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Research Article

Constructing Optimal 2D Variable-Weight OOCs from Semicyclic Group Divisible Designs

Rongcun Qin[1], Hengming Zhao[2*]

[1] School of General Education, Guangxi Vocational University of Agriculture, Nanning, China
[2] School of Mathematics and Statistics, Nanning Normal University, Nanning, China
E-mail: hengmingzh@163.com

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Abstract: The OCDMA network using two Dimensional Variable-Weight Optical Orthogonal Codes (2D VWOOC) can support diverse Quality of Services (QoS) classes and multimedia services, and make the better use of bandwidth resources in fiber optical networks. To simplify practical implementation, the At Most One-Pulse Per Wavelength (AM-OPPW) restriction is often appended to a 2D VWOOC. In this paper, the upper bound on the size of AM-OPPW 2D VWOOCs is derived, semicyclic group divisible designs are introduced to construct AM-OPPW 2D VWOOCs, and an equivalence between optimal AM-OPPW 2D VWOOCs and Semicyclic Group Divisible Designs (SCGDDs) is established. Some direct and recursive constructions for SCGDDs are also presented. Consequently, several infinite families of optimal AM-OPPW 2D VWOOCs are obtained.

Keywords: semicylic group divisible design, two Dimensional (2D) variable-weight optical orthogonal code, optimal construction

MSC: 05B10, 94C30, 11B50

1. Introduction

Optical Code Division Multiple Access (OCDMA) has received much attention as an attractive way of satisfying the need of more reliable and faster communication systems and sharing the huge optical bandwidth among users. A key towards an effective OCDMA system is the choice of optical codes with good correlation properties. As a result, Optical Orthogonal Codes (OOCs) have been applied for OCDMA, see for example [1–4]. A one Dimensional Constant-Weight Optical Orthogonal Code (1D CWOOC) is a collection of 0, 1 sequences with constant weight having good correction properties. In recent years, 1D CWOOCs have been studied extensively and abundant research results have been obtained, the interested reader can refer to [5–11] and references therein. When 1D CWOOCs are used for multimedia applications, their correlation properties can be change. To overcome this problem, Yang introduced multimedia OCDMA communication system employing variable-weight OOCs (1D Variable-Weight Optical Orthogonal Codes (VWOOCs)) [12]. The multiple-weight property of the OOCs enables the system to satisfy multiple Quality of Services (QoS) requirement. Clearly, 1D CWOOCs is a special case of 1D VWOOCs. There are some literature on constructions and designs of 1D VWOOCs (see [12–17] and references therein).

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In optical fiber communication systems, one disadvantage of using 1D CWOOCs is that the length of the sequences increases rapidly when the weight of codes or the number of users is increased, which means that a large bandwidth expansion is required if a big number of codewords is needed. In order to reduce this problem, Yang introduced a two dimensional (constant-weight) OOC in [18]. Many researchers have been working on designs and constructions of 2D CWOOCs, see [19–28] and references therein.

As we know, a 2D CWOOC can not support multiple QoS requirement because all codewords have the same weight. A 2D Variable-Weight OOC (2D VWOOC) was proposed in [29] to satisfy multiple QoS requirements. Subsequently, the performance of an OCDMA system employing such a code was analyzed in [30]. When 2D VWOOCs are used for encoding different services, differentiated QoS for a distinct service can be supported. In doing so, the demands of differentiated QoS for different services and distinct subscribers can be satisfied and the usage of optical network resources can be optimized. Therefore, a 2D VWOOC has the potential to be widely applied. There are some results of optimal 2D VWOOCs with weights 3 and 4 in [31]. However, to our knowledge, very little is known about the constructions of optimal 2D VWOOCs under the At Most One Pulse Per Wavelength (AM-OPPW) constraint.

In this paper, we investigates combinatorial constructions for optimal AM-OPPW 2D VWOOCs. We first derive an upper bound on the size of such codes and then introduce Semicyclic Group Divisible Designs (SCGDDs) to construct AM-OPPW 2D VWOOCs. Based on this bound, direct constructions for SCGDDs are proposed via skew starters, while recursive constructions are provided using divisible group designs and incomplete difference matrices. Consequently, several infinite families of optimal AM-OPPW 2D VWOOCs with the weight set $\{3,\ 4\}$ are obtained from direct and recursive constructions.

The rest of this paper is arranged as follows. In Section II, we give some definitions for the related designs and state the fundamental relationship between these designs and optimal AM-OPPW 2D VWOOCs. Some direct constructions are obtained in Section III. In Section IV, some recursive constructions for optimal AM-OPPW 2D VWOOCs are presented by way of semicyclic group divisible designs and incomplete cyclic difference matrices. In Section V, we apply the constructions described in Sections III, IV together with the related design theory results to produce several infinite families of optimal $(n\times m,\ W,\ 1,\ Q)$ AM-OPPW 2D VWOOCs. Finally, we give some concluding remarks in Section VI.

The paper contains many terminologies, for convenience, we summarize some of them in Table 1.

Table 1. Terminologies in this paper

Terminology	Section/Page	Terminology	Section/Page
Two-dimensional Variable-Weight Optical Orthogonal Code (2D VWOOC)	2/3	Holey Packing (HP)	2/4
Group Divisible Design (GDD)	2/5	Semi-cyclic GDD	2/5
Semi-cyclic HP	2/5	Difference Packing (DP)	3/7
Cyclic GDD	3/7	Relative Difference Family (RDF)	3/7
Holey GDD	4/11	Semi-cyclic holey GDD	4/11
Incomplete Difference Matrix (IDM)	4/13		

2. Link between 2D VWOOCs and SCGDDs

This section will establish the equivalence relation between AM-OPPW 2D VWOOCs and SCGDDs. Let $W=\{w_1,\ w_2,\ \ldots,\ w_e\}$ be an ordering of a set consisting of e integers which are greater than 1. Also, let $Q=(q_1,\ q_2,\ \ldots,\ q_e)$ be an e-tuple of positive rational numbers with the condition that $\Sigma_{l=1}^{e}q_l=1$. Without losing generality, we assume that $w_1<w_2<\ldots<w_e$. A two-dimensional $(n\times m,\ W,\ 1,\ Q)$ variable-weight optical orthogonal code (abbreviated as $(n\times m,\ W,\ 1,\ Q)$ 2D VWOOC) is a set of $n\times m$ (0, 1)-matrices (referred to as codewords) that satisfy the following three properties:

(1) Weight Distribution: Every codeword within $\mathscr{C}$ has a weight that belongs to the set W; moreover, q_l represents the ratio of codewords with a weight of w_l, where $1\leq l\leq e$;

(2) Periodic Auto-correlation: For any matrix $(x_{i,\ j}) \in \mathscr{C}$ with weight w_l from the set W and any integer $0 < \tau < m$, it satisfied that $\sum_{i=0}^{n-1} \sum_{j=0}^{m-1} x_{i,\ j} x_{i,\ j \oplus \tau} \leq 1$;

(3) Periodic Cross-correlation: For any two distinct matrices $(x_{i,\ j})$ and $(y_{i,\ j})$ in $\mathscr{C}$ and any integer $0 \leq \tau < m$, it satisfied that $\sum_{i=0}^{n-1} \sum_{j=0}^{m-1} x_{i,\ j} x_{i,\ j \oplus \tau} \leq 1$, where the addition $\oplus$ is reduced modulo m.

For the convenience of practical implementation, the following two additional restrictions on the placement of pulses are often placed in the matrices of a 2D VWOOC, $\mathscr{C}$:

(1) One Pulse Per Wavelength (OPPW): The weight of each row in any matrix belonging to $\mathscr{C}$ is equal to 1.

(2) At Most One Pulse Per Wavelength (AM-OPPW): The weight of each row in any matrix of $\mathscr{C}$ is no more than 1.

In the following, a 2D VWOOC with the OPPW (AM-OPPW) restriction is simply said to be an OPPW (AM-OPPW) 2D VWOOC. One say that Q is normalized if it is written in the form $Q = \left(\frac{a_1}{b},\ \ldots,\ \frac{a_e}{b}\right)$ with gcd $(a_1,\ \ldots,\ a_e) = 1$. Speaking of a balanced $(n \times m,\ W,\ 1)$ AM-OPPW 2D VWOOC we mean an $(n \times m,\ W,\ 1,\ Q)$ AM-OPPW 2D VWOOC with $Q = \left(\frac{1}{e},\ \ldots,\ \frac{1}{e}\right)$. Let $\Psi(n,\ m,\ W,\ Q)$ denote the largest possible cardinality of an $(n \times m,\ W,\ 1,\ Q)$ AM-OPPW 2D VWOOC, the upper bound of $\Psi(n,\ m,\ W,\ Q)$ is obtained as follow.

Lemma 1 If m and n are positive integers such that $n \geq w_l \geq 2,\ l = 1,\ 2,\ \ldots,\ e$, then

$$\Psi(n,\ m,\ W,\ Q) \leq b \left\lfloor \frac{mn(n-1)}{\sum_{l=1}^{e} a_l w_l (w_l - 1)} \right\rfloor, \tag{1}$$

where $Q = \left(\frac{a_1}{b},\ \ldots,\ \frac{a_e}{b}\right)$ is normalized.

Remark The inequality (1) can be obtained from Theorem 2 and inequality (3) in Section 2.

An $(n \times m,\ W,\ 1,\ Q)$ AM-OPPW 2D VWOOC is called optimal if $\Psi(n,\ m,\ W,\ Q)$ reaches the bound (1). What's more, an optimal $(n \times m,\ W,\ 1,\ Q)$ AM-OPPW 2D VWOOC is perfect if

$$\Psi(n,\ m,\ W,\ Q) = \frac{bmn(n-1)}{\sum_{l=1}^{e} a_l w_l (w_l - 1)}. \tag{2}$$

To establish the equivalence, we now introduce some relevant combinatorial design theory. Let v be positive integer. Let K be a set of integers, each of which is not less than 2. A holey packing K-HP with order v is a triple $(V,\ \mathscr{G},\ \mathscr{B})$ where:

(1) V is a v-set of points;

(2) $\mathscr{G}$ is a partition of V into subsets called holes (or called groups);

(3) $\mathscr{B}$ is a collection of k-subsets of V (called blocks) with $k \in K$, such that a block and a group contain at most one common point and every pair of points from distinct group occurs in at most one block.

The group type of the HP is the list $(|G| : G \in \mathscr{G})$. The usual exponential notation will be used to describe types. Therefore, an HP of type $u_1^{h_1} u_2^{h_2} \cdots u_r^{h_r}$ is one in which there are h_i groups of size u_i for each i. For some special values of v, the blocks of $\mathscr{B}$ consist of each pair of points from distinct group exactly once, such the HP is called a group divisible design [32], denoted by K-GDD.

Given positive integers n and m, denoted by Z_m the additive group of integers module m, by I_n the set $\{0,\ 1,\ \ldots,\ n-1\}$ and $V = I_n \times Z_m$. The elements of V are denoted by $(i,\ a)$, where $i \in I_n$ and $a \in Z_m$. A K-HP (or K-GDD) of type m^n based on points set V having group set $\mathscr{G} = \{\{i\} \times Z_m : i \in I_n\}$ and block set $\mathscr{B}$ is said to be semi-cyclic, denoted by K-Semi-Cyclic Holey Packing (SCHP) (or K-SCGDD) of type m^n, if for any $B \in \mathscr{B}$, adding $1 \in Z_m$ successively to the second coordinate of each point of $B \in \mathscr{B}$ modulo m always yields m distinct blocks of B. Since it is difficult to determine the upper bound for the number of base blocks in a K-SCHP of type m^n, the following definitions are needed.

A (W, Q)-SCHP (or (W, Q)-SCGDD) of type m^n is a W-SCHP (or W-SCGDD) of type m^n with the property that the fraction of blocks of size w_l is q_l, $1 \leq l \leq e$. Speaking of a balanced W-SCHP (or W-SCGDD) we mean a (W, Q)-SCHP (or (W, Q)-SCGDD) with $Q = \left(\frac{1}{e}, \ldots, \frac{1}{e}\right)$. A convenient way of viewing a (W, Q)-SCHP of type m^n is from the difference family perspective. Denoted by $\mathscr{A}$ the collection of all base blocks of a (W, Q)-SCHP of type m^n and define the multiset

$$\Delta_{ij}\mathscr{A} = \{b - a \pmod m : (i, a), (j, b) \in A, (i, a) \neq (j, b), A \in \mathscr{A}\}.$$

When $i = j$, $\Delta_{ii}\mathscr{A}$ is the multiset of all pure (i, i)-differences of $\mathscr{A}$. When $i \neq j$, $\Delta_{ij}\mathscr{A}$ is the multiset of all mixed (i, j)-differences of $\mathscr{A}$. For any $(i, j) \in I_n \times I_n$, it is not difficult to check that $\Delta_{ij}\mathscr{A}$ covers every element of Z_m at most once if $i \neq j$ and $\Delta_{ij}\mathscr{A} = \emptyset$ if $i = j$. For some $\mathscr{A}$, $\Delta_{ij}\mathscr{A}$ may be exact, that is, $\Delta_{ij}\mathscr{A} = Z_m$ if $i \neq j$ and $\Delta_{ij}\mathscr{A} = \emptyset$ if $i = j$, in which case the $\mathscr{A}$ forms the collection of all base blocks of a (W, Q)-SCGDD of type m^n. In [33], (W, Q)-SCGDDs of type m^n are used to construct Multiple-Weight Optical Orthogonal Signature Pattern Codes (MWOOSPCs). When $W = \{w\}$ and $Q = (1)$, this SCGDD is briefly denoted by w-SCGDD of type m^n, such an SCGDD is also termed as a $GD^*(w, 1, m; mn)$ in [9]. For the existence of a w-SCGDD of type m^n, the interested reader can refer to [25, 26], and references therein. If $m = 1$, a w-SCGDD of type 1^n is said to be an $(n, w, 1)$ balanced incomplete block design, denoted by $(n, w, 1)$-Balanced Incomplete Block Design (BIBD).

We use $\Phi(W, Q, n, m)$ to denote the maximum number of base blocks among all (W, Q)-SCHPs of type m^n. From the definition of a (W, Q)-SCHP of type m^n, we have the following inequality

$$\frac{\Phi(W, Q, n, m)}{b}(a_1 w_1(w_1 - 1) + a_2 w_2(w_2 - 1) + \cdots + a_e w_e(w_e - 1)) \leq n(n-1)m,$$

where $Q = \left(\frac{a_1}{b}, \ldots, \frac{a_e}{b}\right)$ is normalized. Since $\Phi(W, Q, n, m)$ is divisible by b, we obtain the following result.

Theorem 1 If m and n are positive integers such that $n \geq w_l \geq 2$, $l = 1, 2, \ldots, e$, then

$$\Phi(W, Q, n, m) \leq b \left\lfloor \frac{mn(n-1)}{\sum_{l=1}^{e} a_l w_l (w_l - 1)} \right\rfloor, \tag{3}$$

where $Q = \left(\frac{a_1}{b}, \ldots, \frac{a_e}{b}\right)$ is normalized.

A (W, Q)-SCHP of type m^n is said to be optimal if $\Phi(W, Q, n, m)$ meets the bound (3). By the definition, a (W, Q)-SCGDD of type m^n is optimal because it has $\frac{bmn(n-1)}{\sum_{l=1}^{e} a_l w_l (w_l - 1)}$ base blocks.

Now we are in a position to give the fundamental relationship between SCHPs and AM-OPPW 2D VWOOCs, as stated in the following theorem.

Theorem 2 A (W, Q)-SCHP of type m^n is equivalent to an $(n \times m, W, 1, Q)$ AM-OPPW 2D VWOOC. Furthermore, the SCHP is optimal if and only if the code is optimal.

Proof. Let $\mathscr{A}$ be a set of base blocks of a (W, Q)-SCHP of type m^n on $I_n \times Z_m$ with the group set $\{\{i\} \times Z_m : i \in I_n\}$. For each base block $A = \{(i_1, a_1), (i_2, a_2), \ldots, (i_{w_l}, a_{w_l})\}$ in $\mathscr{A}$, we construct an $n \times m$ (0, 1)-matrix C_A whose nonzero bit positions are exactly indexed by all elements in A. Since each block and any given group contain at most one common point and every pair of points of $I_n \times Z_m$ from distinct groups occurs in at most one block, the derived matrices satisfy the AM-OPPW restriction and the correlation properties for a 2D VWOOC. Hence, the family $\{C_A : A \in \mathscr{A}\}$ forms an $(n \times m,\ W,\ 1,\ Q)$ AM-OPPW 2D VWOOC.

Conversely, let $\mathscr{C}$ be given an $(n \times m,\ W,\ 1,\ Q)$ AM-OPPW 2D VWOOC. For each $n \times m$ (0, 1)-matrix $\mathscr{C}$, we construct an w_l-subset of $I_n \times Z_m$ by taking the index set of its nonzero bit positions, $l = 1,\ 2,\ \dots,\ e$. This creates a collection $\mathscr{A}$ of w_l-subsets of $I_n \times Z_m$. The AM-OPPW restriction and the correlation properties of the code guarantee that $\mathscr{A}$ forms a family of base blocks of a $(W,\ Q)$-SCHP of type m^n.

From the above discussion, if the given a $(W,\ Q)$-SCHP of type m^n is optimal, then the derived an $(n \times m,\ W,\ 1,\ Q)$ AM-OPPW 2D VWOOC must be optimal. And vice versa. □

In order to construct optimal $(n \times m,\ W,\ 1,\ Q)$ AM-OPPW 2D VWOOCs, we need only to construct their corresponding optimal $(W,\ Q)$-SCHPs of type m^n. The following result is a special case of Theorem 2.

Theorem 3 A $(W,\ Q)$-SCGDD of type m^n is equivalent to a perfect $(n \times m,\ W,\ 1,\ Q)$ AM-OPPW 2D VWOOC.

3. Direct constructions

If B is a subset of an additive group G, the list of differences is $\Delta B = \{b - b' : (b,\ b') \in B \times B,\ b \neq b'\}$.

Construction 1 Let m and n be positive integers such that $n \geq w_l \geq 2$, $l = 1,\ 2,\ \dots,\ e$. If $\mathscr{F}$ is a collection of w_l-subsets of $Z_n \times Z_m$ such that its list of differences $\Delta\mathscr{F} = \bigcup_{B \in \mathscr{F}} \Delta B$ covers every element of $(Z_n - \{0\}) \times Z_m$ exactly once and no element of $\{0\} \times Z_m$ occurs in $\Delta\mathscr{F}$, then there exists a $(W,\ Q)$-SCGDD of type m^n.

Proof. For every block $B = \{(i_1,\ a_1),\ (i_2,\ a_2),\ \dots,\ (i_{w_l},\ a_{w_l})\} \in \mathscr{F}$ $(1 \leq l \leq e)$, construct a collection of w_l-subsets of $Z_n \times Z_m$, $\mathscr{A}_B = \{\{(i_1 + j,\ a_1),\ (i_2 + j,\ a_2),\ \dots,\ (i_{w_l} + j,\ a_{w_l})\} : j \in Z_n\}$ where the additive operation is taken in Z_n. Set $\mathscr{A} = \{\mathscr{A}_B : B \in \mathscr{F}\}$, then $\Delta_{\alpha\beta}\mathscr{A} = Z_m$ for any two distinct elements $\alpha,\ \beta$ of Z_n. Hence $\mathscr{A}$ forms a set of base blocks of a $(W,\ Q)$-SCGDD of type m^n based on $Z_n \times Z_m$ with the group set $\{\{i\} \times Z_m : i \in Z_n\}$. □

The following construction is used to get a $(W,\ Q)$-SCGDD of type m^n on Z_{mn}. Let $V = Z_{mn}$ and group set $\mathscr{G} = \{\{i,\ i+n,\ \dots,\ i+(m-1)n\} : i \in Z_n\}$. We take a permutation $\psi : x \longrightarrow x+1 (\mathrm{mod}\ mn)$ on Z_{mn}. Let $\mathscr{F}$ be a collection of subsets of Z_{mn} and $O(\psi,\ \mathscr{F}) = \bigcup_{B \in \mathscr{F}} O(\psi,\ B)$, where $O(\psi,\ B) = \{B + j : j \in Z_{mn}\}$ denotes the block orbit containing block B under ψ. If $O(\psi,\ \mathscr{F})$ is a K-GDD of type m^n on V with group set $\mathscr{G}$, then we say that the design is formed by developing each member of $\mathscr{F}$ by $+1$ (mod mn). A GDD obtained by this manner is said to be cyclic, denoted by K-CGDD. If a block orbit under ψ has mn distinct blocks, then this block orbit is said to be full, otherwise short. From [9], a K-CGDD of type m^n is an m-regular $(K,\ 1;\ mn)$ cyclic packing (briefly m-regular CP$(K,\ 1;\ mn)$, it's also called an $(mn,\ m,\ K,\ 1)$-RDF in [34]) if it contains only full block orbits. Clearly, a $(W,\ Q)$-CGDD of type m^n is an $(mn,\ m,\ W,\ 1,\ Q)$-RDF if it contains only full block orbits. By replacing a k-GDD in Construction 11 of [26] with a $(W,\ Q)$-GDD, we can get the following result.

Construction 2 Let m and n be positive integers such that $n \geq w_l \geq 2$, $l = 1,\ 2,\ \dots,\ e$. If there exists a set of w_l-subsets of Z_{mn}, $\mathscr{F}$, such that a $(W,\ Q)$-GDD of type m^n with the group set $\mathscr{G} = \{\{i,\ i+n,\ \dots,\ i+(m-1)n\} : i \in Z_n\}$ can be formed by developing each w_l-subsets in $\mathscr{F}$ by $+1$ (mod mn), then the GDD is a $(W,\ Q)$-SCGDD of type m^n.

From Construction 2, we have the following corollary.

Corollary 1 Suppose that a $(W,\ Q)$-CGDD of type m^n (or an $(mn,\ m,\ W,\ 1,\ Q)$-RDF) exists, then there exists a $(W,\ Q)$-SCGDD of type m^n.

According to Corollary 1, we can obtain some $(W,\ Q)$-SCGDDs of type m^n from some results on $(mn,\ m,\ W,\ 1,\ Q)$-RDFs. For more detail of $(mn,\ m,\ W,\ 1,\ Q)$-RDFs, the interested reader can refer to [13, 16]. Note that if m and n are coprime, we have $Z_{mn} \cong Z_n \times Z_m$, then Construction 1 is equivalent to Corollary 1.

The following direct constructions for a balanced $\{3,\ 4\}$-SCGDD of type 81^n are based on skew starters. To develop our constructions, we first require the notion of a skew starter.

A skew starter in Z_v is a set of unordered pairs $S = \{\{x_i,\ y_i\} : 1 \leq i \leq (v-1)/2\}$ satisfying the following three properties:

(1) $\{x_i :\ 1 \leq i \leq (v-1)/2\} \cup \{y_i :\ 1 \leq i \leq (v-1)/2\} = Z_v \setminus \{0\}$;

(2) $\{\pm(x_i - y_i):\ 1 \le i \le (v-1)/2\} = Z_v \setminus \{0\}$;
(3) $\{\pm(x_i + y_i):\ 1 \le i \le (v-1)/2\} = Z_v \setminus \{0\}$.

From the definition, a skew starter in Z_v exists only if v is odd. Furthermore, let $X = \{x_i : 1 \le i \le (v-1)/2\}$ and $Y = \{y_i : 1 \le i \le (v-1)/2\}$, we may assume that $X = -Y$, then $X \bigcup (-X) = Y \bigcup (-Y) = X \bigcup Y = Z_v \setminus \{0\}$. Skew starters have been studied extensively for some years. We summarize the existence results on skew starters in Z_v in the following lemma.

Lemma 2 [35] There is a skew starter in Z_v for any positive integer v such that gcd $(v,\ 6) = 1$, and v is not divisible by 5 or is divisible by 25.

Lemma 3 Suppose that a skew starter in Z_n exists, then there is a balanced $\{3,\ 4\}$-SCGDD of type 81^n.

Proof. Suppose that $S = \left\{\{x_i,\ y_j\} : i = 1,\ 2,\ \ldots,\ \dfrac{n-1}{2}\right\}$ is a skew starter in Z_n. Consider eighteen subsets of $Z_n \times Z_{81}$ of the following form:

$B_1 = \{(x,\ 0),\ (y,\ 0),\ (x+y,\ 10),\ (0,\ 16)\},\ B_2 = \{(0,\ 0),\ (x,\ 3),\ (x+y,\ 6)\}$,
$B_3 = \{(x,\ 0),\ (y,\ 30),\ (x+y,\ 79)\},\ B_4 = \{(x,\ 0),\ (-y,\ 31),\ (-x,\ 48),\ (y,\ 76)\}$,
$B_5 = \{(x,\ 0),\ (y,\ 37),\ (x+y,\ 46)\},\ B_6 = \{(x,\ 0),\ (-y,\ 19),\ (-x,\ 57),\ (y,\ 61)\}$,
$B_7 = \{(x,\ 0),\ (-y,\ 13),\ (-x,\ 14),\ (y,\ 73)\},\ B_8 = \{(x,\ 0),\ (-y,\ 7),\ (-x,\ 25),\ (y,\ 54)\}$,
$B_9 = \{(x,\ 0),\ (y,\ 11),\ (x+y,\ 23)\},\ B_{10} = \{(x,\ 0),\ (y,\ 15),\ (x+y,\ 41)\}$,
$B_{10+i} = (-1,\ 1) \cdot B_{2+i},\ 1 \le i \le 8$.
We have $\bigcup_{i=1}^{18} \Delta B_i = \bigcup_{j \in Z_{81}} \Delta_j \times \{j\}$, where
$\Delta_j = \{x-y,\ y-x\},\ j \in \{0,\ 1,\ 5,\ 8,\ 11,\ 15,\ 17,\ 18,\ 20,\ 27,\ 30,\ 37,\ 38\}$;
$\Delta_j = \{x+y,\ -x-y\},\ j \in \{4,\ 6,\ 7,\ 13,\ 19,\ 22,\ 28,\ 29,\ 31\}$;
$\Delta_j = \{y,\ -y\},\ j \in \{2,\ 23,\ 35,\ 40\}$; $\Delta_j = \{x,\ -x\},\ j \in \{9,\ 12,\ 32\}$;
$\Delta_j = \{2x,\ -2x\},\ j \in \{14,\ 24,\ 25,\ 26,\ 33\}$; $\Delta_j = \{2y,\ -2y\},\ j \in \{21,\ 34,\ 36,\ 39\}$;
$\Delta_3 = \Delta_{10} = \{x,\ y\}$; $\Delta_{16} = \{-x,\ -y\}$; $\Delta_j = -\Delta_{81-j},\ 41 \le j \le 80$.
Set $\mathscr{B} = \{B_i : 1 \le i \le 18,\ \{x,\ y\} \in S\}$, we have

$$\Delta\mathscr{B} = \bigcup_{j \in Z_{81}} \bigcup_{\{x,\ y\} \in S} \Delta_j \times \{j\} = \bigcup_{j \in Z_{81}} (Z_n - \{0\}) \times \{j\} = (Z_n - \{0\}) \times Z_{81}.$$

So, a balanced $\{3,\ 4\}$-SCGDD of type 81^n is from Construction 1. □

4. Recursive constructions

In this section, we will present some recursive constructions for $(W,\ Q)$-SCGDDs. The recursive construction of GDDs frequently employs both the filling technique and Wilson's Fundamental Construction (see [36]). In a related development, Yin [9] demonstrated that similar techniques are applicable to the construction of cyclic packings. Building upon this foundation, we now extend these principles to establish an analogous construction for $(W,\ Q)$-SCGDDs.

Construction 3 If there are a k-SCGDD of type g^n and a $(W,\ Q)$-SCGDD of type m^k, then there exists a $(W,\ Q)$-SCGDD of type $(gm)^n$.

Proof. Let $\mathscr{A}$ be a set of base blocks of the given k-SCGDD of type g^n based on $I_n \times Z_g$ with the group set $\{\{i\} \times Z_g : i \in I_n\}$. For $i,\ j \in I_n$ and $i \ne j$, we have

$$\Delta_{ij}\mathscr{A} = \{b - a : (i,\ a),\ (j,\ b) \in A,\ A \in \mathscr{A}\} = Z_g.$$

Let $\mathscr{B}$ be a set of base blocks of a (W, Q)-SCGDD of type m^k based on $I_k \times Z_m$ with the group set $\{\{i\} \times Z_m : i \in I_k\}$. For $i, j \in I_k$ and $i \neq j$, we have

$$\Delta_{ij}\mathscr{B} = \{b - a : (i, a), (j, b) \in B, B \in \mathscr{B}\} = Z_m.$$

For each $A = \{(i_1, a_1), (i_2, a_2), \ldots, (i_k, a_k)\} \in \mathscr{A}$ and each $B = \{(j_1, b_1), (j_2, b_2), \ldots, (j_{w_l}, b_{w_l})\} \in \mathscr{B}$, we construct

$$A_B = \{(i_{j_1}, a_{j_1} + gb_1), (i_{j_2}, a_{j_2} + gb_2), \ldots, (i_{j_{w_l}}, a_{j_{w_l}} + gb_{w_l})\}.$$

Let $\mathscr{F}_A = \bigcup_{B \in \mathscr{B}} A_B$. For $i_{j_\alpha}, i_{j_\beta} \in I_n$, we have

$$\Delta_{i_{j_\alpha} i_{j_\beta}} \mathscr{F}_A = \{a_{j_\beta} - a_{j_\alpha} + g(b_\beta - b_\alpha) : (i_{j_\alpha}, a_{j_\alpha} + gb_\alpha), (i_{j_\beta}, a_{j_\beta} + gb_\beta) \in A_B, B \in \mathscr{B}\}$$

$$= \{a_{j_\beta} - a_{j_\alpha} + gr : (i_{j_\alpha}, a_{j_\alpha}), (i_{j_\beta}, a_{j_\beta}) \in A, r \in \Delta_{j_\alpha j_\beta}\mathscr{B}\}$$

$$= \{a_{j_\beta} - a_{j_\alpha} + gr : (i_{j_\alpha}, a_{j_\alpha}), (i_{j_\beta}, a_{j_\beta}) \in A, r \in Z_m\}.$$

Set $\mathscr{F} = \bigcup_{A \in \mathscr{A}} \mathscr{F}_A$, then

$$\Delta_{i_{j_\alpha} i_{j_\beta}} \mathscr{F} = \bigcup_{A \in \mathscr{A}} \Delta_{i_{j_\alpha} i_{j_\beta}} \mathscr{F}_A$$

$$= \bigcup_{A \in \mathscr{A}} \{a_{j_\beta} - a_{j_\alpha} + gr : (i_{j_\alpha}, a_{j_\alpha}), (i_{j_\beta}, a_{j_\beta}) \in A, r \in Z_m\}$$

$$= \{s + gr : s \in \Delta_{i_{j_\alpha} i_{j_\beta}} \mathscr{A}, r \in Z_m\}$$

$$= \{s + gr : s \in Z_g, r \in Z_m\}.$$

It is readily known that each element in Z_{gm} occurs in $\Delta_{i_{j_\alpha} i_{j_\beta}} \mathscr{F}$ exactly once. So, $\mathscr{F}$ forms a (W, Q)-SCGDD of type $(gm)^n$ based on $I_n \times Z_{gm}$ with the group set $\{\{i\} \times Z_{gm} : i \in I_n\}$. □

A k-SCGDD of type 1^n can be seen as an $(n, k, 1)$-BIBD, we have the following corollary from Construction 3.

Corollary 2 If there are an $(n, k, 1)$-BIBD and a (W, Q)-SCGDD of type m^k, then there exists a (W, Q)-SCGDD of type m^n.

In contrast to Construction 3 which builds upon a k-SCGDD, the following construction starts from a k-GDD, yielding a new result.

Construction 4 If the following designs exist: (1) a k-GDD of type $g_1^{t_1} g_2^{t_2} \cdots g_r^{t_r}$; (2) a (W, Q)-SCGDD of type $(mu)^k$; (3) a (W, Q)-SCGDD of type m^{ug_i} for $i = 1, 2, \ldots, r$. Then there exists a (W, Q)-SCGDD of type m^n, where $n = u(g_1 t_1 + g_2 t_2 + \cdots + g_r t_r)$.

Proof. Let $(V, \mathscr{G}, \mathscr{B})$ be the given k-GDD. For each block $B = \{b_1, b_2, \ldots, b_k\} \in \mathscr{B}$, by hypothesis, let $\mathscr{F}$ be a set of base blocks of a (W, Q)-SCGDD of type $(mu)^k$ over $B \times Z_{mu}$ with the group set $\{\{b_i\} \times Z_{mu} : i \in I_k\}$. For $i, j \in I_k$ and $i \neq j$, we have

$$\Delta_{b_i b_j}\mathscr{F} = \{a_j - a_i : (b_i, a_i), (b_j, a_j) \in F, F \in \mathscr{F}\} = Z_{mu}.$$

For each $F = \{(b_{i_1}, a_1), (b_{i_2}, a_2), \ldots, (b_{i_w}, a_w)\} \in \mathscr{F}$ and $w \in W$, write a_j $(j = 1, 2, \ldots, w)$ uniquely as $h_j m + f_j$, $0 \leq h_j \leq u-1$ and $0 \leq f_i \leq m-1$. We can construct a collection of w-subsets of $B \times Z_u \times Z_m$,

$$\mathscr{A}(B, F) = \{\{(b_{i_1}, h_1 + s, f_1), (b_{i_2}, h_2 + s, f_2), \ldots, (b_{i_w}, h_w + s, f_w)\} : s \in Z_u\},$$

where the sums are all taken modulo u. Let $\mathscr{A}(B) = \{\mathscr{A}(B, F) : F \in \mathscr{F}\}$. For any two distinct elements $\overline{\alpha}, \overline{\beta} \in B \times Z_u$ with $\overline{\alpha} = (b_{i_j}, h_j + s)$, $\overline{\beta} = (b_{i_{j'}}, h_{j'} + s)$ and $j \neq j'$, we have

$$\Delta_{\overline{\alpha}\overline{\beta}}\mathscr{A}(B) = \{f_{j'} - f_j : (\overline{\alpha}, f_j), (\overline{\beta}, f_{j'}) \in \mathscr{A}(B, F), F \in \mathscr{F}\} = Z_m.$$

Now for every group $G \in \mathscr{G}$ of size g_i for $i = 1, 2, \ldots, r$, one construct a (W, Q)-SCGDD of type m^{ug_i} over the point set $G \times Z_u \times Z_m$, with the group set $\{\{x\} \times \{y\} \times Z_m : x \in G, y \in Z_u\}$ and a set of base blocks $\mathscr{A}(G)$. For any two distinct element $\overline{\alpha}, \overline{\beta} \in G \times Z_u$, we have

$$\Delta_{\overline{\alpha}\overline{\beta}}\mathscr{A}(G) = Z_m.$$

Set $\mathscr{A} = \{\mathscr{A}(B) : B \in \mathscr{B}\} \bigcup \{\mathscr{A}(G) : G \in \mathscr{G}\}$. For any two distinct elements $\overline{\alpha}, \overline{\beta} \in V \times Z_u$, where $\overline{\alpha} = (i, a)$ and $\overline{\beta} = (j, b)$. If $\{i, j\} \subset B \in \mathscr{B}$, we have $\Delta_{\overline{\alpha}\overline{\beta}}\mathscr{A} = \Delta_{\overline{\alpha}\overline{\beta}}\mathscr{A}(B) = Z_m$; Otherwise, $\Delta_{\overline{\alpha}\overline{\beta}}\mathscr{A} = \Delta_{\overline{\alpha}\overline{\beta}}\mathscr{A}(G) = Z_m$. So, $\mathscr{A}$ forms a set of base blocks of a (W, Q)-SCGDD of type m^n with the point set $V \times Z_u \times Z_m$ and the group set $\{\{x\} \times \{y\} \times Z_m : x \in V, y \in Z_u\}$. □

Applying Construction 4 with $r = 1$, $g_1 = 1$ and $t_1 = k$, we immediately reach the following result.

Corollary 3 If there are a (W, Q)-SCGDD of type $(mu)^k$ and a (W, Q)-SCGDD of type m^u, then there exists a (W, Q)-SCGDD of type m^{uk}.

A holey (W, Q)-GDD (briefly (W, Q)-HGDD) of type (n, u^t) is defined to be a quadruple $(V, \mathscr{H}, \mathscr{G}, \mathscr{B})$ satisfying the following five properties:

(1) V is an nut-set of points;

(2) $\mathscr{G} = \{G_1, G_2, \ldots, G_n\}$ is a partition of V into n groups of ut points each;

(3) $\mathscr{H} = \{H_1, H_2, \ldots, H_t\}$ is another partition of V into t holes of nu points each such that $|G_i \bigcap H_j| = u$ for each i and each j;

(4) $\mathscr{B}$ is a family of w_l-subsets (called blocks) of V, $w_l \in W$, such that any block do not contain two distinct points from any group or hole, but contains any other pair of distinct points of V exactly once;

(5) The fraction of blocks of size w_l is q_l, $l = 1, 2, \ldots, e$.

Suppose that $V = I_n \times Z_{ut}$, $\mathscr{G} = \{\{i\} \times Z_{ut} : i \in I_n\}$ and $\mathscr{H} = \{I_n \times \{j, t + j, \ldots, (u-1)t + j\} : j \in Z_t\}$. A (W, Q)-HGDD of type (n, u^t) on points set V having group set $\mathscr{G}$, hole set $\mathscr{H}$ and block set $\mathscr{B}$ is said to be semi-cyclic, denoted by (W, Q)-SCHGDD of type (n, u^t), if for any $B \in \mathscr{B}$, adding $1 \in Z_{ut}$ successively to the second coordinate of each point of $B \in \mathscr{B}$ modulo ut always yields ut distinct blocks of $\mathscr{B}$. Suppose that $\mathscr{A}$ is the family of all base blocks of a (W, Q)-SCHGDD of type (n, u^t). For any $(i, j) \in I_n \times I_n$, it is not difficult to check that $\Delta_{ij}\mathscr{A} = Z_{ut} \setminus \{0, t, \ldots, (u-1)t\}$ if $i \neq j$

and $\Delta_{ij}\mathscr{A}=\emptyset$ if $i=j$. When $W=\{w\}$ and $Q=(1)$, this SCHGDD is simply denoted by w-SCHGDD of type $(n,\ u^t)$. The interested reader can refer to [37, 38] for recent papers on w-SCHGDD of type $(n,\ u^t)$ with $w=3$. A $(W,\ Q)$-SCHGDD of type $(n,\ u^t)$ can be referred as a special $(W,\ Q)$-SCHP of type $(ut)^n$, then the following result is clear.

Lemma 4 Let m and n be positive integers such that $n\geq w_l\geq 2,\ l=1,\ 2,\ \ldots,\ e$. If $n(n-1)u<\sum_{l=1}^{e}a_lw_l(w_l-1)$, then a $(W,\ Q)$-SCHGDD of type $(n,\ u^t)$ is optimal, where $Q=\left(\frac{a_1}{b},\ \ldots,\ \frac{a_e}{b}\right)$ is normalized.

Proof. Since $Q=\left(\frac{a_1}{b},\ \ldots,\ \frac{a_e}{b}\right)$ is normalized, the number of base blocks of a $(W,\ Q)$-SCHGDD of type $(n,\ u^t)$ is $N=\frac{bn(n-1)(ut-u)}{\sum_{l=1}^{e}a_lw_l(w_l-1)}$. If $n(n-1)u<\sum_{l=1}^{e}a_lw_l(w_l-1)$, then $\frac{n(n-1)u}{\sum_{l=1}^{e}a_lw_l(w_l-1)}<1$. Note that $n(n-1)(ut-u)$ is divisible by $\sum_{l=1}^{e}a_lw_l(w_l-1)$, we have

$$N=\frac{bn(n-1)(ut-u)}{\sum_{l=1}^{e}a_lw_l(w_l-1)}$$

$$=b\left(\frac{n(n-1)ut}{\sum_{l=1}^{e}a_lw_l(w_l-1)}-\frac{n(n-1)u}{\sum_{l=1}^{e}a_lw_l(w_l-1)}\right)$$

$$=b\left\lfloor\frac{n(n-1)ut}{\sum_{l=1}^{e}a_lw_l(w_l-1)}\right\rfloor.$$

So, a $(W,\ Q)$-SCHGDD of type $(n,\ u^t)$ is optimal from Theorem 1. □

Construction 5 If there exist a $(W,\ Q)$-SCHGDD of type $(n,\ u^t)$ and an optimal $(W,\ Q)$-SCHP of type u^n, then there eixsts an optimal $(W,\ Q)$-SCHP of type $(ut)^n$. Furthermore, if an optimal $(W,\ Q)$-SCHP of type u^n is a $(W,\ Q)$-SCGDD of type u^n, then there is a $(W,\ Q)$-SCGDD of type $(ut)^n$.

Proof. Suppose that $\mathscr{F}_1$ and $\mathscr{F}_2$ are the sets of base blocks of a $(W,\ Q)$-SCHGDD of type $(n,\ u^t)$ based on $I_n\times Z_{ut}$ and an optimal $(W,\ Q)$-SCHP of type u^n based on $I_n\times Z_u$, respectively. Then, for $i,\ j\in I_n$ and $i\neq j$, we have $\Delta_{ij}\mathscr{F}_1=Z_{ut}\setminus\{0,\ t,\ \ldots,\ (u-1)t\}$ and $\Delta_{ij}\mathscr{F}_2=Z_u\setminus H$, where H is the differences not covered by $\Delta_{ij}\mathscr{F}_2$. For each $B=\{(i_1,\ a_1),\ (i_2,\ a_2),\ \ldots,\ (i_w,\ a_w)\}\in\mathscr{F}_2$, $w\in W$, let

$$B'=\{(i\_1,\ a\_1t),\ (i\_2,\ a\_2t),\ \ldots,\ (i\_w,\ a\_wt)\}(-,\ \mathrm{mod}\ ut).$$

Let $\mathscr{F}_3=\{B':B\in\mathscr{F}_2\}$, then $\Delta_{ij}\mathscr{F}_3=tZ_{ut}\setminus(t\cdot H)$. Set $\mathscr{F}=\mathscr{F}_1\bigcup\mathscr{F}_3$, we have $\Delta_{ij}\mathscr{F}=\Delta_{ij}\mathscr{F}_1\bigcup\Delta_{ij}\mathscr{F}_3=Z_{ut}\setminus(t\cdot H)$, so $\mathscr{F}$ forms the set of base blocks of an optimal $(W,\ Q)$-SCHP of type $(ut)^n$.

The second part is clear according to the above construction. □

The following recursive constructions are based on incomplete cyclic difference matrices, so we first give the definition of a incomplete cyclic difference matrix.

Let G be a additive group of order v and H a subgroup of order h in G. An H-regular $(G,\ k;\ 1)$-incomplete difference matrix is a $k\times(v-h)$ matrix $D=(d_{ij})$, $0\leq i\leq k-1$, $0\leq j\leq v-h-1$, with entries from G, such that for any $0\leq i<j\leq k-1$, the multi-set $\{d_{il}-d_{jl}:0\leq l\leq v-h-1\}$ contains every element of $G\setminus H$ exactly once. If $G=Z_v$ and H is a subgroup of order h in Z_v, then $H=\{0,\ v/h,\ \ldots,\ (h-1)v/h\}$. We usually write an H-regular $(G,\ k;\ 1)$-incomplete difference matrix over Z_v as h-regular $(v,\ k;\ 1)$-Incomplete Difference Matrix (ICDM) if $|H|=h$. When $H=\emptyset$ or $h=0$, an h-regular $(v,\ k;\ 1)$-ICDM over Z_v is termed as $(v,\ k;\ 1)$-CDM. From [39], a $(v,\ k;\ 1)$-CDM is equivalent to a k-SCGDD of type v^k. (Incomplete) cyclic difference matrices have been studied extensively, we summarize some results on (incomplete) cyclic difference matrices in the following Lemma.

Lemma 5 The following difference matrices exist:

(1) There exists an $(m,\ 4;\ 1)$-CDM for any odd integer $m\geq 5$ and $\gcd(m,\ 27)\neq 9$.

(2) There exist a 2-regular $(2^n,\ 4;\ 1)$-ICDM for any integer $n \geq 3$ and a 2-regular $(m,\ 4;\ 1)$-ICDM for $m = 12,\ 18,\ 24$.

Proof. The two results come from [40, 41], respectively. □

Construction 6 If there exist a $(W,\ Q)$-SCGDD of type g^n and an h-regular $(v,\ k;\ 1)$-ICDM, then a $(W,\ Q)$-SCHGDD of type $(n,\ (gh)^{v/h})$ exists, where $k = \max\{w_1,\ w_2,\ \ldots,\ w_e\}$.

Proof. Suppose that $\mathscr{A}$ be the set of base blocks of a $(W,\ Q)$-SCGDD of type g^n based on $I_n \times Z_g$ with group set $\{\{i\} \times Z_g : i \in I_n\}$, then $\Delta_{ij}\mathscr{A} = Z_g$ for $i,\ j \in I_n$ and $i \neq j$. Let $D = (d_{ij})$ be an h-regular $(v,\ k;\ 1)$-ICDM for $0 \leq i \leq k-1$ and $0 \leq j \leq v-h-1$, then $\{d_{\beta j} - d_{\alpha j} : j = 0,\ 1,\ \ldots,\ v-h-1\} = Z_v \setminus \{0,\ v/h,\ \ldots,\ (h-1)v/h\}$ for any two distinct rows $\alpha,\ \beta$. It is easy to see that an h-regular $(v,\ w;\ 1)$-ICDM exists for $w \leq k$. For each base block $B = \{(i_1,\ a_1),\ (i_2,\ a_2),\ \ldots,\ (i_w,\ a_w)\} \in \mathscr{A}$, $w \in W$, we construct

$$\mathscr{F}_B = \{\{(i_1,\ d_{1j} + a_1 v),\ (i_2,\ d_{2j} + a_2 v),\ \ldots,\ (i_w,\ d_{wj} + a_w v)\} : j = 0,\ 1,\ \ldots,\ v-h-1\}.$$

For $1 \leq \alpha < \beta \leq w$, we have

$$\Delta_{i_\alpha i_\beta}\mathscr{F}_B = \{d_{\beta j} - d_{\alpha j} + (a_\beta - a_\alpha)v : (i_\alpha,\ a_\alpha),\ (i_\beta,\ a_\beta) \in B,\ j = 0,\ 1,\ \ldots,\ v-h-1\}$$

$$= \{s + (a_\beta - a_\alpha)v : (i_\alpha,\ a_\alpha),\ (i_\beta,\ a_\beta) \in B,\ s \in Z_v \setminus \{0,\ v/h,\ \ldots,\ (h-1)v/h\}\}.$$

Let $\mathscr{F} = \bigcup_{B \in A} \mathscr{F}_B$. It is calculated that

$$\Delta_{i_\alpha i_\beta}\mathscr{F} = \bigcup_{B \in \mathscr{A}} \Delta_{i_\alpha i_\beta}\mathscr{F}_B$$

$$= \bigcup_{B \in \mathscr{A}} \{s + (a_\beta - a_\alpha)v : (i_\alpha,\ a_\alpha),\ (i_\beta,\ a_\beta) \in B,\ s \in Z_v \setminus \{0,\ v/h,\ \ldots,\ (h-1)v/h\}\}$$

$$= \{s + rv : r \in \Delta_{i_\alpha i_\beta}\mathscr{A},\ s \in Z_v \setminus \{0,\ v/h,\ \ldots,\ (h-1)v/h\}\}$$

$$= \{s + rv : r \in Z_g,\ s \in Z_v \setminus \{0,\ v/h,\ \ldots,\ (h-1)v/h\}\}$$

$$= Z_{gv} \setminus \{0,\ v/h,\ \ldots,\ (gh-1)v/h\}.$$

So, $\mathscr{F}$ forms a set of base blocks of a $(W,\ Q)$-SCHGDD of type $(n,\ (gh)^{v/h})$ over $I_n \times Z_{gv}$ with group set $\{\{i\} \times Z_{gv} : i \in I_n\}$ and hole set $\{I_n \times \{j,\ j+v/h,\ \ldots,\ j+(gh-1)v/h\} : j \in Z_{v/h}\}$. □

By replacing an h-regular $(v,\ k;\ 1)$-ICDM with a $(v,\ k;\ 1)$-CDM in Construction 6, we are not difficult to get the following result.

Construction 7 If there exist a $(W,\ Q)$-SCGDD of type g^n and a $(v,\ k;\ 1)$-CDM, then a $(W,\ Q)$-SCGDD of type $(gv)^n$ exists, where $k = \max\{w_1,\ w_2,\ \ldots,\ w_e\}$.

5. The proof of main results

In this section, we present four infinite classed of AM-OPPW 2D VWOOCs which achieve the bound (2).

5.1 *Perfect* $\left(n\times 4m,\ \{3,\ 4\},\ 1,\ \left(\frac{2}{3},\ \frac{1}{3}\right)\right)$ *AM-OPPW 2D VWOOCs*

In this subsection, leveraging existing results and the recursive constructions given in Section 4, we prove that perfect $\left(n\times 4m,\ \{3,\ 4\},\ 1,\ \left(\frac{2}{3},\ \frac{1}{3}\right)\right)$ AM-OPPW 2D VWOOCs exist for $n\geq 4,\ (n-1)m\equiv 0 \pmod 3$ and $n(n-1)m\equiv 0 \pmod{12}$.

Lemma 6 [26] If m is odd and $n\geq 5$ such that $(n-1)m\equiv 0 \pmod 3$ and $n(n-1)m\equiv 0 \pmod{12}$, then there exists a 4-SCGDD of type m^n.

Lemma 7 There is a $\left(\{3,\ 4\},\ \left(\frac{2}{3},\ \frac{1}{3}\right)\right)$-SCGDD of type $(4\cdot 2^a)^4$ for any integer $a\geq 0$.

Proof. If $a=0$, the conclusion is from the proof of Theorem 1.1 in [33].

If $a=1,\ 2$, by computer, a set of base blocks of a $\left(\{3,\ 4\},\ \left(\frac{2}{3},\ \frac{1}{3}\right)\right)$-SCGDD of type $(4\cdot 2^a)^4$ is displayed in Appendix I.

If $a\geq 3$. A $\left(\{3,\ 4\},\ \left(\frac{2}{3},\ \frac{1}{3}\right)\right)$-SCGDD of type 4^4 is from the above and a 2-regular $(2^a,\ 4;\ 1)$-ICDM is from Lemma 5, we have a $\left(\{3,\ 4\},\ \left(\frac{2}{3},\ \frac{1}{3}\right)\right)$-SCHGDD of type $(4,\ 8^{2^{a-1}})$ from Construction 6. So, the conclusion is from Construction 5 by using a $\left(\{3,\ 4\},\ \left(\frac{2}{3},\ \frac{1}{3}\right)\right)$-SCGDD of type 8^4. □

Lemma 8 There is a $\left(\{3,\ 4\},\ \left(\frac{2}{3},\ \frac{1}{3}\right)\right)$-SCGDD of type $(4\cdot 3^a)^4$ for any integer $a\geq 1$.

Proof. If $a=1,\ 2$, a set of base blocks of a $\left(\{3,\ 4\},\ \left(\frac{2}{3},\ \frac{1}{3}\right)\right)$-SCGDD of type $(4\cdot 3^a)^4$ is displayed in Appendix I.

If $a\geq 3$. A $\left(\{3,\ 4\},\ \left(\frac{2}{3},\ \frac{1}{3}\right)\right)$-SCGDD of type 4^4 is from Lemma 7 and a $(3^a,\ 4;\ 1)$-CDM is from Lemma 5, we have the conclusion from Construction 7. □

Lemma 9 There exists a $\left(\{3,\ 4\},\ \left(\frac{2}{3},\ \frac{1}{3}\right)\right)$-SCGDD of type $(12\cdot 2^a)^4$ for any integer $a\geq 0$.

Proof. If $a=0$, the conclusion comes from Lemma 8.

If $a=1$. Let $\mathscr{A}=\{\{0,34,71\},\{0,22,83,85\},\{0,21,39,90\},\ \{0,41,58,67\},\{0,19,65\},\{0,30,53\},\{0,14,95\},\{0,7,93\},\{0,5,47\}\}$, $\mathscr{A}$ forms a $\left(96,\ 24,\ \{3,\ 4\},\ 1,\ \left(\frac{2}{3},\ \frac{1}{3}\right)\right)$-RDF, so we have a $\left(\{3,\ 4\},\ \left(\frac{2}{3},\ \frac{1}{3}\right)\right)$-SCGDD of type 24^4 from Corollary 1.

If $a=2$, a set of base blocks of a $\left(\{3,\ 4\},\ \left(\frac{2}{3},\ \frac{1}{3}\right)\right)$-SCGDD of type 36^4 is displayed in Appendix I.

If $a\geq 3$. A $\left(\{3,\ 4\},\ \left(\frac{2}{3},\ \frac{1}{3}\right)\right)$-SCGDD of type 12^4 is from above and a 2-regular $(2^a,\ 4;\ 1)$-ICDM is from Lemma 5, we have a $\left(\{3,\ 4\},\ \left(\frac{2}{3},\ \frac{1}{3}\right)\right)$-SCHGDD of type $(4,\ 24^{2^{a-1}})$ from Construction 6. So, the conclusion is from Construction 5 by using a $\left(\{3,\ 4\},\ \left(\frac{2}{3},\ \frac{1}{3}\right)\right)$-SCGDD of type 24^4. □

Lemma 10 There is a $\left(\{3,\ 4\},\ \left(\frac{2}{3},\ \frac{1}{3}\right)\right)$-SCGDD of type $(36\cdot 2^a)^4$ for any integer $a\geq 0$.

Proof. If $a=0$, the conclusion comes from Lemma 9.

If $a = 1,\ 2$. A $\left(\{3,\ 4\},\ \left(\frac{2}{3},\ \frac{1}{3}\right)\right)$-SCGDD of type $(4\cdot 2^{a-1})^4$ exists from Lemma 7 and a 2-regular $(18,\ 4;\ 1)$-ICDM is from Lemma 5, we have a $\left(\{3,\ 4\},\ \left(\frac{2}{3},\ \frac{1}{3}\right)\right)$-SCHGDD of type $(4,\ (4\cdot 2^a)^9)$ from Construction 6. So, the conclusion is from Construction 5 by using a $\left(\{3,\ 4\},\ \left(\frac{2}{3},\ \frac{1}{3}\right)\right)$-SCGDD of type $(4\cdot 2^a)^4$ in Lemma 7.

If $a \geq 3$. A $\left(\{3,\ 4\},\ \left(\frac{2}{3},\ \frac{1}{3}\right)\right)$-SCGDD of type 36^4 is from above and a 2-regular $(2^a,\ 4;\ 1)$-ICDM is from Lemma 5, we have a $\left(\{3,\ 4\},\ \left(\frac{2}{3},\ \frac{1}{3}\right)\right)$-SCHGDD of type $(4,\ 72^{2^{a-1}})$ from Construction 6. So, the conclusion comes from Construction 5 by using a $\left(\{3,\ 4\},\ \left(\frac{2}{3},\ \frac{1}{3}\right)\right)$-SCGDD of type 72^4. □

Lemma 11 There exists a $\left(\{3,\ 4\},\ \left(\frac{2}{3},\ \frac{1}{3}\right)\right)$-SCGDD of type $(4m)^4$ for any integer $m \geq 1$.

Proof. Write $m = 2^a 3^b m_1$, where $a \geq 0$, $b \geq 0$ are integers and $\gcd(6,\ m_1) = 1$.

If $a \geq 0$ and $b = 0,\ 1,\ 2$. A $\left(\{3,\ 4\},\ \left(\frac{2}{3},\ \frac{1}{3}\right)\right)$-SCGDD of type $(4\cdot 2^a 3^b)^4$ is from Lemmas 7, 9, 10, and an $(m_1,\ 4;\ 1)$-CDM is from Lemma 5, we have the conclusion from Construction 7.

If $a \geq 0$ and $b \geq 3$. A $\left(\{3,\ 4\},\ \left(\frac{2}{3},\ \frac{1}{3}\right)\right)$-SCGDD of type $(4\cdot 2^a)^4$ is from Lemma 7 and a $(3^b m_1,\ 4;\ 1)$-CDM is from Lemma 5, the conclusion is obtained from Construction 7. □

Theorem 4 If m and n are positive integers such that $n \geq 4$, $(n-1)m \equiv 0 \pmod 3$ and $n(n-1)m \equiv 0 \pmod{12}$, then there exists a perfect $\left(n \times 4m,\ \{3,\ 4\},\ 1,\ \left(\frac{2}{3},\ \frac{1}{3}\right)\right)$ AM-OPPW 2-D VWOOC.

Proof. If $n = 4$. A $\left(\{3,\ 4\},\ \left(\frac{2}{3},\ \frac{1}{3}\right)\right)$-SCGDD of type $(4m)^4$ is from Lemma 11, the conclusion is obtained from Theorem 3.

If $n \geq 5$. Write $m = 2^a m_1$, where $a \geq 0$ is an integer and m_1 is odd. A 4-SCGDD of type m_1^n is from Lemma 6 and a $\left(\{3,\ 4\},\ \left(\frac{2}{3},\ \frac{1}{3}\right)\right)$-SCGDD of type $(4\cdot 2^a)^4$ is from Lemma 7, we have a $\left(\{3,\ 4\},\ \left(\frac{2}{3},\ \frac{1}{3}\right)\right)$-SCGDD of type $(4\cdot 2^a m_1)^n$ from Construction 3. So, the conclusion comes from Theorem 3. □

5.2 *Perfect balanced* $(n \times 6m,\ \{3,\ 4\},\ 1)$ *AM-OPPW 2D VWOOCs*

Lemma 12 There exists a balanced $\{3,\ 4\}$-SCGDD of type $(6\cdot 2^a)^4$ for any integer $a \geq 0$.

Proof. If $a = 0,\ 1,\ 2$, a set of base blocks of a balanced $\{3,\ 4\}$-SCGDD of type $(6\cdot 2^a)^4$ is displayed in Appendix II.

If $a \geq 3$. A balanced $\{3,\ 4\}$-SCGDD of type 6^4 is from the above and a 2-regular $(2^a,\ 4;\ 1)$-ICDM is from Lemma 5, we have a balanced $\{3,\ 4\}$-SCHGDD of type $(4,\ 12^{2^{a-1}})$ from Construction 6. So, the conclusion comes from Construction 5 by using a balanced $\{3,\ 4\}$-SCGDD of type 12^4. □

Lemma 13 There is a balanced $\{3,\ 4\}$-SCGDD of type $(6\cdot 3^a)^4$ for any integer $a \geq 1$.

Proof. If $a = 1,\ 2$, a set of base blocks of a balanced $\{3,\ 4\}$-SCGDD of type $(6\cdot 3^a)^4$ is displayed in Appendix II.

If $a \geq 3$. A $\{3,\ 4\}$-SCGDD of type 6^4 is from Lemma 12 and a $(3^a,\ 4;\ 1)$-CDM is from Lemma 5, the conclusion is obtained from Construction 7. □

Lemma 14 There is a balanced $\{3,\ 4\}$-SCGDD of type $(18\cdot 2^a)^4$ for any integer $a \geq 0$.

Proof. If $a = 0$, the conclusion comes from Lemma 13.

If $a = 1$. Let $\mathscr{A} = \{0, 21, 43, 90\}$, $\{0, 6, 13, 15\}$, $\{0, 18, 57, 127\}$, $\{0, 73, 83\}$, $\{0, 34, 111\}$, $\{0, 98, 117\}$, $\{0, 29, 30, 79\}$, $\{0, 59, 121\}$, $\{0, 99, 102, 113\}$, $\{0, 53, 58\}$, $\{0, 25, 51, 106\}$, $\{0, 37, 78\}$, $\mathscr{A}$ forms a balanced $(144,\ 36,\ \{3,\ 4\},\ 1)$-RDF, we have a balanced $\{3,\ 4\}$-SCGDD of type 36^4 from Corollary 1.

If $a = 2$. A balanced $\{3, 4\}$-SCGDD of type 6^4 exists from Lemma 12 and a 2-regular $(12, 4; 1)$-ICDM is from Lemma 5, then a balanced $\{3, 4\}$-SCHGDD of type $(4, 12^6)$ exists from Construction 6. Hence, the conclusion comes from Construction 5 by using a balanced $\{3, 4\}$-SCGDD of type 12^4 in Lemma 12.

If $a \geq 3$. A balanced $\{3, 4\}$-SCGDD of type 18^4 is from the above and a 2-regular $(2^a, 4; 1)$-ICDM is from Lemma 5, we have a balanced $\{3, 4\}$-SCHGDD of type $(4, 36^{2^{a-1}})$ from Construction 6. Therefore, the conclusion is from Construction 5 by using a balanced $\{3, 4\}$-SCGDD of type 36^4. □

Lemma 15 There exists a balanced $\{3, 4\}$-SCGDD of type $(54 \cdot 2^a)^4$ for any integer $a \geq 0$.

Proof. If $a = 0$, the conclusion comes from Lemma 14.

If $a = 1, 2$. A balanced $\{3, 4\}$-SCGDD of type $(6 \cdot 2^{a-1})^4$ exists from Lemma 12 and a 2-regular $(18, 4; 1)$-ICDM is from Lemma 5, we have a balanced $\{3, 4\}$-SCHGDD of type $(4, (6 \cdot 2^a)^9)$ from Construction 6. Hence, the conclusion is from Construction 5 by using a balanced $\{3, 4\}$-SCGDD of type $(6 \cdot 2^a)^4$ in Lemma 12.

If $a \geq 3$. A balanced $\{3, 4\}$-SCGDD of type 54^4 is from the above and a 2-regular $(2^a, 4; 1)$-ICDM is from Lemma 5, we have a balanced $\{3, 4\}$-SCHGDD of type $(4, 108^{2^{a-1}})$ from Construction 6. Hence, the conclusion comes from Construction 5 by using a balanced $\{3, 4\}$-SCGDD of type 108^4. □

Lemma 16 There is a balanced $\{3, 4\}$-SCGDD of type $(6m)^4$ for any integer $m \geq 1$.

Proof. Write $m = 2^a 3^b m_1$, where $a \geq 0$, $b \geq 0$ are integers and $\gcd(6, m_1) = 1$.

If $a \geq 0$ and $b = 0, 1, 2$. A balanced $\{3, 4\}$-SCGDD of type $(6 \cdot 2^a 3^b)^4$ is from Lemmas 12, 14, 15, and an $(m_1, 4; 1)$-CDM is from Lemma 5, the conclusion is obtained from Construction 7.

If $a \geq 0$ and $b \geq 3$. A balanced $\{3, 4\}$-SCGDD of type $(6 \cdot 2^a)^4$ is from Lemma 12 and a $(3^b m_1, 4; 1)$-CDM is from Lemma 5, the conclusion is from Construction 7. □

Theorem 5 If m and n are positive integers such that $n \geq 4$, $(n-1)m \equiv 0 \pmod 3$ and $n(n-1)m \equiv 0 \pmod{12}$, then there exists a perfect balanced $(n \times 6m, \{3, 4\}, 1)$ AM-OPPW 2-D VWOOC.

Proof. If $n = 4$. A balanced $\{3, 4\}$-SCGDD of type $(6m)^4$ is from Lemma 16, the conclusion is obtained from Theorem 3.

If $n \geq 5$. Write $m = 2^a m_1$, where $a \geq 0$ is an integer and m_1 is odd. A 4-SCGDD of type m_1^n is from Lemma 6 and a balanced $\{3, 4\}$-SCGDD of type $(6 \cdot 2^a)^4$ is from Lemma 12, a balanced $\{3, 4\}$-SCGDD of type $(6 \cdot 2^a m_1)^n$ is obtained from Construction 3. So, the conclusion comes from Theorem 3. □

5.3 *Perfect balanced* $(n \times 9m, \{3, 4\}, 1)$ *AM-OPPW 2D VWOOCs*

In this subsection, by drawing upon the direct and recursive constructions presented in Sections 3 and 4, respectively, we establish the existence of perfect balanced $(n \times 9m, 3, 4, 1)$ AM-OPPW 2D VWOOCs. This result holds for all positive integers m and n greater than 1 that satisfy $gcd(n, 6) = 1$.

Lemma 17 If $n > 1$ is an integer such that $\gcd(n, 6) = 1$, then there exists a balanced $\{3, 4\}$-SCGDD of type g^n, where $g = 9, 18, 27, 36, 54$.

Proof. If $g = 9$, a balanced $(9n, 9, \{3, 4\}, 1)$-RDF is from [16], then the conclusion comes from Corollary 1. If $g = 18, 27$, a balanced $(gn, g, \{3, 4\}, 1)$-RDF is from [42, 43], then the conclusion comes from Corollary 1. If $g = 36, 54$, a balanced $(gn, g, \{3, 4\}, 1)$-RDF is from [44], then the conclusion is from Corollary 1. □

Lemma 18 If $n > 1$ is an integer such that $\gcd(n, 6) = 1$, then there exists a balanced $\{3, 4\}$-SCGDD of type $(9 \cdot 2^a)^n$, where $a \geq 0$ is an integer.

Proof. If $a = 0, 1, 2$, a balanced $\{3, 4\}$-SCGDD of type $(9 \cdot 2^a)^n$ is from Lemma 17.

If $a \geq 3$. A balanced $\{3, 4\}$-SCGDD of type 9^n is from the above and a 2-regular $(2^a, 4; 1)$-ICDM is from Lemma 5, we have a balanced $\{3, 4\}$-SCHGDD of type $(n, 18^{2^{a-1}})$ from Construction 6. So, the conclusion comes from Construction 5, where the existence of a balanced $\{3, 4\}$-SCGDD of type 18^n. □

Lemma 19 If $n > 1$ is an integer such that $\gcd(n, 6) = 1$, then there exists a balanced $\{3, 4\}$-SCGDD of type $(81)^n$.

Proof. Write $n = 5^a n_1$, where $a \geq 0$ is an integer and n_1 is not divisible by 5. When $a \neq 1$, a skew starter in Z_n exists from Lemma 2, we have the conclusion from Lemma 3. When $a = 1$, a balanced $\{3, 4\}$-SCGDD of type $(81)^{n_1}$ from the above and a $(5, 4; 1)$-CDM exists from Lemma 5, we have a balanced $\{3, 4\}$-SCGDD of type $(81 \cdot 5)^{n_1}$ from

Construction 7. A balanced $(405,\ 81,\ \{3,\ 4\},\ 1)$-RDF is displayed below, we have a balanced $\{3,\ 4\}$-SCGDD of type 81^5 from Corollary 1. So, a balanced $\{3,\ 4\}$-SCGDD of type 81^{5n_1} is obtained from Corollary 3.

$\{0, 162, 394\}$, $\{0, 233, 311\}$, $\{0, 1, 198, 262\}$, $\{0, 2, 186\}$, $\{0, 77, 364\}$, $\{0, 43, 356\}$, $\{0, 38, 396\}$, $\{0, 139, 202, 318\}$, $\{0, 303, 334\}$, $\{0, 3, 99, 322\}$, $\{0, 191, 248, 267\}$, $\{0, 199, 253\}$, $\{0, 37, 81, 369\}$, $\{0, 51, 174, 187\}$, $\{0, 114, 142, 338\}$, $\{0, 16, 62, 74\}$, $\{0, 213, 379\}$, $\{0, 24, 252\}$, $\{0, 279, 297, 376\}$, $\{0, 193, 227, 249\}$, $\{0, 6, 23, 307\}$, $\{0, 112, 154\}$, $\{0, 7, 344\}$, $\{0, 27, 151, 299\}$, $\{0, 101, 204\}$, $\{0, 131, 258\}$, $\{0, 8, 171, 237\}$, $\{0, 129, 238\}$, $\{0, 211, 259, 372\}$, $\{0, 69, 158, 286\}$, $\{0, 84, 137, 401\}$, $\{0, 169, 391\}$, $\{0, 59, 373\}$, $\{0, 21, 294, 333\}$, $\{0, 52, 159, 241\}$, $\{0, 122, 256\}$. □

Lemma 20 If $n > 1$ is an integer such that $\gcd(n,\ 6) = 1$, then there exists a balanced $\{3,\ 4\}$-SCGDD of type $(9 \cdot 3^a)^n$, where $a \geq 0$ is an integer.

Proof. If $a = 1,\ 2$, the conclusion comes from Lemmas 17 and 19.

If $a \geq 3$. A balanced $\{3,\ 4\}$-SCGDD of type 9^n is from Lemma 17 and a $(3^a,\ 4;\ 1)$-CDM is from Lemma 5, we have the conclusion from Construction 7. □

Lemma 21 There exists a balanced $\{3,\ 4\}$-SCGDD of type $(27 \cdot 2^a)^n$ for any integer $a \geq 0$.

Proof. If $a = 0,\ 1$, the conclusion is from Lemma 17.

If $a = 2$. A balanced $\{3,\ 4\}$-SCGDD of type 9^n exists from Lemma 17 and a 2-regular $(12,\ 4;\ 1)$-ICDM is from Lemma 5, we have a balanced $\{3,\ 4\}$-SCHGDD of type $(n,\ 18^6)$ from Construction 6. Hence, the conclusion is from Construction 5 by using a balanced $\{3,\ 4\}$-SCGDD of type 18^n in Lemma 17.

If $a \geq 3$. A balanced $\{3,\ 4\}$-SCGDD of type 27^n exists and a 2-regular $(2^a,\ 4;\ 1)$-ICDM is from Lemma 5, we have a balanced $\{3,\ 4\}$-SCHGDD of type $(n,\ 54^{2^{a-1}})$ from Construction 6. Hence, the conclusion is from Construction 5 by using a balanced $\{3,\ 4\}$-SCGDD of type 54^n. □

Lemma 22 There exists a balanced $\{3,\ 4\}$-SCGDD of type $(81 \cdot 2^a)^n$ for any integer $a \geq 0$.

Proof. If $a = 0$, the conclusion is from Lemma 19.

If $a = 1,\ 2$. A balanced $\{3,\ 4\}$-SCGDD of type g^n exists from Lemma 17, $g = 9,\ 18$, and a 2-regular $(18,\ 4;\ 1)$-ICDM is from Lemma 5, we have a balanced $\{3,\ 4\}$-SCHGDD of type $(n,\ (2g)^9)$ from Construction 6. Hence, the conclusion is from Construction 5 by using a balanced $\{3,\ 4\}$-SCGDD of type $(2g)^n$ in Lemma 17.

If $a \geq 3$. A balanced $\{3,\ 4\}$-SCGDD of type 54^4 is from the above and a 2-regular $(2^a,\ 4;\ 1)$-ICDM is from Lemma 5, a balanced $\{3,\ 4\}$-SCHGDD of type $(4,\ 108^{2^{a-1}})$ is obtained from Construction 6. Hence, the conclusion is from Construction 5 by using a balanced $\{3,\ 4\}$-SCGDD of type 108^4. □

Theorem 6 If m and $n > 1$ are positive integers such that $\gcd(n,\ 6) = 1$, then there exists a perfect balanced $(n \times 9m,\ \{3,\ 4\},\ 1)$ AM-OPPW 2-D VWOOC.

Proof. Write $m = 2^a 3^b m_1$, where $a \geq 0$, $b \geq 0$ are integers and $\gcd(6,\ m_1) = 1$.

If $a \geq 0$ and $b = 0,\ 1,\ 2$. A balanced $\{3,\ 4\}$-SCGDD of type $(9 \cdot 2^a 3^b)^n$ is from Lemmas 18, 21, 22, and an $(m_1,\ 4;\ 1)$-CDM is from Lemma 5, we have a balanced $\{3,\ 4\}$-SCGDD of type $(9 \cdot 2^a 3^b m_1)^n$. So, the conclusion is obtained from Theorem 3.

If $a \geq 0$ and $b \geq 3$. A balanced $\{3,\ 4\}$-SCGDD of type $(9 \cdot 2^a)^n$ is from Lemma 18 and a $(3^b m_1,\ 4;\ 1)$-CDM is from Lemma 5, we have a balanced $\{3,\ 4\}$-SCGDD of type $(9 \cdot 2^a 3^b m_1)^n$ from Construction 7. So, the conclusion is from Theorem 3. □

5.4 *Perfect* $\left(n \times 10m,\ \{3,\ 4\},\ 1,\ \left(\frac{1}{3},\ \frac{2}{3}\right)\right)$ *AM-OPPW 2D VWOOCs*

Lemma 23 There is a $\left(\{3,\ 4\},\ \left(\frac{1}{3},\ \frac{2}{3}\right)\right)$-SCGDD of type $(10 \cdot 2^a)^4$ for any integer $a \geq 0$.

Proof. If $a = 0,\ 1,\ 2$, a balanced $\left(40 \cdot 2^a,\ 10 \cdot 2^a,\ \{3,\ 4\},\ 1,\ \left(\frac{1}{3},\ \frac{2}{3}\right)\right)$-RDF is constructed in Appendix III, then the conclusion is from Corollary 1.

If $a \geq 3$. A $\left(\{3,\ 4\},\ \left(\frac{1}{3},\ \frac{2}{3}\right)\right)$-SCGDD of type 10^4 exists and a 2-regular $(2^a,\ 4;\ 1)$-ICDM is from Lemma 5, similar to the proof of Lemma 7, the conclusion is obtained. □

Lemma 24 There is a $\left(\{3,\ 4\},\ \left(\frac{1}{3},\ \frac{2}{3}\right)\right)$-SCGDD of type $(10\cdot 3^a)^4$ for any integer $a \geq 1$.

Proof. If $a = 1,\ 2$, a balanced $\left(40\cdot 3^a,\ 10\cdot 3^a,\ \{3,\ 4\},\ 1,\ \left(\frac{1}{3},\ \frac{2}{3}\right)\right)$-RDF is constructed in Appendix III, then the conclusion is obtained from Corollary 1.

If $a \geq 3$. A $\left(\{3,\ 4\},\ \left(\frac{1}{3},\ \frac{2}{3}\right)\right)$-SCGDD of type 10^4 is from Lemma 23 and a $(3^a,\ 4;\ 1)$-CDM is from Lemma 5, we have the conclusion from Construction 7. □

Lemma 25 There exists a $\left(\{3,\ 4\},\ \left(\frac{1}{3},\ \frac{2}{3}\right)\right)$-SCGDD of type $(30\cdot 2^a)^4$ for any integer $a \geq 0$.

Proof. If $a = 0$, the conclusion is from Lemma 24.

If $a = 1$. A balanced $\left(240,\ 60,\ \{3,\ 4\},\ 1,\ \left(\frac{1}{3},\ \frac{2}{3}\right)\right)$-RDF is constructed below, we have a $\left(\{3,\ 4\},\ \left(\frac{1}{3},\ \frac{2}{3}\right)\right)$-SCGDD of type 60^4 from Corollary 1.

$\{0, 186, 189, 199\}, \{0, 30, 173, 207\}, \{0, 46, 221\}, \{0, 47, 121, 122\}, \{0, 81, 90, 235\}, \{0, 123, 181\},$
$\{0, 91, 98, 201\}, \{0, 27, 174, 229\}, \{0, 62, 83, 133\}, \{0, 73, 79\}, \{0, 102, 191\}, \{0, 42, 77, 183\},$
$\{0, 82, 113, 135\}, \{0, 129, 131, 146\}, \{0, 14, 139, 217\}, \{0, 25, 70\}, \{0, 18, 61, 87\}, \{0, 29, 114\}.$

If $a = 2$. A $\left(\{3,\ 4\},\ \left(\frac{1}{3},\ \frac{2}{3}\right)\right)$-SCGDD of type 10^4 is from Lemma 23 and a 2-regular $(12,\ 4;\ 1)$-ICDM is from Lemma 5, we have a $\left(\{3,\ 4\},\ \left(\frac{1}{3},\ \frac{2}{3}\right)\right)$-SCHGDD of type $(4,\ 20^6)$ from Construction 6. Hence, the conclusion is from Construction 5 by using a $\left(\{3,\ 4\},\ \left(\frac{1}{3},\ \frac{2}{3}\right)\right)$-SCGDD of type 20^4 in Lemma 23.

If $a \geq 3$. A $\left(\{3,\ 4\},\ \left(\frac{1}{3},\ \frac{2}{3}\right)\right)$-SCGDD of type 10^4 is from above and a 2-regular $(2^a,\ 4;\ 1)$-ICDM is from Lemma 5, we have a $\left(\{3,\ 4\},\ \left(\frac{1}{3},\ \frac{2}{3}\right)\right)$-SCHGDD of type $(4,\ 20^{2^{a-1}})$ from Construction 6. Hence, the conclusion is from Construction 5 by using a $\left(\{3,\ 4\},\ \left(\frac{1}{3},\ \frac{2}{3}\right)\right)$-SCGDD of type 20^4. □

Lemma 26 There exists a $\left(\{3,\ 4\},\ \left(\frac{1}{3},\ \frac{2}{3}\right)\right)$-SCGDD of type $(90\cdot 2^a)^4$ for any integer $a \geq 0$.

Proof. If $a = 0$, the conclusion is from Lemma 24.

If $a = 1,\ 2$. A $\left(\{3,\ 4\},\ \left(\frac{1}{3},\ \frac{2}{3}\right)\right)$-SCGDD of type $(10\cdot 2^a)^4$ is from Lemma 23 and a 2-regular $(18,\ 4;\ 1)$-ICDM is from Lemma 5, a $\left(\{3,\ 4\},\ \left(\frac{1}{3},\ \frac{2}{3}\right)\right)$-SCHGDD of type $(4,\ (10\cdot 2^a)^9)$ is obtained from Construction 6. Hence, the conclusion is from Construction 5 by using a $\left(\{3,\ 4\},\ \left(\frac{1}{3},\ \frac{2}{3}\right)\right)$-SCGDD of type $(10\cdot 2^a)^4$ in Lemma 23.

If $a \geq 3$. A $\left(\{3,\ 4\},\ \left(\frac{1}{3},\ \frac{2}{3}\right)\right)$-SCGDD of type 90^4 exists and a 2-regular $(2^a,\ 4;\ 1)$-ICDM is from Lemma 5. Similar to the proof of Lemma 7, the conclusion is obtained. □

Lemma 27 There exists a $\left(\{3,\ 4\},\ \left(\frac{1}{3},\ \frac{2}{3}\right)\right)$-SCGDD of type $(10m)^4$ for any integer $m \geq 1$.

Proof. Write $m = 2^a 3^b m_1$, where $a \geq 0$, $b \geq 0$ are integers and $\gcd(6,\ m_1) = 1$.

If $a \geq 0$ and $b = 0,\ 1,\ 2$. A $\left(\{3,\ 4\},\ \left(\frac{1}{3},\ \frac{2}{3}\right)\right)$-SCGDD of type $(10\cdot 2^a 3^b)^4$ is from Lemmas 23, 25, 26, and an $(m_1,\ 4;\ 1)$-CDM is from Lemma 5, we have the conclusion from Construction 7.

If $a \geq 0$ and $b \geq 3$. A $\left(\{3,\ 4\},\ \left(\frac{1}{3},\ \frac{2}{3}\right)\right)$-SCGDD of type $(10\cdot 2^a)^4$ is from Lemma 23 and a $(3^b m_1,\ 4;\ 1)$-CDM is from Lemma 5, the conclusion is obtained from Construction 7. □

Theorem 7 If m and n are positive integers such that $n \geq 4$, $(n-1)m \equiv 0 \pmod 3$ and $n(n-1)m \equiv 0 \pmod{12}$, then there exists a perfect $\left(n\times 10m,\ \{3,\ 4\},\ 1,\ \left(\frac{1}{3},\ \frac{2}{3}\right)\right)$ AM-OPPW 2-D VWOOC.

Proof. If $n = 4$. A $\left(\{3, 4\}, \left(\frac{1}{3}, \frac{2}{3}\right)\right)$-SCGDD of type $(10m)^4$ is from Lemma 27, the conclusion is from Theorem 3.

If $n \geq 5$. Write $m = 2^a m_1$, where $a \geq 0$ is an integer and m_1 is odd. A 4-SCGDD of type m_1^n is from Lemma 6 and a $\left(\{3, 4\}, \left(\frac{1}{3}, \frac{2}{3}\right)\right)$-SCGDD of type $(10 \cdot 2^a)^4$ is from Lemma 23, a $\left(\{3, 4\}, \left(\frac{1}{3}, \frac{2}{3}\right)\right)$-SCGDD of type $(10 \cdot 2^a m_1)^n$ is obtianed Construction 3. So, the conclusion is from Theorem 3. □

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Conflict of interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix I

The desired base blocks of a $\left(\{3, 4\}, \left(\frac{2}{3}, \frac{1}{3}\right)\right)$-SCGDDs of type $(4 \cdot g^a)^4$ are displayed below.

$(a, g) = (1, 2)$: the following blocks by $(+1 \pmod 4, -)$.

{(0, 0), (1, 0), (2, 1), (3, 5)}, {(0, 0), (1, 2), (2, 0)}, {(0, 0), (1, 5), (2, 4)}.

$(a, g) = (2, 2)$:

{(1, 0), (2, 2), (3, 6)}, {(0, 3), (2, 8), (3, 9)}, {(0, 10), (1, 1), (2, 5)}, {(0, 9), (1, 3), (2, 10), (3, 4)},
{(0, 7), (2, 1), (3, 6)}, {(1, 3), (2, 8), (3, 7)}, {(0, 2), (1, 0), (2, 0), (3, 10)}, {(0, 4), (1, 9), (2, 3), (3, 6)},
{(0, 0), (1, 9), (3, 5)}, {(0, 11), (1, 3), (2, 1), (3, 8)}, {(0, 6), (1, 8), (3, 10)}, {(0, 2), (1, 1), (2, 10), (3, 0)},
{(0, 1), (1, 9), (2, 10)}, {(0, 8), (1, 3), (2, 11)}, {(0, 0), (1, 0), (3, 3)}, {(0, 0), (2, 0), (3, 0)},
{(0, 0), (1, 1), (2, 4), (3, 1)}, {(1, 0), (2, 11), (3, 7)}.

$(a, g) = (1, 3)$:

{(0, 9), (1, 11), (3, 9)}, {(0, 3), (1, 9), (2, 6)}, {(0, 3), (1, 2), (2, 3)}, {(0, 7), (1, 5), (2, 5), (3, 9)},
{(0, 1), (2, 0), (3, 10)}, {(0, 9), (1, 0), (2, 5), (3, 0)}, {(0, 8), (1, 8), (3, 9)}, {(1, 9), (2, 0), (3, 0)},
{(1, 0), (2, 10), (3, 6)}, {(0, 2), (1, 9), (2, 4), (3, 6)}, {(0, 8), (1, 5), (2, 1), (3, 4)}, {(1, 11), (2, 10), (3, 7)},
{(0, 1), (2, 2), (3, 8)}, {(0, 9), (2, 1), (3, 2)}, {(0, 7), (1, 0), (2, 4)}, {(0, 9), (1, 1), (2, 3), (3, 8)},
{(0, 3), (1, 11), (3, 1)}, {(0, 0), (1, 1), (2, 7), (3, 6)}.

$(a, g) = (2, 3)$:

{(0, 25), (1, 35), (2, 32), (3, 10)}, {(0, 20), (1, 17), (2, 11), (3, 11)}, {(0, 13), (1, 35), (2, 7), (3, 3)},
{(0, 23), (2, 35), (3, 3)}, {(1, 33), (2, 21), (3, 10)}, {(0, 3), (1, 28), (2, 14)}, {(1, 14), (2, 23), (3, 17)},
{(0, 6), (1, 15), (2, 4)}, {(0, 7), (1, 26), (2, 9), (3, 8)}, {(0, 11), (1, 19), (3, 7)}, {(1, 3), (2, 23), (3, 13)},
{(0, 27), (1, 6), (2, 17), (3, 22)}, {(0, 32), (1, 22), (2, 21), (3, 18)}, {(0, 21), (1, 3), (2, 21), (3, 0)},
{(0, 5), (1, 22), (2, 26), (3, 12)}, {(0, 6), (1, 20), (2, 1), (3, 29)}, {(0, 33), (1, 21), (2, 21), (3, 8)},
{(1, 21), (2, 13), (3, 20)}, {(0, 8), (1, 12), (3, 14)}, {(0, 30), (1, 15), (3, 34)}, {(0, 12), (2, 9), (3, 30)},
{(0, 9), (1, 11), (2, 13), (3, 6)}, {(0, 20), (1, 26), (2, 0), (3, 34)}, {(0, 11), (1, 9), (2, 24), (3, 0)},
{(0, 35), (1, 4), (3, 33)}, {(0, 1), (1, 13), (3, 13)}, {(0, 33), (2, 3), (3, 14)}, {(0, 21), (1, 15), (2, 2)},
{(1, 34), (2, 5), (3, 15)}, {(1, 24), (2, 14), (3, 16)}, {(0, 22), (1, 15), (3, 16)}, {(0, 0), (1, 27), (3, 13)},
{(0, 10), (1, 10), (2, 13), (3, 30)}, {(0, 32), (1, 9), (2, 14), (3, 15)}, {(0, 19), (1, 3), (2, 15), (3, 28)},
{(0, 26), (1, 25), (2, 18)}, {(0, 15), (1, 18), (3, 23)}, {(1, 7), (2, 13), (3, 19)}, {(0, 1), (1, 8), (2, 24)},
{(0, 13), (1, 5), (2, 6)}, {(0, 29), (2, 1), (3, 17)}, {(0, 6), (2, 16), (3, 34)}, {(0, 34), (1, 9), (2, 7)},
{(0, 5), (2, 4), (3, 7)}, {(0, 9), (1, 5), (3, 12)}, {(0, 21), (2, 26), (3, 21)}, {(0, 35), (1, 22), (2, 18)},
{(0, 30), (1, 10), (2, 31)}, {(1, 25), (2, 2), (3, 10)}, {(0, 23), (2, 7), (3, 16)}, {(1, 15), (2, 10), (3, 29)},
{(0, 6), (1, 1), (2, 28), (3, 16)}, {(0, 19), (1, 20), (2, 34), (3, 18)}, {(0, 0), (2, 14), (3, 5)}.

Appendix II

The desired base blocks of a balanced $(\{3,\ 4\})$-SCGDD of type $(6 \cdot g^{a})^{4}$ are displayed below.

$(a,\ g) = (0,\ 2)$:

$\{$(0, 0), (1, 0), (2, 0), (3, 0)$\}$, $\{$(0, 0), (1, 2), (2, 1), (3, 5)$\}$, $\{$(1, 0), (2, 3), (3, 5)$\}$, $\{$(0, 0), (1, 4), (3, 2)$\}$,
$\{$(0, 0), (1, 3), (2, 5), (3, 4)$\}$, $\{$(0, 0), (1, 5), (2, 3)$\}$, $\{$(0, 0), (1, 1), (2, 2), (3, 3)$\}$, $\{$(0, 0), (2, 4), (3, 1)$\}$.

$(a,\ g) = (1,\ 2)$: the following base blocks by $(+1 \pmod 4,\ -)$.

$\{$(0, 0), (1, 0), (2, 1), (3, 3)$\}$, $\{$(0, 0), (1, 3), (2, 2), (3, 8)$\}$, $\{$(0, 0), (1, 5), (2, 0)$\}$, $\{$(0, 0), (1, 8), (2, 6)$\}$.

$(a,\ g) = (2,\ 2)$: the following base blocks by $(+1 \pmod 4,\ -)$.

$\{$(0, 0), (1, 6), (2, 9), (3, 7)$\}$, $\{$(0, 0), (1, 13), (3, 6)$\}$, $\{$(0, 0), (1, 2), (2, 2), (3, 17)$\}$,
$\{$(0, 0), (1, 17), (3, 9)$\}$, $\{$(0, 0), (1, 4), (2, 0), (3, 8)$\}$, $\{$(0, 0), (1, 5), (2, 12)$\}$.

$(a,\ g) = (1,\ 3)$: the following base blocks by $(+1 \pmod 4,\ -)$.

$\{$(0, 0), (1, 12), (2, 8), (3, 7)$\}$, $\{$(0, 0), (1, 5), (2, 12), (3, 14)$\}$, $\{$(0, 0), (1, 10), (2, 1), (3, 17)$\}$,
$\{$(0, 0), (2, 4), (3, 12)$\}$, $\{$(0, 0), (1, 15), (2, 15)$\}$, $\{$(0, 0), (1, 3), (2, 16)$\}$.

$(a,\ g) = (2,\ 3)$: the following base blocks by $(+1 \pmod 4,\ -)$.

$\{$(0, 0), (1, 7), (2, 44), (3, 14)$\}$, $\{$(0, 0), (1, 36), (2, 39), (3, 33)$\}$, $\{$(0, 0), (1, 10), (2, 41), (3, 12)$\}$,
$\{$(0, 0), (1, 6), (2, 4)$\}$, $\{$(0, 0), (2, 18), (3, 27)$\}$, $\{$(0, 0), (1, 16), (2, 24)$\}$, $\{$(0, 0), (1, 49), (2, 34)$\}$,
$\{$(0, 0), (1, 0), (2, 12), (3, 25)$\}$, $\{$(0, 0), (1, 50), (2, 22), (3, 50)$\}$, $\{$(0, 0), (1, 2), (2, 46), (3, 3)$\}$,
$\{$(0, 0), (1, 15), (2, 38), (3, 1)$\}$, $\{$(0, 0), (1, 34), (2, 26), (3, 7)$\}$, $\{$(0, 0), (1, 30), (2, 48), (3, 35)$\}$,
$\{$(0, 0), (1, 14), (3, 49)$\}$, $\{$(0, 0), (1, 20), (2, 11)$\}$, $\{$(0, 0), (1, 1), (2, 23)$\}$, $\{$(0, 0), (1, 43), (3, 22)$\}$,
$\{$(0, 0), (1, 33), (2, 17)$\}$.

Appendix III

The desired of $\left(40\cdot g^{a},\ 10\cdot g^{a},\ \{3,\ 4\},\ 1,\ \left(\frac{1}{3},\ \frac{2}{3}\right)\right)$-RDFs are displayed below.

$(a,\ g)=(0,\ 2)$: {0, 1, 3, 26}, {0, 5, 11, 18}, {0, 9, 19}.

$(a,\ g)=(1,\ 2)$: {0, 21, 51, 74}, {0, 35, 69}, {0, 13, 38, 75}, {0, 1, 3, 66}, {0, 9, 19, 58}, {0, 7, 33}.

$(a,\ g)=(2,\ 2)$:
{0, 105, 106, 119}, {0, 51, 110}, {0, 82, 149}, {0, 3, 9, 98}, {0, 27, 138, 153}, {0, 74, 91, 121},
{0, 103, 129}, {0, 21, 115, 158}, {0, 35, 81, 142}, {0, 19, 77, 150}, {0, 33, 118, 123}, {0, 25, 63}.

$(a,\ g)=(1,\ 3)$: {0, 18, 71, 93}, {0, 10, 91, 117}, {0, 21, 82}, {0, 19, 54, 105}, {0, 2, 57, 115},
{0, 41, 78, 111}, {0, 11, 25}, {0, 1, 31, 74}, {0, 6, 23}.

$(a,\ g)=(2,\ 3)$:
{0, 159, 170, 189}, {0, 278, 305, 351}, {0, 25, 178, 263}, {0, 66, 67, 157}, {0, 111, 237},
{0, 141, 279}, {0, 133, 150, 195}, {0, 61, 198, 291}, {0, 34, 87, 345}, {0, 31, 105, 206},
{0, 271, 334}, {0, 38, 51, 253}, {0, 10, 113}, {0, 134, 229, 243}, {0, 114, 295}, {0, 29, 35},
{0, 127, 290, 313}, {0, 33, 119, 310}, {0, 225, 302, 323}, {0, 2, 7, 149}, {0, 94, 151, 261},
{0, 54, 183, 289}, {0, 115, 281}, {0, 139, 161, 282}, {0, 318, 321}, {0, 18, 59, 205}, {0, 43, 118}.