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Research Article

On the Quaternion-Valued Fractional Differential Equation with Hyers-Ulam Stability

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Abstract: This paper investigates the Hyers-Ulam stability of linear Quaternion-Valued Fractional Differential Equations (QVFDEs) in both homogeneous and non-homogeneous forms. By exploiting the correspondence between quaternion modules and vector 2-norms, we transform QVFDEs into equivalent real fractional differential systems. Within this framework, new theoretical results on Hyers-Ulam and generalized Hyers-Ulam stability are established, supported by rigorous proofs and illustrative examples. These results not only reinforce the theoretical foundations of fractional stability analysis but also extend its applicability to systems characterized by quaternionic structures. The study offers valuable insights for modeling and analyzing phenomena in fields such as robotics, control theory, signal processing, and quantum mechanics, where quaternion-based representations naturally arise. Overall, this work contributes to the broader understanding of stability in fractional systems and opens avenues for future research on nonlinear and delayed quaternionic fractional models.

Keywords: quaternion module, Hyers-Ulam stability, fractional differential equations

MSC: 39A12, 37N25, 54C15

1. Introduction

Fractional Calculus (FC) generalizes classical calculus by extending the concepts of derivatives and integrals to arbitrary, non-integer orders. The idea dates back to the 17th century, when Leibniz and L'Hôpital speculated about the meaning of a derivative of fractional order. Its formal development emerged in the 19th century through the works of Riemann, Liouville, and Grünwald, who independently introduced rigorous definitions of fractional integrals and derivatives [1–11].

Over time, FC has found wide-ranging applications across disciplines such as engineering, physics, chemistry, economics, and finance. It has proven particularly powerful in modeling complex systems, including electrical circuits,

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signal processing, and viscoelastic materials. Moreover, FC plays a crucial role in describing anomalous diffusion and stochastic processes, where traditional integer-order calculus proves inadequate.

In recent decades, interest in FC has grown substantially. Researchers have expanded its theoretical foundations, developed new fractional operators, and introduced advanced numerical methods. These contributions have broadened the scope of FC, especially in solving fractional differential equations, reinforcing its importance as a versatile and powerful tool in engineering, applied mathematics, and the sciences [12–21]. The growing integration of FC with modern applied sciences has led to the exploration of fractional models in fields such as control theory, neural networks, and dynamical systems, where fractional-order derivatives capture real-world phenomena more accurately than integer-order models.

The Quaternion-Valued Differential Equations (QVDEs) and the Quaternion-Valued Fractional Differential Equations (QVFDEs) form new classes of differential equations distinguished from Ordinary Differential Equations (ODEs) by their underlying algebraic structure. Owing to the non-abelian property of quaternion algebra, the solution sets of linear homogeneous QVDEs differ significantly from those of the ODEs. In particular, they do not constitute a linear vector space but rather a right-free module [22]. This characteristic introduces analytical complexity, since quaternion multiplication is non-commutative, demanding reformulations and specialized analytical tools to study existence, uniqueness, and stability properties.

The quaternion system, first introduced by Hamilton [23], generalizes complex numbers and provides a powerful framework for representing orientation and rotation in three-dimensional mechanics. Its non-commutative multiplication describes the quotient of two vectors in space. Recently, QVDEs and QVFDEs have gained traction in diverse fields including protein dynamics, neural networks, and quantum mechanics [24, 25]. Beyond biology, quaternion systems offer notable advantages in engineering and physics, especially for modeling orientation and rotational dynamics, where angular dependencies between modules define system orientation [26]. The combination of quaternionic structures with fractional calculus therefore opens a new analytical pathway to study dynamic systems that exhibit both rotational and memory-dependent behavior, such as those found in robotics, aerospace control, and signal processing.

Regarding stability, the concept of Hyers-Ulam stability, originally posed by Ulam and formalized by Hyers [27], has been extended to QVDEs and QVFDEs. Hyers-Ulam stability provides approximate solutions for functional and differential equations, ensuring that if a function approximately satisfies a given equation, then a true solution exists close to it. This concept has become fundamental in the study of dynamic systems where perturbations, modeling errors, or uncertainties are inevitable. For instance, Liu et al. [28] studied Hyers-Ulam stability of linear Caputo-Fabrizio fractional differential equations via the Laplace transform, while Aruldass et al. [29] investigated second-order differential equations using the Mahgoub transform. These studies illustrate that Hyers-Ulam stability serves as a bridge between approximate and exact solutions, providing theoretical assurance of robustness for fractional systems.

Key theoretical advances in quaternion systems include the development of a Wronskian and Liouville formula for QDEs, established by Kou and Xia [22], along with eigenvalue-based methods for constructing fundamental matrices and solving linear QDEs. Xia et al. [22, 30] extended this framework to non-homogeneous QDEs, providing algorithms based on the variation of constants formula and analyzing stability in periodic quaternion systems. Similarly, Kou et al. [31] explored fundamental matrices with multiple eigenvalues for linear ODE systems. Further, Yang et al. [32] employed weakly Picard operators and fixed-point methods to establish abstract Hyers-Ulam stability results for QVFDEs, while Wang et al. [33] transformed first-order QVDEs into real differential systems to analyze stability. In parallel, Li et al. [34] investigated fractional-order neural networks, deriving Hyers-Ulam stability and Hyers-Ulam stability results using sequence approximation techniques.

Despite these advancements, the study of Hyers-Ulam stability in quaternion-valued fractional systems remains limited. The interaction between quaternionic algebra and fractional operators introduces new analytical challenges not present in classical systems. The non-commutative structure of quaternions complicates direct extensions of real or complex-valued stability theories, making it necessary to construct equivalent real formulations that preserve quaternionic dynamics. Therefore, developing a rigorous stability framework for QVFDEs is essential to ensure the reliability and robustness of quaternionic fractional models used in applied contexts.

This work is motivated by the need to deepen the understanding of stability properties in fractional differential equations that incorporate quaternionic dynamics. By combining the theories of FC and quaternion algebra, we aim to formulate generalized stability conditions that account for both nonlocality and rotation-dependent behavior. The established results contribute to the mathematical foundation of QVFDEs and support their application in fields such as robotics, control theory, signal analysis, and quantum mechanics, where precision and stability of solutions are critical.

Consider a homogeneous QVFDE

$$D^{\varepsilon}g(\alpha) = \mu g(\alpha), \quad \alpha \in J, \tag{1}$$

and non-homogeneous QVFDE

$$D^{\varepsilon}g(\alpha) = \mu g(\alpha) + \theta(\alpha), \quad \alpha \in J, \tag{2}$$

where $J = (\sigma, 0] \subseteq (-\infty, 0]$, $\theta\colon J \to \mathbb{F}$ is a continuous function (i.e., map), $\mu = \delta_0 + \delta_1 e_1 + \delta_2 e_2 + \delta_3 e_3 \in \mathbb{F}$ is a quaternion constant, and $g\colon J \to \mathbb{F}$ is a differentiable map. Consider the generalized QVFDE (1) and (2) when $\mu\colon J \to \mathbb{F}$ is a map.

The study of Hyers-Ulam stability for QVFDEs is motivated by the need to model complex systems that exhibit both memory effects and rotational dynamics. Fractional calculus provides a framework for describing nonlocal and hereditary behaviors, while quaternion algebra effectively represents three-dimensional rotations and coupled vector systems. Combining these two mathematical structures enables a more accurate analysis of phenomena in engineering, physics, and applied sciences. Establishing Hyers-Ulam stability ensures that approximate solutions remain close to exact ones under small perturbations, which is crucial for system reliability. Moreover, the non-commutative nature of quaternions introduces analytical challenges that require new stability criteria and transformation techniques. This work addresses these challenges, extending fractional stability theory to quaternionic systems with potential applications in robotics, control theory, signal processing, and quantum mechanics. This paper is organized into five main sections, each contributing to the development and validation of the proposed stability framework. Section 1 presents the background of fractional calculus and quaternion algebra, outlining their significance in modeling systems with memory and rotational properties while highlighting the research gap in the stability analysis of quaternion-valued fractional systems. Section 2 provides the essential mathematical foundations, including the definitions of fractional derivatives, quaternion algebra, and Hyers-Ulam stability, which form the basis for the subsequent theoretical results. Section 3 constitutes the core contribution of the paper, where new theorems on Hyers-Ulam and generalized Hyers-Ulam stability are rigorously established for both homogeneous and non-homogeneous linear QVFDEs using real-valued system transformations and norm-based techniques. Section 4 demonstrates the practical relevance of the theoretical findings through illustrative examples, confirming the validity and robustness of the derived stability conditions. Finally, Section 5 summarizes the main outcomes and outlines future research directions, including the extension of the analysis to nonlinear, delayed, and numerically solvable quaternionic fractional systems.

2. Preliminaries

This section provides the necessary background material to support the subsequent analysis. We begin with a brief overview of fractional-order calculus, highlighting the essential definitions and operators that will be employed throughout this work. Next, we recall fundamental aspects of quaternion algebra, with particular emphasis on its non-commutative structure and its implications for quaternion-valued systems. Finally, we review the concept of Hyers-Ulam stability, outlining its origins, definitions, and relevance to the study of fractional and quaternion-valued differential equations. Let $\mathbb{R}$ be the real numbers field, $\mathbb{C}$ be the complex numbers field and $\mathbb{F}$ denotes to the quaternion set. A quaternion element is a sum of three imaginary parts and one real part, that is, a quaternion $\delta \in \mathbb{F}$ is given by

$$\delta = \delta_0 + \delta_1 e_1 + \delta_2 e_2 + \delta_3 e_3$$

where $\delta_0, \delta_1, \delta_2, \delta_3 \in \mathbb{R}$ and the imaginary roots e_1, e_2, e_3 are imaginary units that satisfy:

$$e_1 e_2 = -e_2 e_1 = e_3, \; e_2 e_3 = -e_3 e_2 = e_1, \; e_3 e_1 = -e_1 e_3 = e_2,$$

$$e_1^2 = e_2^2 = e_3^2 = e_1 e_2 e_3 = -1.$$

The set:

$$\mathbb{Q} = \{\delta_0 + \delta_1 e_1 + \delta_2 e_2 + \delta_3 e_3 \colon \; \delta_0, \delta_1, \delta_2, \delta_3 \in \mathbb{R}\}$$

is called the quaternion set. Denote a 4-dimensional real vector by $\rho = (\delta_0, \delta_1, \delta_2, \delta_3)^T \in \mathbb{R}^4$ and the vector norm of ρ defined as

$$||\rho||_2 = |\rho| = \sqrt{\rho \bar{\rho}} = \sqrt{\delta_0^2 + \delta_1^2 + \delta_2^2 + \delta_3^2}. \tag{3}$$

Let $J = (\sigma, 0] \subseteq (-\infty, 0]$. The set of all quaternion-valued functions $g \colon J \to \mathbb{F}$ is denoted by $\mathbb{JF}$ and the fractional derivative of $g \in \mathbb{JF}$ is given by

$$D^{\varepsilon} g(\alpha) = D^{\varepsilon} g_0(\alpha) + D^{\varepsilon} g_0(\alpha) e_1 + D^{\varepsilon} g_1(\alpha) e_2 + D^{\varepsilon} g_2(\alpha) e_3. \tag{4}$$

The real matrix Λ of size 4×4 is given by

$$\Lambda = \delta_0 E + \Upsilon, \tag{5}$$

where

$$\Upsilon = \begin{bmatrix} 0 & -\delta_1 & -\delta_2 & -\delta_3 \\ \delta_1 & 0 & -\delta_3 & \delta_2 \\ \delta_2 & \delta_3 & 0 & -\delta_1 \\ \delta_3 & -\delta_2 & \delta_1 & 0 \end{bmatrix}$$

and E is a unit matrix such that the numbers $\delta_1, \delta_2, \delta_3 \in \mathbb{R}$ don't equal to zero.

Remark 1 In (5), recall [35] that

1. The matrix Υ is diagonalizable;
2. The eigenvalues of the matrix Υ are given by

$$\mu_1 = i\sqrt{\delta_1^2+\delta_2^2+\delta_3^2}, \quad \mu_2 = i\sqrt{\delta_1^2+\delta_2^2+\delta_3^2}$$

$$\mu_3 = -i\sqrt{\delta_1^2+\delta_2^2+\delta_3^2}, \quad \mu_4 = -i\sqrt{\delta_1^2+\delta_2^2+\delta_3^2}$$

where $i=\sqrt{-1}$ is an imaginary unit in complex numbers.

Theorem 1 [33] In (5), the eigenvalues of matrix Λ are given by

$$\mu_{\Lambda_1} = \delta_0+\mu_1, \quad \mu_{\Lambda_2} = \delta_0+\mu_2,$$

$$\mu_{\Lambda_3} = \delta_0+\mu_3, \quad \mu_{\Lambda_4} = \delta_0+\mu_4.$$

Remark 2 The two norms $||.||_k$ and $||.||$ are compatible if $||\Lambda y|| \le ||\Lambda||_k||y||$ for all $\Lambda \in \mathbb{C}^{n\times n}$ and $y \in \mathbb{C}^n$, where $||.||_k$ is a matrix norm on $\mathbb{C}^{n\times n}$ and $||.||$ is a norm on $\mathbb{C}^n$.

Throughout this paper, for matrix $\Lambda \in \mathbb{C}^{n\times n}$, we define the 2-norm as

$$||\Lambda||_2 = \max\{\sqrt{\mu}\colon \mu \ \text{ is an eigenvalue of a matrix } \Lambda^*\Lambda\}, \tag{6}$$

where $\Lambda^* = \overline{\Lambda}^T = \overline{\Lambda^T}$ represents the conjugate transpose of matrix Λ.

Lemma 1 [33] For a matrix Υ in (5) and a matrix $\Lambda \in \mathbb{C}^{n\times n}$, $||e^{\Lambda t}||_2 \le e^{\Upsilon t}$.

Let $T\colon \Sigma \to \mathbb{B}$ be any operator of any nonempty subset Σ of a Banach space $(\mathbb{B}, ||.||)$ into $\mathbb{B}$. Consider the two equations

$$T(\varepsilon) = 0, \quad \varepsilon \in \Sigma \tag{7}$$

and

$$T(\varepsilon) \le \gamma. \tag{8}$$

Definition 1 [36] If for any solution ν of (8) there is another solution y of the operator equation (7) such that

$$|\nu - y| \le c\gamma$$

then the equation (7) is called an Hyers-Ulam stable, where c is a constant dependent on T.

Definition 2 [36] If for any $\gamma > 0$ and for each solution ν of (8) there is another solution y of (7) such that

$$|\nu - y| \leq \varphi(\gamma)$$

then the equation (7) is called generalized Hyers-Ulam stable, where φ: $\mathbb{R}^+ \to \mathbb{R}^+$ is a non decreasing function and continuous at 0 with $\varphi(0) = 0$.

For the backgrounds of Fractional Differential Operators (FDOs), see [37–39].

Definition 3 Let $0 < \varepsilon < 1$. The left $\varepsilon-$order FDO of integrable function h on J is given by

$$_aI_\alpha^\varepsilon h(\alpha) = \frac{1}{\Gamma(\varepsilon)} \int_a^\alpha (\alpha - s)^{\varepsilon - 1} h(s) ds \tag{9}$$

The right $\varepsilon-$order FDO of h is given by

$$_\alpha I_b^\varepsilon h(\alpha) = \frac{1}{\Gamma(\varepsilon)} \int_\alpha^b (s - \alpha)^{\varepsilon - 1} h(s) ds \tag{10}$$

where Γ is the gamma function given by

$$\Gamma(\varepsilon) = \int_0^\infty e^{-\alpha} s^{\varepsilon - 1} ds.$$

Definition 4 The left $\varepsilon-$order Riemann-Liouville FDO of h is given by

$$_aD_\alpha^\varepsilon h(\alpha) = \frac{d^n}{d\alpha^n} {_aD_\alpha^{-(n-\varepsilon)}} h(\alpha) = \frac{d^n}{d\alpha^n} {_aI_\alpha^{n-\varepsilon}} h(\alpha) \tag{11}$$

$$_\alpha D_b^\varepsilon h(\alpha) = \frac{d^n}{d\alpha^n} {_\alpha D_b^{-(n-\varepsilon)}} h(\alpha) = \frac{d^n}{d\alpha^n} {_\alpha I_b^{n-\varepsilon}} h(\alpha) \tag{12}$$

Definition 5 Let $\varepsilon > 0,$ and $n - 1 < \varepsilon \leq n$ $(n \in \mathbb{N})$. The left $\varepsilon-$order Caputo FDO of h is given by

$$^C_aD_\alpha^\varepsilon h(\alpha) = {_aI_\alpha^{n-\varepsilon}} D^n h(\alpha) = \frac{1}{\Gamma(n - \varepsilon)} \int_0^\alpha (\alpha - s)^{n - \varepsilon - 1} h^{(n)}(s) ds. \tag{13}$$

The right $\varepsilon-$order Caputo FDO of h is given by

$$^C_\alpha D_b^\varepsilon h(\alpha) = -{_\alpha I_b^{n-\varepsilon}} D^n h(\alpha) = \frac{-1}{\Gamma(n - \varepsilon)} \int_\alpha^b (s - \alpha)^{n - \varepsilon - 1} h^{(n)}(s) ds. \tag{14}$$

Definition 6 In Caputo sense, the FDO of h is given for $0 < \varepsilon < 1$ as the left $\varepsilon-$order Caputo FDO

$$ {}_{a}^{C}D_{\alpha}^{\varepsilon}h(\alpha) = {}_{a}I_{\alpha}^{1-\varepsilon}h'(\alpha) = \frac{1}{\Gamma(1-\varepsilon)}\int_{0}^{\alpha}(\alpha-s)^{-\varepsilon}h^{'}(s)ds. \tag{15} $$

The right $\varepsilon-$order Caputo FDO of h is given by

$$ {}_{\alpha}^{C}D_{b}^{\varepsilon}h(\alpha) = -{}_{\alpha}I_{b}^{1-\varepsilon}h'(\alpha) = \frac{-1}{\Gamma(1-\varepsilon)}\int_{\alpha}^{b}(s-\alpha)^{-\varepsilon}h^{'}(s)ds. \tag{16} $$

3. Main results

In this section, we discuss the Hyers-Ulam stability and generalized Hyers-Ulam stability of certain linear QVFDEs. Specifically, we first analyze the homogeneous linear QVFDE (1) under two cases: when $\mu = 1$ and when $\mu \neq 1$. We then turn our attention to the homogeneous linear QVFDE (2). Finally, we extend the discussion to the generalized Hyers-Ulam stability of (1) as well as the generalized Hyers-Ulam stability of the non-homogeneous linear QVFDE (2).

Theorem 2 Let $\mu = 1$ in (1). If for any $\gamma > 0$ and every differentiable map g: $J \to \mathbb{F}$ with

$$ |D^{\varepsilon}g(\alpha) - g(\alpha)| \leq \gamma $$

then there is a solution g_0: $J \to \mathbb{F}$ of (1) such that

$$ |g(\alpha) - g_0(\alpha)| \leq \gamma c, $$

where c is a constant.

Proof. Let $z(\alpha) = D^{\varepsilon}g(\alpha) - g(\alpha)$, then $|z(\alpha)| \leq \gamma$ and

$$ D^{\varepsilon}g(\alpha) = g(\alpha) + z(\alpha) \tag{17} $$

By (4), we have

$$ \begin{aligned} D^{\varepsilon}g(\alpha) =& D^{\varepsilon}g_0(\alpha) + D^{\varepsilon}g_1(\alpha)e_1 + D^{\varepsilon}g_2(\alpha)e_2 + D^{\varepsilon}g_3(\alpha)e_3 \\ =& g_0(\alpha) + g_1(\alpha)e_1 + g_2(\alpha)e_2 + g_3(\alpha)e_3 + z_0(\alpha) + z_1(\alpha)e_1 + z_2(\alpha)e_2 + z_3(\alpha)e_3 \\ =& (g_0(\alpha) + z_0(\alpha)) + (g_1(\alpha) + z_1(\alpha))e_1 + (g_2(\alpha) + z_2(\alpha))e_2 + (g_3(\alpha) + z_3(\alpha))e_3. \end{aligned} $$

Hence

$$ D^{\varepsilon}g_{\kappa}(\alpha) = g_{\kappa}(\alpha) + z_{\kappa}(\alpha), \quad \kappa = 0,\ 1,\ 2,\ 3. \tag{18} $$

and the solution of (18) is given by

$$g_\kappa(\alpha) = C_\kappa e^{\frac{\alpha^\varepsilon}{\varepsilon!}} + \frac{1}{\Gamma(\varepsilon)} \int_\alpha^0 (s-\alpha)^{\varepsilon-1} e^{\frac{(s-\alpha)^\varepsilon}{\varepsilon!}} z_\kappa(s) ds, \quad \kappa = 0,\, 1,\, 2,\, 3,$$

where $C_\kappa = g_\kappa(0) \in \mathbb{R}$. Moreover, derive the solution of (17) is

$$\begin{aligned}
g_\kappa(\alpha) =& g_0(\alpha) + g_1(\alpha)e_1 + g_2(\alpha)e_2 + g_3(\alpha)e_3 \\
=& C_1 e^{\frac{\alpha^\varepsilon}{\varepsilon!}} + \frac{1}{\Gamma(\varepsilon)} \int_\alpha^0 (s-\alpha)^{\varepsilon-1} e^{\frac{(s-\alpha)^\varepsilon}{\varepsilon!}} z_1(s) ds \\
&+ \left(C_2 e^{\frac{\alpha^\varepsilon}{\varepsilon!}} + \frac{1}{\Gamma(\varepsilon)} \int_\alpha^0 (s-\alpha)^{\varepsilon-1} e^{\frac{(s-\alpha)^\varepsilon}{\varepsilon!}} z_2(s) ds \right) e_1 \\
&+ \left(C_3 e^{\frac{\alpha^\varepsilon}{\varepsilon!}} + \frac{1}{\Gamma(\varepsilon)} \int_\alpha^0 (s-\alpha)^{\varepsilon-1} e^{\frac{(s-\alpha)^\varepsilon}{\varepsilon!}} z_3(s) ds \right) e_2 \\
&+ \left(C_4 e^{\frac{\alpha^\varepsilon}{\varepsilon!}} + \frac{1}{\Gamma(\varepsilon)} \int_\alpha^0 (s-\alpha)^{\varepsilon-1} e^{\frac{(s-\alpha)^\varepsilon}{\varepsilon!}} z_4(s) ds \right) e_3 \\
=& e^{\frac{\alpha^\varepsilon}{\varepsilon!}} \Big(C_1 + C_2 + C_3 + C_3 \Big) \\
&+ \frac{1}{\Gamma(\varepsilon)} \int_\alpha^0 (s-\alpha)^{\varepsilon-1} e^{\frac{(s-\alpha)^\varepsilon}{\varepsilon!}} \Big(z_1(s) + z_1(\alpha)e_1 + z_2(\alpha)e_2 + z_3(\alpha)e_3 \Big) ds \\
=& \Delta e^{\frac{\alpha^\varepsilon}{\varepsilon!}} + \frac{1}{\Gamma(\varepsilon)} \int_\alpha^0 (s-\alpha)^{\varepsilon-1} e^{\frac{(s-\alpha)^\varepsilon}{\varepsilon!}} z(s) ds,
\end{aligned} \tag{19}$$

where $\Delta = C_1 + C_2 + C_3 + C_3 = g(0) \in \mathbb{F}$. Note that

$$\lim_{r \to -\infty} \frac{g(\alpha)}{e^{\frac{\alpha^\varepsilon}{\varepsilon!}}} = g(0) + \frac{1}{\Gamma(\varepsilon)} \int_{-\infty}^0 (s-\alpha)^{\varepsilon-1} e^{\frac{s^\varepsilon}{\varepsilon!}} z(s) ds$$

exists, since $z(s) \leq \gamma$. Let

$$g_0(\alpha) = e^{\frac{\alpha^\varepsilon}{\varepsilon!}} g(0) + \frac{1}{\Gamma(\varepsilon)} \int_{-\infty}^0 (s-\alpha)^{\varepsilon-1} e^{\frac{(s-\alpha)^\varepsilon}{\varepsilon!}} z(s) ds \tag{20}$$

It is clear that when $\mu = 1$, $g_0(\alpha)$ is a solution to QVFDE (1).

By combining (19) and (20), we have

$$
\begin{aligned}
|g(\alpha)-g_0(\alpha)| =& \left|\frac{1}{\Gamma(\varepsilon)}\int_{-\infty}^{0}(s-\alpha)^{\varepsilon-1}e^{\frac{(s-\alpha)^{\varepsilon}}{\varepsilon!}}z(s)ds\right| \\
\leq& \frac{1}{\Gamma(\varepsilon)}\int_{-\infty}^{0}\left|(s)^{\varepsilon-1}e^{\frac{s^{\varepsilon}}{\varepsilon!}}z(\alpha-s)ds\right|ds \\
\leq& \frac{\gamma}{\Gamma(\varepsilon)}\int_{-\infty}^{0}\left|s^{\varepsilon-1}e^{\frac{s^{\varepsilon}}{\varepsilon!}}\right|ds.
\end{aligned}
$$

By using Mathematica 5 porgram we get for $\varepsilon > 0$

$$
\int_{-\infty}^{0} s^{\varepsilon-1}e^{\frac{s^{\varepsilon}}{\varepsilon!}}ds = \frac{\varepsilon!}{\varepsilon\log[e]} = \frac{(\varepsilon-1)!}{\log[e]}.
$$

Then

$$
|g(\alpha)-g_0(\alpha)| \leq \frac{\gamma}{\Gamma(\varepsilon)}\int_{-\infty}^{0}\left|s^{\varepsilon-1}e^{\frac{s^{\varepsilon}}{\varepsilon!}}\right|ds \leq \frac{\gamma}{\Gamma(\varepsilon)}\frac{(\varepsilon-1)!}{\log[e]} = \gamma\frac{1}{\log[e]} \leq \gamma c
$$

It immediately follows that $|g(\alpha)-g_0(\alpha)| \leq \gamma c$, where $c = \dfrac{1}{\log[e]}$. □

Corollary 1 If $\mu = 1$ then the QVFDE (1) is an Hyers-Ulam stable.

Now we study the Hyers-Ulam stability of homogeneous linear QVFDE (1) when $\mu \neq 1$.

Theorem 3 In the QVFDE (1), there is $M > 0$, where for any differentiable map g: $J \to \mathbb{F}$ and for any $\gamma > 0$ and every with

$$
|D^{\varepsilon}g(\alpha)-\mu g(\alpha)| \leq \gamma
$$

there is a solution g_0: $J \to \mathbb{F}$ of (1) such that

$$
|g(\alpha)-g_0(\alpha)| \leq M\gamma.
$$

Proof. Let $z(\alpha) = D^{\varepsilon}g(\alpha)-\mu g(\alpha)$, then $|z(\alpha)| \leq \gamma$ and $D^{\varepsilon}g(\alpha) = \mu g(\alpha)+z(\alpha)$ By (4), we have

$$
\begin{aligned}
D^{\varepsilon}g(\alpha) =& D^{\varepsilon}g_0(\alpha)+D^{\varepsilon}g_1(\alpha)e_1+D^{\varepsilon}g_2(\alpha)e_2+D^{\varepsilon}g_3(\alpha)e_3 \\
=& (\delta_0+\delta_1e_1+\delta_2e_2+\delta_3e_3)(g_0(\alpha)+g_1(\alpha)e_1+g_2(\alpha)e_2+g_3(\alpha)e_3) \\
&+z_0(\alpha)+z_1(\alpha)e_1+z_2(\alpha)e_2+z_3(\alpha)e_3
\end{aligned}
$$

$$=\delta_0 g_0(\alpha)-\delta_1 g_1(\alpha)-\delta_2 g_2(\alpha)-\delta_3 g_3(\alpha)$$

$$+(\delta_1 g_0(\alpha)+\delta_0 g_1(\alpha)-\delta_3 g_2(\alpha)\delta_2 g_3(\alpha))e_1$$

$$+(\delta_2 g_0(\alpha)+\delta_3 g_1(\alpha)+\delta_0 g_2(\alpha)-\delta_1 g_3(\alpha))e_2$$

$$+(\delta_3 g_0(\alpha)-\delta_2 g_1(\alpha)+\delta_1 g_2(\alpha)+\delta_0 g_3(\alpha))e_3$$

$$+z_0(\alpha)+z_1(\alpha))e_1+z_2(\alpha))e_2+z_3(\alpha)e_3. \tag{21}$$

Clearly, (21) is equivalent to the following system

$$\begin{bmatrix} D^{\varepsilon}g_0(\alpha) \\ D^{\varepsilon}g_1(\alpha) \\ D^{\varepsilon}g_2(\alpha) \\ D^{\varepsilon}g_3(\alpha) \end{bmatrix} = \begin{bmatrix} \delta_0 & -\delta_1 & -\delta_2 & -\delta_3 \\ \delta_1 & \delta_0 & -\delta_3 & \delta_2 \\ \delta_2 & \delta_3 & \delta_0 & -\delta_1 \\ \delta_3 & -\delta_2 & \delta_1 & \delta_0 \end{bmatrix} \begin{bmatrix} g_0(\alpha) \\ g_1(\alpha) \\ g_2(\alpha) \\ g_3(\alpha) \end{bmatrix} + \begin{bmatrix} z_0(\alpha) \\ z_1(\alpha) \\ z_2(\alpha) \\ z_3(\alpha) \end{bmatrix} \tag{22}$$

Let $x=(g_0, g_1, g_2, g_3)^T,\ z=(z_0, z_1, z_2, z_3)^T$, and

$$\Lambda = \begin{bmatrix} \delta_0 & -\delta_1 & -\delta_2 & -\delta_3 \\ \delta_1 & \delta_0 & -\delta_3 & \delta_2 \\ \delta_2 & \delta_3 & \delta_0 & -\delta_1 \\ \delta_3 & -\delta_2 & \delta_1 & \delta_0 \end{bmatrix}.$$

Then the system (22) is equivalent to $D^{\varepsilon}x(\alpha)=\Lambda x(\alpha)+z(\alpha)$, and its solution is given by

$$x(\alpha)=e^{\frac{\Lambda\alpha^{\varepsilon}}{\varepsilon!}}x(0)+\frac{1}{\Gamma(\varepsilon)}\int_{\alpha}^{0}(s-\alpha)^{\varepsilon-1}e^{\frac{\Lambda(s-\alpha)^{\varepsilon}}{\varepsilon!}}z(s)ds, \tag{23}$$

where $x(0)=(g_0(0), g_1(0), g_2(0), g_3(0))\in\mathbb{R}$. In addition, the QVFDE (1) can be represented as $D^{\varepsilon}x(\alpha)=\Lambda x(\alpha),\ \alpha\in J$. Let

$$x_0(\alpha)=e^{\frac{\Lambda\alpha^{\varepsilon}}{\varepsilon!}}x(0),\ \ \alpha\in J \tag{24}$$

It is clear that $x(0) = (g_0(0),\, g_1(0),\, g_2(0),\, g_3(0))$ is another solution of QVFDE (1). By Theorem 1 and (6), the 2-norm of Λ is

$$\|\Lambda\|_2 = \Lambda^{\mathsf{T}}\Lambda$$

$$= \begin{bmatrix} \delta_0 & \delta_1 & \delta_2 & \delta_3 \\ -\delta_1 & \delta_0 & -\delta_3 & -\delta_2 \\ -\delta_2 & -\delta_3 & \delta_0 & \delta_1 \\ -\delta_3 & \delta_2 & -\delta_1 & \delta_0 \end{bmatrix} \begin{bmatrix} \delta_0 & -\delta_1 & -\delta_2 & -\delta_3 \\ \delta_1 & \delta_0 & -\delta_3 & \delta_2 \\ \delta_2 & \delta_3 & \delta_0 & -\delta_1 \\ \delta_3 & -\delta_2 & \delta_1 & \delta_0 \end{bmatrix}$$

$$= \begin{bmatrix} \delta_0^2+\delta_1^2+\delta_2^2+\delta_3^2 & 0 & 0 & 0 \\ 0 & \delta_0^2+\delta_1^2+\delta_2^2+\delta_3^2 & 0 & 0 \\ 0 & 0 & \delta_0^2+\delta_1^2+\delta_2^2+\delta_3^2 & 0 \\ 0 & 0 & 0 & \delta_0^2+\delta_2^2+\delta_2^2+\delta_3^2 \end{bmatrix}.$$

since the eigenvalue of a matrix Λ is $det(\Lambda^T\Lambda - \mu E) = 0$, then

$$||\Lambda||_2 = \max\{\sqrt{\mu}\colon \mu \text{ is an eigenvalue of a matrix } \Lambda^T\Lambda\} = \sqrt{\delta_0^2+\delta_2^2+\delta_2^2+\delta_3^2}.$$

Therefore, set

$$M = \frac{(\varepsilon-1)!(e^{\frac{\Lambda s^{\varepsilon}}{\Gamma[1+\varepsilon]}}\varepsilon - 1)}{s\Lambda \log[e]}.$$

Then by (23), (24) and Remark 2, Lemma 1, we get

$$\begin{aligned} ||x(\alpha) - x_0(\alpha)||_2 \leq & \frac{1}{\Gamma(\varepsilon)} \int_{\alpha}^{0} ||(s-\alpha)^{\varepsilon-1} e^{\frac{\Lambda(s-\alpha)^{\varepsilon}}{\varepsilon!}} x(s)||_2 ds \\ \leq & \frac{1}{\Gamma(\varepsilon)} \int_{\alpha}^{0} ||(s-\alpha)^{\varepsilon-1} e^{\frac{\Lambda(s-\alpha)^{\varepsilon}}{\varepsilon!}} ||_2 ||x(s)||_2 ds \\ \leq & \frac{1}{\Gamma(\varepsilon)} \int_{\alpha}^{0} ||s^{\varepsilon-1} e^{\frac{\Lambda s^{\varepsilon}}{\varepsilon!}} ||_2 ||x(s-\alpha)||_2 ds \end{aligned}$$

$$\leq \frac{\gamma}{\Gamma(\varepsilon)} \int_{\alpha}^{0} ||s^{\varepsilon-1} e^{\frac{\Lambda s^{\varepsilon}}{\varepsilon!}}||_2 ds$$

$$\leq \frac{\gamma}{\Gamma(\varepsilon)} \int_{\alpha}^{0} s^{\varepsilon-1} e^{\frac{\Lambda s^{\varepsilon}}{\varepsilon!}} ds$$

By using Mathematica 5 porgram

$$\int_{\alpha}^{0} s^{\varepsilon-1} e^{\frac{\Lambda s^{\varepsilon}}{\varepsilon!}} ds = \frac{-\varepsilon! + e^{\frac{\Lambda\alpha^{\varepsilon}}{\Gamma[1+\varepsilon]}} \varepsilon\Gamma[\varepsilon]}{t\Lambda\varepsilon \log[e]} \leq \frac{(\varepsilon-1)!(-1+\varepsilon e^{\frac{\Lambda s^{\varepsilon}}{\Gamma[1+\varepsilon]}})}{s\Lambda \log[e]} = \frac{(\varepsilon-1)!(\varepsilon e^{\frac{\Lambda s^{\varepsilon}}{\Gamma[1+\varepsilon]}} - 1)}{s\Lambda \log[e]}$$

Then

$$|g(\alpha) - g_0(\alpha)| \leq \frac{\gamma}{\Gamma(\varepsilon)} \frac{(\varepsilon-1)!(-1+\varepsilon e^{\frac{\Lambda s^{\varepsilon}}{\Gamma[1+\varepsilon]}})}{s\Lambda \log[e]} \leq \frac{\gamma}{\Gamma(\varepsilon)} \frac{(\varepsilon-1)!(\varepsilon e^{\frac{\Lambda s^{\varepsilon}}{\Gamma[1+\varepsilon]}}\varepsilon - 1)}{s\Lambda \log[e]} \leq \gamma \frac{\varepsilon e^{\frac{\Lambda s^{\varepsilon}}{\Gamma[1+\varepsilon]}} - 1}{s\Lambda \log[e]}$$

It immediately follows that

$$|g(\alpha) - g_0(\alpha)| \leq \gamma \frac{\varepsilon e^{\frac{\Lambda s^{\varepsilon}}{\Gamma[1+\varepsilon]}} - 1}{s\Lambda \log[e]} = M\gamma,$$

where $M = \gamma \dfrac{\varepsilon e^{\frac{\Lambda s^{\varepsilon}}{\Gamma[1+\varepsilon]}} - 1}{s\Lambda \log[e]}$. Finally, by (3), we obtain $|g(\alpha) - g_0(\alpha)| \leq \gamma M$. □

Corollary 2 The QVFDE (1) is an Hyers-Ulam stable for all $-\infty < \sigma < 0$.

The following theorem shows the Hyers-Ulam stability of non-homogeneous linear QVFDE (2).

Theorem 4 In the QVFD (2), there is $M > 0$ where for any differentiable map $g\colon J \to \mathbb{F}$ and for any $\gamma > 0$ with

$$|D^{\varepsilon} g(\alpha) - \mu g(\alpha) - \theta(\alpha)| \leq \gamma$$

there is a solution $g_0\colon J \to \mathbb{F}$ of (2) such that

$$|g(\alpha) - g_0(\alpha)| \leq M\gamma.$$

Proof. Let $z(\alpha) = D^{\varepsilon} g(\alpha) - \mu g(\alpha) - \theta(\alpha)$, then $|z(\alpha)| \leq \gamma$ and $D^{\varepsilon} g(\alpha) = \mu g(\alpha) + \theta(\alpha) + z(\alpha)$ By (4), we have

$$
\begin{aligned}
D^{\varepsilon}g(\alpha) =& D^{\varepsilon}g_0(\alpha)+D^{\varepsilon}g_1(\alpha)e_1+D^{\varepsilon}g_2(\alpha)e_2+D^{\varepsilon}g_3(\alpha)e_3\\
=& \delta_0 g_0(\alpha)-\delta_1 g_1(\alpha)-\delta_2 g_2(\alpha)-\delta_3 g_3(\alpha)\\
&+(\delta_1 g_0(\alpha)+\delta_0 g_1(\alpha)-\delta_3 g_2(\alpha)+\delta_2 g_3(\alpha))e_1\\
&+(\delta_2 g_0(\alpha)+\delta_3 g_1(\alpha)+\delta_0 g_2(\alpha)-\delta_1 g_3(\alpha))e_2\\
&+(\delta_3 g_0(\alpha)-\delta_2 g_1(\alpha)+\delta_1 g_2(\alpha)+\delta_0 g_3(\alpha))e_3\\
&+(z_0(\alpha)+\theta_0(\alpha))+(z_1(\alpha)+\theta_1(\alpha))e_1+(z_2(\alpha)+\theta_2(\alpha))e_2\\
&+(z_3(\alpha)+\theta_3(\alpha))e_3
\end{aligned} \tag{25}
$$

Clearly, (25) is equivalent to the following system

$$
\begin{bmatrix} D^{\varepsilon}g_0(\alpha)\\ D^{\varepsilon}g_1(\alpha)\\ D^{\varepsilon}g_2(\alpha)\\ D^{\varepsilon}g_3(\alpha)\end{bmatrix}
=
\begin{bmatrix} \delta_0 & -\delta_1 & -\delta_2 & -\delta_3\\ \delta_1 & \delta_0 & -\delta_3 & \delta_2\\ \delta_2 & \delta_3 & \delta_0 & -\delta_1\\ \delta_3 & -\delta_2 & \delta_1 & \delta_0 \end{bmatrix}
\begin{bmatrix} g_0(\alpha)\\ g_1(\alpha)\\ g_2(\alpha)\\ g_3(\alpha)\end{bmatrix}
+
\begin{bmatrix} z_0(\alpha)+\theta_0(\alpha)\\ z_1(\alpha)+\theta_1(\alpha)\\ z_2(\alpha)+\theta_2(\alpha)\\ z_3(\alpha)+\theta_3(\alpha)\end{bmatrix} \tag{26}
$$

Let $x=(g_0,\, g_1,\, g_2,\, g_3)^T,\ y=(z_0+\theta_0,\, z_1+\theta_1,\, z_2+\theta_2,\, z_3+\theta_3)^T$, and

$$
\Lambda=\begin{bmatrix} \delta_0 & -\delta_1 & -\delta_2 & -\delta_3\\ \delta_1 & \delta_0 & -\delta_3 & \delta_2\\ \delta_2 & \delta_3 & \delta_0 & -\delta_1\\ \delta_3 & -\delta_2 & \delta_1 & \delta_0 \end{bmatrix}.
$$

Then system of (26) can be written as $D^{\varepsilon}x(\alpha)=\Lambda x(\alpha)+y(\alpha)$, and the solution for system (26) is

$$
x(\alpha)=e^{\frac{\Lambda\alpha^{\varepsilon}}{\varepsilon!}}x(0)+\frac{1}{\Gamma(\varepsilon)}\int_{\alpha}^{0}(s-\alpha)^{\varepsilon-1}e^{\frac{\Lambda(s-\alpha)^{\varepsilon}}{\varepsilon!}}y(s)ds, \tag{27}
$$

In addition, QVFDE (2) can be represented as $D^{\varepsilon}x(\alpha)=\Lambda x(\alpha)+\theta(\alpha)$. Let

$$x_0(\alpha)=e^{\frac{\Lambda\alpha^{\varepsilon}}{\varepsilon!}}x(0)+\frac{1}{\Gamma(\varepsilon)}\int_{\alpha}^{0}(s-\alpha)^{\varepsilon-1}e^{\frac{\Lambda(s-\alpha)^{\varepsilon}}{\varepsilon!}}\theta(s)ds \tag{28}$$

where $x(0)=(g_0(0),\, g_1(0),\, g_2(0),\, g_3(0))\in\mathbb{R}^4$. Clearly $x_0=(g_0,\, g_1,\, g_2,\, g_3)^t$ is a solution to QVFDE (2). By (27), (28), Remark 2 and Lemma 1,

$$\begin{aligned}
||x(\alpha)-x_0(\alpha)||_2 \leq & \frac{1}{\Gamma(\varepsilon)}\int_{\alpha}^{0}||(s-\alpha)^{\varepsilon-1}e^{\frac{\Lambda(s-\alpha)^{\varepsilon}}{\varepsilon!}}z(s)||_2ds\\
\leq & \frac{1}{\Gamma(\varepsilon)}\int_{\alpha}^{0}||(s-\alpha)^{\varepsilon-1}e^{\frac{\Lambda(s-\alpha)^{\varepsilon}}{\varepsilon!}}||_2||z(s)||_2ds\\
\leq & \frac{1}{\Gamma(\varepsilon)}\int_{\alpha}^{0}||s^{\varepsilon-1}e^{\frac{\Lambda s^{\varepsilon}}{\varepsilon!}}||_2||z(s-\alpha)||_2ds\\
\leq & \frac{\gamma}{\Gamma(\varepsilon)}\int_{\alpha}^{0}||s^{\varepsilon-1}e^{\frac{\Lambda s^{\varepsilon}}{\varepsilon!}}||_2ds\\
\leq & \frac{\gamma}{\Gamma(\varepsilon)}\int_{\alpha}^{0}s^{\varepsilon-1}e^{\frac{\Lambda s^{\varepsilon}}{\varepsilon!}}ds\\
\leq & \frac{\gamma}{\Gamma(\varepsilon)}\frac{-\varepsilon!+\varepsilon e^{\frac{\Lambda\alpha^{\varepsilon}}{\Gamma[1+\varepsilon]}}\Gamma[\varepsilon]}{t\Lambda\varepsilon\log[e]}\\
\leq & \frac{\gamma}{\Gamma(\varepsilon)}\frac{(\varepsilon-1)!(\varepsilon e^{\frac{\Lambda\alpha^{\varepsilon}}{\Gamma[1+\varepsilon]}}-1)}{t\Lambda\log[e]}\\
\leq & \gamma\frac{\varepsilon e^{\frac{\sqrt{\delta_0^2+\delta_2^2+\delta_2^2+\delta_3^2}s^{\varepsilon}}{\Gamma[1+\varepsilon]}}-1}{s\sqrt{\delta_0^2+\delta_2^2+\delta_2^2+\delta_3^2}\log[e]}\\
\leq & \gamma M
\end{aligned}$$

where $r\in J=(\sigma,\, 0]\subseteq(-\infty,\, 0]$ and

$$M=\frac{\varepsilon e^{\frac{\sqrt{\delta_0^2+\delta_2^2+\delta_2^2+\delta_3^2}s^{\varepsilon}}{\Gamma[1+\varepsilon]}}-1}{s\sqrt{\delta_0^2+\delta_2^2+\delta_2^2+\delta_3^2}\log[e]}.$$

Finally, by (3), we obtain $|g(\alpha)-g_0(\alpha)| \leq \gamma M$. The proof is complete. □

Corollary 3 The QVFD (2) is an Hyers-Ulam stable for all $-\infty < \sigma < 0$.

The following theorem shows the generalized Hyers-Ulam stability of homogeneous linear QVFDE (1) where $\mu: J \to \mathbb{F}$ is a map.

Theorem 5 Let $\phi: J \to (-\infty, 0]$ and $\mu: J \to \mathbb{F}$ be a map. Then for every differential map $g: J \to \mathbb{F}$ with

$$|D^{\varepsilon} g(\alpha) - \mu g(\alpha)| \leq \phi, \quad \alpha \in J,$$

there is a solution $g_0: J \to \mathbb{F}$ of QVFDE (1) such that

$$|g(\alpha) - \mu g_0(\alpha)| \leq \frac{||X(\alpha)||_2}{\Gamma(\varepsilon)} \int_{\alpha}^{0} (s-\alpha)^{\varepsilon-1} ||X(s)^{-1}||_2 \phi(s) ds, \tag{29}$$

where $X(\alpha)$ is a fundamental matrix of $D^{\varepsilon} x(\alpha) = \Lambda(\alpha) x(\alpha)$.

Proof. Let $z(\alpha) = D^{\varepsilon} g(\alpha) - \mu g(\alpha)$, then $|z(\alpha)| \leq \phi$ and $D^{\varepsilon} g(\alpha) = \mu g(\alpha) + z(\alpha)$.

By (4), we have

$$\begin{aligned} D^{\varepsilon} g(\alpha) =& D^{\varepsilon} g_0(\alpha) + D^{\varepsilon} g_1(\alpha) e_1 + D^{\varepsilon} g_2(\alpha) e_2 + D^{\varepsilon} g_3(\alpha) e_3 \\ =& \delta_0(\alpha) g_0(\alpha) - b(\alpha) g_1(\alpha) - \delta_2(\alpha) g_2(\alpha) - \delta_3(\alpha) g_3(\alpha) \\ &+ (\delta_1(\alpha) g_0(\alpha) + \delta_0(\alpha) g_1(\alpha) - \delta_3(\alpha) g_2(\alpha) + \delta_2(\alpha) g_3(\alpha)) e_1 \\ &+ (\delta_2(\alpha) g_0(\alpha) + \delta_3(\alpha) g_1(\alpha) + \delta_0(\alpha) g_2(\alpha) - \delta_1(\alpha) g_3(\alpha)) e_2 \\ &+ (\delta_3(\alpha) g_0(\alpha) - \delta_2(\alpha) g_1(\alpha) + \delta_1(\alpha) g_2(\alpha) + \delta_0(\alpha) g_3(\alpha)) e_3 \\ &+ z_0(\alpha) + z_1(\alpha) e_1 + z_2(\alpha) e_2 + z_3(\alpha) e_3 \end{aligned} \tag{30}$$

It is clear that (30) is equivalent to the following system

$$\begin{bmatrix} D^{\varepsilon} g_0(\alpha) \\ D^{\varepsilon} g_1(\alpha) \\ D^{\varepsilon} g_2(\alpha) \\ D^{\varepsilon} g_3(\alpha) \end{bmatrix} = \begin{bmatrix} \delta_0(\alpha) & -\delta_1(\alpha) & -\delta_2(\alpha) & -\delta_3(\alpha) \\ \delta_1(\alpha) & \delta_0(\alpha) & -\delta_3(\alpha) & \delta_2(\alpha) \\ \delta_2(\alpha) & \delta_3(\alpha) & \delta_0(\alpha) & -\delta_1(\alpha) \\ \delta_3(\alpha) & -\delta_2(\alpha) & \delta_1(\alpha) & \delta_0(\alpha) \end{bmatrix} \begin{bmatrix} g_0(\alpha) \\ g_1(\alpha) \\ g_2(\alpha) \\ g_3(\alpha) \end{bmatrix} + \begin{bmatrix} z_0(\alpha) \\ z_1(\alpha) \\ z_2(\alpha) \\ z_3(\alpha) \end{bmatrix} \tag{31}$$

Let $x = (g_0, g_1, g_2, g_3)^T$, $z = (z_0, z_1, z_2, z_3)^T$, and

$$\Lambda = \begin{bmatrix} \delta_0(\alpha) & -\delta_1(\alpha) & -\delta_2(\alpha) & -\delta_3(\alpha) \\ \delta_1(\alpha) & \delta_0(\alpha) & -\delta_3(\alpha) & \delta_2(\alpha) \\ \delta_2(\alpha) & \delta_3(\alpha) & \delta_0(\alpha) & -\delta_1(\alpha) \\ \delta_3(\alpha) & -\delta_2(\alpha) & b(\alpha) & \delta_0(\alpha) \end{bmatrix}.$$

Then system (31) has the form $D^{\varepsilon}x(\alpha) = \Lambda(\alpha)x(\alpha) + z(\alpha)$ and the solution

$$x(\alpha) = X(\alpha)\xi + \frac{X(\alpha)}{\Gamma(\varepsilon)} \int_{\alpha}^{0} (s-\alpha)^{\varepsilon-1} X(s)^{-1} z(s)ds, \tag{32}$$

where $\xi \in \mathbb{R},\ X(\alpha) = e^{\frac{1}{\Gamma(\varepsilon+1)} \int_{\alpha}^{0} (s-\alpha)^{\varepsilon} \Lambda(s)ds}$. In addition, when $\mu\colon J \to \mathbb{F}$ is a map, the QVFDE (1) can be represented as $D^{\varepsilon}x(\alpha) = \Lambda x(\alpha)$. Let

$$x_0(\alpha) = X(\alpha)\xi \tag{33}$$

By (32), (33) and Remark 2, we get

$$\begin{aligned} ||x(\alpha) - x_0(\alpha)||_2 &\leq || \frac{X(\alpha)}{\Gamma(\varepsilon)} \int_{\alpha}^{0} (s-\alpha)^{\varepsilon-1} X(s)^{-1} z(s)ds||_2 ds \\ &\leq || \frac{X(\alpha)}{\Gamma(\varepsilon)} ||_2 || \int_{\alpha}^{0} (s-\alpha)^{\varepsilon-1} X(s)^{-1} z(s)ds||_2 \\ &\leq || \frac{X(\alpha)}{\Gamma(\varepsilon)} ||_2 \int_{\alpha}^{0} ||(s-\alpha)^{\varepsilon-1} X(s)^{-1}||_2 ||z(s)||_2 ds \\ &\leq || \frac{X(\alpha)}{\Gamma(\varepsilon)} ||_2 \int_{\alpha}^{0} ||(s-\alpha)^{\varepsilon-1} X(s)^{-1}||_2 \phi(s)ds. \end{aligned}$$

Finally, by (3), we obtain (29). The proof is complete. □

Corollary 4 The QVFDE (1) is generalized Hyers-Ulam stable for all any map $\mu\colon J \to \mathbb{F}$.

Finally, we give the generalized Hyers-Ulam stability of non-homogeneous linear QVFDE (2) where $\mu\colon J \to \mathbb{F}$ is a map.

Theorem 6 Let $\phi\colon J \to (-\infty, 0]$ and $\mu\colon J \to \mathbb{F}$ be a map. Then for every differential map $g\colon J \to \mathbb{F}$ with

$$|D^{\varepsilon}g(\alpha) - \mu g(\alpha) - \theta(\alpha)| \leq \phi, \ \ \alpha \in J,$$

there is a solution g_0: $J \to \mathbb{F}$ of QVFDE (2) such that

$$|g(\alpha) - \mu g_0(\alpha) - \theta(\alpha)| \le \frac{||X(\alpha)||_2}{\Gamma(\varepsilon)} \int_\alpha^0 (s-\alpha)^{\varepsilon-1} ||X(s)^{-1}||_2 \phi(s) ds, \tag{34}$$

where $X(\alpha)$ is a fundamental matrix of $D^\varepsilon x(\alpha) = \Lambda(\alpha) x(\alpha)$.

Proof. Let $z(\alpha) = D^\varepsilon g(\alpha) - \mu g(\alpha) - \theta(\alpha)$, then $|z(\alpha)| \le \phi$ and $D^\varepsilon g(\alpha) = \mu g(\alpha) + \theta(\alpha) + z(\alpha)$.

By (4), we have

$$\begin{aligned} D^\varepsilon g(\alpha) =& D^\varepsilon g_0(\alpha) + D^\varepsilon g_1(\alpha) e_1 + D^\varepsilon g_2(\alpha) e_2 + D^\varepsilon g_3(\alpha) e_3 \\ =& \delta_0(\alpha) g_0(\alpha) - \delta_1(\alpha) g_1(\alpha) - \delta_2(\alpha) g_2(\alpha) - \delta_3(\alpha) g_3(\alpha) \\ &+ (\delta_1(\alpha) g_0(\alpha) + \delta_0(\alpha) g_1(\alpha) - \delta_3(\alpha) g_2(\alpha) + \delta_2(\alpha) g_3(\alpha)) e_1 \\ &+ (\delta_2(\alpha) g_0(\alpha) + \delta_3(\alpha) g_1(\alpha) + \delta_0(\alpha) g_2(\alpha) - \delta_1(\alpha) g_3(\alpha)) e_2 \\ &+ (\delta_3(\alpha) g_0(\alpha) - \delta_2(\alpha) g_1(\alpha) + \delta_1(\alpha) g_2(\alpha) + \delta_0(\alpha) g_3(\alpha)) e_3 \\ &+ (\theta_0(\alpha) + z_0(\alpha)) + (\theta_1(\alpha) + z_1(\alpha)) e_1 + (\theta_2(\alpha) + z_2(\alpha)) e_2 \\ &+ (\theta_3(\alpha) + z_3(\alpha)) e_3. \end{aligned} \tag{35}$$

Clearly, (35) is equivalent to the following system

$$\begin{bmatrix} D^\varepsilon g_0(\alpha) \\ D^\varepsilon g_1(\alpha) \\ D^\varepsilon g_2(\alpha) \\ D^\varepsilon g_3(\alpha) \end{bmatrix} = \begin{bmatrix} a(\alpha) & -\delta_1(\alpha) & -\delta_2(\alpha) & -\delta_3(\alpha) \\ \delta_1(\alpha) & a(\alpha) & -\delta_3(\alpha) & \delta_2(\alpha) \\ \delta_2(\alpha) & \delta_3(\alpha) & a(\alpha) & -\delta_1(\alpha) \\ \delta_3(\alpha) & -\delta_2(\alpha) & \delta_1(\alpha) & a(\alpha) \end{bmatrix} \begin{bmatrix} g_0(\alpha) \\ g_1(\alpha) \\ g_2(\alpha) \\ g_3(\alpha) \end{bmatrix} + \begin{bmatrix} z_0(\alpha) + \theta_0(\alpha) \\ z_1(\alpha) + \theta_1(\alpha) \\ z_2(\alpha) + \theta_2(\alpha) \\ z_3(\alpha) + \theta_3(\alpha) \end{bmatrix}. \tag{36}$$

Let $x=(g_0, g_1, g_2, g_3)^T$, $y=(z_0(\alpha)+\theta_0(\alpha), z_1(\alpha)+\theta_1(\alpha), z_2(\alpha)+\theta_2(\alpha), z_3(\alpha)+\theta_3(\alpha))^T$, and

$$\Lambda=\begin{bmatrix} \delta_0(\alpha) & -\delta_1(\alpha) & -\delta_2(\alpha) & -\delta_3(\alpha) \\ \delta_1(\alpha) & \delta_0(\alpha) & -\delta_3(\alpha) & \delta_2(\alpha) \\ \delta_2(\alpha) & \delta_3(\alpha) & \delta_0(\alpha) & -\delta_1(\alpha) \\ \delta_3(\alpha) & -\delta_2(\alpha) & \delta_1(\alpha) & \delta_0(\alpha) \end{bmatrix}.$$

Then system (36) has the form $D^{\varepsilon}x(\alpha)=\Lambda(\alpha)x(\alpha)+y(\alpha)$ and the solution

$$x(\alpha)=X(\alpha)\xi+\frac{1}{\Gamma(\varepsilon)}\frac{X(\alpha)}{\Gamma(\varepsilon)}\int_{\alpha}^{0}(s-\alpha)^{\varepsilon-1}X(s)^{-1}y(s)ds, \tag{37}$$

where $\xi\in\mathbb{R}$, $X(\alpha)=e^{\frac{1}{\Gamma(\varepsilon+1)}\int_{\alpha}^{0}(s-\alpha)^{\varepsilon}\Lambda(s)ds}$. In addition, when $\mu\colon J\to\mathbb{F}$ is a map, the QVFDE (2) can be represented as $D^{\varepsilon}x(\alpha)=\Lambda x(\alpha)+\theta(\alpha)$. Let

$$x_0(\alpha)=X(\alpha)\xi+\frac{1}{\Gamma(\varepsilon)}\frac{X(\alpha)}{\Gamma(\varepsilon)}\int_{\alpha}^{0}(s-\alpha)^{\varepsilon-1}X(s)^{-1}y(s)ds \tag{38}$$

Clearly, $x_0=(g_0, g_1, g_2, g_3)^T$ is a solution to QVFDE (2). Next, by (37), (38) and Remark 2, we get

$$\begin{aligned} ||x(\alpha)-x_0(\alpha)||_2 &\le ||\frac{X(\alpha)}{\Gamma(\varepsilon)}\int_{\alpha}^{0}(s-\alpha)^{\varepsilon-1}X(s)^{-1}z(s)ds||_2 ds \\ &\le ||\frac{X(\alpha)}{\Gamma(\varepsilon)}||_2||\int_{\alpha}^{0}(s-\alpha)^{\varepsilon-1}X(s)^{-1}z(s)ds||_2 \\ &\le ||\frac{X(\alpha)}{\Gamma(\varepsilon)}||_2\int_{\alpha}^{0}||(s-\alpha)^{\varepsilon-1}X(s)^{-1}||_2||z(s)||_2 ds \\ &\le ||\frac{X(\alpha)}{\Gamma(\varepsilon)}||_2\int_{\alpha}^{0}||(s-\alpha)^{\varepsilon-1}X(s)^{-1}||_2\phi(s)ds. \end{aligned}$$

Finally, by (3), we obtain (34). The proof is complete. □

Corollary 5 The QVFDE (2) is generalized Hyers-Ulam stable for any map $\mu\colon J\to\mathbb{F}$.

4. Some applications

This work on QVFDEs with UH-stability has several applications across diverse fields. In robotics, it aids in modeling and controlling the orientation and motion of robotic arms and drones, where quaternions simplify rotational calculations.

In computer graphics, it enhances animations and 3D simulations by accurately representing rotations and transformations. In physics, it supports the analysis of complex systems, such as particle dynamics and wave propagation, where fractional calculus captures non-local effects. In signal processing, it enables the development of advanced filtering and time-series analysis techniques leveraging fractional derivatives. In control systems, it facilitates the design of robust controllers for fractional dynamic systems, ensuring stability and performance under varying conditions. In biological systems, it provides tools for modeling phenomena such as anomalous diffusion, where classical approaches may be inadequate. In this section, we illustrate applications of the theorems introduced above.

Example 1 Consider the QVFDE

$$D^{\varepsilon}g(\alpha) = (-e_1 - e_2 - e_3)g(\alpha), \quad g(0) = e_1 + e_2, \quad \alpha \in J. \tag{39}$$

Let $z(\alpha) = D^{\varepsilon}g(\alpha) - (-e_1 - e_2 - e_3)g(\alpha)$. Then $|z(\alpha)| \leq \gamma$ and

$$D^{\varepsilon}g(\alpha) = (-e_1, -e_2, -e_3)g(\alpha) + z(\alpha).$$

Then (39) can be given as

$$\begin{bmatrix} D^{\varepsilon}g_0(\alpha) \\ D^{\varepsilon}g_1(\alpha) \\ D^{\varepsilon}g_2(\alpha) \\ D^{\varepsilon}g_3(\alpha) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & 1 \\ -1 & 0 & 1 & -1 \\ -1 & -1 & 0 & 1 \\ -1 & 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} g_0(\alpha) \\ g_1(\alpha) \\ g_2(\alpha) \\ g_3(\alpha) \end{bmatrix} + \begin{bmatrix} z_0(\alpha) \\ z_1(\alpha) \\ z_2(\alpha) \\ z_3(\alpha) \end{bmatrix} \tag{40}$$

Let $x = (g_0, g_1, g_2, g_3)^T$, $z = (z_0(\alpha), z_1(\alpha), z_2(\alpha), z_3(\alpha))^T$, and

$$\Lambda = \begin{bmatrix} 0 & 1 & 1 & 1 \\ -1 & 0 & 1 & -1 \\ -1 & -1 & 0 & 1 \\ -1 & 1 & -1 & 0 \end{bmatrix}.$$

Note that the eigenvalues of Λ are $\mu_1, \mu_3 = \sqrt{3}i$ and $\mu_2, \mu_4 = -\sqrt{3}i$. The solution for system (40) is

$$x(\alpha) = e^{\frac{\Lambda\alpha^{\varepsilon}}{\varepsilon!}}(0, 1, 1, 0)^T + \frac{1}{\Gamma(\varepsilon)}\int_{\alpha}^{0}(s-\alpha)^{\varepsilon-1}e^{\frac{\Lambda(s-\alpha)^{\varepsilon}}{\varepsilon!}}z(s)ds \tag{41}$$

In addition, QVFDE (39) can be represented as $D^{\varepsilon}x(\alpha) = \Lambda x(\alpha)$ Let

$$x_0(\alpha) = e^{\frac{\Lambda\alpha^\varepsilon}{\varepsilon!}}(0, 1, 1, 0)^T. \tag{42}$$

Clearly $x_0 = (g_0, g_1, g_2, g_3)^t$ is a solution to QVFDE (39). Note that the 2-norm of Λ is $||\Lambda||_2 = \sqrt{3}$. By combining (41) and (42), we get

$$\begin{aligned}
||x(\alpha) - x_0(\alpha)||_2 \leq & \frac{1}{\Gamma(\varepsilon)} \int_\alpha^0 ||(s-\alpha)^{\varepsilon-1} e^{\frac{\Lambda(s-\alpha)^\varepsilon}{\varepsilon!}} z(s)||_2 ds \\
\leq & \frac{1}{\Gamma(\varepsilon)} \int_\alpha^0 ||(s-\alpha)^{\varepsilon-1} e^{\frac{\Lambda(s-\alpha)^\varepsilon}{\varepsilon!}}||_2 ||z(s)||_2 ds \\
\leq & \frac{1}{\Gamma(\varepsilon)} \int_\alpha^0 ||s^{\varepsilon-1} e^{\frac{\Lambda s^\varepsilon}{\varepsilon!}}||_2 ||z(s-\alpha)||_2 ds \\
\leq & \frac{\gamma}{\Gamma(\varepsilon)} \int_\alpha^0 ||s^{\varepsilon-1} e^{\frac{\Lambda s^\varepsilon}{\varepsilon!}}||_2 ds \\
\leq & \frac{\gamma}{\Gamma(\varepsilon)} \int_\alpha^0 s^{\varepsilon-1} e^{\frac{\Lambda s^\varepsilon}{\varepsilon!}} ds \\
\leq & \frac{\gamma}{\Gamma(\varepsilon)} \frac{-\varepsilon! + \varepsilon e^{\frac{\Lambda\alpha^\varepsilon}{\Gamma[1+\varepsilon]}} \Gamma[\varepsilon]}{t\Lambda\varepsilon \log[e]} \\
\leq & \frac{\gamma}{\Gamma(\varepsilon)} \frac{(\varepsilon-1)!(\varepsilon e^{\frac{\Lambda\alpha^\varepsilon}{\Gamma[1+\varepsilon]}} - 1)}{t\Lambda \log[e]} \\
\leq & \gamma \frac{\varepsilon e^{\frac{\sqrt{3}s^\varepsilon}{\Gamma[1+\varepsilon]}} - 1}{s\sqrt{3}\log[e]}
\end{aligned}$$

where $r \in J = (\sigma, 0] \subseteq (-\infty, 0]$. Finally, by (3) and $\varepsilon = 0.5$, we obtain

$$|g(\alpha) - g_0(\alpha)| \leq \gamma \frac{0.5 e^{\frac{\sqrt{3}s^{0.5}}{\Gamma[1.5]}} - 1}{s\sqrt{3}\log[e]}.$$

Note that QVFDE (39) is Hyers-Ulam stable when $\sigma > -\infty$.

Example 2 Consider the QVFDE is following:

$$D^\varepsilon g(\alpha) = (1 + e_1 + e_2) g(\alpha) + (e_1 + e_3)\alpha, \quad g(0) = i + j, \quad \alpha \in J. \tag{43}$$

Let $z(\alpha) = D^{\varepsilon}g(\alpha) - (1+e_1+e_2)g(\alpha) - (i+k)\alpha$. Then $|z(\alpha)| \leq \gamma$ and

$$D^{\varepsilon}g(\alpha) = z(\alpha) + (1+e_1+e_2)g(\alpha) + (e_1+e_3)\alpha.$$

Then (43) can be given as

$$\begin{bmatrix} D^{\varepsilon}g_0(\alpha) \\ D^{\varepsilon}g_1(\alpha) \\ D^{\varepsilon}g_2(\alpha) \\ D^{\varepsilon}g_3(\alpha) \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 & -1 \\ 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & -1 \\ 1 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} g_0(\alpha) \\ g_1(\alpha) \\ g_2(\alpha) \\ g_3(\alpha) \end{bmatrix} + \begin{bmatrix} z_0(\alpha) \\ z_1(\alpha)+\alpha \\ z_2(\alpha) \\ z_3(\alpha)+\alpha \end{bmatrix} \tag{44}$$

Let $x = (g_0, g_1, g_2, g_3)^T$, $y = (z_0(\alpha), z_1(\alpha)+\alpha, z_2(\alpha), z_3(\alpha)+\alpha)^T$, and

$$\Lambda = \begin{bmatrix} 1 & -1 & 0 & -1 \\ 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & -1 \\ 1 & 0 & 1 & 1 \end{bmatrix}.$$

Note that the eigenvalues of Λ are μ_1, $\mu_3 = 1+\sqrt{2}i$ and μ_2, $\mu_4 = 1-\sqrt{2}i$. The solution for system (44) is

$$x(\alpha) = e^{\frac{\Lambda\alpha^{\varepsilon}}{\varepsilon!}}(0, 1, 1, 0)^T + \frac{1}{\Gamma(\varepsilon)}\int_{\alpha}^{0}(s-\alpha)^{\varepsilon-1}e^{\frac{\Lambda(s-\alpha)^{\varepsilon}}{\varepsilon!}}y(s)ds \tag{45}$$

In addition, QVFDE (43) can be represented as $D^{\varepsilon}x(\alpha) = \Lambda x(\alpha) + \theta(\alpha)$. Let

$$x_0(\alpha) = e^{\frac{\Lambda\alpha^{\varepsilon}}{\varepsilon!}}(0, 1, 1, 0)^T = \frac{1}{\Gamma(\varepsilon)}\int_{\alpha}^{0}(s-\alpha)^{\varepsilon-1}e^{\frac{\Lambda(s-\alpha)^{\varepsilon}}{\varepsilon!}}\theta(s)ds. \tag{46}$$

Clearly $x_0 = (g_0, g_1, g_2, g_3)^t$ is a solution to QVFDE (43). So the 2-norm of Λ is $||\Lambda||_2 = \sqrt{3}$. By combining (45) and (46), we get

$$\begin{aligned} ||x(\alpha) - x_0(\alpha)||_2 \leq & \frac{1}{\Gamma(\varepsilon)}\int_{\alpha}^{0}||(s-\alpha)^{\varepsilon-1}e^{\frac{\Lambda(s-\alpha)^{\varepsilon}}{\varepsilon!}}(y(s)-\theta(s))||_2 ds \\ \leq & \frac{1}{\Gamma(\varepsilon)}\int_{\alpha}^{0}||(s-\alpha)^{\varepsilon-1}e^{\frac{\Lambda(s-\alpha)^{\varepsilon}}{\varepsilon!}}||_2||z(s)||_2 ds \end{aligned}$$

$$\leq \frac{1}{\Gamma(\varepsilon)} \int_{\alpha}^{0} ||s^{\varepsilon-1} e^{\frac{\Lambda s^{\varepsilon}}{\varepsilon!}}||_2 ||z(s-\alpha)||_2 ds$$

$$\leq \frac{\gamma}{\Gamma(\varepsilon)} \int_{\alpha}^{0} ||s^{\varepsilon-1} e^{\frac{\Lambda s^{\varepsilon}}{\varepsilon!}}||_2 ds$$

$$\leq \frac{\gamma}{\Gamma(\varepsilon)} \int_{\alpha}^{0} s^{\varepsilon-1} e^{\frac{\Lambda s^{\varepsilon}}{\varepsilon!}} ds$$

$$\leq \frac{\gamma}{\Gamma(\varepsilon)} \frac{-\varepsilon! + \varepsilon e^{\frac{\Lambda \alpha^{\varepsilon}}{\Gamma[1+\varepsilon]}} \Gamma[\varepsilon]}{t \Lambda \varepsilon \log[e]}$$

$$\leq \frac{\gamma}{\Gamma(\varepsilon)} \frac{(\varepsilon-1)!(\varepsilon e^{\frac{\Lambda \alpha^{\varepsilon}}{\Gamma[1+\varepsilon]}} - 1)}{t \Lambda \log[e]}$$

$$\leq \gamma \frac{\varepsilon e^{\frac{\sqrt{3} s^{\varepsilon}}{\Gamma[1+\varepsilon]}} - 1}{s\sqrt{3} \log[e]}$$

where $r \in J = (\sigma, 0] \subseteq (-\infty, 0]$. Finally, by (3) and $\varepsilon = 0.5$, we obtain

$$|g(\alpha) - g_0(\alpha)| \leq \gamma \frac{0.5 e^{\frac{\sqrt{3} s^{0.5}}{\Gamma[1.5]}} - 1}{s\sqrt{3} \log[e]}.$$

Hence the QVFDE (43) is Hyers-Ulam stable when $\sigma > -\infty$.

5. Conclusion

In this paper, we have investigated the Hyers-Ulam stability of linear Quaternion-Valued Fractional Differential Equations (QVFDEs) in both homogeneous and non-homogeneous cases. By transforming QVFDEs into equivalent real fractional systems, we established several new stability results, including generalized Hyers-Ulam stability, and verified them through illustrative examples. These findings enrich the theory of fractional differential equations in the quaternion setting and highlight the role of quaternion algebra in stability analysis. For future research, this work can be extended in several directions. One promising avenue is the study of Hyers-Ulam stability for nonlinear QVFDEs and systems with time delays. Another is the exploration of numerical methods tailored to quaternionic fractional systems, enabling practical simulations in applied fields such as physics, engineering, and quantum mechanics. Furthermore, investigating stability in more generalized fractional operators, such as the Atangana-Baleanu or Caputo-Fabrizio derivatives, could broaden the applicability of the results.

Author contributions

Conceptualization: F.H.D., M.A.; Methodology: F.H.D., M.A., A.D.; Validation: F.H.D., M.A., A.D., H.A.O; Investigation: F.H.D., M.A., A.S.; Resources: F.H.D., M.A.; Writing-original draft: A.S., A.D., H.A.O., S.D.; Writing-review and editing: A.S., H.A.O., S.D. All authors have read and agreed to the published version of the manuscript.

Conflict of interest

The authors declare no competing financial interest.

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