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Research Article

On a Novel Class of Special Polynomials: Central Bell-Based Type 2 Euler Polynomials Associated with Umbral Calculus

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Abstract: In this work, the authors define the central Bell-based central factorial polynomials of the second kind and examine many of their fundamental formulae, properties, and relations, some of which are derived using the umbral calculus technique. Then, the authors introduce the central Bell-based type 2 Euler polynomials of order α that extend the concepts of central Bell polynomials and type 2 Euler polynomials. For these polynomials, the authors derive diverse formulas, relations, and identities, such as some summation formulas, an addition formula, two partial derivative properties, a recurrence relation, two explicit formulas, and two summation formulas covering central factorial numbers of the second kind and central Bell polynomials. Moreover, the authors investigate two implicit summation formulas for central Bell-based type 2 Euler polynomials of order α utilizing some series manipulation methods. Also, the authors develop useful identities of symmetry for the central Bell-based type 2 Euler polynomials of order α. Furthermore, the authors obtain several interesting formulas of the central Bell-based type 2 Euler polynomials of order α arising from umbral calculus to possess alternative ways of obtaining our consequences and also some new consequences. Finally, the authors provide a determinantal representation for central Bell-based type 2 Euler polynomials.

Keywords: central Bell polynomials, central factorial numbers of the second kind, type 2 Euler polynomials, mixed-type polynomials, partial differential equations, differential equations

MSC: 11B83, 11S80, 05A19

1. Introduction

Some well-known special functions, polynomials, and numbers (e.g., hypergeometric function, Riemann zeta function; Euler polynomials, Hermite polynomials, Laguerre polynomials, Bell polynomials, Bernoulli polynomials; Stirling numbers and central factorial numbers [1–33]) have piqued the interest of several mathematicians in recent years because of their diverse use, from combinatorics and number theory to other fields of applied mathematics. With these, the literature has a variety of variations and generalizations of functions, polynomials, and numbers. Intricate patterns and useful rules make polynomial sequences essential in many fields, such as computer science, mathematics, and engineering. Their widespread

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applications make them essential mathematical tools. For example, engineering utilizes them for signal processing and control systems, physics utilizes them to approximate complex phenomena, and computer science utilizes them for algorithm design and cryptography. Diverse famous special polynomial sequences, such as Bernoulli, Bell, Fibonacci, Hermite, Euler, Gould-Hopper, etc., have deeply fruitful algebraic and analytic properties. The construction of several variations of the special polynomials was achieved by combining the concepts of two or three special functions, numbers, or polynomials. For instance, Lagrange-based Apostol-Genocchi, Apostol-Euler, and Apostol-Bernoulli polynomials, the Hermite-based Euler, Genocchi, and Bernoulli polynomials, the Gould-Hopper-based Bernoulli polynomials, Bell-based Bernoulli and Euler polynomials, degenerate Bell-Hermite-based Bernoulli polynomials, Hermite-Bell-based Euler polynomials, and degenerate Bell-based Euler polynomials were constructed by mixing the definitions of Bell, Hermite, Gould-Hopper, Euler, Lagrange, Genocchi, and Bernoulli polynomials, and many of their properties and applications were investigated profoundly, cf. [8–12, 15, 16, 26, 31, 33] and take a look at the references that are cited. In the last half-decade years, the Bell-based Bernoulli polynomials of order $\alpha \in \mathbb{C}$ were defined as follows (cf. [8])

$$\sum_{m=0}^{\infty} {}_{\phi}B_m^{(\alpha)}(x, z)\frac{t^m}{m!} = \left(\frac{t}{e^t - 1}\right)^{\alpha} e^{xt+z(e^t-1)} \quad (|t| < 2\pi)$$

and the Bell-based Euler polynomials of order $\alpha \in \mathbb{C}$ were defined as follows (cf. [15])

$$\sum_{m=0}^{\infty} {}_{\phi}E_m^{(\alpha)}(x, z)\frac{t^m}{m!} = \left(\frac{2}{e^t + 1}\right)^{\alpha} e^{xt+z(e^t-1)} \quad (|t| < \pi).$$

For the mentioned polynomials, various identities and formulas covering addition formulas, partial differentiation rules, summation formulas, and correlations with the usual Bell (exponential) polynomials and the Stirling numbers of the second kind were obtained deeply. Their many formulas were derived from the umbral calculus.

The Hermite-Bell based Euler polynomials of order $\alpha \in \mathbb{C}$ are defined by the following exponential generating function (cf. [11]):

$$\sum_{n=0}^{\infty} {}_{HB}E_n^{(\alpha)}(x, y, z)\frac{t^n}{n!} = \left(\frac{2}{e^t + 1}\right)^{\alpha} e^{xt+yt^2+z(e^t-1)} \quad (|t| < \pi).$$

Recently, by mixing type 2 Bernoulli polynomials and central Bell polynomials, the central Bell-based type 2 Bernoulli polynomials of order $\alpha \in \mathbb{C}$ were considered by (cf. [9]):

$$\sum_{m=0}^{\infty} {}_{CB}b_m^{(\alpha)}(x; z)\frac{t^m}{m!} = \left(\frac{t}{e^{\frac{t}{2}} - e^{-\frac{t}{2}}}\right)^{\alpha} e^{xt+z\left(e^{\frac{t}{2}}-e^{-\frac{t}{2}}\right)} \quad (|t| < 2\pi). \tag{1}$$

Then, several relations and formulas, including partial derivation rules, addition formulas, summation formulas, and correlations with the central Bell polynomials and the central factorial numbers of the second kind, were derived profoundly (cf. [9]). Motivated and inspired by the definition of the central Bell-based type 2 Bernoulli polynomials of order α in (1), in this study, we consider the central Bell-based type 2 Euler polynomials of order α by mixing the type 2 Euler polynomials and the central Bell polynomials. The paper aims to analyze the connections between the type 2 Euler polynomials and the central Bell polynomials and derive many relations associated with them. We investigate diverse relations, identities, and formulas, some of which will be derived by using the umbral calculus technique. These include five summation formulas in Theorems 6, 13, and 14; an addition formula in Theorem 7; two partial derivative properties in Theorem 8; a recurrence relation in Theorem 9, and two explicit formulas in Theorems 10 and 11. Also, we derive an implicit summation formula in Theorem 16 and two identities of symmetry for the central Bell-based type 2 Euler polynomials of order α in Theorems 17 and 18. In addition, the authors obtain diverse interesting formulas of the central Bell-based type 2 Euler polynomials of order α arising from umbral calculus methods. Lastly, the authors give a determinantal representation for central Bell-based type 2 Euler polynomials. Before going into these, the authors first define the central Bell-based central factorial polynomials of the second kind and provide some of their properties (given in Theorems 1-5) by the following section.

For $\varpi \in \mathbb{N}_0$, the Central Factorial Polynomials (abbreviated with *CFP*) $T(m, \varpi : x)$ and Central Factorial Numbers (abbreviated with *CFN*) $T(m, \varpi)$ of the second kind are defined as follows (cf. [1, 2, 4, 9, 13, 16–20, 24, 28, 29, 32]):

$$\sum_{m=0}^{\infty} T(m, \varpi : x) \frac{t^m}{m!} = \frac{\left(e^{\frac{t}{2}} - e^{-\frac{t}{2}}\right)^{\varpi}}{\varpi!} e^{xt} \text{ and } \sum_{m=0}^{\infty} T(m, \varpi) \frac{t^m}{m!} = \frac{\left(e^{\frac{t}{2}} - e^{-\frac{t}{2}}\right)^{\varpi}}{\varpi!}. \tag{2}$$

The numbers $T(m, \varpi)$ are computed as follows, for $m \in \mathbb{N}_0$:

$$x^m = \sum_{\varpi=0}^{m} T(m, \varpi) x^{[\varpi]}, \tag{3}$$

where the notation $x^{[\varpi]}$ termed as the central factorial of x equals to $\prod_{j=0}^{\varpi-1} \left(x - \frac{\varpi-1}{2} + j\right)$ with $x^{[0]} = 1$, cf. [1, 2, 4, 9, 13, 16–20, 24, 28, 29, 32].

Central factorial numbers have a closer association with Stirling numbers than with other well-known special numbers, such as trigonometric functions, Euler numbers, Bernoulli numbers, and their inverses. Different perspectives have been used to study the properties of these numbers (see Wang). Central factorial numbers are crucial in various branches of mathematics: in numerical analysis, in interpolation theory, in finite difference calculus, in approximation theory, in some expansions of trigonometric convolution integrals.

The Bivariate Central Bell Polynomials (abbreviated with *BCBP*), denoted by $\phi_m^{(c)}(x; z)$, one-variable Central Bell Polynomials (abbreviated with *CBP*), denoted by $\phi_m^{(c)}(x)$ and classical Central Bell Numbers (abbreviated with *CBN*), denoted by $\phi_m^{(c)}$, are defined as follows (cf. [1, 4, 9, 17, 19]):

$$\sum_{m=0}^{\infty} \phi_m^{(c)}(x; z) \frac{t^m}{m!} = e^{xt + z\left(e^{\frac{t}{2}} - e^{-\frac{t}{2}}\right)}, \tag{4}$$

$$\sum_{m=0}^{\infty} \phi_m^{(c)}(x) \frac{t^m}{m!} = e^{x\left(e^{\frac{t}{2}} - e^{-\frac{t}{2}}\right)}, \tag{5}$$

and

$$\sum_{m=0}^{\infty} \phi_m^{(c)} \frac{t^m}{m!} = e^{\left(e^{\frac{t}{2}} - e^{-\frac{t}{2}}\right)}, \tag{6}$$

respectively. We note that $\phi_m^{(c)}(0; z) := \phi_m^{(c)}(z)$ and $\phi_m^{(c)}(0; 1) = \phi_m^{(c)}(1) = \phi_m^{(c)}$. We observe from (2), (5) and (4) that

$$\phi_m^{(c)}(x; z) = \sum_{\varpi=0}^{m} T(m, \varpi : x) z^{\varpi}$$

and

$$\phi_m^{(c)}(z) = \sum_{\varpi=0}^{m} T(m, \varpi) z^{\varpi}, \tag{7}$$

(cf. [1, 4, 9, 17, 19]).

The familiar Euler polynomials $\mathfrak{E}_m^{(\alpha)}(x)$ of order α are defined as follows (cf. [11, 12, 15, 23, 26, 27, 31]):

$$\sum_{m=0}^{\infty} \mathfrak{E}_m^{(\alpha)}(x) \frac{t^m}{m!} = \left(\frac{2}{e^t+1}\right)^{\alpha} e^{xt} \quad (|t| < \pi). \tag{8}$$

Setting $x = 0$ in (9), we get $\mathfrak{E}_m^{(\alpha)} := \mathfrak{E}_m^{(\alpha)}(0)$ known as the usual Euler numbers of order α. The numbers $\mathfrak{E}_m := \mathfrak{E}_m^{(1)}$ and the polynomials $\mathfrak{E}_m(x) := \mathfrak{E}_m^{(1)}(x)$ are termed the classical Euler numbers and polynomials, respectively.

The Type 2 Euler Polynomials (abbreviated with *T2EPα*) $E_m^{(\alpha)}(x)$ of order α are defined by (cf. [5, 13, 18, 20]):

$$\sum_{m=0}^{\infty} E_m^{(\alpha)}(x) \frac{t^m}{m!} = \left(\frac{2}{e^{\frac{t}{2}}+e^{-\frac{t}{2}}}\right)^{\alpha} e^{xt} \quad (|t| < \pi). \tag{9}$$

Setting $x = 0$ in (9), we get $E_m^{(\alpha)} := E_m^{(\alpha)}(0)$ known as the type 2 Euler numbers of order α. The numbers $E_m := E_m^{(1)}$ and the polynomials $E_m(x) := E_m^{(1)}(x)$ are termed the classical Type 2 Euler Numbers and Polynomials (abbreviated with *T2EN* and *T2EP*), respectively.

The Type 2 Bernoulli Polynomials (abbreviated with *T2BPα*) $b_m^{(\alpha)}(x)$ of order α are given by (cf. [5, 9, 18, 20]):

$$\sum_{m=0}^{\infty} b_m^{(\alpha)}(x) \frac{t^m}{m!} = \left(\frac{t}{e^{\frac{t}{2}}-e^{-\frac{t}{2}}}\right)^{\alpha} e^{xt} \quad (|t| < 2\pi).$$

Setting $x = 0$ in (9), we get $b_m^{(\alpha)}(0) := b_m^{(\alpha)}$ known as the type 2 Bernoulli numbers of order α. The numbers $b_m^{(1)} := b_m$ and the polynomials $b_m^{(1)}(x) := b_m(x)$ are termed the classical Type 2 Bernoulli Numbers and Polynomials (abbreviated with *T2BN* and *T2BP*), respectively.

2. Central Bell-based central factorial polynomials of the second kind

In this part, the authors define the central Bell-based central factorial polynomials of the second kind and examine some of their formulas and identities.

Definition 1 The Central Bell-Based Central Factorial Polynomials of the Second Kind (abbreviated with *CBBCFPSK*) are defined as follows:

$$\frac{1}{\varpi!}\left(e^{\frac{t}{2}}-e^{-\frac{t}{2}}\right)^{\varpi}e^{xt+z\left(e^{\frac{t}{2}}-e^{-\frac{t}{2}}\right)}=\sum_{m=0}^{\infty}{}_{CB}T(m,\,\varpi;\,x,\,z)\frac{t^m}{m!}. \tag{10}$$

Remark 1 Choosing $x=0$ in (10), we obtain Central Bell-Central Factorial Polynomials of the Second Kind (abbreviated with *CBCFPSK*), which are also a novel extension of the numbers and polynomials in (2) as follows:

$$\frac{1}{\varpi!}\left(e^{\frac{t}{2}}-e^{-\frac{t}{2}}\right)^{\varpi}e^{z\left(e^{\frac{t}{2}}-e^{-\frac{t}{2}}\right)}=\sum_{m=0}^{\infty}{}_{CB}T(m,\,\varpi;\,z)\frac{t^m}{m!}. \tag{11}$$

Remark 2 If we take $z=0$ in (10), we find the polynomials $T\,(m,\,\varpi\colon x)$ in (2).

Remark 3 If we take $x=z=0$ in (10), the polynomials *CBBCFPSK* become the numbers $T(m,\,\varpi)$ in (2).

We start by providing the following theorem. We provide a summation formula for *CBBCFPSK* related to *BCBP* as follows.

Theorem 1 The following equality

$${}_{CB}T\,(m,\,\varpi;\,x,\,z)=\frac{1}{\varpi!}\sum_{j=0}^{m}\binom{m}{j}\sum_{l=0}^{\varpi}\binom{\varpi}{l}(-1)^{\varpi-l}\left(l-\frac{\varpi}{2}\right)^{j}\phi_{m-j}^{(c)}(x,\,z) \tag{12}$$

is valid for $m\geq\varpi\geq 0$.

Proof. Using (2), (4) and (10), we have

$$\begin{aligned}
\sum_{m=0}^{\infty}{}_{CB}T(m,\,\varpi;\,x,\,z)\frac{t^m}{m!}&=\frac{1}{\varpi!}\left(e^{\frac{t}{2}}-e^{-\frac{t}{2}}\right)^{\varpi}e^{xt+z\left(e^{\frac{t}{2}}-e^{-\frac{t}{2}}\right)}\\
&=\left(\frac{1}{\varpi!}\sum_{l=0}^{\varpi}\binom{\varpi}{l}(-1)^{\varpi-l}e^{l-\frac{\varpi}{2}}\right)\left(\sum_{m=0}^{\infty}\phi_{m}^{(c)}(x,\,z)\frac{t^m}{m!}\right)\\
&=\frac{1}{\varpi!}\left(\sum_{j=0}^{\infty}\sum_{l=0}^{\varpi}\binom{\varpi}{l}(-1)^{\varpi-l}\left(l-\frac{\varpi}{2}\right)^{j}\frac{t^j}{j!}\right)\left(\sum_{m=0}^{\infty}\phi_{m}^{(c)}(x,\,z)\frac{t^m}{m!}\right)\\
&=\sum_{m=0}^{\infty}\left(\frac{1}{\varpi!}\sum_{j=0}^{m}\binom{m}{j}\sum_{l=0}^{\varpi}\binom{\varpi}{l}(-1)^{\varpi-l}\left(l-\frac{\varpi}{2}\right)^{j}\phi_{m-j}^{(c)}(x,\,z)\right)\frac{t^m}{m!}.
\end{aligned}$$

Equating the coefficients of t^m, we obtain the result (12). □

Corollary 1 A relation including *CBCFPSK* and *CBP* is valid for $m\geq\varpi\geq 0$:

$${}_{CB}T(m,\,\varpi;\,z)=\frac{1}{\varpi!}\sum_{j=0}^{m}\binom{m}{j}\sum_{l=0}^{\varpi}\binom{\varpi}{l}(-1)^{\varpi-l}\left(l-\frac{\varpi}{2}\right)^{j}\phi_{m-j}^{(c)}(z).$$

We give two summation formulas for *CBBCFPSK* related to *CBCFPSK* and *CBP* as follows.

Theorem 2 The following equalities hold for $m \geq \varpi \geq 0$:

$$ {}_{CB}T(m, \varpi; x, z) = \sum_{\kappa=0}^{m} \binom{m}{\kappa} {}_{CB}T(\kappa, \varpi; z) x^{m-\kappa}, $$

and

$$ {}_{CB}T(m, \varpi; x, z) = \frac{1}{\varpi!} \sum_{j=0}^{m} \binom{m}{j} \sum_{l=0}^{\varpi} \binom{\varpi}{l} (-1)^{\varpi-l} \left(x + l - \frac{\varpi}{2} \right)^{j} \phi_{m-j}^{(c)}(z). $$

Proof. The proofs can be completed similarly to Theorem 1. □

We give two addition formulas for *CBBCFPSK* as follows.

Theorem 3 The following equalities are valid for $m \geq \varpi \geq 0$:

$$ {}_{CB}T(m, \varpi; x_1 + x_2, z) = \sum_{\kappa=0}^{m} \binom{m}{\kappa} {}_{CB}T(\kappa, \varpi; x_1, z) x_2^{m-\kappa}, \tag{13} $$

and

$$ {}_{CB}T(m, \varpi; x, z_1 + z_2) = \sum_{\kappa=0}^{m} \binom{m}{\kappa} {}_{CB}T(\kappa, \varpi; x, z_1) \phi_{m-\kappa}^{(c)}(z_2). \tag{14} $$

Proof. By changing x with $x_1 + x_2$ and z with $z_1 + z_2$ in (10), respectively, we can easily acquire (13) and (14). Therefore, we omit the details. □

We put an addition formula for *CBBCFPSK* as follows.

Theorem 4 The following equality is valid for $m, \varpi_1, \varpi_2 \in \mathbb{N}_0$ with $m \geq \varpi_1$ and $m \geq \varpi_2$:

$$ {}_{CB}T(m, \varpi_1 + \varpi_2; x_1 + x_2, z_1 + z_2) = \frac{\varpi_1! \varpi_2!}{(\varpi_1 + \varpi_2)!} \sum_{j=0}^{m} \binom{m}{j} {}_{CB}T(m - j, \varpi_1; x_1, z_1)\, {}_{CB}T(j, \varpi_2; x_2, z_2). \tag{15} $$

Proof. By using (2) and (10), we observe that

$$ \sum_{m=0}^{\infty} {}_{CB}T(m, \varpi_1 + \varpi_2; x_1 + x_2, z_1 + z_2) \frac{t^m}{m!} $$

$$ = \frac{1}{(\varpi_1 + \varpi_2)!} \left(e^{\frac{t}{2}} - e^{-\frac{t}{2}} \right)^{\varpi_1 + \varpi_2} e^{(x_1 + x_2)t + (z_1 + z_2)(e^{\frac{t}{2}} - e^{-\frac{t}{2}})} $$

$$ = \frac{\varpi_1! \varpi_2!}{(\varpi_1 + \varpi_2)!} \frac{1}{\varpi_1!} \left(e^{\frac{t}{2}} - e^{-\frac{t}{2}} \right)^{\varpi_1} e^{x_1 t + z_1 (e^{\frac{t}{2}} - e^{-\frac{t}{2}})} \frac{1}{\varpi_2!} \left(e^{\frac{t}{2}} - e^{-\frac{t}{2}} \right)^{\varpi_2} e^{x_2 t + z_2 (e^{\frac{t}{2}} - e^{-\frac{t}{2}})} \frac{1}{\varpi_2!} $$

$$= \frac{\varpi_1!\varpi_2!}{(\varpi_1+\varpi_2)!} \sum_{m=0}^{\infty} {}_{CB}T(m, \varpi_1; x, z) \frac{t^m}{m!} \sum_{m=0}^{\infty} {}_{CB}T(m, \varpi_2; x_2, z_2) \frac{t^m}{m!}$$

$$= \frac{\varpi_1!\varpi_2!}{(\varpi_1+\varpi_2)!} \sum_{m=0}^{\infty} \sum_{j=0}^{m} \binom{m}{j} {}_{CB}T(m-j, \varpi_1; x, z) {}_{CB}T(j, \varpi_2; x_2, z_2) \frac{t^m}{m!}.$$

Comparing the coefficients of t^m, we get (15). □

We provide a correlation between *CBBCFPSK* and *BCBP* as follows.

Theorem 5 The following relation holds for $m, \varpi \in \mathbb{N}_0$ with $m \geq \varpi$:

$$\sum_{j=0}^{m} \binom{m}{j} {}_{CB}T(j, \varpi; x, z) \phi_{m-j}^{(c)}(-x, -z) = \sum_{l=0}^{\varpi} \binom{\varpi}{l} \frac{(-1)^{\varpi-l}}{\varpi!} \left(l - \frac{\varpi}{2}\right)^m. \tag{16}$$

Proof. In view of (2), (4) and (10), we have

$$\frac{1}{\varpi!}\left(e^{\frac{t}{2}} - e^{-\frac{t}{2}}\right)^{\varpi} = e^{xt+z(e^{\frac{t}{2}}-e^{-\frac{t}{2}})} e^{-xt-z(e^{\frac{t}{2}}-e^{-\frac{t}{2}})} \frac{\left(e^{\frac{t}{2}} - e^{-\frac{t}{2}}\right)^{\varpi}}{\varpi!}$$

$$\frac{1}{\varpi!} \sum_{m=0}^{\infty} \sum_{l=0}^{\varpi} \binom{\varpi}{l} (-1)^{\varpi-l} \left(l - \frac{\varpi}{2}\right)^m \frac{t^m}{m!} = \left(\sum_{m=0}^{\infty} \phi_m^{(c)}(-x, -z) \frac{t^m}{m!}\right) \left(\sum_{j=0}^{\infty} {}_{CB}T(j, \varpi; x, z) \frac{t^j}{j!}\right)$$

$$= \sum_{m=0}^{\infty} \left(\sum_{j=0}^{m} \binom{m}{j} {}_{CB}T(j, \varpi; x, z) \phi_{m-j}^{(c)}(-x, -z)\right) \frac{t^m}{m!}.$$

Equating the coefficients of t^m, we acquire the desired result (16). □

3. Central Bell-based type 2 euler polynomials of order α

In this part, the authors define the central Bell-based type 2 Euler polynomials of higher order and analyze several relations and formulas, including partial derivation rules, addition formulas, summation formulas, and correlations with the central Bell polynomials and the central factorial numbers of the second kind.

Here, we consider central Bell-based type 2 Euler polynomials of order α below.

Definition 2 The Central Bell-Based Type 2 Euler Polynomials (abbreviated with *CBBT2EP*α) of order $\alpha \in \mathbb{C}$ are considered by

$$\sum_{m=0}^{\infty} {}_{CB}E_m^{(\alpha)}(x; z) \frac{t^m}{m!} = \left(\frac{2}{e^{\frac{t}{2}} + e^{-\frac{t}{2}}}\right)^{\alpha} e^{xt+z\left(e^{\frac{t}{2}}-e^{-\frac{t}{2}}\right)} \quad (|t| < \pi). \tag{17}$$

Several particular circumstances of the central Bell-based type 2 Euler polynomials of order α are examined below.

Remark 4 When $x = 0$ in (17), we acquire Type 2 Central Bell-Euler Polynomials (abbreviated with *T2CBEP*α) ${}_{CB}E_m^{(\alpha)}(z)$ of order α, which are also novel extensions of the type 2 Euler numbers of order α in (9), given below:

$$\sum_{m=0}^{\infty} {}_{CB}E_m^{(\alpha)}(z)\frac{t^m}{m!}=\left(\frac{2}{e^{\frac{t}{2}}+e^{-\frac{t}{2}}}\right)^{\alpha} e^{z\left(e^{\frac{t}{2}}-e^{-\frac{t}{2}}\right)} \quad (|t|<\pi). \tag{18}$$

Remark 5 Upon letting $z=0$ in (17), *CBBT2EP*α become Type 2 Euler Polynomials (abbreviated with *T2EP*α) $E_m^{(\alpha)}(x)$ of order α in (9).

Remark 6 When $z=0$ and $\alpha=1$, *CBBT2EP*α become the Type 2 Euler Polynomials (abbreviated with *T2EP*) $E_m(x)$ in (9).

Remark 7 For $\alpha=1$ in (17), we acquire ${}_{CB}E_m^{(1)}(x;z):={}_{CB}E_m(x;z)$ which are termed the Central Bell-Based Type 2 Euler Polynomials (abbreviated with *CBBT2EP*).

Multifarious properties of *CBBT2EP*α are analyzed below. Here are three summation formulas for *CBBT2EP*α related to *T2EN*α, *BCBP*, *T2EP*α, *CBP* and *T2CBEP*α as follows.

Theorem 6 The following equalities are correct for $m\in\mathbb{N}_0$:

$${}_{CB}E_m^{(\alpha)}(x;z)=\sum_{p=0}^{m}\binom{m}{p}E_p^{(\alpha)}\phi_{m-p}^{(c)}(x;z), \tag{19}$$

$${}_{CB}E_m^{(\alpha)}(x;z)=\sum_{p=0}^{m}\binom{m}{p}E_p^{(\alpha)}(x)\phi_{m-p}^{(c)}(z), \tag{20}$$

$${}_{CB}E_m^{(\alpha)}(x;z)=\sum_{p=0}^{m}\binom{m}{p}{}_{CB}E_p^{(\alpha)}(z)x^{m-p}. \tag{21}$$

Proof. We compute using (9), (4), (17) and (18) that

$$\begin{aligned}\sum_{m=0}^{\infty} {}_{CB}E_m^{(\alpha)}(x;z)\frac{t^m}{m!}&=\left(\frac{2}{e^{\frac{t}{2}}+e^{-\frac{t}{2}}}\right)^{\alpha} e^{xt+z\left(e^{\frac{t}{2}}-e^{-\frac{t}{2}}\right)}\\&=\sum_{m=0}^{\infty}E_m^{(\alpha)}\frac{t^m}{m!}\sum_{m=0}^{\infty}\phi_m^{(c)}(x;z)\frac{t^m}{m!}\\&=\sum_{m=0}^{\infty}\left[\sum_{p=0}^{m}\binom{m}{p}E_p^{(\alpha)}\phi_{m-p}^{(c)}(x;z)\right]\frac{t^m}{m!},\end{aligned}$$

which implies the formula in (19). The proof of (20) and (21) can be done similarly. □

We provide an addition formula for *CBBT2EP*α as follows.

Theorem 7 The following relationship

$${}_{CB}E_m^{(\alpha_1+\alpha_2)}(x_1+x_2;z_1+z_2)=\sum_{p=0}^{m}\binom{m}{p}{}_{CB}E_p^{(\alpha_1)}(x_1;z_1)\;{}_{CB}E_{m-p}^{(\alpha_2)}(x_2;z_2) \tag{22}$$

hold for $m \in \mathbb{N}_0$.

Proof. We compute from (17) that

$$\sum_{m=0}^{\infty} {}_{CB}E_m^{(\alpha_1+\alpha_2)}(x_1+x_2;\, z_1+z_2)\frac{t^m}{m!} = \left(\frac{2}{e^{\frac{t}{2}}+e^{-\frac{t}{2}}}\right)^{\alpha_1+\alpha_2} e^{(x_1+x_2)t+(z_1+z_2)\left(e^{\frac{t}{2}}-e^{-\frac{t}{2}}\right)}$$

$$= \left(\frac{2}{e^{\frac{t}{2}}+e^{-\frac{t}{2}}}\right)^{\alpha_1} e^{x_1t+z_1\left(e^{\frac{t}{2}}-e^{-\frac{t}{2}}\right)} \left(\frac{2}{e^{\frac{t}{2}}+e^{-\frac{t}{2}}}\right)^{\alpha_2} e^{x_2t+z_2\left(e^{\frac{t}{2}}-e^{-\frac{t}{2}}\right)}$$

$$= \sum_{m=0}^{\infty} {}_{CB}E_m^{(\alpha_1)}(x_1;\, z_1)\frac{t^m}{m!}\sum_{m=0}^{\infty} {}_{CB}E_m^{(\alpha_2)}(x_2;\, z_2)\frac{t^m}{m!}$$

$$= \sum_{m=0}^{\infty}\sum_{p=0}^{m}\binom{m}{p} {}_{CB}E_p^{(\alpha_1)}(x_1;\, z_1)\ {}_{CB}E_{m-p}^{(\alpha_2)}(x_2;\, z_2)\frac{t^m}{m!},$$

which means the claimed equality (22). □

Some of the particular circumstances of Theorem 7 are provided as follows:

$${}_{CB}E_m^{(\alpha)}(x+1;\, z) = \sum_{p=0}^{m}\binom{m}{p} {}_{CB}E_p^{(\alpha)}(x;\, z),$$

$${}_{CB}E_m^{(\alpha)}(x;\, z+1) = \sum_{p=0}^{m}\binom{m}{p} {}_{CB}E_m^{(\alpha)}(x;\, z_1)\,\phi_m^{(c)},$$

$${}_{CB}E_m^{(\alpha_1+\alpha_2)}(x;\, z) = \sum_{p=0}^{m}\binom{m}{p} {}_{CB}E_p^{(\alpha_1)}(x;\, z)\,E_{m-p}^{(\alpha_2)},$$

where the first formula is a generalization of the formula for type 2 Euler polynomials provided by (cf. [18, 20]):

$$E_m(x+1) = \sum_{p=0}^{m}\binom{m}{p}E_p(x).$$

We provide the difference operator formulas for $CBBT2EP\alpha$ as follows.

Theorem 8 The derivative operator formulas for ${}_{CB}E_m^{(\alpha)}(x;\, z)$:

$$\frac{\partial}{\partial x}\,{}_{CB}E_m^{(\alpha)}(x;\, z) = m\,{}_{CB}E_{m-1}^{(\alpha)}(x;\, z) \tag{23}$$

and

$$\frac{\partial}{\partial z}\,{}_{CB}E_m^{(\alpha)}(x;z) = {}_{CB}E_m^{(\alpha)}\left(x+\frac{1}{2};z\right) - {}_{CB}E_m^{(\alpha)}\left(x-\frac{1}{2};z\right) \tag{24}$$

are valid for $m \in \mathbb{N}$.

Proof. Depending on the following equations

$$\frac{\partial}{\partial x} e^{xt+z\left(e^{\frac{t}{2}}-e^{-\frac{t}{2}}\right)} = te^{xt+z\left(e^{\frac{t}{2}}-e^{-\frac{t}{2}}\right)},$$

and

$$\frac{\partial}{\partial z} e^{xt+z\left(e^{\frac{t}{2}}-e^{-\frac{t}{2}}\right)} = \left(e^{\frac{t}{2}}-e^{-\frac{t}{2}}\right) e^{xt+z\left(e^{\frac{t}{2}}-e^{-\frac{t}{2}}\right)} = e^{\left(x+\frac{1}{2}\right)t+z\left(e^{\frac{t}{2}}-e^{-\frac{t}{2}}\right)} - e^{\left(x-\frac{1}{2}\right)t+z\left(e^{\frac{t}{2}}-e^{-\frac{t}{2}}\right)},$$

the proofs are easily done. We omit the details. □

Two relations for *CBBT2EP* associated with *BCBP* are given as follows.

Theorem 9 The following equalities are valid for $m \in \mathbb{N}_0$.

$$\phi_m^{(c)}(x;z) = \frac{{}_{CB}E_m\left(x+\frac{1}{2};z\right) + {}_{CB}E_m\left(x-\frac{1}{2};z\right)}{2} \tag{25}$$

and

$$\phi_m^{(c)}(x;z) = \sum_{p=0}^{m} \binom{m}{p}\, {}_{CB}E_{m-p}(x;z)\left(\frac{1}{2^{p+1}} - \frac{1}{(-2)^{p+1}}\right).$$

Proof. Depending on the following equality

$$e^{xt+z\left(e^{\frac{t}{2}}-e^{-\frac{t}{2}}\right)} = \frac{e^{\frac{t}{2}}+e^{-\frac{t}{2}}}{2} \sum_{m=0}^{\infty} {}_{CB}E_m(x;z)\frac{t^m}{m!},$$

the proofs are readily completed, using (17). We omit the details. □

Remark 8 The result (25) is an extension of the identity for $E_m(x)$ provided by

$$x^m = \frac{E_m\left(x+\frac{1}{2}\right) + E_m\left(x-\frac{1}{2}\right)}{2}.$$

Two summation formulas for *CBBT2EP* associated with the usual Euler numbers and polynomials, *CBP* and *BCBP*, are given as follows.

Theorem 10 The following equalities

$$ {}_{CB}E_m(x;z) = \sum_{v=0}^{m} \binom{m}{v} \mathfrak{E}_{m-v} \phi_v^{(c)} \left(x + \frac{1}{2}; z \right) $$

and

$$ {}_{CB}E_m(x;z) = \sum_{v=0}^{m} \binom{m}{v} \mathfrak{E}_{m-v} \left(x + \frac{1}{2} \right) \phi_v^{(c)}(z) $$

hold for $m \in \mathbb{N}_0$.

Proof. Utilizing Definition 2, (8) and (4), we obtain

$$ \sum_{m=0}^{\infty} {}_{CB}E_m(x;z) \frac{t^m}{m!} = \frac{2e^{xt}}{e^{\frac{t}{2}} + e^{-\frac{t}{2}}} e^{z\left(e^{\frac{t}{2}} - e^{-\frac{t}{2}}\right)} $$

$$ = \frac{2}{e^t + 1} e^{\left(x + \frac{1}{2}\right)t} e^{z\left(e^{\frac{t}{2}} - e^{-\frac{t}{2}}\right)} $$

$$ = \sum_{m=0}^{\infty} \mathfrak{E}_m \frac{t^m}{m!} \sum_{m=0}^{\infty} \phi_m^{(c)} \left(x + \frac{1}{2}; z \right) \frac{t^m}{m!} $$

$$ = \sum_{m=0}^{\infty} \left(\sum_{v=0}^{m} \binom{m}{v} \mathfrak{E}_{m-v} \phi_v^{(c)} \left(x + \frac{1}{2}; z \right) \right) \frac{t^m}{m!}, $$

which implies the first assertion. The other can be proved similarly. □

An explicit formula for *CBBT2EP* is provided by the following theorem.

Theorem 11 An explicit formula for *CBBT2EP*

$$ {}_{CB}E_m(x;z) = \sum_{\varpi=0}^{\infty} \sum_{p=0}^{\infty} \sum_{v=0}^{p} \binom{p}{v} \frac{z^p}{p!} (-1)^{\varpi+v} 2^{1-m} (2x + 1 + 2\varpi + p - 2v)^m $$

holds for $m \in \mathbb{N}_0$ and $e^t < 1$.

Proof. By means of Definition 2, we derive

$$ \sum_{m=0}^{\infty} {}_{CB}E_m(x;z) \frac{t^m}{m!} = \frac{2}{e^{\frac{t}{2}} + e^{-\frac{t}{2}}} e^{z\left(e^{\frac{t}{2}} - e^{-\frac{t}{2}}\right)} e^{xt} $$

$$ = 2 \sum_{p=0}^{\infty} z^p \frac{\left(e^{\frac{t}{2}} - e^{-\frac{t}{2}}\right)^p}{p!} \sum_{\varpi=0}^{\infty} (-1)^{\varpi} e^{\left(x + \frac{1}{2} + \varpi\right)t} $$

$$= 2\sum_{p=0}^{\infty}\frac{z^p}{p!}\sum_{v=0}^{p}\binom{p}{v}\sum_{\varpi=0}^{\infty}(-1)^{\varpi+v}e^{(2x+1+2\varpi+p-2v)\frac{t}{2}}$$

$$= \sum_{m=0}^{\infty}\sum_{\varpi=0}^{\infty}\sum_{p=0}^{\infty}\sum_{v=0}^{p}\binom{p}{v}\frac{z^p}{p!}(-1)^{\varpi+v}2^{1-m}(2x+1+2\varpi+p-2v)^m\frac{t^m}{m!},$$

which implies the requested formula. □

A summation formula for *CBBT2EPα* related to *T2EPα* and *CFN* is provided by the following theorem.

Theorem 12 The following formula for ${}_{CB}E_m^{(\alpha)}(x;z)$

$${}_{CB}E_m^{(\alpha)}(x;z) = \sum_{v=0}^{m}\sum_{p=0}^{v}\binom{m}{v}z^p\mathfrak{E}_{m-v}^{(\alpha)}\left(x+\frac{\alpha}{2}\right)T(v,p)$$

holds for $m \in \mathbb{N}_0$.

Proof. By means of Definition 2, we derive

$$\sum_{m=0}^{\infty}{}_{CB}E_m^{(\alpha)}(x;z)\frac{t^m}{m!} = \left(\frac{2}{e^{\frac{t}{2}}+e^{-\frac{t}{2}}}\right)^{\alpha}e^{z\left(e^{\frac{t}{2}}-e^{-\frac{t}{2}}\right)}e^{xt}$$

$$= \left(\frac{2}{e^t+1}\right)^{\alpha}e^{\left(x+\frac{\alpha}{2}\right)t}\sum_{p=0}^{\infty}z^p\frac{\left(e^{\frac{t}{2}}-e^{-\frac{t}{2}}\right)^p}{p!}$$

$$= \sum_{m=0}^{\infty}\mathfrak{E}_m^{(\alpha)}\left(x+\frac{\alpha}{2}\right)\frac{t^m}{m!}\sum_{m=0}^{\infty}\sum_{p=0}^{m}z^pT(m,p)\frac{t^m}{m!}$$

$$= \sum_{m=0}^{\infty}\sum_{v=0}^{m}\sum_{p=0}^{v}\binom{m}{v}z^p\mathfrak{E}_{m-v}^{(\alpha)}\left(x+\frac{\alpha}{2}\right)T(v,p)\frac{t^m}{m!},$$

which gives the claimed formula. □

Two direct consequences of Theorem 11 are as follows:

$${}_{CB}E_m(x;z) = \sum_{v=0}^{m}\sum_{p=0}^{v}\binom{m}{v}z^p\mathfrak{E}_{m-v}\left(x+\frac{1}{2}\right)T(v,p)$$

and

$${}_{CB}E_m^{(\alpha)}(z) = \sum_{v=0}^{m}\sum_{p=0}^{v}\binom{m}{v}z^p\mathfrak{E}_{m-v}^{(\alpha)}\left(\frac{\alpha}{2}\right)T(v,p).$$

Another summation formula for *CBBT2EPα* related to *BCBP* is provided by the following theorem.

Theorem 13 The following formula

$$\phi_m^{(c)}(x;z) = 2^{-\alpha} \sum_{\varpi=0}^{\alpha} \sum_{\upsilon=0}^{m} \binom{m}{\upsilon} \left(\frac{\alpha}{2} - \varpi\right)^{m-v} {}_{CB}E_{\upsilon}^{(\alpha)}(x;z) \tag{26}$$

hold for m, $\alpha \in \mathbb{N}_0$.

Proof. Utilizing Definition 2 and (4), based on the following computations

$$e^{xt+z\left(e^{\frac{t}{2}}-e^{-\frac{t}{2}}\right)} = 2^{-\alpha}\left(e^{\frac{t}{2}}+e^{-\frac{t}{2}}\right)^{\alpha} \sum_{m=0}^{\infty} {}_{CB}E_m^{(\alpha)}(x;z)\frac{t^m}{m!}$$

$$= 2^{-\alpha} \sum_{\varpi=0}^{\alpha} e^{\left(\frac{\alpha}{2}-\varpi\right)t} \sum_{m=0}^{\infty} {}_{CB}E_m^{(\alpha)}(x;z)\frac{t^m}{m!}$$

$$= 2^{-\alpha} \sum_{\varpi=0}^{\alpha} \sum_{m=0}^{\infty} \left(\frac{\alpha}{2}-\varpi\right)^m \frac{t^m}{m!} \sum_{m=0}^{\infty} {}_{CB}E_m^{(\alpha)}(x;z)\frac{t^m}{m!}$$

$$= 2^{-\alpha} \sum_{m=0}^{\infty} \sum_{\varpi=0}^{\alpha} \sum_{\upsilon=0}^{m} \binom{m}{\upsilon} \left(\frac{\alpha}{2}-\varpi\right)^{m-v} {}_{CB}E_{\upsilon}^{(\alpha)}(x;z)\frac{t^m}{m!},$$

thus, the proof is completed. □

A summation formula for *CBBT2EPα* related to *T2CBEPα* and *CFP* is provided by the following theorem.

Theorem 14 The following correlation

$${}_{CB}E_m^{(\alpha)}(x;z) = \sum_{\upsilon=0}^{m} \sum_{p=0}^{\upsilon} \binom{m}{\upsilon} (2x)_p T\left(\upsilon,\, p\colon \frac{p-2x}{2}\right) {}_{CB}E_{m-\upsilon}^{(\alpha)}(z) \tag{27}$$

holds for $m \in \mathbb{N}_0$.

Proof. Utilizing Definition 2 and (2) and (18), we obtain

$$\sum_{m=0}^{\infty} {}_{CB}E_m^{(\alpha)}(x;z)\frac{t^m}{m!} = \left(e^{\frac{t}{2}} - e^{-\frac{t}{2}} + e^{-\frac{t}{2}}\right)^{2x} \left(\frac{2}{e^{\frac{t}{2}}+e^{-\frac{t}{2}}}\right)^{\alpha} e^{z\left(e^{\frac{t}{2}}-e^{-\frac{t}{2}}\right)}$$

$$= \left(\frac{2}{e^{\frac{t}{2}}+e^{-\frac{t}{2}}}\right)^{\alpha} e^{z\left(e^{\frac{t}{2}}-e^{-\frac{t}{2}}\right)} \sum_{p=0}^{\infty} (2x)_p \frac{\left(e^{\frac{t}{2}}-e^{-\frac{t}{2}}\right)^p}{p!} e^{t\frac{(p-2x)}{2}}$$

$$= \sum_{m=0}^{\infty} {}_{CB}E_m^{(\alpha)}(z)\frac{t^m}{m!} \sum_{m=0}^{\infty} \sum_{p=0}^{m} (2x)_p T\left(m,\, p\colon \frac{p-2x}{2}\right) \frac{t^m}{m!}$$

$$= \sum_{m=0}^{\infty}\sum_{\upsilon=0}^{m}\sum_{p=0}^{\upsilon}\binom{m}{\upsilon}(2x)_p T\left(\upsilon,\, p\colon \frac{p-2x}{2}\right)\, {}_{CB}E_{m-\upsilon}^{(\alpha)}(z)\frac{t^m}{m!},$$

which implies the desired equality (27). □

Another summation formula for *CBBT2EP*α is given by the following theorem.

Theorem 15 The following correlation

$${}_{CB}E_m^{(\alpha)}(x;z) = \sum_{p=0}^{2x}\binom{2x}{p}(-1)^{2x-p}2^p\, {}_{CB}E_m^{(\alpha-p)}\left(\frac{p}{2}-x,\, z\right) \tag{28}$$

holds for $m \in \mathbb{N}_0$.

Proof. Utilizing Definition 2 and equation (18), we obtain

$$\sum_{m=0}^{\infty} {}_{CB}E_m^{(\alpha)}(x;z)\frac{t^m}{m!} = \left(\frac{2}{e^{\frac{t}{2}}+e^{-\frac{t}{2}}}\right)^{\alpha} e^{z\left(e^{\frac{t}{2}}-e^{-\frac{t}{2}}\right)}\left(e^{\frac{t}{2}}+e^{-\frac{t}{2}}-e^{-\frac{t}{2}}\right)^{2x}$$

$$= \left(\frac{2}{e^{\frac{t}{2}}+e^{-\frac{t}{2}}}\right)^{\alpha} e^{z\left(e^{\frac{t}{2}}-e^{-\frac{t}{2}}\right)}\sum_{p=0}^{2x}\binom{2x}{p}(-1)^{2x-p}\left(e^{\frac{t}{2}}+e^{-\frac{t}{2}}\right)^{p} e^{t\frac{(p-2x)}{2}}$$

$$= \sum_{p=0}^{2x}\binom{2x}{p}(-1)^{2x-p}2^p\left(\frac{2}{e^{\frac{t}{2}}+e^{-\frac{t}{2}}}\right)^{\alpha-p} e^{t\frac{(p-2x)}{2}+z\left(e^{\frac{t}{2}}-e^{-\frac{t}{2}}\right)}$$

$$= \sum_{p=0}^{2x}\binom{2x}{p}(-1)^{2x-p}2^p\sum_{m=0}^{\infty} {}_{CB}E_m^{(\alpha-p)}\left(\frac{p}{2}-x,\, z\right)\frac{t^m}{m!}$$

$$= \sum_{m=0}^{\infty}\sum_{\upsilon=0}^{m}\sum_{p=0}^{\upsilon}\binom{m}{\upsilon}(2x)_p T\left(\upsilon,\, p\colon \frac{p-2x}{2}\right)\, {}_{CB}E_{m-\upsilon}^{(\alpha)}(z)\frac{t^m}{m!},$$

which means the asserted equality (28). □

We provide the following series of equalities (cf. [8, 9]):

$$\sum_{m,\,\varpi=0}^{\infty} f(m+\varpi)\frac{x^m}{m!}\frac{z^{\varpi}}{\varpi!} = \sum_{N=0}^{\infty} f(N)\frac{(x+z)^N}{N!}, \tag{29}$$

and

$$\sum_{p=0}^{\infty}\sum_{\upsilon=0}^{p} A(\upsilon,\, p-\upsilon) = \sum_{p,\,\upsilon=0}^{\infty} A(\upsilon,\, p). \tag{30}$$

Two implicit summation formulas for $CBBT2EP\alpha$ are provided by the following theorem.

Theorem 16 The following equalities hold for $p,\, \nu \in \mathbb{N}_0$:

$$ {}_{CB}E^{(\alpha)}_{p+\upsilon}(x;z) = \sum_{m,\,\varpi=0}^{p,\,\upsilon} \binom{p}{m}\binom{\upsilon}{\varpi}\, {}_{CB}E^{(\alpha)}_{p+\upsilon-m-\varpi}(x;z)\,(x-\xi)^{m+\varpi} \tag{31} $$

and

$$ {}_{CB}E^{(\alpha)}_{p+\upsilon}(x;z) \sum_{m,\,\varpi=0}^{p,\,\upsilon} \binom{p}{m}\binom{\upsilon}{\varpi}\, {}_{CB}E^{(\alpha)}_{p+\upsilon-m-\varpi}(x;\xi)\,\phi^{(c)}_{m+\varpi}(z-\xi). $$

Proof. We get by changing t by $t+u$ in (17) and utilizing (29) that

$$ e^{-\xi(t+u)} \sum_{p,\,\upsilon=0}^{\infty} {}_{CB}E^{(\alpha)}_{p+\upsilon}(\xi;z)\frac{t^p}{p!}\frac{u^\upsilon}{\upsilon!} = \left(\frac{2}{e^{\frac{t+u}{2}}+e^{-\frac{t+u}{2}}}\right)^{\alpha} e^{z\left(e^{\frac{t+u}{2}}-e^{-\frac{t+u}{2}}\right)}, $$

and

$$ e^{-x(t+u)} \sum_{p,\,\upsilon=0}^{\infty} {}_{CB}E^{(\alpha)}_{p+\upsilon}(x;z)\frac{t^p}{p!}\frac{u^\upsilon}{\upsilon!} = \left(\frac{2}{e^{\frac{t+u}{2}}+e^{-\frac{t+u}{2}}}\right)^{\alpha} e^{z\left(e^{\frac{t+u}{2}}-e^{-\frac{t+u}{2}}\right)}, $$

which means the following equality

$$ e^{(x-\xi)(t+u)} \sum_{p,\,\upsilon=0}^{\infty} {}_{CB}E^{(\alpha)}_{p+\upsilon}(\xi;z)\frac{t^p}{p!}\frac{u^\upsilon}{\upsilon!} = \sum_{p,\,\upsilon=0}^{\infty} {}_{CB}E^{(\alpha)}_{p+\upsilon}(x;z)\frac{t^p}{p!}\frac{u^\upsilon}{\upsilon!}. $$

Therefore, utilizing (30), we have

$$ \sum_{p,\,\upsilon=0}^{\infty} {}_{CB}E^{(\alpha)}_{p+\upsilon}(x;z)\frac{t^p}{p!}\frac{u^\upsilon}{\upsilon!} = \sum_{m,\,\varpi=0}^{\infty}(x-\xi)^{m+\varpi}\frac{t^m}{m!}\frac{u^\varpi}{\varpi!} \sum_{p,\,\upsilon=0}^{\infty} {}_{CB}E^{(\alpha)}_{p+\upsilon}(\xi;z)\frac{t^p}{p!}\frac{u^\upsilon}{\upsilon!} $$

$$ = \sum_{p,\,\upsilon=0}^{\infty}\sum_{m,\,\varpi=0}^{p,\,\upsilon} \frac{(x-\xi)^{m+\varpi}\; {}_{CB}E^{(\alpha)}_{p+\upsilon-m-\varpi}(x;z)}{m!\varpi!(p-\upsilon)!(\upsilon-\varpi)!} t^p u^\upsilon, $$

which means the claimed formula (31). The other can be similarly proved. Thus, we omit it. □

Corollary 2 Putting $\upsilon = 0$ in Theorem 16, we have

$$ {}_{CB}E^{(\alpha)}_{p}(x;z) = \sum_{m=0}^{p}\binom{p}{m}\, {}_{CB}E^{(\alpha)}_{p-m}(\xi;z)\,(x-\xi)^m $$

and

$$ {}_{CB}E_p^{(\alpha)}(x;z)\sum_{m=0}^{p}\binom{p}{m}\,{}_{CB}E_{p-m}^{(\alpha)}(x;\xi)\,\phi_m^{(c)}(z-\xi). $$

A symmetric identity for *CBBT2EP*α is provided by the following theorem.

Theorem 17 Let $a,b\in\mathbb{N}$ and $m\geq 0$. We have

$$ \sum_{p=0}^{m}\binom{m}{p}\,{}_{CB}E_{m-p}^{(\alpha)}(bx;z)\,{}_{CB}E_p^{(\alpha)}(ax;z)\,a^{m-p}b^{p}=\sum_{p=0}^{m}\binom{m}{p}\,{}_{CB}E_p^{(\alpha)}(bx;z)\,{}_{CB}E_{m-p}^{(\alpha)}(ax;z)\,a^{p}b^{m-p}. \tag{32} $$

Proof. Choose

$$ \Upsilon=\left(\frac{2}{e^{\frac{at}{2}}+e^{-\frac{at}{2}}}\frac{2}{e^{\frac{bt}{2}}+e^{-\frac{bt}{2}}}\right)^{\alpha}e^{2abxt+z\left(e^{\frac{at}{2}}+e^{\frac{bt}{2}}-e^{-\frac{at}{2}}-e^{-\frac{bt}{2}}\right)}. $$

We compute two expansions of Υ:

$$ \Upsilon=\sum_{m=0}^{\infty}{}_{CB}E_m^{(\alpha)}(bx;z)\frac{(at)^m}{m!}\sum_{m=0}^{\infty}{}_{CB}E_m^{(\alpha)}(ax;z)\frac{(bt)^m}{m!} $$

$$ =\sum_{m=0}^{\infty}\sum_{p=0}^{m}\binom{m}{p}\,{}_{CB}E_{m-p}^{(\alpha)}(bx;z)\,{}_{CB}E_p^{(\alpha)}(ax;z)\,a^{m-p}b^{p}\frac{t^m}{m!} $$

and similarly

$$ \Upsilon=\sum_{m=0}^{\infty}\sum_{p=0}^{m}\binom{m}{p}\,{}_{CB}E_p^{(\alpha)}(bx;z)\,{}_{CB}E_{m-p}^{(\alpha)}(ax;z)\,a^{p}b^{m-p}\frac{t^m}{m!}, $$

which mean the claimed identity (32). □

Here is another symmetric identity for ${}_{CB}E_m^{(\alpha)}(x;z)$ given below.

Theorem 18 Let $a,b\in\mathbb{N}$ and $m\geq 0$. We have

$$ \sum_{p=0}^{m}\sum_{i=0}^{b-1}\sum_{j=0}^{a-1}\binom{m}{p}(-1)^{i+j}\,{}_{CB}E_{m-p}^{(\alpha)}(ax_2;z)\,a^{p}b^{m-p} $$

$$ \times\,{}_{CB}E_p^{(\alpha)}\left(b(x_1-1)+i+\frac{1}{2}+\frac{b}{a}\left(j+\frac{1}{2}\right);z\right) $$

$$= \sum_{p=0}^{m}\sum_{i=0}^{b-1}\sum_{j=0}^{a-1}\binom{m}{p}(-1)^{i+j}\ {}_{CB}E_{m-p}^{(\alpha)}(bx_1;z)\ a^{m-p}b^{p}$$

$$\times\ {}_{CB}E_{p}^{(\alpha)}\left(a(x_2-1)+j+\frac{1}{2}+\frac{a}{b}\left(i+\frac{1}{2}\right);z\right). \tag{33}$$

Proof. Let

$$\Psi = \frac{2^{2\alpha}\left(e^{\frac{abt}{2}}+e^{-\frac{abt}{2}}\right)^{2} e^{ab(x_1+x_2)t+z\left(e^{\frac{at}{2}}+e^{\frac{bt}{2}}-e^{-\frac{at}{2}}-e^{-\frac{bt}{2}}\right)}}{\left(e^{\frac{at}{2}}+e^{-\frac{at}{2}}\right)^{\alpha+1}\left(e^{\frac{bt}{2}}+e^{-\frac{bt}{2}}\right)^{\alpha+1}}$$

$$= \left(\frac{e^{\frac{abt}{2}}+e^{-\frac{abt}{2}}}{e^{\frac{at}{2}}+e^{-\frac{at}{2}}}\right) e^{abx_1t+z\left(e^{\frac{at}{2}}-e^{-\frac{at}{2}}\right)}\left(\frac{2}{e^{\frac{at}{2}}+e^{-\frac{at}{2}}}\right)^{\alpha}$$

$$\times\left(\frac{e^{\frac{abt}{2}}+e^{-\frac{abt}{2}}}{e^{\frac{bt}{2}}+e^{-\frac{bt}{2}}}\right) e^{abx_2t+z\left(e^{\frac{bt}{2}}-e^{-\frac{bt}{2}}\right)}\left(\frac{2}{e^{\frac{bt}{2}}+e^{-\frac{bt}{2}}}\right)^{\alpha}.$$

Utilizing

$$\frac{e^{\frac{abt}{2}}+e^{-\frac{abt}{2}}}{e^{\frac{at}{2}}+e^{-\frac{at}{2}}} = \sum_{i=0}^{b-1}(-1)^{i} e^{at\left(i+\frac{1-b}{2}\right)} \text{ and } \frac{e^{\frac{abt}{2}}+e^{-\frac{abt}{2}}}{e^{\frac{bt}{2}}+e^{-\frac{bt}{2}}} = \sum_{j=0}^{a-1}(-1)^{j} e^{bt\left(j+\frac{1-a}{2}\right)},$$

we observe that

$$\Psi = \left(\frac{2}{e^{\frac{at}{2}}+e^{-\frac{at}{2}}}\right)^{\alpha} e^{abx_1t+z\left(e^{\frac{at}{2}}-e^{-\frac{at}{2}}\right)}\sum_{i=0}^{b-1}(-1)^{i} e^{at\left(i+\frac{1-b}{2}\right)}\left(\frac{2}{e^{\frac{bt}{2}}+e^{-\frac{bt}{2}}}\right)^{\alpha} e^{abx_2t+z\left(e^{\frac{bt}{2}}-e^{-\frac{bt}{2}}\right)}\sum_{j=0}^{a-1}(-1)^{j} e^{bt\left(j+\frac{1-a}{2}\right)}$$

$$= \sum_{i=0}^{b-1}\sum_{j=0}^{a-1}(-1)^{i+j}\left(\frac{2}{e^{\frac{at}{2}}+e^{-\frac{at}{2}}}\right)^{\alpha} e^{\left(b(x_1-1)+i+\frac{1}{2}+\frac{b}{a}\left(j+\frac{1}{2}\right)\right)at+z\left(e^{\frac{at}{2}}-e^{-\frac{at}{2}}\right)}\left(\frac{2}{e^{\frac{bt}{2}}+e^{-\frac{bt}{2}}}\right)^{\alpha} e^{ax_2bt+z\left(e^{\frac{bt}{2}}-e^{-\frac{bt}{2}}\right)}$$

$$= \sum_{m=0}^{\infty}\sum_{p=0}^{m}\sum_{i=0}^{b-1}\sum_{j=0}^{a-1}\binom{m}{p}(-1)^{i+j}\ {}_{CB}E_{p}^{(\alpha)}\left(b(x_1-1)+i+\frac{1}{2}+\frac{b}{a}\left(j+\frac{1}{2}\right);z\right)\ {}_{CB}E_{m-p}^{(\alpha)}(ax_2;z)\,a^{p}b^{m-p}\frac{t^{m}}{m!},$$

and in the same way,

$$\Psi = \sum_{m=0}^{\infty}\sum_{p=0}^{m}\sum_{i=0}^{b-1}\sum_{j=0}^{a-1}\binom{m}{p}(-1)^{i+j}\ {}_{CB}E_{m-p}^{(\alpha)}(bx_1;z)_{CB}E_{p}^{(\alpha)}\left(a(x_2-1)+j+\frac{1}{2}+\frac{a}{b}\left(i+\frac{1}{2}\right);z\right)\ a^{m-p}b^{p}\frac{t^{m}}{m!},$$

which imply the desired identity (33). □

4. CBBT2EP α associated with umbral calculus

Mathematicians have been studying and examining the formulas of special polynomials obtained from umbral calculus in recent years, cf. [5–7, 11, 18–20, 22]. The concept of umbral calculus is briefly reviewed as follows, taken from the references [6–8, 15, 21–23, 27].

Let $\mathscr{F}$ be the set of all formal power series in the variable t over $\mathbb{C}$ in conjunction with

$$\mathscr{F}=\left\{f \mid f(t)=\sum_{\varpi=0}^{\infty} a_{\varpi}\frac{t^{\varpi}}{\varpi!}, \quad (a_{\varpi}\in\mathbb{C})\right\}.$$

Let $\mathbb{P}_m=\{p(x)\in\mathbb{C}[x] : \deg p(x)\leq m\}$ and let $\mathbb{P}^*$ be the vector space of all linear functionals on $\mathbb{P}$. The symbol $\langle L|p(x)\rangle$ implies the action of a linear functional L on the polynomial $p(x)$ in the umbral calculus, which is a linear property on $\mathbb{P}^*$:

$$\langle L+M|p(x)\rangle=\langle L|p(x)\rangle+\langle M|p(x)\rangle$$

and

$$\langle cL|p(x)\rangle=c\langle L|p(x)\rangle$$

for $c\in\mathbb{C}$.

The formal power series

$$f(t)=\sum_{\varpi=0}^{\infty} a_{\varpi}\frac{t^{\varpi}}{\varpi!} \tag{34}$$

introduces a linear functional on $\mathbb{P}$ by setting

$$\langle f(t)|x^m\rangle=a_m \quad (m\geq 0). \tag{35}$$

Choosing $f(t)=t^{\varpi}$ in (34) and (35) provides

$$\langle t^{\varpi}|x^m\rangle=m!\delta_{m,\varpi}, \quad (m,\varpi\geq 0) \tag{36}$$

where

$$\delta_{m,\varpi} = \begin{cases} 1, & \text{if } m = \varpi, \\ 0, & \text{if } m \neq \varpi. \end{cases}$$

Indeed, any linear functional L in $\mathbb{P}^*$ has the form (34). Namely, because of

$$f_L(t) = \sum_{\varpi=0}^{\infty} \langle L|x^{\varpi}\rangle \frac{t^{\varpi}}{\varpi!},$$

we get

$$\langle f_L(t)|x^m\rangle = \langle L|x^m\rangle,$$

and therefore as linear functionals $L = f_L(t)$. In addition, the map $L \to f_L(t)$ is a vector space isomorphism from $\mathbb{P}^*$ onto $\mathscr{F}$. Henceforth, $\mathscr{F}$ will show not only the algebra of formal power series in t but also the vector space of all linear functionals on $\mathbb{P}$, and hence, an element $f(t)$ of $\mathscr{F}$ will be considered as not only a formal power series but also a linear functional. It can be observed from (35) that

$$\left\langle e^{zt}|x^m\right\rangle = z^m \tag{37}$$

and so

$$\left\langle e^{zt}|p(x)\right\rangle = p(z) \quad (p(x) \in \mathbb{P}).$$

The order $o(f(t))$ of a power series $f(t)$ is the smallest integer ϖ for which the coefficient of t^{ϖ} does not disappear. If $o(f(t)) = 0$, then $f(t)$ is termed an invertible series. A series $f(t)$ for which $o(f(t)) = 1$ will be termed a delta series (cf. [6–8, 15, 21–23, 27]).

We utilize the symbol t^{ϖ} for the ϖ-th derivative operator on $\mathbb{P}$ provided by:

$$t^{\varpi}x^m = \begin{cases} \frac{m!}{(m-\varpi)!}x^{m-\varpi}, & \varpi \leq m, \\ 0, & \varpi > m. \end{cases}$$

If $g(t)$, $f(t)$ are in $\mathscr{F}$, then

$$\langle f(t)g(t)|p(x)\rangle = \langle f(t)|g(t)p(x)\rangle = \langle g(t)|f(t)p(x)\rangle \tag{38}$$

for all polynomials $p(x)$. Note that, for all polynomials $p(x)$ and for all $f(t)$ in $\mathscr{F}$,

$$p(x)=\sum_{\varpi=0}^{\infty}\langle t^{\varpi}|p(x)\rangle\frac{x^{\varpi}}{\varpi!} \text{ and } f(t)=\sum_{\varpi=0}^{\infty}\langle f(t)|x^{\varpi}\rangle\frac{t^{\varpi}}{\varpi!}. \tag{39}$$

Utilizing (39), we get

$$p^{(\varpi)}(x):=D^{\varpi}p(x)=\sum_{\kappa=\varpi}^{\infty}\frac{\langle t^{\kappa}|p(x)\rangle}{\kappa!}x^{\kappa-\varpi}\prod_{s=1}^{\varpi}(\kappa-s+1)$$

providing

$$p^{(\varpi)}(0)=\langle t^{\varpi}|p(x)\rangle \quad \text{and} \quad \langle 1|p^{(\varpi)}(x)\rangle=p^{(\varpi)}(0). \tag{40}$$

Hence, from (40), we notice that

$$t^{\varpi}p(x)=p^{(\varpi)}(x). \tag{41}$$

Let $g(t)\in\mathscr{F}$ be an invertible series and $f(t)\in\mathscr{F}$ be a delta series. Then, there exists a unique sequence $s_m(x)$ of polynomials fulfilling the characteristic below:

$$\langle g(t)f(t)^{\varpi}|s_m(x)\rangle=m!\delta_{m,\,\varpi} \qquad (m,\,\varpi\geq 0) \tag{42}$$

which is termed an orthogonality condition for any Sheffer sequence, see [6–8, 15, 21–23, 27].

The sequence $s_m(x)$ is termed the Sheffer sequence for the pair of $(g(t),\,f(t))$, or this $s_m(x)$ is Sheffer for $(g(t),\,f(t))$, which is indicated by $s_m(x)\sim(g(t),\,f(t))$.

Let $s_m(x)$ be Sheffer for $(g(t),\,f(t))$. Then, for any polynomial $p(x)$, and for any $h(t)$ in $\mathscr{F}$, we have

$$p(x)=\sum_{\varpi=0}^{\infty}\langle g(t)f(t)^{\varpi}|p(x)\rangle\frac{s_{\varpi}(x)}{\varpi!} \text{ and } h(t)=\sum_{\varpi=0}^{\infty}\langle h(t)|s_{\varpi}(x)\rangle\frac{g(t)f(t)^{\varpi}}{\varpi!}. \tag{43}$$

Moreover, for all x in $\mathbb{C}$, the sequence $s_m(x)$ is Sheffer for $(g(t),\,f(t))$ if and only if

$$\sum_{m=0}^{\infty}s_m(x)\frac{t^m}{m!}=\frac{e^{x\overline{f}(t)}}{g(\overline{f}(t))}, \tag{44}$$

where $f\left(\overline{f}(t)\right)=\overline{f}(f(t))=t$.

An essential characteristic of the Sheffer sequence $s_m(x)$ possessing $(g(t),\,t)$ is the Appell sequence. It is also termed Appell for $g(t)$ in conjunction with the outcome below:

$$s_m(x) = \frac{1}{g(t)} x^m \Leftrightarrow t s_m(x) = m s_{m-1}(x). \tag{45}$$

A more significant characteristic of the Sheffer sequence $s_m(x)$ is provided by

$$\sum_{m=0}^{\infty} s_m(x) \frac{t^m}{m!} = \frac{e^{xt}}{g(t)} \Leftrightarrow s_m(x) \text{ is Appell for } g(t) \quad (x \in \mathbb{C}).$$

The theory of umbral calculus has been utilized in the studies of several special polynomials with various generalizations by many authors recently, cf. [6–8, 15, 21–23, 27].

Recall from (17) that

$$\sum_{m=0}^{\infty} {}_{CB}E_m(x; z) \frac{t^m}{m!} = \frac{2}{e^{\frac{t}{2}} + e^{-\frac{t}{2}}} e^{xt + z\left(e^{\frac{t}{2}} - e^{-\frac{t}{2}}\right)}. \tag{46}$$

As t goes to 0 in (46) gives ${}_{CB}E(x; z) = 1$. This implies that the generating function of the polynomials ${}_{CB}E_m(x; z)$ is invertible.

We now provide multifarious formulas of central Bell-based type 2 Euler polynomials arising from umbral calculus. We readily observe from (44) and (45) that

$${}_{CB}E_m(x; z) \sim \left(\frac{e^{\frac{t}{2}} + e^{-\frac{t}{2}}}{2} e^{-z\left(e^{\frac{t}{2}} - e^{-\frac{t}{2}}\right)}, t \right) \tag{47}$$

and

$$t \; {}_{CB}E_m(x; z) = m \; {}_{CB}E_{m-1}(x; z). \tag{48}$$

It can be seen from (48) that ${}_{CB}E_m(x; z)$ is Appell for $\frac{e^{\frac{t}{2}} + e^{-\frac{t}{2}}}{2} e^{-z\left(e^{\frac{t}{2}} - e^{-\frac{t}{2}}\right)}$.

By (36) and (46), we get

$$\begin{aligned} {}_{CB}E_m(x; z) &= \frac{2}{e^{\frac{t}{2}} + e^{-\frac{t}{2}}} e^{z\left(e^{\frac{t}{2}} - e^{-\frac{t}{2}}\right)} x^m = e^{z\left(e^{\frac{t}{2}} - e^{-\frac{t}{2}}\right)} E_m(x) \\ &= \sum_{\varpi=0}^{m} \binom{m}{\varpi} \phi_{\varpi}^{(c)}(z) E_{m-\varpi}(x) \end{aligned} \tag{49}$$

which coincides with the consequence (23). By (41) and (46), we observe that

$$ {}_{CB}E_m(x;z) = \frac{2}{e^{\frac{t}{2}}+e^{-\frac{t}{2}}} e^{z\left(e^{\frac{t}{2}}-e^{-\frac{t}{2}}\right)} x^m = \frac{2}{e^{\frac{t}{2}}+e^{-\frac{t}{2}}} \phi_m^{(c)}(x;z) $$

$$ = \sum_{\varpi=0}^{\infty} \frac{E_\varpi}{\varpi!} t^\varpi \phi_m^{(c)}(x;z) = \sum_{\varpi=0}^{m} \binom{m}{\varpi} E_\varpi \phi_{m-\varpi}^{(c)}(x;z). $$

Suppose that

$$ \mathfrak{h}(t,z) = \mathfrak{g}(t)\mathfrak{h}(z) = \frac{e^{\frac{t}{2}}+e^{-\frac{t}{2}}}{2} e^{-z\left(e^{\frac{t}{2}}-e^{-\frac{t}{2}}\right)}. $$

The following theorem indicates that any polynomial can be written a linear combination of *CBBT2EP* and a sequence c_m derived below.

Theorem 19 For all $p(x) \in \mathbb{P}_m = \{q(x) \in \mathbb{C}[x] : \deg q(x) \le m\}$, there exist constants $c_0, c_1, \ldots, c_m$ such that $p(x) = \sum_{\varpi=0}^{m} c_\varpi \, {}_{CB}E_\varpi(x;z)$, where

$$ c_\varpi = \frac{1}{\varpi!} \left\langle \mathfrak{h}(t,z) t^\varpi \mid p(x) \right\rangle. \tag{50} $$

Proof. By (42), (44) and (47), we observe that

$$ \left\langle \mathfrak{h}(t,z) t^\varpi \mid {}_{CB}E_m(x;z) \right\rangle = m!\delta_{m,\varpi} \qquad (m, \varpi \ge 0), $$

which means

$$ \left\langle \mathfrak{h}(t,z) t^\varpi \mid p(x) \right\rangle = \sum_{\kappa=0}^{m} c_\kappa \left\langle \mathfrak{h}(t,z) t^\varpi \middle| {}_{CB}E_m(x;z) \right\rangle $$

$$ = \sum_{\kappa=0}^{m} c_\kappa \kappa! \delta_{\kappa,\varpi} = \varpi! c_\varpi, $$

which provides the assertion in (50). □

CBBT2EP can be derived by the following operational rule.

Theorem 20 The following identity holds for $m \in \mathbb{N}_0$:

$$ {}_{CB}E_m(x;z) = e^{z\left(e^{\frac{t}{2}}-e^{-\frac{t}{2}}\right)} E_m(x). \tag{51} $$

Proof. We observe from (49) that

$$\mathfrak{h}(-z)E_m(x)=\mathfrak{h}(-z)\frac{2}{e^{\frac{t}{2}}+e^{-\frac{t}{2}}}x^m,$$

which yields the claimed relation (51). □

We get a linear functional rule as follows.

Theorem 21 The following equality holds for $p(x)\in\mathbb{P}$:

$$\langle\,\mathfrak{h}(t,z)|\,p(x)\rangle=\frac{1}{2}\sum_{\varpi=0}^{\infty}\sum_{\kappa=0}^{\varpi}(-z)^{\varpi}\binom{\varpi}{\kappa}(-1)^{\kappa}\left(p\left(\frac{\varpi}{2}+\kappa+\frac{1}{2}\right)+p\left(\frac{\varpi}{2}+\kappa-\frac{1}{2}\right)\right).$$

Proof. By (37), we get

$$\langle\,\mathfrak{h}(t,z)|x^m\rangle=\frac{1}{2}\left\langle\sum_{\varpi=0}^{\infty}(-z)^{\varpi}\left(e^{\frac{t}{2}}-e^{-\frac{t}{2}}\right)^{\varpi}\left(e^{\frac{t}{2}}+e^{-\frac{t}{2}}\right)\middle|x^m\right\rangle$$

$$=\frac{1}{2}\sum_{\varpi=0}^{\infty}\sum_{\kappa=0}^{\varpi}(-z)^{\varpi}\binom{\varpi}{\kappa}(-1)^{\kappa}\left\langle\left(e^{\left(\frac{\varpi}{2}+\kappa+\frac{1}{2}\right)t}+e^{\left(\frac{\varpi}{2}+\kappa-\frac{1}{2}\right)t}\right)|x^m\right\rangle$$

$$=\frac{1}{2}\sum_{\varpi=0}^{\infty}\sum_{\kappa=0}^{\varpi}(-z)^{\varpi}\binom{\varpi}{\kappa}(-1)^{\kappa}\left(\left(\frac{\varpi}{2}+\kappa+\frac{1}{2}\right)^m+\left(\frac{\varpi}{2}+\kappa-\frac{1}{2}\right)^m\right),$$

which yields the desired consequence. □

We get a linear functional rule associated with *BCBP* as follows.

Theorem 22 The following equality holds for $m\in\mathbb{N}$:

$$\langle\,\mathfrak{h}(t,z)|x^m\rangle=\frac{\phi_m^{(c)}\left(\frac{1}{2};-z\right)+\phi_m^{(c)}\left(-\frac{1}{2};-z\right)}{2}.$$

Proof. By (37), we acquire

$$\langle\,\mathfrak{h}(t,z)|x^m\rangle=\frac{1}{2}\left\langle\left(e^{\frac{t}{2}}+e^{-\frac{t}{2}}\right)e^{-z\left(e^{\frac{t}{2}}-e^{-\frac{t}{2}}\right)}\middle|x^m\right\rangle$$

$$=\frac{1}{2}\left\langle\sum_{\varpi=0}^{\infty}\frac{\left(\phi_{\varpi}^{(c)}\left(\frac{1}{2};-z\right)+\phi_{\varpi}^{(c)}\left(-\frac{1}{2};-z\right)\right)}{\varpi!}t^{\varpi}|x^m\right\rangle$$

$$=\frac{1}{2}\sum_{\varpi=0}^{\infty}\frac{\left(\phi_{\varpi}^{(c)}\left(\frac{1}{2};-z\right)+\phi_{\varpi}^{(c)}\left(-\frac{1}{2};-z\right)\right)}{\varpi!}m!\delta_{m,\varpi}$$

$$= \frac{\phi_m^{(c)}\left(\frac{1}{2}; -z\right) + \phi_m^{(c)}\left(-\frac{1}{2}; -z\right)}{2},$$

which ends the proof. □

An integral representation for *CBBT2EP* is provided using a linear functional for *CBBT2EP* by the following theorem.

Theorem 23 (Integral representation) Let $m \in \mathbb{N}$. Then we have

$$\langle \mathfrak{h}(t, z) |\ {}_{CB}E_m(x; z) \rangle = \frac{1}{2} \sum_{\varpi=0}^{\infty} \sum_{\kappa=0}^{\varpi} (-z)^{\varpi} \binom{\varpi}{\kappa} \frac{(-1)^{\kappa}}{m-1} \int_{\kappa}^{\kappa+1} {}_{CB}E_{m-1}\left(\frac{\varpi - 1}{2} + u; z\right) du.$$

Proof. By utilizing (37) and (48), we get

$$\begin{aligned}
&\langle \mathfrak{h}(t, z) |\ {}_{CB}E_m(x; z) \rangle \\
&= \frac{1}{2} \left\langle \sum_{\varpi=0}^{\infty} (-z)^{\varpi} \left(e^{\frac{t}{2}} - e^{-\frac{t}{2}}\right)^{\varpi} \left(e^{\frac{t}{2}} + e^{-\frac{t}{2}}\right) \middle|\ {}_{CB}E_m(x; z) \right\rangle \\
&= \frac{1}{2} \sum_{\varpi=0}^{\infty} \sum_{\kappa=0}^{\varpi} (-z)^{\varpi} \binom{\varpi}{\kappa} (-1)^{\kappa} \left\langle \left(e^{\left(\frac{\varpi}{2} + \kappa + \frac{1}{2}\right)t} + e^{\left(\frac{\varpi}{2} + \kappa - \frac{1}{2}\right)t} \right) |\ {}_{CB}E_m(x; z) \right\rangle \\
&= \frac{1}{2} \sum_{\varpi=0}^{\infty} \sum_{\kappa=0}^{\varpi} (-z)^{\varpi} \binom{\varpi}{\kappa} (-1)^{\kappa} \left({}_{CB}E_m\left(\frac{\varpi}{2} + \kappa + \frac{1}{2}; z\right) - {}_{CB}E_m\left(\frac{\varpi}{2} + \kappa - \frac{1}{2}; z\right) \right) \\
&= \frac{1}{2} \sum_{\varpi=0}^{\infty} \sum_{\kappa=0}^{\varpi} (-z)^{\varpi} \binom{\varpi}{\kappa} \frac{(-1)^{\kappa}}{m-1} \int_{\kappa}^{\kappa+1} {}_{CB}E_{m-1}\left(\frac{\varpi - 1}{2} + u; z\right) du,
\end{aligned}$$

which completes the proof. □

The following theorem is beneficial for deriving any polynomial by combining *CBBT2EP* in a linear combination.

Theorem 24 The following equality holds for $p(x) \in \mathbb{P}_m$:

$$p(x) = \sum_{\varpi=0}^{m} \sum_{j=0}^{\infty} \sum_{\kappa=0}^{j} \binom{j}{\kappa} {}_{CB}E_{\varpi}(x; z) \frac{z^j}{\varpi!} \frac{(-1)^{j+\kappa}}{2} \left(p^{(\varpi)}\left(\frac{j+1}{2} + \kappa\right) + p^{(\varpi)}\left(\frac{j-1}{2} + \kappa\right) \right). \tag{52}$$

Proof. We observe from Theorem 19 that

$$p(x) = \sum_{\varpi=0}^{m} b_{\varpi}\ {}_{CB}E_{\varpi}(x; z)$$

with

$$\left\langle \mathfrak{h}(t,z)t^{\varpi} | p(x) \right\rangle = \varpi! b_{\varpi}. \tag{53}$$

Hence, from (53), we have

$$b_{\varpi} = \frac{1}{\varpi!} \left\langle \mathfrak{h}(t,z)t^{\varpi} | p(x) \right\rangle$$

$$= \frac{1}{\varpi!} \sum_{j=0}^{\infty} \sum_{\kappa=0}^{j} \frac{(-z)^j}{2} \binom{j}{\kappa} (-1)^{\kappa} \left\langle \left(e^{\left(\frac{j}{2}+\kappa+\frac{1}{2}\right)t} + e^{\left(\frac{j}{2}+\kappa-\frac{1}{2}\right)t} \right) \middle| t^{\varpi} p(x) \right\rangle$$

$$= \frac{1}{\varpi!} \sum_{j=0}^{\infty} \sum_{\kappa=0}^{j} \frac{(-z)^j}{2} \binom{j}{\kappa} (-1)^{\kappa} \left\langle \left(e^{\left(\frac{j}{2}+\kappa+\frac{1}{2}\right)t} + e^{\left(\frac{j}{2}+\kappa-\frac{1}{2}\right)t} \right) \middle| p^{(\varpi)}(x) \right\rangle$$

$$= \frac{1}{\varpi!} \sum_{j=0}^{\infty} \sum_{\kappa=0}^{j} \frac{(-z)^j}{2} \binom{j}{\kappa} (-1)^{\kappa} \left(p^{(\varpi)} \left(\frac{j+1}{2} + \kappa \right) + p^{(\varpi)} \left(\frac{j-1}{2} + \kappa \right) \right).$$

Therefore, the proof is completed. □

Recall from (17) that Central Bell-Based Type 2 Euler Polynomials (abbreviated with *CBBT2EPr*) of order $r \in \mathbb{N}_0$ are provided by

$$\sum_{m=0}^{\infty} {}_{CB}E_m^{(r)}(x;z) \frac{t^m}{m!} = \left(\frac{2}{e^{\frac{t}{2}} + e^{-\frac{t}{2}}} \right)^r e^{xt+z\left(e^{\frac{t}{2}} - e^{-\frac{t}{2}}\right)} \quad (|t| < \pi). \tag{54}$$

If t goes to 0 on the above, it gives ${}_{CB}E_m^{(r)}(x;z) = 1$, which yields an invertible series for generating function of the polynomials ${}_{CB}E_m^{(r)}(x;z)$.

Let

$$\mathfrak{h}^r(t,z) = \mathfrak{g}^r(t)\mathfrak{h}(z) = \frac{\left(e^{\frac{t}{2}} + e^{-\frac{t}{2}}\right)^r}{2^r} e^{-z\left(e^{\frac{t}{2}} - e^{-\frac{t}{2}}\right)}.$$

We see that $\mathfrak{h}^r(t,z)$ is an invertible series. By (54), ${}_{CB}E_m^{(r)}(x;z)$ is an Appell for $\mathfrak{h}^r(t,z)$. Thus, from (45), we observe that

$${}_{CB}E_m^{(r)}(x;z) = \frac{1}{\mathfrak{h}^r(t,z)} x^m, \tag{55}$$

and

$$t\ {}_{CB}E_m^{(r)}(x;z) = m\ {}_{CB}E_{m-1}^{(r)}(x;z).$$

Thus, we have

$${}_{CB}E_m^{(r)}(x;z) \sim (\mathfrak{h}^r(t,z), t).$$

If $f_1(t)$, ..., $f_j(t)$ are in $\mathscr{F}$, then

$$\langle f_1(t)...f_j(t)|x^m\rangle = \sum_{i_1+i_2+...+i_j=m} \binom{m}{i_1, ..., i_j} \langle f_1(t)|x^{i_1}\rangle ... \langle f_j(t)|x^{i_j}\rangle,$$

where

$$\binom{m}{i_1, \cdots, i_r} = \frac{m!}{i_1! \cdots i_r!}.$$

Also, by (35) and (54), we acquire

$$\left\langle \frac{2^r}{\left(e^{\frac{t}{2}} + e^{-\frac{t}{2}}\right)^r} e^{wt+z\left(e^{\frac{t}{2}} - e^{-\frac{t}{2}}\right)} \middle| x^m \right\rangle = {}_{CB}E_m^{(r)}(w;z) = \sum_{\kappa=0}^{m} \binom{m}{\kappa} {}_{CB}E_\kappa^{(r)}(z)\, w^{m-\kappa}. \tag{56}$$

A product formula for *T2EPr* is provided by the following theorem.
Theorem 25 The following equality holds for $m \in \mathbb{N}$:

$${}_{CB}E_m^{(r)}(z) = \sum_{i_1+\cdots+i_r=m} \binom{m}{i_1, \cdots, i_r} {}_{CB}E_{i_r}(z) \prod_{j=1}^{r-1} E_{i_j}. \tag{57}$$

Proof. Also, we observe that

$$\left\langle \frac{2^r}{\left(e^{\frac{t}{2}} + e^{-\frac{t}{2}}\right)^r} e^{z\left(e^{\frac{t}{2}} - e^{-\frac{t}{2}}\right)} \middle| x^m \right\rangle$$

$$= \left\langle \frac{2}{e^{\frac{t}{2}} + e^{-\frac{t}{2}}} \times \cdots \times \frac{2}{e^{\frac{t}{2}} + e^{-\frac{t}{2}}} \frac{2e^{z\left(e^{\frac{t}{2}} - e^{-\frac{t}{2}}\right)}}{e^{\frac{t}{2}} + e^{-\frac{t}{2}}} \middle| x^m \right\rangle$$

$$= \sum_{i_1+\cdots+i_r=m} \binom{m}{i_1, \cdots, i_r} {}_{CB}E_{i_r}(z) \times E_{i_1} \times \cdots \times E_{i_{r-1}}. \tag{58}$$

By using (56), we have

$$\left\langle \frac{2^r}{\left(e^{\frac{t}{2}} + e^{-\frac{t}{2}}\right)^r} e^{z\left(e^{\frac{t}{2}} - e^{-\frac{t}{2}}\right)} | x^m \right\rangle = {}_{CB}E_m^{(r)}(z). \tag{59}$$

Therefore, the assertion (57) can be seen from (58) and (59). □

Taking $q(x) = {}_{CB}E_m^{(r)}(x; z) \in \mathbb{P}_m$ in Theorem 24, we give a special case below.

Corollary 3 The following equality holds for $m, r \in \mathbb{N}_0$:

$${}_{CB}E_m^{(r)}(x; z) = \sum_{\varpi=0}^{m} \sum_{j=0}^{\infty} \sum_{\kappa=0}^{j} \binom{j}{\kappa} \binom{m}{\varpi} {}_{CB}E_{\varpi}(x; z) z^j (-1)^{j+\kappa} {}_{CB}E_m^{(r-1)}\left(\frac{j-1}{2} + \kappa; z\right).$$

Proof. By Theorem 24, we observe that

$${}_{CB}E_m^{(r)}(x; z) = \sum_{\varpi=0}^{m} \sum_{j=0}^{\infty} \sum_{\kappa=0}^{j} \binom{j}{\kappa} {}_{CB}E_{\varpi}(x; z) \frac{z^j}{\varpi!} \frac{(-1)^{j+\kappa}}{2} \binom{m}{\varpi}$$

$$\times \left({}_{CB}E_{m-\varpi}^{(r)}\left(\frac{j+1}{2} + \kappa; z\right) + {}_{CB}E_{m-\varpi}^{(r)}\left(\frac{j-1}{2} + \kappa; z\right)\right). \tag{60}$$

From (54), we have

$$\sum_{m=0}^{\infty} \left({}_{CB}E_m^{(r)}\left(\frac{j+1}{2} + \kappa; z\right) + {}_{CB}E_m^{(r)}\left(\frac{j-1}{2} + \kappa; z\right)\right) \frac{t^m}{m!}$$

$$= \left(\frac{2}{e^{\frac{t}{2}} + e^{-\frac{t}{2}}}\right)^r e^{\left(\frac{j-1}{2} + \kappa\right) + z\left(e^{\frac{t}{2}} - e^{-\frac{t}{2}}\right)} \left(e^{\frac{t}{2}} + e^{-\frac{t}{2}}\right)$$

$$= 2\left(\frac{2}{e^{\frac{t}{2}} + e^{-\frac{t}{2}}}\right)^{r-1} e^{\left(\frac{j-1}{2} + \kappa\right) + z\left(e^{\frac{t}{2}} - e^{-\frac{t}{2}}\right)}$$

$$= 2\sum_{m=0}^{\infty} {}_{CB}E_m^{(r-1)}\left(\frac{j-1}{2} + \kappa; z\right) \frac{t^m}{m!},$$

which means

$$ {}_{CB}E_m^{(r)}\left(\frac{j+1}{2}+\kappa;z\right)+{}_{CB}E_m^{(r)}\left(\frac{j-1}{2}+\kappa;z\right)=2\,{}_{CB}E_m^{(r-1)}\left(\frac{j-1}{2}+\kappa;z\right). \tag{61} $$

From (60) and (61), the proof is completed. □

Theorem 26 The following relation holds for $m \in \mathbb{N}_0$:

$$ e^{z\left(e^{\frac{t}{2}}-e^{-\frac{t}{2}}\right)}E_m^{(r)}(x) = {}_{CB}E_m^{(r)}(x;z). \tag{62} $$

Proof. From (55), we get

$$ e^{z\left(e^{\frac{t}{2}}-e^{-\frac{t}{2}}\right)}E_m^{(r)}(x) = e^{z\left(e^{\frac{t}{2}}-e^{-\frac{t}{2}}\right)}\left(\frac{2}{e^{\frac{t}{2}}+e^{-\frac{t}{2}}}\right)^r x^m, $$

which means the claimed relationship (62). □

Using the following theorem, one can obtain any polynomial by combining the central Bell-based type 2 Euler polynomials of order r in a linear way.

Theorem 27 For $m \in \mathbb{N}_0$, we have

$$ q(x) = \sum_{\varpi=0}^{m} b_\varpi^r\,{}_{CB}E_\varpi^{(r)}(x;z) \in \mathbb{P}_m, $$

where

$$ b_\varpi^r = \sum_{j=0}^{\infty}\sum_{u=0}^{j}\sum_{\kappa=0}^{r}\binom{r}{\kappa}\frac{j!}{\varpi!u!}(-z)^j T(j,u)\,q^{(\varpi+u)}\left(\frac{r}{2}-\kappa\right). $$

Proof. Assume that

$$ q(x) = \sum_{\varpi=0}^{m} b_\varpi^r\,{}_{CB}E_\varpi^{(r)}(x;z) \in \mathbb{P}_m. \tag{63} $$

To find the coefficient b_ϖ^r, similar to proof of Theorem 24, we first observe that

$$ \left\langle \mathfrak{h}^r(t,x)t^\varpi | q(x)\right\rangle = \sum_{\kappa=0}^{m} b_\kappa^r \left\langle \mathfrak{h}^r(t,x)t^\varpi \middle| {}_{CB}E_m^{(r)}(x;z)\right\rangle $$

$$ = \sum_{\kappa=0}^{m} b_\kappa^r \kappa!\delta_{\kappa,\varpi} = \varpi! b_\varpi^r. $$

For $\varpi \geq r$, then

$$b_{\varpi}^{r} = \frac{1}{\varpi!} \left\langle \mathfrak{h}^{r}(t, x) t^{\varpi} | q(x) \right\rangle$$

$$= \frac{1}{\varpi!} \left\langle \sum_{j=0}^{\infty} j! (-z)^{j} \left(e^{\frac{t}{2}} + e^{-\frac{t}{2}}\right)^{r} \frac{\left(e^{\frac{t}{2}} - e^{-\frac{t}{2}}\right)^{j}}{j!} t^{\varpi} | q(x) \right\rangle$$

$$= \frac{1}{\varpi!} \left\langle \sum_{j=0}^{\infty} \sum_{u=0}^{j} \sum_{\kappa=0}^{r} \binom{r}{\kappa} j! (-z)^{j} \frac{T(j, u)}{u!} e^{t\left(\frac{r}{2} - \kappa\right)} t^{\varpi + u} | q(x) \right\rangle$$

$$= \sum_{j=0}^{\infty} \sum_{u=0}^{j} \sum_{\kappa=0}^{r} \frac{j!}{\varpi! u!} \binom{r}{\kappa} (-z)^{j} T(j, u) \left\langle e^{t\left(\frac{r}{2} - \kappa\right)} | q^{(\varpi + u)}(x) \right\rangle$$

$$= \sum_{j=0}^{\infty} \sum_{u=0}^{j} \sum_{\kappa=0}^{r} \binom{r}{\kappa} \frac{j!}{\varpi! u!} (-z)^{j} T(j, u) q^{(\varpi + u)} \left(\frac{r}{2} - \kappa\right).$$

Henceforth, by (63) and coefficient b_{ϖ}^{r}, the proof is done. □

As a last result of this section, we provide the following equality.

Corollary 4 The following summation formula

$$ {}_{CB}E_{m}(x; w) = \sum_{\varpi=0}^{m} \sum_{j=0}^{\infty} \sum_{u=0}^{j} \sum_{\kappa=0}^{r} \binom{r}{\kappa} \frac{j! (m)_{\varpi + u}}{\varpi! u!}$$

$$\times (-z)^{j} T(j, u) \, {}_{CB}E_{m-\varpi-u}\left(\frac{r}{2} - \kappa; w\right) {}_{CB}E_{\varpi}^{(r)}(x; z)$$

holds for $m, r \in \mathbb{N}_0$.

5. Further remarks

We here provide a determinantal representation for *CBBT2EPα*.

Suppose that

$$\Omega(t) = \sum_{m=0}^{\infty} \mu_{m, \alpha} \frac{t^{m}}{m!} = \left[\frac{2}{e^{\frac{t}{2}} + e^{-\frac{t}{2}}}\right]^{-\alpha}, \tag{64}$$

where μ_m is a sequence. Then, we derive from (9) and (64) that

$$\left(\sum_{m=0}^{\infty} E_m^{(\alpha)} \frac{t^m}{m!}\right) \Omega(t) = \sum_{m=0}^{\infty} \left(\sum_{\kappa=0}^{m} \binom{m}{\kappa} E_\kappa^{(\alpha)} \mu_{m-\kappa}\right) \frac{t^m}{m!},$$

which yields

$$\sum_{\kappa=0}^{m} \binom{m}{\kappa} E_\kappa^{(\alpha)} \mu_{m-\kappa} = \begin{cases} 1 \text{ for } m=0, \\ \\ 0 \text{ for } m>0. \end{cases}$$

Therefore, we get

$$\begin{cases} \mu_0 = \frac{1}{E_0^{(\alpha)}}, \\ \\ \mu_m = -\left(\sum_{\kappa=1}^{m} \binom{m}{\kappa} E_\kappa^{(\alpha)} \mu_{m-\kappa}\right). \end{cases}$$

We give a determinantal representation for *CBBT2EPα* as follows.

Theorem 28 We have

$$_{CB}E_0^{(\alpha)}(x;z) = \frac{1}{\mu_0}$$

and

$$_{CB}E_m^{(\alpha)}(x;z) = (-1)^m \begin{vmatrix} 1 & \phi_1^{(c)}(x;z) & \phi_2^{(c)}(x;z) & \cdots & \phi_{m-1}^{(c)}(x;z) & \phi_m^{(c)}(x;z) \\ \mu_0 & \mu_1 & \mu_2 & \cdots & \mu_{m-1} & \mu_m \\ 0 & \mu_0 & \binom{2}{1}\mu_1 & \cdots & \binom{m-1}{1}\mu_{m-2} & \binom{m}{1}\mu_{m-1} \\ 0 & 0 & \ddots & & \binom{m-1}{2}\mu_{m-3} & \binom{m}{2}\mu_{m-2} \\ \vdots & \vdots & & \ddots & \vdots & \vdots \\ \vdots & \vdots & & & & \\ 0 & 0 & \cdots & \cdots & \mu_0 & \binom{m}{m-1}\mu_1 \end{vmatrix}. \tag{65}$$

Proof. It can be observed from (4), (17) and (64) that

$$\sum_{m=0}^{\infty} \phi_m^{(c)}(x; z) \frac{t^m}{m!} = \sum_{m=0}^{\infty} \sum_{\kappa=0}^{m} \binom{m}{\kappa} {}_{CB}E_{\kappa}^{(\alpha)}(x; z) \mu_{m-\kappa} \frac{t^m}{m!},$$

which means the following infinite system of equations in the unknown variables:

$$\phi_0^{(c)}(x; z) = {}_{CB}E_0^{(\alpha)}(x; z) \mu_0,$$

$$\phi_1^{(c)}(x; z) = {}_{CB}E_0^{(\alpha)}(x; z) \mu_1 + {}_{CB}E_1^{(\alpha)}(x; z) \mu_0,$$

$$\vdots$$

$$\phi_m^{(c)}(x; z) = {}_{CB}E_0^{(\alpha)}(x; z) \mu_m + \binom{m}{1} {}_{CB}E_1^{(\alpha)}(x; z) \mu_{m-1} + \cdots + {}_{CB}E_m^{(\alpha)}(x; z) \mu_0.$$

Because of the specific structure of the aforesaid system (lower triangular), we can compute the unknown variables of ${}_{CB}E_m^{(\alpha)}(x; z)$ by exclusively using the first $m+1$ equations. This can be accomplished by employing Cramer's rule, which facilitates the computation of the solution:

$${}_{CB}E_m^{(\alpha)}(x; z) = \frac{1}{(\mu_0)^{m+1}} \begin{vmatrix} \mu_0 & 0 & 0 & 0 & \cdots & \phi_0^{(c)}(x; z) \\ \mu_1 & \mu_0 & 0 & 0 & \cdots & \phi_1^{(c)}(x; z) \\ \mu_2 & \binom{2}{1}\mu_1 & \mu_0 & 0 & \cdots & \phi_2^{(c)}(x; z) \\ \vdots & \vdots & & \ddots & & \vdots \\ \mu_{m-1} & \binom{m-1}{1}\mu_{m-2} & \binom{m-2}{2}\mu_{m-3} & \cdots & \cdots & \phi_{m-1}^{(c)}(x; z) \\ \mu_m & \binom{m}{1}\mu_{m-1} & \binom{m}{2}\mu_{m-2} & \binom{m}{3}\mu_{m-3} & \cdots & \phi_m^{(c)}(x; z) \end{vmatrix},$$

which can also be rewritten as

$$
{}_{CB}E_m^{(\alpha)}(x;z) = \frac{1}{(\mu_0)^{m+1}} \begin{vmatrix} \mu_0 & \mu_1 & \mu_2 & \cdots & \mu_{m-1} & \mu_m \\ 0 & \mu_0 & \binom{2}{1}\mu_1 & \cdots & \binom{m-1}{1}\mu_{m-2} & \binom{m}{1}\mu_{m-1} \\ 0 & 0 & \ddots & & \binom{m-1}{2}\mu_{m-3} & \binom{m}{2}\mu_{m-2} \\ \vdots & \vdots & & \ddots & \vdots & \vdots \\ \vdots & \vdots & & & & \\ 0 & 0 & \cdots & \cdots & \mu_0 & \binom{m}{m-1}\mu_1 \\ 1 & \phi_1^{(c)}(x;z) & \phi_2^{(c)}(x;z) & \cdots & \phi_{m-1}^{(c)}(x;z) & \phi_m^{(c)}(x;z) \end{vmatrix},
$$

which completes the proof. □

By means of (65), it is not difficult to list the central Bell-based type 2 Euler polynomials of order α.

6. Conclusions

In recent years, Duran [8] considered the central Bell-based type 2 Bernoulli polynomials of order α given by

$$
\left(\frac{t}{e^{\frac{t}{2}} - e^{-\frac{t}{2}}}\right)^{\alpha} e^{xt+z\left(e^{\frac{t}{2}} - e^{-\frac{t}{2}}\right)} = \sum_{m=0}^{\infty} {}_{CB}b_m^{(\alpha)}(x;z)\frac{t^m}{m!} \quad (|t| < 2\pi)
$$

and derived many formulas and relations, covering several symmetric properties, derivative properties, summation formulas, and addition formulas. Inspired and motivated by the aforesaid study, in this paper, we have defined the central Bell-based type 2 Euler polynomials of order α provided below

$$
\left(\frac{2}{e^{\frac{t}{2}} + e^{-\frac{t}{2}}}\right)^{\alpha} e^{xt+z\left(e^{\frac{t}{2}} - e^{-\frac{t}{2}}\right)} = \sum_{m=0}^{\infty} {}_{CB}E_m^{(\alpha)}(x;z)\frac{t^m}{m!} \quad (|t| < \pi)
$$

and we have derived diverse formulas and properties covering several derivative properties and summation equalities. In addition, we have obtained multifarious implicit summation formulas and identities of symmetry for the mentioned polynomials. Besides, we have investigated diverse curious formulas of the central Bell-based type 2 Euler polynomials of order α arising from umbral calculus to possess alternative ways of achieving our outcomes. Lastly, we provided a determinantal representation for central Bell-based type 2 Euler polynomials. The consequences acquired in this study are extensions of the lots earlier outcomes, some of which are included in the references [13, 17–20, 29]. We will consider the possibility of analyzing the polynomials discussed in this paper in the context of the monomiality principle for future directions.

Author's contributions

All authors contributed equally to the article.

Conflict of interest

The authors declare no competing financial interest.

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