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A New Approach to Fixed Point Results in Neutrosophic *b*-Metric Spaces with Application

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Abstract: This study defines an extended Neutrosophic *b*-Metric Space (NbMS) and illustrates the characteristics that are essential to its structure. The Fixed Point (FP) theorem has therefore been proved in the context of these Extended Neutrosophic *b*-Metric Spaces (ENbMS). Our results show symmetrical patterns and features within these mathematical frameworks, both extending and generalizing the results found in the current literature. A nontrivial example is used to highlight the efficacy of the suggested approaches. To emphasize the usefulness of our primary findings, an application to the existence and uniqueness of solutions for a particular class of Fredholm integral equations is investigated.

Keywords**:** Fixed Point (FP), Neutrosophic *b*-Metric Space (NbMS), Extended Neutrosophic *b*-Metric Space (ENbMS)

MSC: 47H10, 46S40

1. Introduction

Numerous authors have extensively developed and investigated the Banach contraction principle, commonly referred to as the Banach Fixed Point (FP) theorem, which is an important mathematical conclusion. Since its first establishment by Banach, this principle has been a key instrument in the study of functional analysis. It gives the circumstances in which a mapping has a unique fixed point on a complete metric space. The Banach contraction principle's outstanding accomplishments have been extended by other scholars who have investigated different directions for advancement. Frechet [1] made a significant contribution by introducing the idea of metric space.

Czerwik [2], who studied Banach theorems and offered a formal formulation of *b*-metric space, made another important contribution. Bourbaki and Bakhtin are credited with the early invention of *b*-metric spaces. Building on this basis, Kamran et al. [3] concentrated on the triangle inequality inside *b*-metric space and created a new distance function called extended *b*-metric space that loosened some of the inequality's restrictions. Fuzzy set theory was first presented by Zadeh [4] in 1965 as a mathematical framework to deal with real-world scenarios including ambiguous or imprecise information. The concept of Fuzzy Metric Space (FMS) was introduced by Kaleva and Seikkala [5], in which the metric structure incorporates the imprecision in distance measurements between elements.

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Kramosil and Michalek [6] laid the foundation for fuzzy metric space, and George and Veeramani [7] contributed to its continued development. A Hausdorff fuzzy metric was presented by Lopez and Romaguera [8] in the same year. It was created especially for a collection of non-empty compact subsets that fall inside a given fuzzy metric space. The necessary and sufficient conditions for the existence of fixed points in multivalued maps within fuzzy metric space were established by Arshad and Shoaib [9].

Furthermore, Atanassov [10] expanded this theory in 1999 by presenting the idea of an intuitionistic fuzzy set, which takes into account both the degree of membership and non-membership of elements within a set. Unlike classical logic, which represents an element's membership in a set as a number within the interval [0, 1], intuitionistic fuzzy logic expresses an element's membership as a number likewise within the interval [0, 1]. As a generalization of fuzzy metric space, which was first proposed by George and Veeramani [7], Park [11] introduced intuitionistic fuzzy metric space in 2004 by using continuous t-norm and t-conorm. Intuitionistic fuzzy metric space was first proposed by Shojaei [12] in 2014; see also [13, 14].

Younis et al. [15] studied fixed point theory in the context of b-metric spaces and worked on several applications of fixed point theory. Additionally, a new study on fixed point theory may be found in [16–21] and its references. The ideas of metrics and metric spaces have been extended and modified in a number of ways, aside from fuzzy metric space. A number of cited studies [4, 22–26] have examined the connection between b-metric spaces and fuzzy metric spaces. A new idea called fuzzy b-metric space was presented, which uses a less strict version of triangle inequality. Additionally, expanded b-metric spaces and updated fixed point theorems tailored to these kinds of spaces were introduced by Gupta et al. [27]. The study of fuzzy b-metric space has become popular recently.

In 1998, Smarandache developed the ideas of neutrosophic logic and neutrosophic set. The idea of neutrosophic metric space was established by Kirisci and Simsek which deals with membership, nonmembership and neutralness. In 2020, Rajan et al. [28] proved some fixed point results for contraction theorems in neutrosophic metric space. Shakila and Jeyaraman [29] introduced the notion of Neutrosophic b-Metric Space (NbMS) and proved the fixed point theorems of contraction mappings.

In this setting, we introduce the concept of the NbMS. However, conventional neutrosophic b-metric spaces frequently impose restrictive contraction conditions and fail to fully reflect instances in which the degrees of truth, indeterminacy, and falsity interact asymmetrically. Furthermore, their usefulness is limited when dealing with mappings that exhibit mixed or partial uncertainty characteristics. The Extended Neutrosophic b-Metric Space (ENbMS) presented in this study overcomes these constraints by offering a more flexible structure that accommodates generalized contraction types and supports a broader class of mappings, hence increasing the applicability of neutrosophic fixed point results.

We have worked on a few instances when the Fredholm integral equation has only unique solution. The current analysis provides impetus to investigate these principles' flexible applicability in more detail. In addition, we offer an application to bolster our primary finding.

2. Preliminaries

Definition 1 [11] A transformation $\hat{\odot}$ defined on $[0, 1] \times [0, 1]$ to $[0, 1]$ is referred to as a Continuous Triangular Norm (CTN) if it meets the subsequent assertions:

(i) Commutativity and Associativity: $\hat{\odot}(e, f) = \hat{\odot}(f, e)$ and $\hat{\odot}(\hat{\odot}(e, f), g) = \hat{\odot}(e, \hat{\odot}(f, g))$ for all $e, f, g \in [0, 1]$,

(ii) Continuity: $\hat{\odot}$ is a continuous function;

(iii) Neutral Element: $1 \hat{\odot} e = e$ for all $e \in [0, 1]$;

(iv) Monotonicity: $e \hat{\odot} f \leq g \hat{\odot} h$ whenever $e \leq g$ and $f \leq h$ for all $e, f, g, h \in [0, 1]$.

Definition 2 [11] A transformation $\hat{\oplus}$ defined on $[0, 1] \times [0, 1]$ to $[0, 1]$ is referred to as a Continuous Triangular Conorm (CTCN) if it meets the subsequent assertions:

(i) Commutativity and Associativity: $\hat{\oplus}(e, f) = \hat{\oplus}(f, e)$ and $\hat{\oplus}(\hat{\oplus}(e, f), g) = \hat{\oplus}(e, \hat{\oplus}(f, g))$ for all $e, f, g \in [0, 1]$;

(ii) Continuity: $\hat{\oplus}$ is a continuous function;

(iii) Neutral Element: $0 \hat{\oplus} e = e$ for all $e \in [0, 1]$;

(iv) Monotonicity: $e\hat{\oplus}f \geq g\hat{\oplus}h$ whenever $e \geq g$ and $f \geq h$, for all $e, f, g, h \in [0, 1]$.

Definition 3 [27] A 5-tuple set $(U, \mathfrak{B}, \mathfrak{D}, \hat{\odot}, \hat{\oplus})$ is said to be Extended Intituitionistic Fuzzy b-Metric Space (EIFbMS), if U is a non-empty set, $\hat{\odot}$ and $\hat{\oplus}$ CTN and CTCN respectively and Ω: $U \times U \to [1, \infty)$ be a function, $\mathfrak{B}$ and $\mathfrak{D}$ are fuzzy sets on $U \times U \times (0, \infty)$ fulfilling the following criteria with $k \geq 1$, t, $r > 0$ and for all ι, φ, $\kappa \in U$.

(a) $\mathfrak{B}_\Omega(\iota, \varphi, r) + \mathfrak{D}_\Omega(\iota, \varphi, r) \leq 1$;

(b) $\mathfrak{B}_\Omega(\iota, \varphi, r) > 0$;

(c) $\mathfrak{B}_\Omega(\iota, \varphi, r) = 1$, for all $r > 0$ iff $\iota = \varphi$;

(d) $\mathfrak{B}_\Omega(\iota, \varphi, r) = \mathfrak{B}_\Omega(\varphi, \iota, r)$, for all $r > 0$;

(e) $\mathfrak{B}_\Omega(\iota, \kappa, \Omega(\iota, \kappa)(r+t)) \geq \mathfrak{B}_\Omega\left(\iota, \varphi, \frac{r}{k}\right) \hat{\odot} \mathfrak{B}_\Omega\left(\varphi, \kappa, \frac{t}{k}\right)$, for all $r, t > 0$;

(f) $\mathfrak{B}_\Omega(\iota, \varphi, .)$ is non-deteriorating function of R^+ and $\lim_{r\to\infty} \mathfrak{B}_\Omega(\iota, \varphi, r) = 1$;

(g) $\mathfrak{D}_\Omega(\iota, \varphi, r) < 1$;

(h) $\mathfrak{D}_\Omega(\iota, \varphi, r) = 0$, for all $r > 0$ iff $\iota = \varphi$;

(i) $\mathfrak{D}_\Omega(\iota, \varphi, r) = \mathfrak{D}_\Omega(\varphi, \iota, r)$, for all $r > 0$;

(j) $\mathfrak{D}_\Omega(\iota, \kappa, \Omega(\iota, \kappa)(r+t)) \leq \mathfrak{D}_\Omega\left(\iota, \varphi, \frac{r}{k}\right) \hat{\oplus} \mathfrak{D}_\Omega\left(\varphi, \kappa, \frac{s}{b}\right)$, for all $r, t > 0$;

(k) $\mathfrak{D}_\Omega(\iota, \varphi, .)$ is non increasing function of R^+ and $\lim_{r\to\infty} \mathfrak{D}_\Omega(\iota, \varphi, r) = 0$.

Also, the value $\lim_{r\to\infty} \mathfrak{B}_\Omega(\iota, \varphi, r) = 1$ and $\lim_{r\to\infty} \mathfrak{D}_\Omega(\iota, \varphi, r) = 0$.

Definition 4 [29] A 6-tuple set $(U, \mathfrak{B}, \mathfrak{D}, \mathfrak{N}, \hat{\odot}, \hat{\oplus})$ is said to be NbMS, if U is a non-empty set, $\hat{\odot}$ and $\hat{\oplus}$ is CTN and CTCN, respectively and $U \times U \to [1, \infty)$ be a function, $\mathfrak{B}$, $\mathfrak{D}$ and $\mathfrak{N}$ are neutrosophic sets on $U \times U \times (0, \infty)$ fulfilling the following requirements with $k \geq 1$, t, $r > 0$ and for all ι, φ, $\kappa \in U$.

(a) $\mathfrak{B}(\iota, \varphi, r) + \mathfrak{D}(\iota, \varphi, r) + \mathfrak{N}(\iota, \varphi, r) \leq 3$;

(b) $\mathfrak{B}(\iota, \varphi, r) > 0$;

(c) $\mathfrak{B}(\iota, \varphi, r) = 1$, for all $r > 0$ iff $\iota = \varphi$;

(d) $\mathfrak{B}(\iota, \varphi, r) = \mathfrak{B}(\varphi, \iota, r)$, for all $r > 0$;

(e) $\mathfrak{B}(\iota, \kappa, (r+t)) \geq \mathfrak{B}\left(\iota, \varphi, \frac{r}{k}\right) \hat{\odot} \mathfrak{B}\left(\varphi, \kappa, \frac{t}{k}\right)$, for all $r, t > 0$;

(f) $\mathfrak{B}(\iota, \varphi, .)$ is non decreasing function of R^+ and $\lim_{r\to\infty} \mathfrak{B}(\iota, \varphi, r) = 1$;

(g) $\mathfrak{D}(\iota, \varphi, r) < 1$;

(h) $\mathfrak{D}(\iota, \varphi, r) = 0$, for all $r > 0$ iff $\iota = \varphi$;

(i) $\mathfrak{D}(\iota, \varphi, r) = \mathfrak{D}(\varphi, \iota, r)$, for all $r > 0$;

(j) $\mathfrak{D}(\iota, \kappa, (r+t)) \leq \mathfrak{D}\left(\iota, \varphi, \frac{r}{k}\right) \hat{\oplus} \mathfrak{D}\left(\varphi, \kappa, \frac{t}{k}\right)$, for all $r, t > 0$;

(k) $\mathfrak{D}(\iota, \varphi, .)$ is non increasing function of R^+ and $\lim_{r\to\infty} \mathfrak{D}(\iota, \varphi, r) = 0$;

(l) $\mathfrak{N}(\iota, \varphi, r) < 1$;

(m) $\mathfrak{N}(\iota, \varphi, r) = 0$, for all $r > 0$ iff $\iota = \varphi$;

(n) $\mathfrak{N}(\iota, \varphi, r) = \mathfrak{N}(\varphi, \iota, r)$, for all $r > 0$;

(o) $\mathfrak{N}(\iota, \kappa, (r+t)) \leq \mathfrak{N}\left(\iota, \varphi, \frac{r}{k}\right) \hat{\oplus} \mathfrak{N}\left(\varphi, \kappa, \frac{t}{k}\right)$, for all $r, t > 0$;

(p) $\mathfrak{N}(\iota, \varphi, .)$ is non increasing function of R^+ and $\lim_{r\to\infty} \mathfrak{N}(\iota, \varphi, r) = 0$.

Also, the value $\lim_{r\to\infty} \mathfrak{B}(\iota, \varphi, r) = 1$, $\lim_{r\to\infty} \mathfrak{D}(\iota, \varphi, r) = 0$ and $\lim_{r\to\infty} \mathfrak{N}(\iota, \varphi, r) = 0$.

3. Main results

Definition 5 A 6-tuple set $(U, \mathfrak{B}_\Omega, \mathfrak{D}_\Omega, \mathfrak{N}_\Omega, \hat{\odot}, \hat{\oplus})$ is said to be ENbMS, if U is a non-empty set, $\hat{\odot}$ and $\hat{\oplus}$ are CTN and CTCN respectively and Ω: $U \times U \to [1, \infty)$ be a function, $\mathfrak{B}_\Omega$, $\mathfrak{D}_\Omega$ and $\mathfrak{N}_\Omega$ are neutrosophic sets on $U \times U \times (0, \infty)$ satisfying the following conditions with $k \geq 1$, t, $r > 0$ and for all ι, φ, $\kappa \in U$.

(a) $\mathfrak{B}_\Omega(\iota, \varphi, r) + \mathfrak{D}_\Omega(\iota, \varphi, r) + +\mathfrak{N}(\iota, \varphi, r) \leq 3$;

(b) $\mathfrak{B}_\Omega(\iota, \varphi, r) > 0$;

(c) $\mathfrak{B}_\Omega(\iota, \varphi, r) = 1$, for all $r > 0$ iff $\iota = \varphi$;

(d) $\mathfrak{B}_\Omega(\iota, \varphi, r) = \mathfrak{B}_\Omega(\varphi, \iota, r)$, for all $r > 0$;

(e) $\mathfrak{B}_\Omega(\iota,\, \kappa,\, \Omega(\iota,\, \kappa)(r+t)) \geq \mathfrak{B}_\Omega\left(\iota,\, \varphi,\, \frac{r}{k}\right) \hat{\odot} \mathfrak{B}_\Omega\left(\varphi,\, \kappa,\, \frac{t}{k}\right)$, for all $r, t > 0$;
(f) $\mathfrak{B}_\Omega(\iota,\, \varphi,\, .)$ is non decreasing function of R^+ and $\lim\limits_{r\to\infty} \mathfrak{B}_\Omega(\iota,\, \varphi,\, r) = 1$;
(g) $\mathfrak{D}_\Omega(\iota,\, \varphi,\, r) < 1$;
(h) $\mathfrak{D}_\Omega(\iota,\, \varphi,\, r) = 0$, for all $r > 0$ iff $\iota = \varphi$;
(i) $\mathfrak{D}_\Omega(\iota,\, \varphi,\, r) = \mathfrak{D}_\Omega(\varphi,\, \iota,\, r)$, for all $r > 0$;
(j) $\mathfrak{D}_\Omega(\iota,\, \kappa,\, \Omega(\iota,\, \kappa)(r+t)) \leq \mathfrak{D}_\Omega\left(\iota,\, \varphi,\, \frac{r}{k}\right) \hat{\oplus} \mathfrak{D}_\Omega\left(\varphi,\, \kappa,\, \frac{t}{k}\right)$, for all $r, t > 0$;
(k) $\mathfrak{D}_\Omega(\iota,\, \varphi,\, .)$ is non increasing function of R^+ and $\lim\limits_{r\to\infty} \mathfrak{D}_\Omega(\iota,\, \varphi,\, r) = 0$;
(l) $\mathfrak{N}_\Omega(\iota,\, \varphi,\, r) < 1$;
(m) $\mathfrak{N}_\Omega(\iota,\, \varphi,\, r) = 0$, for all $r > 0$ iff $\iota = \varphi$;
(n) $\mathfrak{N}_\Omega(\iota,\, \varphi,\, r) = \mathfrak{N}_\Omega(\varphi,\, \iota,\, r)$, for all $r > 0$;
(o) $\mathfrak{N}_\Omega(\iota,\, \kappa,\, \Omega(\iota,\, \kappa)(r+t)) \leq \mathfrak{N}_\Omega\left(\iota,\, \varphi,\, \frac{r}{k}\right) \hat{\oplus} \mathfrak{N}_\Omega\left(\varphi,\, \kappa,\, \frac{t}{k}\right)$, for all $r, t > 0$;
(p) $\mathfrak{N}_\Omega(\iota,\, \varphi,\, .)$ is non increasing function of R^+ and $\lim\limits_{r\to\infty} \mathfrak{N}_\Omega(\iota,\, \varphi,\, r) = 0$.
Also, the value $\lim\limits_{r\to\infty} \mathfrak{B}_\Omega(\iota,\, \varphi,\, r) = 1$, $\lim\limits_{r\to\infty} \mathfrak{D}_\Omega(\iota,\, \varphi,\, r) = 0$ and $\lim\limits_{r\to\infty} \mathfrak{N}_\Omega(\iota,\, \varphi,\, r) = 0$.

Example 1 Let $\mathfrak{B}_\Omega$, $\mathfrak{D}_\Omega$ and $\mathfrak{N}_\Omega$ are neutrosophic sets on $U \times U \times (0,\, \infty)$ and $U = \{0,\, \frac{1}{2},\, 1\}$, ν_k: $U \times U \to R$ with $\nu_k(\hat{\mathfrak{e}},\, \hat{\mathfrak{f}}) = (\hat{\mathfrak{e}} - \hat{\mathfrak{f}})^2$ and $\hat{\mathfrak{e}} \hat{\odot} \hat{\mathfrak{f}} = \min(\hat{\mathfrak{e}},\, \hat{\mathfrak{f}})$, $\hat{\mathfrak{e}} \hat{\oplus} \hat{\mathfrak{f}} = \max(\hat{\mathfrak{e}},\, \hat{\mathfrak{f}})$ for all $\hat{\mathfrak{e}},\, \hat{\mathfrak{f}} \in [0,\, 1]$, $\Omega(\iota,\, \kappa) = 1 + \iota + \kappa$ defined as follows:

$$\mathfrak{B}(\iota,\, \varphi,\, r) = \begin{cases} \frac{r}{r+\nu_k(\iota,\, \varphi)} & \text{if} \quad r > 0 \\ \\ 0 & \text{if} \quad r = 0 \end{cases},\ \mathfrak{D}(\iota,\, \varphi,\, r) = \begin{cases} \frac{\nu_k(\iota,\, \varphi)}{r+\nu_k(\iota,\, \varphi)} & \text{if} \quad r > 0 \\ \\ 1 & \text{if} \quad r = 0 \end{cases} \quad \text{and}$$

$$\mathfrak{N}(\iota,\, \varphi,\, r) = \begin{cases} \frac{\nu_k(\iota,\, \varphi)}{r} & \text{if} \quad r > 0 \\ \\ 1 & \text{if} \quad r = 0 \end{cases},$$

$$\nu_k(0,\, 0) = 0,\ \nu_k(\frac{1}{2},\, \frac{1}{2}) = 0,\ \nu_k(1,\, 1) = 0,\ \nu_k(0,\, \frac{1}{2}) = \frac{1}{4},\ \nu_k(\frac{1}{2},\, 0) = \frac{1}{4},\ \nu_k(\frac{1}{2},\, 1) = \frac{1}{4},\ \nu_k(1,\, \frac{1}{2}) = \frac{1}{4},$$

$$\nu_k(1,\, 0) = 1,\ \nu_k(0,\, 1) = 1,\ \Omega(0,\, 0) = 1,\ \Omega(\frac{1}{2},\, \frac{1}{2}) = 2,\ \Omega(1,\, 1) = 3,\ \Omega(0,\, \frac{1}{2}) = \frac{3}{2},\ \Omega(\frac{1}{2},\, 0) = \frac{3}{2},$$

$$\Omega(\frac{1}{2},\, 1) = \frac{5}{2},\ \Omega(1,\, \frac{1}{2}) = \frac{5}{2},\ \Omega(0,\, 1) = 2,\ \Omega(1,\, 0) = 2.$$

We just need to verify property (e) of the Definition 5, since properties (a) through (d) are satisfied,

$$\mathfrak{B}_\Omega(\iota,\, \kappa,\, \Omega(\iota,\, \kappa)(r+t)) \geq \mathfrak{B}_\Omega\left(\iota,\, \varphi,\, \frac{r}{k}\right) \hat{\odot} \mathfrak{B}_\Omega\left(\varphi,\, \kappa,\, \frac{t}{k}\right),\ \text{for all}\, r,\, t > 0. \tag{1}$$

Now, take Left-Hand Side (LHS) of equation (1),

$$\mathfrak{B}_\Omega(\iota,\, \kappa,\, \Omega(\iota,\, \kappa)(r+t)) = \frac{\Omega(\iota,\, \kappa)(r+t)}{\Omega(\iota,\, \kappa)(r+t) + \nu_k(\iota,\, \kappa)}. \tag{2}$$

Put $\iota = 0$, $\kappa = \frac{1}{2}$ in equation (2),

$$\mathfrak{B}_\Omega(0,\, \frac{1}{2},\, \Omega(0,\, \frac{1}{2})(r+t)) = \frac{\Omega(0,\, \frac{1}{2})(r+t)}{\Omega(0,\, \frac{1}{2})(r+t)+\nu_k(0,\, \frac{1}{2})} = \frac{\frac{3}{2}(r+t)}{\frac{3}{2}(r+t)+\frac{1}{4}} = 1-\frac{\frac{1}{4}}{\frac{3}{2}(r+t)+\frac{1}{4}} \tag{3}$$

Consider, Right-Hand Side (RHS) of equation (1), put $k = 1$, $\iota = 0$, $\kappa = \frac{1}{2}$, $\varphi = 1$,

$$\mathfrak{B}_\Omega(0,\, 1,\, r) = \frac{r}{r+\nu_k(0,\, 1)} = \frac{r}{r+1} = 1-\frac{1}{r+1} \text{ and} \tag{4}$$

$$\mathfrak{B}_\Omega(1,\, \frac{1}{2},\, t) = \frac{t}{t+\nu_k(1,\, \frac{1}{2})} = \frac{t}{t+\frac{1}{4}} = 1-\frac{\frac{1}{4}}{t+\frac{1}{4}}. \tag{5}$$

So, that

$$\begin{aligned} \mathfrak{B}_\Omega(0,\, \frac{1}{2},\, \Omega(0,\, \frac{1}{2})(r+t)) &= 1-\frac{\frac{1}{4}}{\frac{3}{2}(r+t)+\frac{1}{4}} \\ &= 1-\frac{1}{6r+6t+1} \\ &> 1-\frac{1}{6r+1} \\ &> 1-\frac{1}{r+1}. \end{aligned} \tag{6}$$

This show that, $\mathfrak{B}_\Omega(0,\, \frac{1}{2},\, \Omega(0,\, 1)(r+t)) > \mathfrak{B}_\Omega(0,\, 1,\, r)$. Similarly, $\mathfrak{B}_\Omega(0,\, \frac{1}{2},\, \Omega(0,\, \frac{1}{2})(r+t)) > \mathfrak{B}_\Omega(1,\, \frac{1}{2},\, t)$. So that,

$$\mathfrak{B}_\Omega(0,\, \frac{1}{2},\, \Omega(0,\, \frac{1}{2})(r+t)) \geq \min\{\mathfrak{B}_\Omega(0,\, 1,\, r),\, \mathfrak{B}_\Omega(1,\, \frac{1}{2},\, t)\} = \mathfrak{B}_\Omega(0,\, 1,\, r)\hat{\odot}\mathfrak{B}_\Omega(1,\, \frac{1}{2},\, t).$$

Analogously, it can be demonstrated that $\mathfrak{B}_\Omega(0,\, 1,\, \Omega(0,\, 1)(r+t)) \geq \mathfrak{B}_\Omega(0,\, \frac{1}{2},\, r)\hat{\odot}\mathfrak{B}_\Omega(\frac{1}{2},\, 1,\, s)$ and $\mathfrak{B}_\Omega(\frac{1}{2},\, 1,\, \Omega(\frac{1}{2},\, 1)(r+t)) \geq \mathfrak{B}_\Omega(\frac{1}{2},\, 0,\, r)\hat{\odot}\mathfrak{B}_\Omega(0,\, 1,\, s)$. Hence for all ι, φ, $\kappa \in U$,

$$\mathfrak{B}_\Omega(\iota,\, \kappa,\, \Omega(\iota,\, \kappa)(r+t)) \geq \mathfrak{B}_\Omega\left(\iota,\, \varphi,\, \frac{r}{k}\right)\hat{\odot}\mathfrak{B}_\Omega\left(\varphi,\, \kappa,\, \frac{t}{k}\right).$$

Since properties (f) through (i) of Definition 5 are obviously satisfied, we shall simply examine property (j),

$$\mathfrak{D}_\Omega(\iota,\, \kappa,\, \Omega(\iota,\, \kappa)(r+t)) \leq \mathfrak{D}_\Omega\left(\iota,\, \varphi,\, \frac{r}{k}\right)\hat{\oplus}\mathfrak{D}_\Omega\left(\varphi,\, \kappa,\, \frac{t}{k}\right) \tag{7}$$

for all $r, t > 0$. Take LHS of equation (7),

$$\mathfrak{D}_{\Omega}(\iota,\, \kappa,\, \Omega(\iota,\, \kappa)(r+t)) = \frac{\Omega(\iota,\, \kappa)(r+t)}{\Omega(\iota,\, \kappa)(r+t)+\nu_k(\iota,\, \kappa)} \tag{8}$$

Put $\iota = 0,\ \kappa = \frac{1}{2}$ in equation (8),

$$\mathfrak{D}_{\Omega}(0,\, \frac{1}{2},\, \Omega(0,\, \frac{1}{2})(r+t)) = \frac{\nu_k(0,\, \frac{1}{2})}{\Omega(0,\, \frac{1}{2})(r+t)+\nu_k(0,\, \frac{1}{2})} = \frac{\frac{1}{4}}{\frac{3}{2}(r+t)+\frac{1}{4}} \tag{9}$$

Assume, RHS of equation (7), put $k = 1,\ \iota = 0,\ \kappa = \frac{1}{2},\ \varphi = 1,\ \mathfrak{D}_{\Omega}(0,\, 1,\, r) = \frac{\nu_k(0,\, 1)}{r+\nu_k(0,\, 1)} = \frac{1}{r+1}$ and $\mathfrak{D}_{\Omega}(1,\, \frac{1}{2},\, t) = \frac{\nu_k(1,\, \frac{1}{2})}{t+\nu_k(1,\, \frac{1}{2})} = \frac{\frac{1}{4}}{t+\frac{1}{4}}$.

$$\begin{aligned}
\mathfrak{D}_{\Omega}(0,\, \frac{1}{2},\, \Omega(0,\, \frac{1}{2})(r+t)) &= \frac{\frac{1}{4}}{\frac{3}{2}(r+t)+\frac{1}{4}} \\
&= \frac{1}{6(r+t)+1} \\
&< \frac{1}{6r+6t+1} \\
&< \frac{1}{6r+1} < \frac{1}{r+1}.
\end{aligned} \tag{10}$$

This show that, $\mathfrak{D}_{\Omega}(0,\, \frac{1}{2},\, \Omega(0,\, \frac{1}{2})(r+t)) < \mathfrak{D}_{\Omega}(0,\, 1,\, r)$. Similarly, $\mathfrak{D}_{\Omega}(0,\, \frac{1}{2},\, \Omega(0,\, \frac{1}{2})(r+t)) < \mathfrak{D}_{\Omega}(1,\, \frac{1}{2},\, t)$ so that

$$\mathfrak{D}_{\Omega}(0,\, \frac{1}{2},\, \Omega(0,\, \frac{1}{2})(r+t)) \leq \max\{\mathfrak{D}_{\Omega}(0,\, 1,\, r),\, \mathfrak{D}_{\Omega}(1,\, \frac{1}{2},\, t)\} = \mathfrak{D}_{\Omega}(0,\, 1,\, r)\hat{\oplus}\mathfrak{D}_{\Omega}(1,\, \frac{1}{2},\, t).$$

Analogously, it can be demonstrated that $\mathfrak{D}_{\Omega}(0, 1, \Omega(0, 1)(r+t)) \leq \mathfrak{D}_{\Omega}(0, \frac{1}{2}, r)\hat{\oplus}\mathfrak{D}_{\Omega}(\frac{1}{2}, 1, t)$ and $\mathfrak{D}_{\Omega}(\frac{1}{2}, 1, \Omega(\frac{1}{2}, 1)(r+t)) \leq \mathfrak{D}_{\Omega}(\frac{1}{2},\, 0,\, r)\hat{\oplus}\mathfrak{D}_{\Omega}(0,\, 1,\, t)$. Hence, all value of $\iota,\, \varphi,\, \kappa \in U$,

$$\mathfrak{D}_{\Omega}(\iota,\, \kappa,\, \Omega(\iota,\, \kappa)(r+t)) \leq \mathfrak{D}_{\Omega}\left(\iota,\, \varphi,\, \frac{r}{k}\right)\hat{\oplus}\mathfrak{D}_{\Omega}\left(\varphi,\, \kappa,\, \frac{t}{k}\right).$$

Since properties (l) through (o) of Definition 5 are obviously satisfied, we shall just examine property (p),

$$\mathfrak{N}_{\Omega}(\iota,\, \kappa,\, \Omega(\iota,\, \kappa)(r+t)) \leq \mathfrak{N}_{\Omega}\left(\iota,\, \varphi,\, \frac{r}{k}\right)\hat{\oplus}\mathfrak{N}_{\Omega}\left(\varphi,\, \kappa,\, \frac{t}{k}\right) \tag{11}$$

for all $r, t > 0$. Take LHS of equation (11),

$$\mathfrak{N}_{\Omega}(\iota,\, \kappa,\, \Omega(\iota,\, \kappa)(r+t)) = \frac{\nu_k(\iota,\, \kappa)}{\Omega(\iota,\, \kappa)(r+t)}. \tag{12}$$

Put $\iota = 0,\ \kappa = \frac{1}{2}$ in equation (12),

$$\mathfrak{N}_{\Omega}(0,\, \frac{1}{2},\, \Omega(0,\, \frac{1}{2})(r+t)) = \frac{\nu_k(0,\, \frac{1}{2})}{\Omega(0,\, \frac{1}{2})(r+t)} = \frac{\frac{1}{4}}{\frac{3}{2}(r+t)} \tag{13}$$

Take, RHS of equation (11), put $k = 1,\ \iota = 0,\ \kappa = \frac{1}{2},\ \varphi = 1,\ \mathfrak{N}_{\Omega}\left(\iota,\, \varphi,\, \frac{r}{k}\right) = \mathfrak{N}_{\Omega}(0,\, 1,\, r) = \frac{\nu_k(0,\, 1)}{r} = \frac{1}{r}$ and $\mathfrak{N}_{\Omega}\left(\iota,\, \varphi,\, \frac{t}{k}\right) = \mathfrak{N}_{\Omega}\left(1,\, \frac{1}{2},\, t\right) = \frac{\nu_k(1,\, \frac{1}{2})}{t} = \frac{1}{4t},\ \mathfrak{D}_{\Omega}(0,\, \frac{1}{2},\, \Omega(0,\, \frac{1}{2})(r+t)) = \frac{\frac{1}{4}}{\frac{3}{2}(r+t)} = \frac{1}{6r+6t} < \frac{1}{6r} < \frac{1}{r}$. Then, $(U,\, \mathfrak{B}_{\Omega},\, \mathfrak{D}_{\Omega},\, \mathfrak{N}_{\Omega},\, \hat{\odot},\, \hat{\oplus})$ is an (ENbMS).

Example 2 Let $U = [0,\, 1],\ \nu(\iota,\, \varphi) = (\iota - \varphi)^2,\ \hat{\odot}(a, b) = \min\{a, b\},\ \hat{\oplus}(a, b) = \max\{a, b\},\ \Omega(\iota,\, \varphi) = 1 + \iota + \varphi,\ k = 2$. Define, for $(r > 0)$,

$$\mathfrak{B}_{\Omega}(\iota,\, \varphi,\, r) = \frac{r}{r + \nu(\iota,\, \varphi)},\ \mathfrak{D}_{\Omega}(\iota,\, \varphi,\, r) = \frac{\nu(\iota,\, \varphi)}{1 + \nu(\iota,\, \varphi) + r},\ \mathfrak{N}_{\Omega}(\iota,\, \varphi,\, r) = \frac{\nu(\iota,\, \varphi)}{2 + \nu(\iota,\, \varphi) + r}.$$

Condition (e):

$$\mathfrak{B}_{\Omega}(\iota,\, \kappa,\, \Omega(\iota,\, \kappa)(r+t)) \geq \mathfrak{B}_{\Omega}\left(\iota,\, \varphi,\, \frac{r}{k}\right) \hat{\odot} \mathfrak{B}_{\Omega}\left(\varphi,\, \kappa,\, \frac{t}{k}\right)$$

Since $\hat{\odot} = min$, the RHS becomes $\min\left\{\frac{\frac{r}{k}}{\frac{r}{k} + \nu(\iota,\, \varphi)},\, \frac{\frac{t}{k}}{\frac{t}{k} + \nu(\varphi,\, \kappa)}\right\}$
Thus condition (e) is equivalent to proving:

$$\frac{\Omega(\iota,\, \kappa)(r+t)}{\Omega(\iota,\, \kappa)(r+t) + \nu(\iota,\, \kappa)} \geq \min\left\{\frac{r}{r + k\nu(\iota,\, \varphi)},\, \frac{t}{t + k\nu(\varphi,\, \kappa)}\right\}.$$

$$\Omega(\iota,\, \kappa) = 1 + \iota + \kappa \geq 1$$

So,

$$\Omega(\iota,\, \kappa)(r+t) \geq r + t.$$

$$\nu(\iota,\, \kappa) = (\iota - \kappa)^2 \leq \left(|\iota - \varphi| + |\varphi - \kappa|\right)^2$$

$$\Rightarrow \nu(\iota,\, \kappa) \leq 2\nu(\iota,\, \varphi) + 2\nu(\varphi,\, \kappa)$$

Denominator of LHS grows slowly, numerator grows faster:

$$\frac{r+t}{r+t+\nu(\iota,\,\kappa)} \geq \min\left\{\frac{r}{r+\nu(\iota,\,\varphi)},\,\frac{t}{t+\nu(\varphi,\,\kappa)}\right\}$$

Multiplying numerator and denominator by (k = 2) preserves inequality.
Thus:

$$\mathfrak{B}_\Omega(\iota,\,\kappa,\,\Omega(\iota,\,\kappa)(r+t)) \geq \mathfrak{B}_\Omega\left(\iota,\,\varphi,\,\frac{r}{k}\right)\hat{\odot}\mathfrak{B}_\Omega\left(\varphi,\,\kappa,\,\frac{t}{k}\right)$$

Condition (e) holds.
Condition (j):

$$\mathfrak{D}_\Omega(\iota,\,\kappa,\,\Omega(\iota,\,\kappa)(r+t)) \leq \mathfrak{D}_\Omega\left(\iota,\,\varphi,\,\frac{r}{k}\right)\hat{\oplus}\mathfrak{D}_\Omega\left(\varphi,\,\kappa,\,\frac{t}{k}\right)$$

Since $\hat{\oplus} = \max$, this becomes:

$$\frac{\nu(\iota,\,\kappa)}{1+\nu(\iota,\,\kappa)+\Omega(\iota,\,\kappa)(r+t)} \leq \max\left\{\frac{\nu(\iota,\,\varphi)}{1+\nu(\iota,\,\varphi)+\frac{r}{k}},\,\frac{\nu(\varphi,\,\kappa)}{1+\nu(\varphi,\,\kappa)+\frac{t}{k}}\right\}.$$

The LHS denominator:

$$1+\nu(\iota,\,\kappa)+\Omega(\iota,\,\kappa)(r+t)$$

is much larger than either denominator on the RHS because:

$$\Omega(\iota,\,\kappa) \geq 1$$

$$\Rightarrow \Omega(\iota,\,\kappa)(r+t) \geq r+t$$

Thus:

$$\frac{\nu(\iota,\,\kappa)}{1+\nu(\iota,\,\kappa)+\Omega(\iota,\,\kappa)(r+t)} \leq \frac{\nu(\iota,\,\kappa)}{1+\nu(\iota,\,\kappa)+r}$$

and using

$$\nu(\iota,\,\kappa) \leq \nu(\iota,\,\varphi)+\nu(\varphi,\,\kappa)$$

the numerator on LHS cannot exceed both numerators on RHS.

Therefore the fraction is always $\geq$ the maximum. Condition (j) holds.

Condition (o):

$$\mathfrak{N}_\Omega(\iota,\, \kappa,\, \Omega(\iota,\, \kappa)(r+t)) \leq \mathfrak{N}_\Omega\left(\iota,\, \varphi,\, \frac{r}{k}\right) \hat{\oplus} \mathfrak{N}_\Omega\left(\varphi,\, \kappa,\, \frac{t}{k}\right)$$

Same reasoning as (j), since

$$\mathfrak{N}_\Omega(\iota,\, \varphi,\, r) = \frac{\nu(\iota,\, \varphi)}{2+\nu(\iota,\, \varphi)+r}$$

has the same numerator structure, a larger denominator than $\mathfrak{D}_\Omega$

Thus:

$$\frac{\nu(\iota,\, \kappa)}{2+\nu(\iota,\, \kappa)+\Omega(\iota,\, \kappa)(r+t)} \leq \max\left\{\frac{\nu(\iota,\, \varphi)}{2+\nu(\iota,\, \varphi)+\frac{r}{k}},\, \frac{\nu(\varphi,\, \kappa)}{2+\nu(\varphi,\, \kappa)+\frac{t}{k}}\right\}$$

Condition (o) holds.

Lemma 1 Let $\{u_p\}$ be a sequence in ENbMS $(U,\, \mathfrak{B}_\Omega,\, \mathfrak{D}_\Omega,\, \mathfrak{N}_\Omega,\, \hat{\odot},\, \hat{\oplus})$. Suppose that there exists $\Delta \in \left(0,\, \frac{1}{k}\right)$ such that

$$\mathfrak{B}_\Omega(u_{p-1},\, u_p,\, r) \geq \mathfrak{B}_\Omega(u_{p-1},\, v_p,\, \frac{r}{\Delta}),\; p \in \mathbb{N} \tag{14}$$

$$\mathfrak{D}_\Omega(u_{p-1},\, u_p,\, r) \leq \mathfrak{D}_\Omega(u_{p-1},\, u_p,\, \frac{r}{\Delta}),\; p \in \mathbb{N} \text{ and} \tag{15}$$

$$\mathfrak{N}_\Omega(u_{p-1},\, u_p,\, r) \leq \mathfrak{N}_\Omega(u_{p-1},\, u_p,\, \frac{r}{\Delta}),\; p \in \mathbb{N}. \tag{16}$$

Also, there exist $u_0,\, u_1 \in U$, $r > 0$ and $d \in (0,\, 1)$ such that

$$\begin{gathered}\lim_{p\to\infty} \Sigma_{i=p}^{\infty} \mathfrak{B}_\Omega\left(u_0,\, u_1,\, \frac{r}{d^i}\right) = 1 \\ \lim_{p\to\infty} \Sigma_{i=p}^{\infty} \mathfrak{D}_\Omega\left(u_0,\, u_1,\, \frac{r}{d^i}\right) = 0 \text{ and} \\ \lim_{p\to\infty} \Sigma_{i=p}^{\infty} \mathfrak{N}_\Omega\left(u_0,\, u_1,\, \frac{r}{d^i}\right) = 0.\end{gathered} \tag{17}$$

Then $\{u_n\}$ is a Cauchy sequence.

Proof. For each $r > 0$, assuming $p > q > p_0$, since $\mathfrak{B}_\Omega$ is k-non decreasing, $\mathfrak{D}_\Omega$ is k-non increasing and $\mathfrak{N}_\Omega$ is k-non increasing. The series $\Sigma_{i=1}^{\infty} \rho^i$ is convergent. Hence, there is $p_0 \in \mathbb{N}$ such that $\Sigma_{i=1}^{\infty} \rho^i < 1$ for all $p > p_0$, $\rho \in (0,\, 1)$. Now,

$$\begin{aligned}\mathfrak{B}_\Omega(u_p,\, u_{p+q},\, r) &\geq \mathfrak{B}_\Omega\left(u_p,\, u_{p+q},\, \frac{r\Sigma_{i=p}^{p+q-1}\rho^i}{k}\right)\\ &\geq \mathfrak{B}_\Omega\left(u_p,\, u_{p+1},\, \frac{r\rho^p}{k}\right)\hat{\odot}\mathfrak{B}_\Omega\left(u_{p+1},\, u_{p+q},\, \frac{r\Sigma_{i=p+1}^{p+q-1}\rho^i}{k^2}\right)\\ &\geq \mathfrak{B}_\Omega\left(u_p,\, u_{p+1},\, \frac{r\rho^p}{k}\right)\hat{\odot}\mathfrak{B}_\Omega\left(u_{p+1},\, u_{p+2},\, \frac{r\rho^{p+1}}{k^2}\right)\ldots\hat{\odot}\mathfrak{B}_\Omega\left(u_{p+q-1},\, u_{p+q},\, \frac{r\rho^{p+q-1}}{k^q}\right),\end{aligned}$$

$$\begin{aligned}\mathfrak{D}_\Omega(u_p,\, u_{p+q},\, r) &\leq \mathfrak{D}_\Omega\left(u_p,\, u_{p+q},\, \frac{r\Sigma_{i=p}^{p+q-1}\rho^i}{k}\right)\\ &\leq \mathfrak{D}_\Omega\left(u_p,\, u_{p+1},\, \frac{r\rho^p}{k}\right)\hat{\oplus}\mathfrak{D}_\Omega\left(u_{p+1},\, u_{p+q},\, \frac{r\Sigma_{i=p+1}^{p+q-1}\rho^i}{k^2}\right)\\ &\leq \mathfrak{D}_\Omega\left(u_p,\, u_{p+1},\, \frac{r\rho^p}{k}\right)\hat{\oplus}\mathfrak{D}_\Omega\left(u_{p+1},\, u_{p+2},\, \frac{r\rho^{p+1}}{k^2}\right)\ldots\hat{\oplus}\mathfrak{D}_\Omega\left(u_{p+q-1},\, u_{p+q},\, \frac{r\rho^{p+q-1}}{k^q}\right),\end{aligned}$$

and

$$\begin{aligned}\mathfrak{N}_\Omega(u_p,\, u_{p+q},\, r) &\leq \mathfrak{N}_\Omega\left(u_p,\, u_{p+q},\, \frac{r\Sigma_{i=p}^{p+q-1}\rho^i}{k}\right)\\ &\leq \mathfrak{N}_\Omega\left(u_p,\, u_{p+1},\, \frac{r\rho^p}{k}\right)\hat{\oplus}\mathfrak{N}_\Omega\left(u_{p+1},\, u_{p+q},\, \frac{r\Sigma_{i=p+1}^{p+q-1}\rho^i}{k^2}\right)\\ &\leq \mathfrak{N}_\Omega\left(u_p,\, u_{p+1},\, \frac{r\rho^p}{k}\right)\hat{\oplus}\mathfrak{N}_\Omega\left(u_{p+1},\, u_{p+2},\, \frac{r\rho^{p+1}}{k^2}\right)\ldots\hat{\oplus}\mathfrak{N}_\Omega\left(u_{p+q-1},\, u_{p+q},\, \frac{r\rho^{p+q-1}}{k^q}\right).\end{aligned}$$

From (11), (12) and (13) we have

$$\mathfrak{B}_\Omega(u_{p-1},\, u_p,\, r) \geq \mathfrak{B}_\Omega\left(u_{p-1},\, u_p,\, \frac{r}{\Delta}\right),\ p\in\mathbb{N}$$

$$\mathfrak{D}_\Omega(u_{p-1},\, u_p,\, r) \leq \mathfrak{D}_\Omega\left(u_{p-1},\, u_p,\, \frac{r}{\Delta}\right),\ p\in\mathbb{N},$$

$$\mathfrak{N}_\Omega(u_{p-1},\, u_p,\, r) \leq \mathfrak{N}_\Omega\left(u_{p-1},\, u_p,\, \frac{r}{\Delta}\right),\ p\in\mathbb{N}$$

and if $p > q$ and $k > 1$, we have

$$
\begin{aligned}
\mathfrak{B}_\Omega(u_p,\, u_{p+q},\, r) &\geq \mathfrak{B}_\Omega\left(u_0,\, u_1,\, \frac{r\rho^p}{k^2\Delta^p}\right)\hat{\odot}\mathfrak{B}_\Omega\left(u_0,\, u_1,\, \frac{r\rho^{p+1}}{k^3\Delta^{p+1}}\right)\hat{\odot}\ldots\mathfrak{B}_\Omega\left(u_0,\, u_1,\, \frac{r\rho^{p+q-1}}{k^m\Delta^{p+q-1}}\right)\\
&\geq \Sigma_{i=p}^{p+q-1}\mathfrak{B}_\Omega\left(u_0,\, u_1,\, \frac{r\rho^i}{k^{i-p+2}\Delta^i}\right)\\
&\geq \Sigma_{i=p}^{p+q-1}\mathfrak{B}_\Omega\left(u_0,\, u_1,\, \frac{r\rho^i}{k^i\Delta^i}\right)\\
&\geq \Sigma_{i=p}^{\infty}\mathfrak{B}_\Omega\left(u_0,\, u_1,\, \frac{r}{d^i}\right),
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{D}_\Omega(u_p,\, u_{p+q},\, r) &\leq \mathfrak{D}_\Omega\left(u_0,\, u_1,\, \frac{r\rho^p}{k^2\Delta^p}\right)\hat{\oplus}\mathfrak{D}_\Omega\left(u_0,\, u_1,\, \frac{r\rho^{n+1}}{k^3\Delta^{p+1}}\right)\hat{\oplus}\ldots\mathfrak{D}_\Omega\left(u_0,\, u_1,\, \frac{r\rho^{p+q-1}}{k^m\Delta^{p+q-1}}\right)\\
&\leq \Sigma_{i=p}^{p+q-1}\mathfrak{D}_\Omega\left(u_0,\, u_1,\, \frac{r\rho^i}{k^{i-p+2}\Delta^i}\right)\\
&\leq \Sigma_{i=p}^{p+q-1}\mathfrak{D}_\Omega\left(u_0,\, u_1,\, \frac{r\rho^i}{k^i\Delta^i}\right)\\
&\leq \Sigma_{i=p}^{\infty}\mathfrak{D}_\Omega\left(u_0,\, u_1,\, \frac{r}{d^i}\right),\ \text{and}
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{N}_\Omega(u_p,\, u_{p+q},\, r) &\leq \mathfrak{N}_\Omega\left(u_0,\, u_1,\, \frac{r\rho^p}{k^2\Delta^p}\right)\hat{\oplus}\mathfrak{N}_\Omega\left(u_0,\, u_1,\, \frac{r\rho^{n+1}}{k^3\Delta^{p+1}}\right)\hat{\oplus}\ldots\mathfrak{N}_\Omega\left(u_0,\, u_1,\, \frac{r\rho^{p+q-1}}{k^m\Delta^{p+q-1}}\right)\\
&\leq \Sigma_{i=p}^{p+q-1}\mathfrak{N}_\Omega\left(u_0,\, u_1,\, \frac{r\rho^i}{k^{i-p+2}\Delta^i}\right)\\
&\leq \Sigma_{i=p}^{p+q-1}\mathfrak{N}_\Omega\left(u_0,\, u_1,\, \frac{r\rho^i}{k^i\Delta^i}\right)\\
&\leq \Sigma_{i=p}^{\infty}\mathfrak{N}_\Omega\left(u_0,\, u_1,\, \frac{r}{d^i}\right),
\end{aligned}
$$

where $d = \frac{k\Delta}{\rho}$. As $d \in (0,\, 1)$, from (13), we get $\{u_n\}$ is a Cauchy sequence. □

Theorem 1 Let $(U,\, \mathfrak{B}_\Omega,\, \mathfrak{D}_\Omega,\, \mathfrak{N}_\Omega,\, \hat{\odot},\, \hat{\oplus})$ is a complete ENbMS with the mapping Ω: $U \times U \to [1,\, \infty)$ and for all $\iota,\, \varphi \in U$, such that

$$\lim_{P\to\infty}\mathfrak{B}_\Omega(\iota,\, \varphi,\, r) = 1, \tag{18}$$

$$\lim_{p\to\infty} \mathfrak{D}_{\Omega}(\iota,\, \varphi,\, r) = 0 \text{ and} \tag{19}$$

$$\lim_{p\to\infty} \mathfrak{N}_{\Omega}(\iota,\, \varphi,\, r) = 0.$$

Suppose that the mapping $\sigma\colon U \to U$ fulfills the following relations

$$\mathfrak{B}_{\Omega}(\sigma\iota,\, \sigma\varphi,\, \pi r) \geq \mathfrak{B}_{\Omega}(\iota,\, \varphi,\, r), \tag{20}$$

$$\mathfrak{D}_{\Omega}(\sigma\iota,\, \sigma\varphi,\, \pi r) \leq \mathfrak{D}_{\Omega}(\iota,\, \varphi,\, r) \text{ and} \tag{21}$$

$$\mathfrak{N}_{\Omega}(\sigma\iota,\, \sigma\varphi,\, \pi r) \leq \mathfrak{N}_{\Omega}(\iota,\, \varphi,\, r), \tag{22}$$

where $\pi \in (0,\, 1)$. Let us take an arbitrary $\tau_0 \in U$, p, $\mu \in \mathbb{N}$, we have $\Omega(\tau_p,\, \tau_{p+\mu}) < \frac{1}{\pi}$, where $\tau_p = \sigma^p \tau_0$. Then the mapping σ has a unique fixed point.

Proof. Let's begin at any point $\tau_0 \in U$ and creative a sequence $\{\tau_p\}$ using an iterative procedure $\tau_p = \sigma^p \tau_0$, $p \in \mathbb{N}$. For p, $r > 0$, by repeatedly applying the contractive condition from equation (19), we get

$$\begin{aligned}
\mathfrak{B}_{\Omega}(\tau_p,\, \tau_{p+1},\, \pi r) &= \mathfrak{B}_{\Omega}(\sigma\tau_{p-1},\, \sigma\tau_p,\, \pi r)\\
&\geq \mathfrak{B}_{\Omega}(\tau_{p-1},\, \tau_n,\, r)\\
&\geq \mathfrak{B}_{\Omega}\left(\tau_{p-2},\, \tau_{p-1},\, \frac{r}{\pi}\right)\\
&\geq \mathfrak{B}_{\Omega}\left(\tau_{p-3},\, \tau_{p-2},\, \frac{r}{\pi^2}\right)\\
&\;\;\vdots\\
&\geq \mathfrak{B}_{\Omega}\left(\tau_0,\, \tau_1,\, \frac{r}{\pi^{p-1}}\right).
\end{aligned}$$

So, we have

$$\mathfrak{B}_{\Omega}(\tau_p,\, \tau_{p+1},\, \pi r) \geq \mathfrak{B}_{\Omega}\left(\tau_0,\, \tau_1,\, \frac{r}{\pi^{p-1}}\right). \tag{23}$$

For any $\mu \in \mathbb{N}$, write down $r = \frac{\mu r}{\mu} = \frac{r}{\mu} + \cdots + \frac{r}{\mu}$ and frequently refer to Definition 5(e),

$$\mathfrak{B}_{\Omega}(\tau_p,\, \tau_{p+\mu},\, r) \geq \mathfrak{B}_{\Omega}\left(\tau_p,\, \tau_{p+1},\, \frac{r}{k\mu\varphi(\tau_p,\, \tau_{p+\mu})}\right) \hat{\odot} \mathfrak{B}_{\Omega}\left(\tau_{p+1},\, \tau_{p+2},\, \frac{r}{bq\varphi(\tau_p,\, \tau_{p+\mu})\varphi(\tau_{p+1},\, \tau_{p+\mu})}\right) \hat{\odot}$$

$$\mathfrak{B}_{\Omega}\left(\tau_{p+2},\, \tau_{p+3},\, \frac{r}{k\mu\varphi(\tau_p,\, \tau_{p+\mu})\varphi(\tau_{p+1},\, \tau_{p+\mu})(\tau_{p+2},\, \tau_{p+\mu})}\right)\hat{\odot}\ldots\hat{\odot}$$

$$\mathfrak{B}_{\Omega}\left(\tau_{p+\mu-1},\, \tau_{p+\mu},\, \frac{r}{k\mu\varphi(\tau_p,\, \tau_{p+\mu})\varphi(\tau_{p+1},\, \tau_{p+\mu})(\tau_{p+2},\, \tau_{p+\mu})\ldots(\tau_{p+\mu-1},\, \tau_{p+\mu})}\right). \tag{24}$$

Using equation (23) and Definition 5(e), we obtain

$$\mathfrak{B}_{\Omega}(\tau_p,\, \tau_{p+\mu},\, r) \geq \mathfrak{B}_{\Omega}\left(\tau_0,\, \tau_1,\, \frac{r}{k\mu\varphi(\tau_n,\, \tau_{p+\mu})\pi^p}\right)\hat{\odot}\mathfrak{B}_{\Omega}\left(\tau_0,\, \tau_1,\, \frac{r}{k\mu\varphi(\tau_p,\, tau_{p+\mu})\varphi(\tau_{p+1},\, \tau_{p+\mu})\pi^{p+1}}\right)\hat{\odot}$$

$$\mathfrak{B}_{\Omega}\left(\tau_0,\, \tau_1,\, \frac{r}{k\mu\varphi(\tau_p,\, \tau_{p+\mu})\varphi(\tau_{p+1},\, \tau_{p+\mu})(\tau_{p+2},\, \tau_{p+\mu})\pi^{p+3}}\right)\hat{\odot}\ldots\hat{\odot}$$

$$\mathfrak{B}_{\Omega}\left(\tau_0,\, \tau_1,\, \frac{r}{k\mu\varphi\tau_p,\, \tau_{p+\mu})\varphi(\tau_{p+1},\, \tau_{n+q})(\tau_{p+2},\, \tau_{p+\mu})\ldots(tau_{p+\mu-1},\, tau_{p+\mu})k^{p+\mu-1}}\right).$$

From contractive condition (20),

$$\begin{aligned}
\mathfrak{D}_{\Omega}(\tau_p,\, \tau_{p+1},\, \pi r) &= \mathfrak{D}_{\Omega}(\sigma\tau_{p-1},\, \sigma\tau_p,\, \pi r)\\
&\leq \mathfrak{D}_{\Omega}(\tau_{p-1},\, \tau_p,\, r)\\
&\leq \mathfrak{D}_{\Omega}\left(\tau_{p-2},\, \tau_{p-1},\, \frac{r}{\pi}\right)\\
&\leq \mathfrak{D}_{\Omega}\left(\tau_{p-3},\, \tau_{p-2},\, \frac{r}{\pi^2}\right)\\
&\;\;\vdots\\
&\leq \mathfrak{D}_{\Omega}\left(\tau_0,\, \tau_1,\, \frac{r}{\pi^{p-1}}\right).
\end{aligned}$$

So, we have

$$\mathfrak{D}_{\Omega}(\tau_p,\, \tau_{p+1},\, \mu r) \leq \mathfrak{D}_{\Omega}\left(\tau_0,\, \tau_1,\, \frac{r}{\pi^{p-1}}\right). \tag{25}$$

For any $\mu \in \mathbb{N}$, write down $r = \frac{\mu r}{\mu} = \frac{r}{\mu} + \cdots + \frac{r}{\mu}$ and frequently apply Definition 5(j),

$$\mathfrak{D}_{\Omega}(\tau_p,\, \tau_{p+\mu},\, r) \leq \mathfrak{D}_{\Omega}\left(\tau_p,\, \tau_{p+1},\, \frac{r}{k\mu\varphi(\tau_p,\, \tau_{p+\mu})}\right) \hat{\oplus} \mathfrak{D}_{\Omega}\left(\tau_{p+1},\, \tau_{p+2},\, \frac{r}{k\mu\varphi(\tau_p,\, \tau_{p+\mu})\varphi(\tau_{p+1},\, \tau_{p+\mu})}\right)$$

$$\hat{\oplus} \mathfrak{D}_{\Omega}\left(\tau_{p+2},\, \tau_{p+3},\, \frac{r}{k\mu\varphi(\tau_p,\, \tau_{p+\mu})\varphi(\tau_{p+1},\, \tau_{p+\mu})(\tau_{p+2},\, \tau_{p+q})}\right) \hat{\oplus} \dots$$

$$\hat{\oplus} \mathfrak{D}_{\Omega}\left(\tau_{p+\mu-1},\, \tau_{p+\mu},\, \frac{r}{k\mu\varphi(\tau_p,\, \tau_{p+\mu})\varphi(\tau_{p+1},\, \tau_{p+q})(\tau_{p+2},\, \tau_{p+q})\dots(\tau_{p+\mu-1},\, \tau_{p+\mu})}\right).$$

Using equation (26) and Definition 5(j), we obtain

$$\mathfrak{D}_{\Omega}(\tau_p,\, \tau_{p+\mu},\, r) \leq \mathfrak{D}_{\Omega}\left(\tau_0,\, \tau_1,\, \frac{r}{k\mu\varphi(\tau_p,\, \tau_{p+\mu})\pi^{p}}\right) \hat{\oplus} \mathfrak{D}_{\Omega}\left(\tau_0,\, \tau_1,\, \frac{r}{k\mu\varphi(\tau_p,\, \tau_{p+\mu})\varphi(\tau_{p+1},\, \tau_{p+\mu})\pi^{p+1}}\right)$$

$$\hat{\oplus} \mathfrak{D}_{\Omega}\left(\tau_0,\, \tau_1,\, \frac{r}{k\mu\varphi(\tau_p,\, \tau_{p+\mu})\varphi(\tau_{p+1},\, \tau_{p+\mu})(\tau_{p+2},\, \tau_{p+\mu})\pi^{p+3}}\right) \hat{\oplus} \dots$$

$$\hat{\oplus} \mathfrak{D}_{\Omega}\left(\tau_0,\, \tau_1,\, \frac{r}{k\mu\varphi\tau_p,\, \tau_{p+\mu})\varphi(\tau_{p+1},\, \tau_{p+\mu})(\tau_{p+2},\, \tau_{p+\mu})\dots(\tau_{p+\mu-1},\, \tau_{p+\mu})\pi^{p+\mu-1}}\right).$$

$$\begin{aligned}
\mathfrak{N}_{\Omega}(\tau_p,\, \tau_{p+1},\, \pi r) &= \mathfrak{N}_{\Omega}(\sigma\tau_{p-1},\, \sigma\tau_p,\, \pi r) \\
&\leq \mathfrak{N}_{\Omega}(\tau_{p-1},\, \tau_p,\, r) \\
&\leq \mathfrak{N}_{\Omega}\left(\tau_{p-2},\, \tau_{p-1},\, \frac{r}{\pi}\right) \\
&\leq \mathfrak{N}_{\Omega}\left(\tau_{p-3},\, \tau_{p-2},\, \frac{r}{\pi^2}\right) \\
&\;\;\vdots \\
&\leq \mathfrak{N}_{\Omega}\left(\tau_0,\, \tau_1,\, \frac{r}{\pi^{p-1}}\right).
\end{aligned}$$

$$\mathfrak{N}_{\Omega}(\tau_p,\, \tau_{p+1},\, \mu r) \leq \mathfrak{N}_{\Omega}\left(\tau_0,\, \tau_1,\, \frac{r}{\pi^{p-1}}\right). \tag{26}$$

For any $\mu \in \mathbb{N}$, write down $r = \frac{\mu r}{\mu} = \frac{r}{\mu} + \cdots + \frac{r}{\mu}$and frequently refer to Definition 5(o),

$$\mathfrak{N}_{\Omega}(\tau_p,\, \tau_{p+\mu},\, r) \leq \mathfrak{N}_{\Omega}\left(\tau_p,\, \tau_{p+1},\, \frac{r}{k\mu\varphi(\tau_p,\, \tau_{p+\mu})}\right) \hat{\oplus} \mathfrak{N}_{\Omega}\left(\tau_{p+1},\, \tau_{p+2},\, \frac{r}{k\mu\varphi(\tau_p,\, \tau_{p+\mu})\varphi(\tau_{p+1},\, \tau_{p+\mu})}\right)$$

$$\hat{\oplus}\mathfrak{N}_{\Omega}\left(\tau_{p+2},\, \tau_{p+3},\, \frac{r}{k\mu\varphi(\tau_p,\, \tau_{p+\mu})\varphi(\tau_{p+1},\, \tau_{p+\mu})(\tau_{p+2},\, \tau_{p+q})}\right)\hat{\oplus}\dots$$

$$\hat{\oplus}\mathfrak{N}_{\Omega}\left(\tau_{p+\mu-1},\, \tau_{p+\mu},\, \frac{r}{k\mu\varphi(\tau_p,\, \tau_{p+\mu})\varphi(\tau_{p+1},\, \tau_{p+q})(\tau_{p+2},\, \tau_{p+q})\dots(\tau_{p+\mu-1},\, \tau_{p+\mu})}\right).$$

Using equation (26) and Definition 5(o), we obtain

$$\mathfrak{N}_{\Omega}(\tau_p,\, \tau_{p+\mu},\, r) \leq \mathfrak{N}_{\Omega}\left(\tau_0,\, \tau_1,\, \frac{r}{k\mu\varphi(\tau_p,\, \tau_{p+\mu})\pi^{p}}\right)\hat{\oplus}\mathfrak{N}_{\Omega}\left(\tau_0,\, \tau_1,\, \frac{r}{k\mu\varphi(\tau_p,\, \tau_{p+\mu})\varphi(\tau_{p+1},\, \tau_{p+\mu})\pi^{p+1}}\right)$$

$$\hat{\oplus}\mathfrak{N}_{\Omega}\left(\tau_0,\, \tau_1,\, \frac{r}{k\mu\varphi(\tau_p,\, \tau_{p+\mu})\varphi(\tau_{p+1},\, \tau_{p+\mu})(\tau_{p+2},\, \tau_{p+\mu})\pi^{p+3}}\right)\hat{\oplus}\dots$$

$$\hat{\oplus}\mathfrak{N}_{\Omega}\left(\tau_0,\, \tau_1,\, \frac{r}{k\mu\varphi\tau_p,\, \tau_{p+\mu})\varphi(\tau_{p+1},\, \tau_{p+\mu})(\tau_{p+2},\, \tau_{p+\mu})\dots(\tau_{p+\mu-1},\, \tau_{p+\mu})\pi^{p+\mu-1}}\right).$$

For all $p,\, \mu \in \mathbb{N}$, we know that $\varphi(\tau_p,\, \tau_{p+\mu})\pi < 1$ with $\pi \in (0,\, 1)$.
Now using equation (23) and (26) and $p \to \infty$, we get

$$\lim_{p\to\infty}\mathfrak{B}_{\Omega}(\tau_p,\, \tau_{p+\mu},\, r) = 1\hat{\odot}1\hat{\odot}\hat{\odot}\dots 1 = 1,\ \lim_{p\to\infty}\mathfrak{D}_{\Omega}(\tau_p,\, \tau_{p+\mu},\, r) = 0\hat{\oplus}0\hat{\oplus}0\hat{\oplus}\dots 0 = 0 \text{ and}$$

$$\lim_{p\to\infty}\mathfrak{N}_{\Omega}(\tau_p,\, \tau_{p+\mu},\, r) = 0\hat{\oplus}0\hat{\oplus}0\hat{\oplus}\dots 0 = 0.$$

Therefore $\{\tau_n\}$ is a Cauchy sequence. After $(U,\, \mathfrak{B}_{\Omega},\, \mathfrak{D}_{\Omega}\mathfrak{N}_{\Omega},\, \hat{\odot},\, \hat{\oplus})$ is a complete ENbMS there exists $\tau \in U$ in a way that $\lim\limits_{p\to\infty}\{\tau_p\} = \tau$. We must show τ is the FP of σ.
Using 5(e), 5(j) and 5(o), we get

$$\mathfrak{B}_{\Omega}(\sigma\tau,\, \sigma,\, r) \geq \mathfrak{B}_{\Omega}\left(\sigma\tau,\, \sigma\tau_p,\, \frac{r}{2\varphi(\sigma\tau,\, \tau)}\right)\hat{\odot}\mathfrak{B}_{\Omega}\left(\sigma\tau,\, \tau,\, \frac{r}{2\varphi(\sigma\tau,\, \tau)}\right)$$

$$\geq \mathfrak{B}_{\Omega}\left(\tau,\, tau_p,\, \frac{r}{2\varphi(\sigma\tau,\, \tau)\pi}\right)\hat{\odot}\mathfrak{B}_{\Omega}\left(\tau_{p+1},\, \tau_p,\, \frac{r}{2\varphi(\sigma\tau,\, \tau)}\right)$$

$$\to 1\hat{\odot}1 = 1,\ \text{as}\, p \to \infty,$$

$$\mathfrak{D}_{\Omega}(\sigma\tau,\, \sigma,\, r) \leq \mathfrak{D}_{\Omega}\left(\sigma\tau,\, \sigma\tau_p,\, \frac{r}{2\varphi(\sigma\tau,\, \tau)}\right)\hat{\oplus}\mathfrak{D}_{\Omega}\left(\sigma\tau_p,\, \tau,\, \frac{r}{2\varphi(\sigma\tau,\, \tau)}\right)$$

$$\leq \mathfrak{D}_{\Omega}\left(\tau,\, \tau_p,\, \frac{r}{2\varphi(\sigma\tau,\, \tau)\pi}\right)\hat{\oplus}\mathfrak{D}_{\Omega}\left(tau_{p+1},\, \tau_p,\, \frac{r}{2\varphi(\sigma\tau,\, \tau)}\right)$$

$$\to 0 \hat{\oplus} 0 = 0, \text{ as } p \to \infty \text{ and}$$

$$\mathfrak{N}_\Omega(\sigma\tau, \sigma, r) \leq \mathfrak{N}_\Omega\left(\sigma\tau, \sigma\tau_p, \frac{r}{2\varphi(\sigma\tau, \tau)}\right) \hat{\oplus} \mathfrak{N}_\Omega\left(\sigma\tau_p, \tau, \frac{r}{2\varphi(\sigma\tau, \tau)}\right)$$

$$\leq \mathfrak{N}_\Omega\left(\tau, \tau_p, \frac{r}{2\varphi(\sigma\tau, \tau)\pi}\right) \hat{\oplus} \mathfrak{N}_\Omega\left(\tau_{p+1}, \tau_p, \frac{r}{2\varphi(\sigma\tau, \tau)}\right)$$

$$\to 0 \hat{\oplus} 0 = 0, \text{ as } p \to \infty,$$

$\sigma\tau = \sigma$ so, τ is the FP of σ. To prove uniqueness, consider d is the other FP such that $\sigma d = d$ for arbitrary $d \in U$ then

$$\mathfrak{B}_\Omega(d, \tau, r) = \mathfrak{B}_\Omega(\sigma d, \sigma\tau, r) \geq \mathfrak{B}_\Omega\left(d, \tau, \frac{r}{\pi}\right)$$

$$= \mathfrak{B}_\Omega\left(\sigma d, \sigma\tau, \frac{r}{\pi}\right)$$

$$\geq \mathfrak{B}_\Omega\left(d, \tau, \frac{r}{\pi^2}\right) \geq \ldots \mathfrak{B}_\Omega\left(d, \tau, \frac{r}{\pi^p}\right) \to 1 \text{ as } p \to \infty,$$

$$\mathfrak{D}_\Omega(d, \tau, r) = \mathfrak{D}_\Omega(\sigma d, \sigma\tau, r) \leq \mathfrak{D}_\Omega\left(d, \tau, \frac{r}{\pi}\right)$$

$$= \mathfrak{D}_\Omega\left(\sigma d, \sigma\tau, \frac{r}{\pi}\right)$$

$$\leq \mathfrak{D}_\Omega\left(d, \tau, \frac{r}{\pi^2}\right) \leq \ldots \mathfrak{D}_\Omega\left(d, \tau, \frac{r}{\pi^p}\right) \to 0 \text{ as } p \to \infty \text{ and}$$

$$\mathfrak{N}_\Omega(d, \tau, r) = \mathfrak{N}_\Omega(\sigma d, \sigma\tau, r) \leq \mathfrak{N}_\Omega\left(d, \tau, \frac{r}{\pi}\right)$$

$$= \mathfrak{N}_\Omega\left(\sigma d, \sigma\tau, \frac{r}{\pi}\right)$$

$$\leq \mathfrak{N}_\Omega\left(d, \tau, \frac{r}{\pi^2}\right) \leq \ldots \mathfrak{N}_\Omega\left(d, \tau, \frac{r}{\pi^p}\right) \to 0 \text{ as } p \to \infty.$$

Hence, $\tau = d$. This concludes the proof. □

Example 3 Let $U = [0, 1]$ then $\mathfrak{B}_\Omega(\iota, \varphi, r) = \left(\frac{1}{r}\right)^{(\iota-\varphi)^2}$, $\mathfrak{D}_\Omega(\iota, \varphi, r) = 1 - \left(\frac{1}{r}\right)^{(\iota-\varphi)^2}$ and $\mathfrak{N}_\Omega(\iota, \varphi, r) = (r)^{(\iota-\varphi)^2} - 1$ and sequence $\{\tau_p\} = \frac{1}{2p}$ and $\Omega = 1 + \iota + \varphi$. It is simple to confirm that $(U, \mathfrak{B}_\Omega, \mathfrak{D}_\Omega, \mathfrak{N}_\Omega, \hat{\odot}, \hat{\oplus})$ is a complete ENbMS. Let $\sigma\colon U \to U$ be a mapping defined as $\sigma(\iota) = 1 - \iota$ for all $r > 0$, $\pi \in (0, 1)$ we have,

$$\mathfrak{B}_\Omega(\sigma\iota, \sigma\varphi, r) = \mathfrak{B}_\Omega(1-\iota, 1-\varphi, \pi r) = \left(\frac{1}{\pi r}\right)^{(1-\iota-1+\varphi)^2} = \left(\frac{1}{\pi r}\right)^{(\varphi-\iota)^2} = \left(\frac{1}{\pi r}\right)^{(\iota-\varphi)^2}.$$

Now, $\pi \in (0,\, 1)$ i.e., implies $\pi r < r$ because $r > 0$ implies $\frac{1}{\pi r} > \frac{1}{r}$. So,

$$\mathfrak{B}_\Omega(1-\iota,\, 1-\varphi,\, \pi r) > \left(\frac{1}{r}\right)^{(\iota-\varphi)^2} = \mathfrak{B}_\Omega(\iota,\, \varphi,\, r)$$

$$\mathfrak{D}_\Omega(\sigma\iota,\, \sigma\varphi,\, r) = \mathfrak{D}_\Omega(1-\iota,\, 1-\varphi,\, \pi r)$$

$$= 1 - \left(\frac{1}{\pi r}\right)^{(1-\iota-1+\varphi)^2}$$

$$= 1 - \left(\frac{1}{\pi r}\right)^{(\varphi-\iota)^2}$$

$$= 1 - \left(\frac{1}{\pi r}\right)^{(\iota-\varphi)^2} \quad \text{and}$$

$$\mathfrak{D}_\Omega(1-\iota,\, 1-\varphi,\, \pi r) < 1 - \left(\frac{1}{r}\right)^{(\iota-\varphi)^2} = \mathfrak{D}_\Omega(\iota,\, \varphi,\, r) \text{ and}$$

$$\mathfrak{N}_\Omega(\sigma\iota,\, \sigma\varphi,\, r) = \mathfrak{N}_\Omega(1-\iota,\, 1-\varphi,\, \pi r)$$

$$= \left(\frac{1-\left(\frac{1}{\pi r}\right)^{(\iota-\varphi)^2}}{\left(\frac{1}{\pi r}\right)^{(\iota-\varphi)^2}}\right)$$

$$= (\pi r)^{(\iota-\varphi)^2} - 1.$$

Now, $\pi \in (0,\, 1)$ implies $\pi r < r$ because $r > 0$

$$\frac{1}{\pi r} > \frac{1}{r}$$

$$\frac{1}{\pi r}^{(\iota-\varphi)^2} > \left(\frac{1}{r}\right)^{(\iota-\varphi)^2}$$

$$-\left(\frac{1}{\pi r}\right)^{(\iota-\varphi)^2} < -\left(\frac{1}{r}\right)^{(\iota-\varphi)^2}$$

$$1-\left(\frac{1}{\pi r}\right)^{(\iota-\varphi)^2} < 1-\left(\frac{1}{r}\right)^{(\iota-\varphi)^2}$$

$$\left(\frac{1-\left(\frac{1}{\pi r}\right)^{(\iota-\varphi)^2}}{\left(\frac{1}{\pi r}\right)^{(\iota-\varphi)^2}}\right) < \left(\frac{1-\left(\frac{1}{r}\right)^{(\iota-\varphi)^2}}{\left(\frac{1}{r}\right)^{(\iota-\varphi)^2}}\right)$$

$$(\pi r)^{(\iota-\varphi)^2}-1 < (r)^{(\iota-\varphi)^2}-1.$$

So, from above equation $\mathfrak{N}_\Omega(1-\iota,\, 1-\varphi,\, \pi r) = (\pi r)^{(\iota-\varphi)^2}-1 = \mathfrak{N}_\Omega(\iota,\, \varphi,\, r)$, $\Omega(\tau_p.\tau_{p+1}) < \frac{1}{\pi}$ satisfy so σ has a unique fixed point and that point is $\frac{1}{2}$.

4. Application to fredholm integral equations

A class of integral equations planted by the Swedish mathematician Erik Ivar Fredholm is called Fredholm integral equations. In such equations an unknown is to be estimated and such equations are expressed in the form

$$\Psi(w) = g(w) + \gamma \int_a^b K(w,\, t)\Psi(t)dt, \tag{27}$$

where $g(w)$ is a given function while $\Psi(w)$ is the unknown function to be estimated, $K(w,\, t)$ is the kernel, and γ is Fredholm parameter. Fredholm integral equations are quite significant and appear in various fields including mathematical and physical sciences such as potential theory, signal processing, and quantum mechanics.

One of the key aspects is the solvability of Fredholm integral equations and characteristics of their solutions. Solutions of these equations depends on the properties of the kernel as well as the interval of integration and solutions may exhibit different behaviors, including uniqueness, existence, and convergence properties. Among several solution techniques, one of the most fundamental approach for solving these equations is the fixed point theory in which contraction mappings play vital roles to estimate the existence and uniqueness of the solutions. The well-known Banach contraction principle and other FP theorems provide a key platform to examine these nonlinear integral equations. and numerical computations.

Let $U = C([x,\, y]^2,\, \mathbb{R})$ be the set of all continuous real-valued functions defined on the interval $[x,\, y] \times [x,\, y]$. Now, we let the fuzzy integral equation

$$\iota(w) = y(j) + \gamma \int_x^y F(w,\, j)\iota(j)dj,\; \forall w,\, j \in [x,\, y] \tag{28}$$

where $\beta > 0$ is a triangular shaped fuzzy number, $y(j)$ is a fuzzy function of $j \in [x,\, y]$, and $F \in U$. For all $\iota,\, \varphi \in U$ and $r > 0$, define $\mathfrak{B}_\Omega$, $\mathfrak{D}_\Omega$ and $\mathfrak{N}_\Omega$ by

$$\mathfrak{B}_\Omega(\iota(j),\, \varphi(j),\, r) = \sup_{j\in[x,\, y]} \frac{r}{r+\max\{\iota(j),\, \varphi(j)\}^2}$$

$$\mathfrak{D}_\Omega(\iota(j),\, \varphi(j),\, r) = 1 - \sup_{j\in[x,\, y]} \frac{r}{r+\max\{\iota(j),\, \varphi(j)\}^2}$$

and

$$\mathfrak{N}_\Omega(\iota(j),\, \varphi(j),\, r) = \frac{1}{\sup_{j \in [x,\, y]} \frac{r}{r + \max\{\iota(j),\, \varphi(j)\}^2}} - 1,$$

with the CTN and CTCN defined by

$$\iota \hat{\odot} \varphi = \iota . \varphi \ \text{ and } \ \iota \hat{\oplus} \varphi = \max(\iota,\, \varphi).$$

Define Ω: $U \times U \to [1,\, \infty)$ by

$$\Omega(\iota,\, \varphi) = \begin{cases} 1 & \text{if } \iota = \varphi; \\ \\ 1 + \max\{\iota,\, \varphi\}, & \text{otherwise.} \end{cases}$$

Then $(U, \mathfrak{B}_\Omega, \mathfrak{D}_\Omega, \mathfrak{N}_\Omega, \hat{\odot}, \hat{\oplus})$ is a complete ENbMS. Assume that $\max\{F(w,\, j)\iota(j), F(w,\, j)\varphi(j)\} < C \max\{\iota(j),\, \varphi(j)\}$ for $\iota,\, \varphi \in U$, $\pi \in (0,\, 1)$, and for all $C \geq 1$, $w,\, j \in [x,\, y]$. Also consider $\gamma \int_x^y \frac{dj}{\sqrt{2}} \leq \pi < 1$. Then the fuzzy integral equation (28) has a unique solution.

Proof. Define σ: $U \to U$ by

$$\sigma\iota(w) = y(j) + \gamma \int_x^y F(w,\, j)\iota(j) dj,\ \forall w,\, j \in [x,\, y].$$

Now for all $\iota,\, \varphi \in U$, we obtain

$$\begin{aligned} \mathfrak{B}_\Omega(\sigma\iota(j),\, \sigma\varphi(j),\, \pi r) &= \sup_{j \in [x,\, y]} \frac{\pi r}{\pi r + \max\{\sigma\iota(j),\, \sigma\varphi(j)\}^2} \\ &= \sup_{j \in [x,\, y]} \frac{\pi r}{\pi r + \max(y(j) + \gamma \int_x^y F(w,\, j)\iota(j)dj,\, y(j) + \gamma \int_x^y F(w,\, j)\iota(j)dj)^2} \\ &= \sup_{j \in [x,\, y]} \frac{\pi r}{\pi r + \max(\beta \int_x^y F(w,\, j)\iota(j)dj,\, \beta \int_x^y F(w,\, j)\iota(j)dj)^2} \\ &= \sup_{j \in [x,\, y]} \frac{\pi r}{\pi r + \max\{F(w,\, j)\iota(j),\, F(w,\, j)\varphi(j)\}^2} \\ &> \sup_{j \in [x,\, y]} \frac{r}{r + C \max\{\iota(j),\, \varphi(j)\}^2} \\ &> \mathfrak{B}_\Omega(\iota(j),\, \varphi(j),\, r). \end{aligned}$$

$$\begin{aligned}
\mathfrak{D}_\Omega(\sigma\iota(j),\, \sigma\varphi(j),\, \pi r) &= 1 - \sup_{j\in[x,\, y]} \frac{\pi r}{\pi r + \max\{\sigma\iota(j),\, \sigma\varphi(j)\}^2} \\
&= 1 - \sup_{j\in[x,\, y]} \frac{\pi r}{\pi r + \max(y(j) + \gamma \int_x^y F(w,\, j)\iota(j)dj,\, y(j) + \gamma \int_x^y F(w,\, j)\iota(j)dj)^2} \\
&= 1 - \sup_{j\in[x,\, y]} \frac{\pi r}{\pi r + \max(\gamma \int_x^y F(w,\, j)\iota(j)dj,\, \gamma \int_x^y F(w,\, j)\iota(j)dj)^2} \\
&= 1 - \sup_{j\in[x,\, y]} \frac{\pi r}{\pi r + \max\{F(w,\, j)\iota(j),\, F(w,\, j)\varphi(j)\}^2} \\
&< 1 - \sup_{j\in[x,\, y]} \frac{r}{r + C\max\{\iota(j),\, \varphi(j)\}^2} \\
&< \mathfrak{D}_\Omega(\iota(j),\, \varphi(j),\, r)
\end{aligned}$$

$$\begin{aligned}
\mathfrak{N}_\Omega(\sigma\iota(j),\, \sigma\varphi(j),\, \pi r) &= \left(\frac{1 - \sup_{j\in[x,\, y]} \frac{\pi r}{\pi r + \max\{\sigma\iota(j),\, \sigma\varphi(j)\}^2}}{\sup_{j\in[x,\, y]} \frac{\pi r}{\pi r + \max\{\sigma\iota(j),\, \sigma\varphi(j)\}^2}} \right) \\
&= \left(\frac{1 - \sup_{j\in[x,\, y]} \frac{\pi r}{\pi r + \max(\gamma \int_x^y F(w,\, j)\iota(j)dj,\, \gamma \int_x^y F(w,\, j)\iota(j)dj)^2}}{\sup_{j\in[x,\, y]} \frac{\pi r}{\pi r + \max(\beta \int_x^y F(w,\, j)\iota(j)dj,\, \beta \int_x^y F(w,\, j)\iota(j)dj)^2}} \right) \\
&= \left(\frac{1 - \sup_{j\in[x,\, y]} \frac{\pi r}{\pi r + \max\{F(w,\, j)\iota(j),\, F(w,\, j)\varphi(j)\}^2}}{\sup_{j\in[x,\, y]} \frac{\pi r}{\pi r + \max\{F(w,\, j)\iota(j),\, F(w,\, j)\varphi(j)\}^2}} \right) \\
&< \frac{1}{\sup_{j\in[x,\, y]} \frac{r}{r + C\max\{\iota(j),\, \varphi(j)\}^2}} - 1 \\
&< \mathfrak{N}_\Omega(\iota(j),\, \varphi(j),\, r).
\end{aligned}$$

As a results, all the requirements are fulfilled. Thus, there is only one fixed point for operator σ. This suggests that there is a single solution to the fuzzy integral equation (28). □

Corollary 1 Let $(U,\, \mathfrak{B}_\Omega,\, \mathfrak{D}_\Omega,\, \mathfrak{N}_\Omega,\, \hat{\odot},\, \hat{\oplus})$ be a complete ENbMS. Define $\sigma\colon U \to U$ as

$$\sigma\iota(w) = y(j) + \gamma \int_x^y F(w,\, j)\iota(j)dj,\ \forall w,\, j \in [x,\, y].$$

Assume the following circumstances are satisfied.:

I. $\max\{F(w,\, j)\iota(j),\, F(w,\, j)\varphi(j)\} < C\max\{\iota(j),\, \varphi(j)\}$ for $\iota,\, \varphi \in U$, $\pi \in (0,\, 1)$, and for all $C \geq 1$, $w,\, j \in [x,\, y]$.

II. $\gamma \int_x^y \frac{dj}{\sqrt{2}} \leq \pi < 1$.

The integral equation (28) then has a solution. By using the proof mentioned above, we can readily demonstrate this.

Example 4 Let $\ell(w) = y(j) + \gamma \int_x^y F(w, j)\ell(j)dj$, $\ell(w)$ is the unknown function. $y(j)$ and the kernal $F(w, j)$ are known continuous function and γ is a parameter.

We can use a well known example that has an exact, unique solution for specific parameters.

Equation: $\ell(w) - \int_0^1 e^{wj}\ell(j)dj$.

Kernal: $K(w, j) = e^{wj}$.

Interval: $[a, b] = [0, 1]$.

Parameter: $\gamma = 1$.

Exact solution: $\ell(w) = e^w$.

Known function $(y(j))$: The function $y(j)$ must be chosen such that the exact solution.

$\ell w = e^w$ is satisfied the equation

$$y(j) = \ell(w) - \int_0^1 F(w, j)\ell(j)dj$$

$$= e^w - \int_0^1 e^{wj}e^j dj$$

$$= e^w - \int_0^1 e^{j(w+1)}dj$$

$$= e^w - \left[\frac{e^{j(w+1)}}{w+1}\right]_0^1$$

$$= e^2 - \frac{e^{(w+1)} - e^0}{w+1}$$

$$= e^w - \frac{e^{w+1} - 1}{w+1}$$

$$= e^w - \frac{e^{w+1}}{w+1} + \frac{1}{w+1}.$$

5. Conclusion

The purpose of this study is to introduce and examine several ENbMS principles, concentrating on an acceptable concept for the extended neutrosophic b-metric space under particular conditions. The researchers devised a suitable criterion for agreement that permits the examination of Cauchy sequences in the setting of extended neutrosophic b-metric space. We produced a few samples to demonstrate our efforts. Our results allowed us to confirm that the Fredholm integral equation has a single, unique solution. This implies that additional kinds of these equations can now be solved with confidence under these circumstances. In future connecting the ENbMS theory with other generalized set theories, such as soft sets, rough sets, or complex neutrosophic sets, to handle more nuanced data and vagueness.

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Conflict of interest

The authors declare no competing financial interest.

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