}$$

• If we take $y = 0$ in (20), $\mathbb{T}_{w,F}(x, 0)$ becomes the Tangent-Fibonacci polynomials.

• If we take $x = y = 0$ in (20), $\mathbb{T}_{w,F} = \mathbb{T}_{w,F}(0, 0)$ are called the w^{th} Tangent-Fibonacci numbers.

To derive structural properties of these polynomials, we now prove a series of summation, recurrence, and convolution-type identities.

Theorem 1 The following summation formulas for the polynomials $\mathbb{T}_{w,F}(x, y)$ hold true:

$$\mathbb{T}_{w,F}(x, y) = \sum_{j=0}^{w} \binom{w}{j}_F \mathbb{T}_{j,F}(x+y)_F^{w-j}, \tag{21}$$

$$\mathbb{T}_{w,F}(x,y)=\sum_{j=0}^{w}\binom{w}{j}_F\mathbb{T}_{j,F}(0,y)x^{w-j}, \tag{22}$$

$$\mathbb{T}_{w,F}(x,y)=\sum_{j=0}^{w}\binom{w}{j}_F\mathbb{T}_{w-j,F}(x)(-1)^{\frac{j(j-1)}{2}}y^{j}. \tag{23}$$

Proof. By appropriately applying Eqs. (6) and (11) within the generating function (20), three distinct forms are derived. Subsequently, the F-Cauchy product rule is utilized in the resulting expressions, and by comparing the corresponding powers of t on both sides of the resultant equation, we obtain formulas (21)–(23). ☐

Theorem 2 The following recursive formulas for the two variable Tangent-Fibonacci polynomials $\mathbb{T}_{w,F}(x,y)$ hold true:

$$\frac{\partial_F}{\partial_F x}\{\mathbb{T}_{w,F}(x,y)\}=F_w\mathbb{T}_{w-1,F}(x,y), \tag{24}$$

$$\frac{\partial_F}{\partial_F y}\{\mathbb{T}_{w,F}(x,y)\}=F_w\mathbb{T}_{w-1,F}(x,-y), \tag{25}$$

where $\frac{\partial_F}{\partial_F x}$ is F-partial derivative operator.

Proof. Differentiating generating function (20) with respect to x and y, utilizing Eqs. (16) and (17), and subsequently simplifying through the application of the F-Cauchy product rule formulas yields the results given in (24) and (25). ☐

The subsequent definition delineates the r-Stirling-Fibonacci numbers of the second kind [18]

$$\sum_{k=m}^{\infty}S_{2,r}^{F}(k+r,m+r)\frac{t^k}{F_k!}=e_F^{rt}\frac{(e_F^t-1)^m}{F_m!}. \tag{26}$$

For $r=0$, we have Stirling-Fibonacci numbers of the second kind

$$\sum_{k=m}^{\infty}S_2^F(k,m)\frac{t^k}{F_k!}=\frac{(e_F^t-1)^m}{F_m!}. \tag{27}$$

Theorem 3 Let $w\geq 0$ be an integer, then we have

$$\mathbb{T}_{w,F}(x+_F r,y)=\sum_{k=0}^{w}\sum_{m=0}^{k}\binom{w}{k}_F\mathbb{T}_{w-k,F}(y)(x)_{m,F}\,S_{2,r}^{F}(k+r,m+r), \tag{28}$$

where

$$(x)_{m,F}=F_xF_{x-1}F_{x-2}\cdots F_{x-m+1}.$$

Proof. By virtue of (1), (20), and (28), we have

$$\sum_{w=0}^{\infty}\mathbb{T}_{w,F}(x+_F r,y)\frac{t^w}{F_w!}=\frac{2}{e^{2t}+1}e_F^{rt}E_F^{yt}(e_F^t-1+1)^x$$

$$=\frac{2}{e_F^{2t}+1}e_F^{rt}E_F^{yt}\sum_{m=0}^{\infty}(x)_{m,F}\frac{(e_F^t-1)^m}{F_m!}$$

$$=\frac{1}{e_F^{2t}+1}E_F^{yt}\sum_{m=0}^{\infty}(x)_{m,F}\sum_{k=m}^{\infty}S_{2,r}^F(k+r,m+r)\frac{t^k}{F_k!}$$

$$=\sum_{w=0}^{\infty}\mathbb{T}_{w,F}(y)\frac{t^w}{F_w!}\sum_{k=0}^{\infty}\sum_{m=0}^{k}(x)_{m,F}\,S_{2,r}^F(k+r,m+r)\frac{t^k}{F_k!}$$

$$=\sum_{w=0}^{\infty}\left(\sum_{k=0}^{w}\sum_{m=0}^{k}\binom{w}{k}_F\mathbb{T}_{w-k,F}(y)(x)_{m,F}\,S_{2,r}^F(k+r,m+r)\right)\frac{t^w}{F_w!}.$$

To derive the result, we equate the coefficients of $\frac{t^w}{F_w!}$ on both sides of the last equation. □

Corollary 1 Let $w\geq 0$ be an integer, then we have

$$\mathbb{T}_{w,F}(x,y)=\sum_{k=0}^{w}\sum_{m=0}^{k}\binom{w}{k}\mathbb{T}_{w-k,F}(y)(x)_{m,F}\,S_2^F(k,m). \tag{29}$$

Proof. By setting $r=0$ in Eq. (28), the resultant outcome is obtained. □

Theorem 4 Let $w\geq 0$ be an integer, then

$$\sum_{k=0}^{w}\binom{w}{k}_F\mathbb{T}_{w-k,F}(-1)^k+\mathbb{T}_{w,F}=\begin{cases}2, & \text{if } w=0,\\ 0, & \text{if } w>0.\end{cases}$$

Proof. If $t\neq 0$ in the generating function of the Tangent-Fibonacci number, then we find

$$\sum_{w=0}^{\infty}\mathbb{T}_{w,F}\frac{t^w}{F_w!}\left(\sum_{k=0}^{\infty}2^k\frac{t^k}{F_k!}+1\right)=2,$$

$$\sum_{w=0}^{\infty}\left(\sum_{k=0}^{w}\binom{w}{k}_F\mathbb{T}_{w-k,F}2^k+\mathbb{T}_{w,F}\right)\frac{t^w}{F_w!}=2.$$

Hence, complete proof of the theorem. □

Theorem 5 Let $w\geq 0$ be an integer, then

$$\sum_{k=0}^{w}\binom{w}{k}_{F}\alpha^{w-k}\beta^{k}\mathbb{T}_{w-k,F}(\alpha^{-1}x,y)\mathbb{T}_{k,F}(\beta^{-1}x)=\sum_{k=0}^{w}\binom{w}{k}_{F}\beta^{w-k}\alpha^{k}\mathbb{T}_{w-k,F}(\beta^{-1}x)\mathbb{T}_{k,F}(\alpha^{-1}x,y).$$

Proof. Let

$$A=\frac{4}{(e_F^{2\alpha t}+1)(e_F^{2\beta t}+1)}e_F^{(x+_F\xi)t}E_F^{(\alpha y)t}.$$

Then, we have

$$A=\sum_{w=0}^{\infty}\mathbb{T}_{w,F}(\alpha^{-1}x,y)\frac{(\alpha t)^w}{F_w!}\sum_{k=0}^{\infty}\mathbb{T}_{k,F}(\beta^{-1}x)\frac{(\beta t)^k}{F_k!}$$

$$=\sum_{w=0}^{\infty}\left(\sum_{k=0}^{w}\binom{w}{k}_{F}\alpha^{w-k}\beta^{k}\mathbb{T}_{w-k,F}(\alpha^{-1}x,y)\mathbb{T}_{k,F}(\beta^{-1}x)\right)\frac{t^w}{F_w!}. \tag{30}$$

Similarly, we have

$$A=\sum_{w=0}^{\infty}\mathbb{T}_{w,F}(\beta^{-1}x)\frac{(\beta t)^w}{F_w!}\sum_{k=0}^{\infty}\mathbb{T}_{k,F}(\alpha^{-1}x,y)\frac{(\alpha t)^k}{F_k!}$$

$$=\sum_{w=0}^{\infty}\left(\sum_{k=0}^{w}\binom{w}{k}_{F}\beta^{w-k}\alpha^{k}\mathbb{T}_{w-k,F}(\beta^{-1}x)\mathbb{T}_{k,F}(\alpha^{-1}x,y)\right)\frac{t^w}{F_w!}. \tag{31}$$

Comparing the coefficients of both Eqs. (30) and (31), we obtain the result. □

Corollary 2 Letting $y=0$ in Theorem 5, we have

$$\sum_{k=0}^{w}\binom{w}{k}_{F}\alpha^{w-k}\beta^{k}\mathbb{T}_{w-k,F}(\alpha^{-1}x)\mathbb{T}_{k,F}(\beta^{-1}x)=\sum_{k=0}^{w}\binom{w}{k}_{F}\beta^{w-k}\alpha^{k}\mathbb{T}_{w-k,F}(\beta^{-1}x)\mathbb{T}_{k,F}(\alpha^{-1}x).$$

Theorem 6 Let $w\geq 0$ be an integer, then

$$\mathbb{T}_{w,F}(x,y)=2(x+y)_F^w-\sum_{j=0}^{w}2^j\binom{w}{j}_{F}\mathbb{T}_{w-j,F}(x,y).$$

Proof. Consider the following identity

$$\frac{1}{(e_F^{2t}+1)e_F^{2t}}=\frac{1}{e_F^{2t}+1}-\frac{1}{e_F^{2t}}.$$

Evaluating the following fraction using above identity, we find that

$$\frac{2e_F^{xt}E_F^{yt}}{(e_F^{2t}+1)e_F^{2t}}=\frac{2e_F^{xt}E_F^{yt}}{e_F^{2t}+1}-\frac{2e_F^{xt}E_F^{yt}}{e_F^{2t}}$$

$$\sum_{w=0}^{\infty}\mathbb{T}_{w,F}(x,y)\frac{t^w}{F_w!}=\sum_{j=0}^{\infty}2^j\frac{t^j}{F_j!}\sum_{w=0}^{\infty}\mathbb{T}_{w,F}(x,y)\frac{t^w}{F_w!}-\sum_{w=0}^{\infty}(x+y)_F^w\frac{t^w}{F_w!}$$

$$\sum_{w=0}^{\infty}2\left((x+y)_F^w-\mathbb{T}_{w,F}(x,y)\right)\frac{t^w}{F_w!}=\sum_{w=0}^{\infty}\left(\sum_{j=0}^{w}2^j\binom{w}{j}_F\mathbb{T}_{w-j,F}(x,y)\right)\frac{t^w}{F_w!}.$$

To obtain the result, we equate the coefficients of $\frac{t^w}{F_w!}$ on both sides of the equation. ☐

3. Parametric type extension of the Tangent-Fibonacci polynomials

This section extends the two-variable Tangent-Fibonacci polynomials by integrating parametric trigonometric elements through the application of the golden cosine and sine functions. Let $x, y \in \mathbb{R}$. The Taylor series of the functions $e_F^{xt}\cos_F(yt)$ and $e_F^{xt}\sin_F(yt)$ are given as follows [11]:

$$e_F^{xt}\cos_F(yt)=\sum_{w=0}^{\infty}\mathscr{C}_{w,F}(x,y)\frac{t^w}{F_w!}, \tag{32}$$

and

$$e_F^{xt}\sin_F(yt)=\sum_{w=0}^{\infty}\mathscr{S}_{w,F}(x,y)\frac{t^w}{F_w!}, \tag{33}$$

where

$$\mathscr{C}_{w,F}(x,y)=\sum_{k=0}^{\left\lfloor\frac{w}{2}\right\rfloor}(-1)^k\binom{w}{2k}_F x^{w-2k}y^{2k}, \tag{34}$$

$$\mathscr{S}_{w,F}(x,y)=\sum_{k=0}^{\left\lfloor\frac{w-1}{2}\right\rfloor}(-1)^k\binom{w}{2k+1}_F x^{w-2k-1}y^{2k+1}. \tag{35}$$

Based on the aforementioned definitions of $\mathscr{C}_{w,F}(x,y)$ and $\mathscr{S}_{w,F}(x,y)$, as well as the number of a $\mathbb{T}_{w,F}$, we can specify two-type parameters for Tangent-Fibonacci polynomials as follows:

Definition 2 Two parametric families of the Tangent-Fibonacci polynomials are hereby defined via the following generating functions:

$$\frac{2}{e_F^{2t}+1}e_F^{xt}\cos_F(yt)=\sum_{w=0}^{\infty}\mathbb{T}^{(c)}_{w,F}(x,y)\frac{t^w}{F_w!}, \tag{36}$$

and

$$\frac{2}{e_F^{2t}+1}e_F^{xt}\sin_F(yt)=\sum_{w=0}^{\infty}\mathbb{T}^{(s)}_{w,F}(x,y)\frac{t^w}{F_w!}. \tag{37}$$

Based on the aforementioned definitions, we have arrived at the following principal results.

Theorem 7 The following identities hold true:

$$\mathbb{T}^{(c)}_{w,F}(x,y)=\sum_{l=0}^{w}\binom{w}{l}_F\mathbb{T}_{l,F}\,\mathscr{C}_{w-l,F}(x,y), \tag{38}$$

and

$$\mathbb{T}^{(s)}_{w,F}(x,y)=\sum_{l=0}^{w}\binom{w}{l}_F\mathbb{T}_{l,F}\,\mathscr{S}_{w-l,F}(x,y). \tag{39}$$

Proof. Using (32) and (36), we have

$$\frac{2}{e_F^{2t}+1}e_F^{xt}\cos_F(yt)=\sum_{l=0}^{\infty}\mathbb{T}_{l,F}\frac{t^l}{F_l!}\sum_{w=0}^{\infty}C_{w,F}(x,y)\frac{t^w}{F_w!}$$

$$=\sum_{w=0}^{\infty}\left(\sum_{l=0}^{w}\binom{w}{l}_F\mathbb{T}_{l,F}\,C_{w-l,F}(x,y)\right)\frac{t^w}{F_w!}$$

$$\sum_{w=0}^{\infty}\mathbb{T}^{(c)}_{w,F}(x,y)\frac{t^w}{F_w!}=\sum_{n=0}^{\infty}\left(\sum_{l=0}^{w}\binom{w}{l}_F\mathbb{T}_{l,F}\,C_{w-l,F}(x,y)\right)\frac{t^w}{F_w!}.$$

By equating the coefficients of $\frac{t^w}{F_w!}$ on both sides of the preceding equation, we derive the intended result (38). Eq. (39) can be similarly obtained. □

Theorem 8 The following identities hold true:

$$\mathbb{T}^{(c)}_{w,F}(x+\theta,y)=\sum_{l=0}^{w}(-1)^{\binom{l}{2}}\binom{w}{l}_F\mathbb{T}^{(c)}_{w-l,F}(x,y)\,\theta^l, \tag{40}$$

and

$$\mathbb{T}^{(s)}_{w,\,F}(x+\theta,\,y)=\sum_{l=0}^{w}(-1)^{\binom{l}{2}}\binom{w}{l}_{F}\mathbb{T}^{(s)}_{w-l,\,F}(x,\,y)\,\theta^{l}. \tag{41}$$

Proof. Utilizing (36), we obtain the following functional equation through derivation:

$$\begin{aligned}\sum_{w=0}^{\infty}\mathbb{T}^{(c)}_{w,\,F}(x+\theta,\,y)\frac{t^{w}}{F_{w}!}&=\frac{2}{e_{F}^{2t}+1}e_{F}^{xt}E_{F}^{\theta t}\cos_{F}yt\\ &=\sum_{w=0}^{\infty}\mathbb{T}^{(c)}_{w,\,F}(x,\,y)\frac{t^{w}}{F_{w}!}\sum_{l=0}^{\infty}(-1)^{\binom{l}{2}}\,\theta^{l}\frac{t^{l}}{F_{l}!}\\ &=\sum_{w=0}^{\infty}\left(\sum_{l=0}^{w}(-1)^{\binom{l}{2}}\binom{w}{l}_{F}\mathbb{T}^{(c)}_{w-l,\,F}(x,\,y)\theta^{l}\right)\frac{t^{w}}{F_{w}!}.\end{aligned}$$

By equating the coefficients of t^{w} on both sides of the previously mentioned equation, it becomes evident that

$$\mathbb{T}^{(c)}_{w,\,F}(x+\theta,\,y)=\sum_{l=0}^{w}\binom{w}{l}_{F}(-1)^{\binom{l}{2}}\mathbb{T}^{(c)}_{w-l,\,F}(x,\,y)\theta^{l}.$$

The proof of the result (40) is detailed herein. The assertion (41) can be verified through a comparable method. □

Theorem 9 (F-Derivative Formulas) The following identities hold:

$$\frac{\partial_{F}}{\partial_{F}x}\left(\mathbb{T}^{(c)}_{w,\,F}(x,\,y)\right)=F_{w}\mathbb{T}^{(c)}_{w-1,\,F}(x,\,y), \tag{42}$$

$$\frac{\partial_{F}}{\partial_{F}x}\left(\mathbb{T}^{(s)}_{w,\,F}(x,\,y)\right)=F_{w}\mathbb{T}^{(s)}_{w-1,\,F}(x,\,y), \tag{43}$$

$$\frac{\partial_{F}}{\partial_{F}y}\left(\mathbb{T}^{(c)}_{w,\,F}(x,\,y)\right)=-F_{w}\mathbb{T}^{(s)}_{w-1,\,F}(x,\,y), \tag{44}$$

$$\frac{\partial_{F}}{\partial_{F}y}\left(\mathbb{T}^{(s)}_{w,\,F}(x,\,y)\right)=F_{w}\mathbb{T}^{(c)}_{w-1,\,F}(x,\,y). \tag{45}$$

Proof. Utilising the golden partial derivative operator $\dfrac{\partial_{F}}{\partial_{F}x}$ and using (36) and (18), we have

$$\sum_{w=0}^{\infty}\frac{\partial_{F}}{\partial_{F}x}\left(\mathbb{T}^{(c)}_{w,\,F}(x,\,y)\right)\frac{t^{w}}{F_{w}!}=\frac{\partial_{F}}{\partial_{F}x}\left\{\frac{2}{e_{F}^{2t}+1}e_{F}^{xt}\cos_{F}(yt)\right\}$$

$$= \sum_{w=0}^{\infty} \mathbb{T}^{(c)}_{w,F}(x, y)\frac{t^{w+1}}{F_w!}$$

$$= \sum_{w=1}^{\infty} \mathbb{T}^{(c)}_{w-1,F}(x, y)\frac{t^{w}}{F_{w-1}!}.$$

Since $\frac{\partial_F}{\partial_F x}\left(\mathbb{T}^{(c)}_{0,F}(x, y)\right) = 0$; for $w \geq 1$, by analyzing the coefficients of $\frac{t^w}{F_w!}$ on both sides of the final equation, the desired result is achieved (42). The remaining results can then be easily deduced. □

Theorem 10 The following identities hold true:

$$\mathscr{C}_{w,F}(x, y) = \frac{1}{2}\left[\sum_{k=0}^{w} 2^k \binom{w}{k}_F \mathbb{T}^{(c)}_{w-k,F}(x, y) + \mathbb{T}^{(c)}_{w,F}(x, y)\right], \tag{46}$$

and

$$\mathscr{T}_{w,F}(x, y) = \frac{1}{2}\left[\sum_{k=0}^{w} 2^k \binom{w}{k}_F \mathbb{T}^{(s)}_{w-k,F}(x, y) + \mathbb{T}^{(s)}_{w,F}(x, y)\right]. \tag{47}$$

Proof. By using (36), we have

$$2e_F^{xt}\cos_F(yt) = (e_F^{2t} + 1)\sum_{w=0}^{\infty} \mathbb{T}^{(c)}_{w,F}(x, y)\frac{t^w}{F_w!}$$

$$= \sum_{k=0}^{\infty} 2^k \frac{t^k}{F_k!}\sum_{w=0}^{\infty} \mathbb{S}^{(c)}_{w,F}(x, y)\frac{t^w}{F_w!} + \sum_{w=0}^{\infty} \mathbb{S}^{(c)}_{w,F}(x, y)\frac{t^w}{F_w!}$$

$$= \sum_{w=0}^{\infty}\left(\sum_{k=0}^{w}\binom{w}{k}_F 2^k \mathbb{S}^{(c)}_{w-k,F}(x, y) + \mathbb{S}^{(c)}_{w,F}(x, y)\right)\frac{t^w}{F_w!}.$$

From (32), we get

$$\sum_{w=0}^{\infty} 2\mathscr{C}_{w,F}(x, y)\frac{t^w}{F_w!} = \sum_{w=0}^{\infty}\left(\sum_{k=0}^{w}\binom{w}{k}_F 2^k \mathbb{S}^{(c)}_{w-k,F}(x, y) + \mathbb{S}^{(c)}_{w,F}(x, y)\right)\frac{t^w}{F_w!}.$$

By equating the coefficients of t^w, the desired result can be obtained. A similar derivation can be applied to Eq. (47). □

4. Two variable Tangent-Fibonacci polynomials via matrices form

In this section, we present the two-variable Tangent-Fibonacci polynomials in the form of a lower triangular matrix. Using this matrix, we derive a formula that establishes the relationship between this matrix and the generalized Pascal matrix via Fibonomial coefficients of the first kind.

Definition 3 Let $\mathbb{T}_{w,F}(x, y)$ be the w^{th} two variable Tangent-Fibonacci polynomial. The $(w+1)\times(w+1)$ two variable of Tangent-Fibonacci polynomials matrix $\boldsymbol{T}_{w,F}(x, y) = [s_{i,j}(x, y, F)]$ for $i, j = 1, 2, \cdots, w$ is defined by

$$s_{i,j}(x, y, F) = \begin{cases} \binom{i}{j}_F \mathbb{T}_{i-j,F}(x, y), & \text{if } i \geq j, \\ 0, & \text{if otherwise.} \end{cases} \tag{48}$$

We denote the Tangent-Fibonacci number matrix for $x = y = 0$ as

$$\boldsymbol{T}_{w,F}(0, 0) = \boldsymbol{T}_{w,F}.$$

Theorem 11 The Tangent-Fibonacci polynomial matrix $\boldsymbol{T}_{w,F}(x, y)$ satisfies the following formula

$$\boldsymbol{T}_{w,F}(x +_F \phi, y) = \mathbf{P}_{w,F}[x]\boldsymbol{T}_{w,F}(\phi, y) = \mathbf{P}_{w,F}[\phi]\boldsymbol{T}_{w,F}(x, y),$$

where $\mathbf{P}_{w,F}[x]$ is the generalized Pascal matrix via Fibonomial coefficients of the first kind [33, Page 2, Eq. (7)].

Proof. From Eqs. (12), (20), and Definition 3, taking $i > j$, we have

$$\begin{aligned}
\{\boldsymbol{T}_{w,F}(x +_F \phi, y)\}_{ij} &= \binom{i}{j}\mathbb{T}_{i-j,F}(x +_F \phi, y) \\
&= \binom{i}{j}\sum_{k=0}^{i-j}\binom{i-j}{k}_F \mathbb{T}_{k,F}(\phi, y)x^{i-j-k} \\
&= \sum_{k=j}^{i}\binom{i}{j}_F\binom{i-j}{k-j}_F \mathbb{T}_{k-j,F}(\phi, y)x^{i-k} \\
&= \sum_{k=j}^{i}\binom{i}{k}_F x^{i-k}\binom{k}{j}_F \mathbb{T}_{k-j,F}(\phi, y) \\
&= \mathbf{P}_{w,F}[x]\boldsymbol{T}_{w,F}(\phi, y).
\end{aligned}$$

Hence the proof is completed. □

5. Conclusion

In this article, we introduced the Tangent-Fibonacci polynomials as F-analogues of the classical Tangent polynomials, formulated in two variables within the framework of Golden Calculus. Extending the results presented in the previously published study (see [14]), we established the generating functions of these polynomials and derived several of their fundamental properties, including recurrence relations, summation identities, and derivative formulas. In addition, we defined the Tangent-Fibonacci numbers associated with this polynomial family and examined their principal properties and mutual relationships.

The results obtained in this work offer a unified and generalized viewpoint on the interaction between Tangent-type structures and Fibonacci-based frameworks, thereby opening a new avenue for the investigation of special polynomial families. This approach not only enhances the theoretical understanding of such polynomials within Golden Calculus but also suggests meaningful connections with combinatorics, number theory, approximation theory, and applied mathematical modeling.

We believe that the techniques and methods developed herein provide a solid foundation for future research on broader classes of special polynomials and their corresponding number sequences. In particular, further exploration of parametric extensions, operator-based identities, and matrix representations within the Golden Calculus setting may reveal deeper structural insights and new applications. By integrating Fibonacci-based and Tangent-based analytical perspectives, this study aims to stimulate continued research and contribute to the ongoing development of modern mathematical theory.

In future work, we intend to employ Golden Calculus to construct parametric forms of additional special polynomial families and to derive a wide range of combinatorial identities through their generating functions.

Conflict of interest

The authors declare no competing financial interest.

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