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Research Article

Derivations and Commutativity in Factor Rings over Prime Ideals

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Abstract: In this article, we will assume that N and K are ideals in any ring G, where N is non-zero and K is prime, with K being a proper subset of N. Our main goal is to prove that if the characteristic of G/K is not 2, and G admits derivations f and h that satisfy specific functional identities, then the factor ring G/K is commutative. In some cases, we will show that the range of derivations f and h lies in K. In fact, the created identities connect a non-zero ideal N to a prime ideal K. Moreover, several important consequences and ramifications were derived. Furthermore, we will construct a counterexample to demonstrate that the conclusions of our theorems do not hold without the primeness assumption of the ideal K.

***Keywords*:** prime ideal, derivation, quotient ring

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1. Introduction

Throughout this paper, $Z(G)$ is the center of an associative ring G. For any $a, m \in G$, the symbols $[a, m] = am - ma$ and $(a \circ m) = am + ma$ represent the commutator and anti-commutator, respectively. If $aGm = 0$ leads to either $a = 0$ or $m = 0$ for any $a, m \in G$, then G is called a prime ring. If $aGa = 0$ implies that $a = 0$ for any $a \in G$, then G is called a semiprime ring. A ring G is called 2-torsion free if for any $a \in G$, $2a = 0$ implies $a = 0$. A proper ideal K in G is called prime if, for any two ideals N and M in G, the inclusion of $NM \subseteq K$ implies that either $N \subseteq K$ or $M \subseteq K$. In other words, K is a prime ideal in G if whenever $a, m \in G$ and $aGm \subseteq K$, then either $a \in K$ or $m \in K$. According to the previous definition, another criterion can be derived to characterize a prime ring. Namely, a prime ideal K in a ring G must be equal to zero. As is known, the relationship between a prime ring and an integral domain is as follows: every integral domain is a prime ring. The opposite of this statement only holds when the ring is commutative with an identity. In a commutative ring G, if an ideal K is prime, then the quotient ring G/K is an integral domain, and vice versa.

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A mapping $\gamma : G \longrightarrow G$ is considered additive if it preserves the addition operation, meaning γ satisfies $\gamma(a+m) = \gamma(a)+\gamma(m)$ for a, m in G. The additivity property of mappings plays a crucial role in characterizing many properties of rings, such as commutativity. One of the most significant of these mappings is the derivation, which is defined as an additive mapping $f : G \longrightarrow G$ that satisfies $f(am) = f(a)m + af(m)$ for any a, $m \in G$, usually known as an ordinary derivation on G. A familiar example of a derivation is an inner derivation $f_t : G \longrightarrow G$ determined by a fixed element t, defined as $f_t(a) = [t, a]$ for all $a \in G$. It is easy to verify that the function $f : G \longrightarrow G$ defined by $f(a) = 0$ for $a \in G$ is a derivation, known as a trivial derivation of G. It is obvious that every derivation is an additive function, but the opposite is not true in general. For instance, the identity function $id_G : G \longrightarrow G$ defined by $id_G(a) = a$ for $a \in G$ is an additive function but not a derivation of G. For any non-empty subsets S and T of G, the mapping $\gamma : S \longrightarrow G$ is said to be T-commuting on S if $[\gamma(a), a] \in T$ for all a in S. In particular, if $T = Z(G)$ (resp. $T = \{0\}$), then γ is said to be centralizing (resp. commuting) on the set S.

One of the important initial results that discuss the commutativity of a prime ring that admits a derivation f is Posner's second theorem [1], which states that if a prime ring G admits a derivation f satisfying $[f(a), a] \in Z(G)$ for any $a \in G$, then either G is commutative or $f = 0$. In his work [2], Herstein showed that a prime ring G with $char(G) \neq 2$ is commutative if there is a non-trivial derivation f such that $[f(a), f(m)] = 0$ for all a, m in G. Moreover, he showed that if the characteristic of G is two, then G must be commutative or an order in a simple algebra that is 4-dimensional over its center. In [3], Bell et al. investigated commutativity in rings that admit a derivation, which is a strong commutativity preserving map on a nonzero right ideal N. They obtained that if a semiprime ring G admits a derivation f satisfying $[f(a), f(m)] = [a, m]$ for all $a, m \in N$, then $N \subseteq Z(G)$. In particular, G is commutative if $N = G$. In [4] Ashraf et al. proved that a prime ring G is commutative if and only if it satisfies any one of the following identities: $f(am) \pm am \in Z(G)$, $f(am) \pm ma \in Z(G)$, and $f(a)f(m) \pm am \in Z(G)$ for all $a, m \in N$, where N is a non-zero ideal of G.

In his article [5], Atiya assumed that N is a non-zero ideal in a 2-torsion free semiprime ring G. He proved that G contains a non-zero central ideal if G admits a derivation f satisfying either condition $[f^2(a), f^2(m)] - [a, m] \in Z(G)$, $f([f(a), f(m)]) \pm [a, m] \in Z(G)$ for all $a, m \in N$. Building on previous discussions, diverse researchers have explored similar situations by considering a more general additive mapping $F : G \longrightarrow G$, known as a generalized derivation associated with a derivation f. This mapping is defined such that $F(am) = F(a)m + af(m)$ for $a, m \in G$. It is evident that when $F = f$ in the previous definition, we obtain the concept of a derivation. Therefore, every derivation is a generalized derivation, but the opposite is not always true. In a study by Quadri et al. [6], the behavior of a prime ring G that admits a generalized derivation (F, f) satisfying $F[a, m] - [a, m] = 0$ for all $a, m \in N$ was discussed, where N is a nonzero ideal of G.

In fact, previous outcomes have encouraged the inspection of commutativity in prime and semiprime rings using derivations, generalized derivations, or other appropriate types of additive mappings such as generalized reverse derivations, generalized (α, β)-derivations, generalized multiplicative derivations, or automorphisms that satisfy various algebraic identities operating on G or any suitable subsets thereof. For more details on these investigations, readers can refer to: [7, 8].

The preceding findings have led many researchers to question what occurs if K is not the zero prime ideal in G. This inquiry has been discussed by numerous scholars. For instance, please refer to [9–11].

In [12], Almahdi et al. provided an enhanced version of Posner's second theorem. They proved that if K is a prime ideal of an arbitrary ring G, then either G/K is commutative or $f(G) \subseteq K$ when G admits a derivation f such that $[f(a), a] \in K$ for every $a \in G$.

In the context of two derivations f and h, Mir et al. [10] examined the behavior of a factor ring G/K when f and h satisfy any of the following properties: $[f(a), h(m)] - [a, m] \in K$, $f(a)h(m) - [a, m] \in K$, $f(a) \circ h(m) - a \circ m \in K$, $f(a) \circ h(m) - [a, m] \in K$ for all $a, m \in G$. In [13], Oukhtite et al. assumed two ideals $N \neq 0$ and K in a ring G with $char(G/K) \neq 2$, where K is a prime and $K \subsetneq N$. They examined the relationship between the ring G/K and two derivations f and h that satisfy one of the following properties: $\overline{[f(a), m]} + \overline{[a, h(m)]} - \overline{[a, m]} \in Z(G/K)$, $\overline{(f(a) \circ m)} + \overline{(a \circ h(m))} - \overline{(a \circ m)} \in Z(G/K)$ for all $a, m \in N$.

This paper focuses on investigating the structure of a ring G/K under the action of two derivations, f and h, that satisfy certain algebraic equations. These derivations map a nonzero ideal N to a prime ideal K. More precisely, we will assume that N is a non-zero ideal in an arbitrary ring G and K is a prime ideal in G, provided that K is a proper subset of N. After that, we will prove that if the characteristic of G/K is different from 2, then G/K is commutative. In some cases, the derivations f and h map a ring G into K. Furthermore, various related results have been obtained.

2. Preliminaries

This section will focus on gathering several lemmas that are necessary to proceed with the development of the proofs of our theorems. We will start with the following important fact, which we will use frequently:

Fact 1 Suppose N is a nonzero ideal in any ring G, and K is a prime ideal of G such that $K \subsetneq N$. If $aNm \subseteq K$ for $a, m \in G$, then either $a \in K$ or $m \in K$. In particular, $aN \subseteq K$ or $Na \subseteq K$ implies $a \in K$.

Lemma 1 Let N be a non-zero right (resp., left) ideal in any ring G, and let K be a prime ideal of G such that $K \subsetneq N$. If $f : G \longrightarrow G$ is a derivation on G such that $f(N) \subseteq K$, then $f(G) \subseteq K$.

The proof of the following lemma is obtained directly by reducing $\alpha = \beta = id_G$ in [9, Corollar 1], so we will skip it.

Lemma 2 If K and N are ideals in an arbitrary ring G, where K is prime and $K \subsetneq N$, such that $[a, f(a)] \in K$ for any element $a \in N$, then G/K is commutative or f sends G to the prime ideal.

In 1957, Posner [1, Theorem 1] investigated the connection between a prime ring and the product of two derivations. He proved that if the product of two derivations on a prime ring, with a characteristic not equal to two, is a derivation, then one of the derivations must be zero. Following that, without assuming that the product of two derivations is a derivation, Creedon [14, Theorem 2] proved that if K is a prime ideal of any ring G with $char(G/K) \neq 2$ and if f and h are derivations from G to itself such that $f(h(G)) \subseteq K$, then either $f(G) \subseteq K$ or $h(G) \subseteq K$. Instead of connecting the derivations f and h of the ring G to a prime ideal K in the previous result, we can prove the following lemma when the mentioned derivations relate a nonzero ideal N to a prime ideal K.

Lemma 3 Let K be a prime ideal of a ring G with $char(G/K) \neq 2$, and let N be a non-zero ideal of G with K a proper subset of N. If f and h are derivations from G to itself such that $f(h(N)) \subseteq K$, then either $f(N) \subseteq K$ or $h(N) \subseteq K$. Additionally, either $f(G) \subseteq K$ or $h(G) \subseteq K$.

Proof. By employing similar reasoning as in the proof of Theorem [14, Theorem 2], with necessary adjustments, we can conclude that either $f(N) \subseteq K$ or $h(N) \subseteq K$. Therefore, as demonstrated in Lemma 1, we determine that either $f(G) \subseteq K$ or $h(G) \subseteq K$. □

After considering the previous lemma, it is natural to inquire whether the converse implication still holds. The following lemma settles the debate without any restrictions on the characteristic of the ring:

Lemma 4 Let K be a prime ideal of a ring G, and let N be a non-zero ideal of G such that $K \subsetneq N$. If f and h are derivations from G to itself such that $f(N) \subseteq K$ or $h(N) \subseteq K$, then $f(h(N)) \subseteq K$ and $h(f(N)) \subseteq K$.

Proof. Suppose N is a non-zero ideal of G and h is a derivation on G such that $h(N) \subseteq K$. For all $a, m \in N$, we have $h(am) = h(a)m + ah(m) \in K$. Taking the derivation f in the previous expression, we get $f(h(a))m + h(a)f(m) + f(a)h(m) + af(h(m)) \in K$ for every $a, m \in N$. Using our initial assumption that $h(N) \subseteq K$, we obtain $f(h(a))m + af(h(m)) \in K$. Replacing a by ta in the last expression and using it, we get $f(h(t))am \in K$ for all $a, m, t \in N$. That is, $f(h(t))Nm \subseteq K$ for all $m, t \in N$. Applying Fact 1 along with the fact that $K \subsetneq N$, we conclude that $f(h(N)) \subseteq K$.

Now, by replacing f with h and h with f, we immediately deduce the conclusion $h(f(N)) \subseteq K$, as required. □

Remark 1 It is easy to see that, assuming $char(G/K) \neq 2$, the conclusion of Lemma 4 implies that $f(h(G)) \subseteq K$ and $h(f(G)) \subseteq K$ by using Lemma 3.

Lemma 5 Let G be a ring, N a nonzero ideal, and K a prime ideal of G such that $K \subsetneq N$. If G/K is 2-torsion free and $f : G \longrightarrow G$ is a derivation such that for all $a \in N$, $[f^2(a), a] \in K$, then either G/K is commutative or $f(G) \subseteq K$.

Proof. Since every prime ideal is also a semiprime, it is straightforward to verify that the arguments and tactics used in the proof of [15, Lemma 2.4] remain valid, with some necessary modifications, when considering an identity involved in a non-zero ideal rather than the whole ring. □

The upcoming lemma pertains to semi-prime ideals. We will be using it freely in the next section when discussing prime ideals, as every prime ideal is a semi-prime ideal.

Lemma 6 [11, Theorem 3.3] Let G be a ring with a nonzero ideal N and a semiprime ideal K such that K is a proper subset of N. If G/K is 2-torsion free and f is a derivation on G such that $[f(a), a]f(a) \in K$ for all $a \in N$, then $[f(a), a] \in K$ for all $a \in N$.

3. Main results

Theorem 1 Let $N \neq 0$, K be ideals in a ring G, where K is prime, the characteristic of G/K is not equal to two, and K is a proper subset of N. Then the derivations f and h satisfy the identity $[f^2(a), m] \pm [h(m), a] \in K$ for all a, m in N if and only if either:

(i) $f(G) \subseteq K$ and $h(G) \subseteq K$; or

(ii) G/K is commutative.

Proof. Clearly, if either (i) or (ii) is valid, then $[f^2(a), m] \pm [h(m), a] \in K$ for all $a, m \in N$. Conversely, we assume that

$$[f^2(a), m] \pm [h(m), a] \in K \qquad \text{for all } a, m \in N. \tag{1}$$

By selecting an arbitrary element $t \in N$ and replacing a by at in Equation (1), we get

$$f^2(a)[t, m] + [f^2(a), m]t + [2f(a)f(t), m] + a[f^2(t), m] + [a, m]f^2(t) \pm a[h(m), t] \pm [h(m), a]t \in K.$$

Multiplying the right of Equation (1) by t and comparing it with the former expression, we get

$$f^2(a)[t, m] + [2f(a)f(t), m] + a[f^2(t), m] + [a, m]f^2(t) \pm a[h(m), t] \in K \quad \text{for all } a, m, t \in N. \tag{2}$$

Once again, multiplying the left of Equation (1) by a and compairing it with (2), we get

$$f^2(a)[t, m] + [2f(a)f(t), m] + [a, m]f^2(t) \in K \quad \text{for all } a, m, t \in N. \tag{3}$$

Setting $m = mt$ in Equation (3) and using it, we find

$$m[2f(a)f(t), t] + m[a, t]f^2(t) + [a, m][t, f^2(t)] \in K \quad \text{for all } a, m, t \in N. \tag{4}$$

Substituting m with sm in (4) and applying it, we get

$$[a, s]m[t, f^2(t)] \in K \quad \text{for all } a, m, s, t \in N.$$

That is, $[a, s]N[t, f^2(t)] \subseteq K$ for all $a, s, t \in N$. Using Fact 1, we obtain either $[a, s] \in K$ or $[t, f^2(t)] \in K$ for all $a, s, t \in N$.

In the first case, G/K is commutative. In the second case, we have $[t, f^2(t)] \in K$ for all $t \in N$. Applying Lemma 5, we find that G/K is commutative or $f(G) \subseteq K$. To complete the proof, we temporarily assume that G/K is not commutative. Therefore, $f(G) \subseteq K$. This simplifies Equation (3) to

$$f^2(a)[t, m] + [a, m]f^2(t) \in K \quad \text{for all } a, m, t \in N. \tag{5}$$

By suggesting $t = m$ in (5) and using it, we get $[a, m]f^2(m) \in K$ for all $a, m \in N$. Substituting a with ta in the last expression and using it, we deduce $[t, m]Nf^2(m) \subseteq K$ for all $m, t \in N$. By using Fact 1, either $[t, m] \in K$ and G/K is commutative, which is a contradiction, or $f^2(m) \in K$ for all $m \in N$. If $f^2(m) \in K$ for all $m \in N$, then by using Lemma 3, we get $f(G) \subseteq K$. Therefore, Equation (1) becomes $[h(m), a] \in K$ for all a, m in N. Consequently, utilizing Lemma 2, we can infer that $h(G) \subseteq K$. □

If $K = \{0\}$ in the previous theorem, then G is prime and thus we immediately derive the following corollary:

Corollary 1 Let N be a nonzero ideal of a ring G with a characteristic not equal to two. If ($f \neq 0$ or $h \neq 0$) such that $[f^2(a), m] \pm [h(m), a] = 0$ for all $a, m \in N$, then G is commutative.

The following corollary is derived directly from Theorem 1 by setting $h = f$:

Corollary 2 Let $N \neq 0$, K be ideals of a ring G, where K is prime, the characteristic of G/K is not equal to two, and K is a proper subset of N. Then $[f^2(a), m] \pm [f(m), a] \in K$ for all a, m in N if and only if either G/K is a commutative or $f(G) \subseteq K$.

In Theorem 1, by suggesting h equal the zero derivation 0_G, a version of [15, Lemma 2.4] can be obtained for any two elements a and m in G as follows:

Corollary 3 Let $N \neq 0$, K be ideals of a ring G, where K is prime, the characteristic of G/K is not equal to two, and K is a proper subset of N. If the derivation f satisfies the identity $[f^2(a), m] \in K$ for all $a, m \in N$, then G/K is commutative or $f(G) \subseteq K$.

Theorem 2 Let $N \neq 0$, K be ideals of a ring G, where K is prime, the characteristic of G/K is not equal to two, and K is a proper subset of N. If the derivations f and h satisfy the identity $[f^2(a), m] \pm (h(m) \circ a) \in K$ for all $a, m \in N$, then one of the following conclusions is valid:

(i) $f(G) \subseteq K$ and $h(G) \subseteq K$;

(ii) G/K is commutative and $h(G) \subseteq K$.

Proof. There is no difficulty in proving this by following arguments and tactics analogous to those utilized in the proof of Theorem 1, with some necessary slight adjustments. □

Theorem 3 Let $N \neq 0$, K be ideals of a ring G, where K is prime, the characteristic of G/K is not equal to two, and K is a proper subset of N. Then the derivations f and h satisfy the identity $(f^2(a) \circ m) \pm [h(m), a] \in K$ for all $a, m \in N$ if and only if either:

(i) $f(G) \subseteq K$ and $h(G) \subseteq K$; or

(ii) G/K is commutative and $f(G) \subseteq K$.

Proof. It is easy to check if (i) or (ii) is valid then $(f^2(a) \circ m) \pm [h(m), a] \in K$ for all $a, m \in N$. Conversely, we will assume that

$$(f^2(a) \circ m) \pm [h(m), a] \in K \qquad \text{for all } a, m \in N. \tag{6}$$

In light of the proof of Theorem 1, following similar arguments and tactics, we can obtain either $f(G) \subseteq K$ and $h(G) \subseteq K$ or G/K is commutative. Now suppose that G/K is commutative, then Equation (6) becomes $2mf^2(a) \in K$ for all $a, m \in N$. Replacing a by at in the last expression and using it, we obtain $4mf(a)f(t) \in K$ for all $a, m, t \in N$. As K is prime and $char(G/K) \neq 2$, we deduce $f(a)Nf(t) \subseteq K$ for all $a, t \in N$. Using Fact 1, we conclude that $f(N) \subseteq K$ and so by using Lemma 3, we conclude that $f(G) \subseteq K$. □

Remark 2 If we consider that $h = f$ in Theorem 3, the only conclusion we can draw that satisfies the identity $(f^2(a) \circ m) \pm [f(m), a] \in K$ for all a, m in N is $f(G) \subseteq K$.

By suggesting $h = 0$ in Theorem 3, we can derive an analogous version of [15, Lemma 2.4] for any two elements $a, m \in G$ in the anticommutator case as follows:

Corollary 4 Let $N \neq 0$, K be ideals of a ring G, where K is prime, the characteristic of G/K is not equal to two, and K is a proper subset of N. If the derivation f satisfies the identity $(f^2(a) \circ m) \in K$ for all $a, m \in N$, then $f(G) \subseteq K$.

Theorem 4 Let $N \neq 0$, K be ideals of a ring G, where K is prime, the characteristic of G/K is not equal to two, and K is a proper subset of N. Then the derivations f and h satisfy the identity $f^2([a, m]) \pm h(m \circ a) \in K$ for all $a, m \in N$ if and only if one of the following conclusions is valid: G/K is commutative and $h(G) \subseteq K$ or $f(G) \subseteq K$ and $h(G) \subseteq K$.

Proof. Obviously, if (i) or (ii) is valid, then $f^2([a, m]) \pm h(m \circ a) \in K$ for all $a, m \in N$. Conversely, we assume that for every $a, m \in N$

$$f^2([a, m]) \pm h(m \circ a) \in K. \tag{7}$$

Replacing m by ma in (7) and applying it, we find

$$2f([a, m])f(a) + [a, m]f^2(a) \pm (m \circ a)h(a) \in K. \tag{8}$$

By selecting an arbitrary element $t \in N$ and replacing m by tm in the last expression and using it, we get

$$2f(t)[a, m]f(a) + 2f([a, t])mf(a) + 2[a, t]f(m)f(a) + [a, t]mf^2(a) \mp [t, a]mh(a) \in K. \tag{9}$$

By suggesting $t = a$ in the last relation, we have $2f(a)[a, m]f(a) \in K$ for all a, m in N. Applying the assumption that $char(G/K) \neq 2$, we deduce that $f(a)[a, m]f(a) \in K$ for all $a \in N$. By substituting m with $mf(a)$ in the previous relation and applying it, we get $f(a)m[a, f(a)]f(a) \in K$ for all $a, m \in N$. In particular, $[a, f(a)]f(a)N[a, f(a)]f(a) \subseteq K$ for all $a \in N$. The primeness of K implies that $[a, f(a)]f(a) \in K$ for all $a \in N$. By using Lemma 6, we obtain $[a, f(a)] \in K$ for all $a \in N$. Hence, Lemma 2 implies that either G/K is commutative or $f(G) \subseteq K$.

If $f(G) \subseteq K$, then Equation (9) becomes $[t, m]Nh(a) \subseteq K$ for all $a, t \in N$. Using Fact 1, we get $[t, a] \in K$ or $h(a) \in K$ for all $a, t \in N$. The first case gives G/K is commutative, the other case, we can deduce that $h(G) \subseteq K$ as shown in Lemma 1.

On the other hand, the commutativity of G/K reduces Equation (8) to $2amh(a) \in K$ for all $a, m \in N$. Given that $char(G/K) \neq 2$, this, combined with the primeness of K implies that $h(N) \subseteq K$. Thus, using Lemma 3, we obtain $h(G) \subseteq K$. □

The following result can be derived directly by setting $h = f$ in Theorem 4.

Corollary 5 Let $N \neq 0$, K be ideals of a ring G, where K is prime, the characteristic of G/K is not equal to two, and K is a proper subset of N. If the derivation f satisfies the identity $f^2([a, m]) \pm f(m \circ a) \in K$ for all a, m in N, then $f(G) \subseteq K$.

Theorem 5 Let $N \neq 0$, K be ideals of a ring G, where K is prime, the characteristic of G/K is not equal to two, and K is a proper subset of N. Then the derivations f and h satisfy any one of the following conditions for all a, m in N:

(i) $f^2(a \circ m) \pm [h(m), a] \in K$,

(ii) $f^2(a \circ m) \pm h([m, a]) \in K$.

If and only if G/K is commutative and $f(G) \subseteq K$ or $f(G) \subseteq K$ and $h(G) \subseteq K$.

Proof. Obviously, if (i) or (ii) is valid, then $f^2(a \circ m) \pm h([m, a]) \in K$ for all $a, m \in N$. Conversely, we assume that for every $a, m \in N$

$$f^2(a \circ m) \pm h([m, a]) \in K. \tag{10}$$

Replacing m with ma in (10) and applying it, we obtain

$$2f(a \circ m)f(a) + (a \circ m)f^2(a) \pm m[h(a), a] \pm [m, a]h(a) \in K \quad \text{for all } a, m \in N. \tag{11}$$

By choosing an arbitrary element $t \in N$ and substituting m with tm in Equation (11) and using it, we get

$$2f(t)(a \circ m)f(a) + 2f[a, t]mf(a) + 2[a, t]f(m)f(a) + [a, t]mf^2(a) \pm [t, a]mh(a) \in K. \tag{12}$$

By suggesting $a = t$ in Equation (12), we get $2f(a)(a \circ m)f(a) \in K$ for all a, m in N. Applying the assumption that $char(G/K) \neq 2$, we find $f(a)(a \circ m)f(a) \in K$ for every $a \in N$. By substituting m with $f(a)m$ in the previous relation and applying it, we get $f(a)[a, f(a)]mf(a) \in K$ for all $a, m \in N$. In particular, $f(a)[a, f(a)]Nf(a)[a, f(a)] \subseteq P$ for every $a \in N$. The primeness of K forces that $f(a)[a, f(a)] \in K$ for any $a \in N$ and thus by using Lemma 6, we obtain either G/K is commutative or $f(G) \subseteq K$.

If $f(G) \subseteq K$, then this case reduces Equation (12) to $[t, a]Nh(a) \subseteq K$ for all a, t in N. Using Fact 1, we get $[t, a] \in K$ or $h(a) \in K$ for all $a, t \in N$. Therefore, in the first case, G/K is commutative. In the second case, $h(G) \subseteq K$ as shown in Lemma 1.

On the other hand, the commutativity of a factor ring G/K reduces Equation (12) to $4f(t)amf(a) \in K$ for all $a, m, t \in N$. Since $char(G/K) \neq 2$, $f(t)aNf(a) \subseteq K$ for all $a, t \in N$. The primeness of K along with Fact 1 implies that $f(G) \subseteq K$. □

By putting $h = f$ in Theorem 5, we can prove the following result:

Corollary 6 Let $N \neq 0$, K be ideals of a ring G, where K is prime, the characteristic of G/K is not equal to two, and K is a proper subset of N. Then the derivation f satisfies any one of the following conditions for any a, m in N:

(i) $f^2(a \circ m) \pm [f(m), a] \in K$;

(ii) $f^2(a \circ m) \pm f([m, a]) \in K$ if and only if $f(G) \subseteq K$.

The following proposition is obtained immediately by substituting h with $h \pm id_G$ in Theorem 5. Therefore, we will skip the proof.

Proposition 1 Let $N \neq 0$, K be ideals of a ring G, where K is prime, the characteristic of G/K is not equal to two, and K is a proper subset of N. Then the derivations f and h satisfy any one of the following conditions for any a, m in N:

(i) $f^2(a \circ m) \pm [h(m), a] \pm [m, a] \in K$,

(ii) $f^2(a \circ m) \pm h([m, a]) \pm [m, a] \in K$ if and only if G/K is commutative and $f(G) \subseteq K$.

By employing arguments and tactics similar to the proof of Theorem 5, with some slight modifications, we can readily establish the following theorem:

Theorem 6 Let $N \neq 0$, K be ideals of a ring G, where K is prime, the characteristic of G/K is not equal to two, and K is a proper subset of N. Then the derivations f and h satisfy any one of the following conditions for any a, m in N:

(i) $f^2(a \circ m) \pm (h(m) \circ a) \in K$;

(ii) $f^2(a \circ m) \pm h(m \circ a) \in K$ if and only if $f(G) \subseteq K$ and $h(G) \subseteq K$.

Theorem 7 Let $N \neq 0$, K be ideals of a ring G, where K is prime, the characteristic of G/K is not equal to two, and K is a proper subset of N. Then the derivations f and h satisfy the identity $[f(a), a] \pm ah^2(a) \in K$ for all $a \in N$ if and only if either $h(G) \subseteq K$ and $f(G) \subseteq K$ or G/K is commutative and $h(G) \subseteq K$.

Proof. Verifying the sufficiency of the condition is not difficult. To prove the necessity of the condition, we assume

$$[f(a),\, a] \pm ah^2(a) \in K \quad \text{for all } \; a \in N. \tag{13}$$

Replacing a by $a+m$ in the last relation and applying it, we find

$$[f(a),\, m] + [f(m),\, a] + ah^2(m) + mh^2(a) \in K \quad \text{for all } \; a,\, m \in N. \tag{14}$$

Substituting a by am in Equation (14), we get

$$[f(a),\, m]m + a[f(m),\, m] + [a,\, m]f(m) + a[f(m),\, m] + [f(m),\, a]m$$

$$+ amh^2(m) + mh^2(a)y + 2mh(a)h(m) + mah^2(m) \in K.$$

By multiplying Equation (14) by m from the right hand and subtracting it from the previous relation, we find

$$a[f(m),\, m] + [a,\, m]f(m) + a[f(m),\, m] + amh^2(m) - ah^2(m)m + 2mh(a)h(m) + mah^2(m) \in K. \tag{15}$$

Multiplying Equation (13) by a from the left hand and subtracting it from (15), we get

$$a[f(m),\, m] + [a,\, m]f(m) - ah^2(m)m + 2mh(a)h(m) + mah^2(m) \in K \quad \text{for all } \; a,\, m \in N. \tag{16}$$

By selecting an arbitrary element $t \in N$ and substituting a with ta in Equation (16) and applying it, we obtain

$$[t,\, m]af(m) + 2mh(t)ah(m) - 2tmh(a)h(m) + 2mth(a)h(m) + mtah^2(m) - tmah^2(m) \in K. \tag{17}$$

Taking $t = m$ in (17) and using the assumption that $char(G/K) \neq 2$, we obtain $mh(m)ah(m) \in K$ for all $a,\, m \in N$. In particular, $mh(m)Nh(m) \subseteq K$ for any element m in N. Using Fact 1, we get $mh(m) \in K$ for all $m \in N$. Linearizing the last relation, we get $ah(m) + mh(a) \in K$ for every $a,\, m$ in N. Substituting a with at in the previous relation and utilizing it, we obtain $ath(m) - ah(m)t + mah(t) \in K$ for every $a,\, m,\, t$ in N. Replacing a by na in the last relation and using it, we can deduce that $[m,\, n]Nh(t) \subseteq K$ for every $m,\, n,\, t \in N$. Applying Fact 1, we have either $[m,\, n] \in K$ or $h(t) \in K$ for any $m,\, n,\, t \in N$. The first case yields G/K is commutative. Hence, the commutativity (mod K) simplifies Equation (17) to $2mh(t)ah(m) \in K$ for all $a,\, m,\, t \in N$. Given that $char(G/K) \neq 2$, this implies that $mh(t)ah(m) \in K$ for all $a,\, m,\, t \in N$. Hence, $mh(t)Nh(m) \subseteq K$ for all $m,\, t \in N$.

Therefore, by applying Fact 1 together with the hypotheses that $K \subsetneq N$, we conclude that $h(G) \subseteq K$.

In the second case, we have $h(t) \in K$ for all $t \in N$. This simplifies Equation (17) to $[t,\, m]af(m) + mtah^2(m) - tmah^2(m) \in K$ for all $a,\, m,\, t \in N$. Replacing a by am in the last expression, we get $[t,\, m]amf(m) + mtamh^2(m) - tmamh^2(m) \in K$ for all $a,\, m,\, t \in N$. Multiplying Equation (13) by mta from the left hand and comparing it with the previous expression, we deduce $[t,\, m]amf(m) - mta[f(m),\, m] - tmamh^2(m) \in K$ for all $a, m, t \in N$. Again, by multiplying Equation (13) by tma from the left and adding it to the last relation, we obtain $[t,\, m]amf(m) + [t,\, m]a[f(m),\, m] \in K$ for all $a, m, t \in N$. This can be rewritten as $[t, m]Nmf(m) \subseteq K$ for all $m, t \in N$. By using Fact 1, we find that either $[t, m] \in K$

or $mf(m) \in K$ for each m, t in N. The first case implies that G/K is commutative. In the other case, following the same discussion as above, we can deduce that either G/K is commutative or $f(m) \in K$ for every m in N. If $f(m) \in K$ for every m in N, then Equation (17) becomes $[m, t]Nh^2(m) \subseteq K$ for every $m, t \in N$. As shown in Fact 1, we again get either G/K is commutative or $h^2(m) \subseteq K$ for every $m \in N$. In the alternative case, by setting $f = h$ in Lemma 3 and utilizing it, we can deduce that $h(G) \subseteq K$. □

If we set $h = 0$, it is clear that $h^2 = 0$ and this reduces Theorem 7 to a developed version of the second Posner's theorem as follows:

Corollary 7 Let $N \neq 0$, K be ideals of a ring G, where K is prime, the characteristic of G/K is not equal to two, and K is a proper subset of N. If the derivation f satisfies the identity $[f(a, a] \in K$ for all $a \in N$, then G/K is commutative or $f(G) \subseteq K$.

By substituting h with f in Theorem 7, the following corollary shows that there exists no non-trivial K-derivation f that satisfies the identity $[f(a), a] \pm af^2(a) \in K$ for all $a \in N$.

Corollary 8 Let $N \neq 0$, K be ideals of a ring G, where K is prime, the characteristic of G/K is not equal to two, and K is a proper subset of N. If the derivation f satisfies the identity $[f(a), a] \pm af^2(a) \in K$ for all $a \in N$, then $f(G) \subseteq K$.

In light of Theorem 7, it is common to inquire about the expected outcome when the commutator is replaced with the anticommutator. The following theorem offers a solution to this inquiry. To prove it, we will utilize arguments and techniques analogous to those employed in proving the previous theorem, with some necessary adjustments. Therefore, we will not present the proof here.

Theorem 8 Let $N \neq 0$, K be ideals of a ring G, where K is prime, the characteristic of G/K is not equal to two, and K is a proper subset of N. If the derivations f and h satisfy the identity $(f(a) \circ a) \pm ah^2(a) \in K$ for all $a \in N$, then $h(G) \subseteq K$ and $f(G) \subseteq K$.

If we set $h = 0$, in Theorem 8 we obtain an improved version of the second Posner's theorem in the case of the anticommutator, as below:

Corollary 9 Let $N \neq 0$, K be ideals of a ring G, where K is prime, the characteristic of G/K is not equal to two, and K is a proper subset of N. If the derivations f satisfy the identity $(f(a) \circ a) \in K$ for all $a \in N$, then $f(G) \subseteq K$.

The following example demonstrates that neglecting the primeness hypothesis of the ideal K will render the conclusions of our theorems invalid.

Example 1 Let $G = \{aE_{12} + mE_{22} \mid a, m \in \mathbb{Z}\}$ be a ring, and let $K = \{0\}$ and $N = \{3mE_{12}\}$. Define the functions $f, h : G \longrightarrow G$ as follows:

$$f(aE_{12} + mE_{22}) = 2aE_{12} \qquad \text{and} \qquad h(aE_{12} + mE_{22}) = 4aE_{12}.$$

It is easy to verify that G is a non-commutative ring of characteristic different from 2, N is a non-zero ideal in G, and $K \subsetneq N$. Furthermore, K is not prime in G, because $(mE_{12})^2 = 0 \in K$, but $mE_{12} \notin K$. Additionally, we can check that f and h are derivations on G. Moreover, we can observe that if $A \in N$, then by the definitions of f and h, we find that $f(A) \in N$ and $f^2(A) = h(A) = \begin{pmatrix} 0 & 12a \\ 0 & 0 \end{pmatrix} \in N$. On the other hand, by the definition of the ideal N, it can be seen that $AB = 0$ for all $A, B \in N$. Therefore, all identities in Theorems 1-8 are satisfied because every term in these identities is a product of two elements of N which equals 0, and this value lies in K. However, upon examining the conclusions of these theorems, we find that $G/K \cong G$ is non-commutative, $f(G) \nsubseteq K$, and $h(G) \nsubseteq K$. This asserts that the primeness hypothesis of K in Theorems 1-8 cannot be excluded.

Remark 3 Considering G, N, K, f, and h as in the example above, it is evident that the same arguments still hold true to show that the primeness constraint of K in Proposition 1 and Corollaries 1-9 cannot be removed.

4. Conclusion

Inspired by Posner's first and second theorems as well as their subsequent updates in current literature, our article aims to examine the commutativity of a factor ring G/K. This investigation is conducted when an arbitrary ring G admits two ordinary derivations, f and h, that satisfy specific equations defined on a non-zero two-sided ideal N. These derivations have a range in a prime ideal K such that $K \subsetneq N$. To emphasize the importance of the primeness assumption of ideal K in the hypotheses of our theorems, we have explored a counterexample in the ring of second-order square matrices.

5. Future studies

Our study establishes a wide range for future investigations into the relationship between appropriate additive mappings and the properties of a ring structure. For example, one could consider the following suggestions, or a combination of them, to explore potential conclusions:

(1) Instead of a prime ideal, consider the identities of our theorems included in a semi-prime ideal or the center of G.

(2) Explore more general derivations such as semiderivations, multiplicative generalized derivations, (α, β)-derivations, or generalized (α, β)-derivations, where α and β are automorphisms on G.

(3) Instead of an associative ring, consider a non-associative ring, or instead of an ideal N, use a Lie ideal.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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